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Vacuum entanglement enhancement by a weak gravitational field

M. Cliche[1] and A. Kempf[2]

[1]Department of Physics, Cornell University, Ithaca, New York, USA 14853

[2]Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

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Separate regions in space are generally entangled, even in the vacuum state. It is known that this entanglement can be swapped to separated Unruh-DeWitt detectors, i.e., that the vacuum can serve as a source of entanglement. Here, we demonstrate that, in the presence of curvature, the amount of entanglement that Unruh-DeWitt detectors can extract from the vacuum can be increased.

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I. INTRODUCTION

Two Unruh-DeWitt detectors that interact with a quantum field in the vacuum state have access to a renewable source of entanglement [1–5], namely, by swapping entanglement from the quantum field. In this context, it was recently shown [6] that, in an expanding space-time, the entanglement of the vacuum decreases significantly due to the effects of the Gibbons-Hawking temperature [7]. While this example showed that gravity is able to act as a decohering agent, we will here show that gravity can also act to enhance entanglement-related phenomena. Namely, we will show that a weak gravitational field, such as that caused by a planet, can enhance the extraction of entanglement from the vacuum.

This article is organized as follows. In Sec. II we review the extraction of entanglement with Unruh-DeWitt detectors in Minkowski space-time. In Sec. III we review the Newtonian limit of general relativity, and in Sec. IV we look at Unruh-DeWitt detectors in the presence of weak gravity. Then, in Sec. V we compute the first-order correction to the propagator on the perturbed background. In Sec. VI we calculate explicitly the entanglement between the two detectors near a spherically symmetric star. In Sec. VII, we propose extensions.

We work with the natural units $\hbar = c = G = 1$ and the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. We denote the coordinate time by $x^0 = t$ while the proper time is denoted by τ. Wherever necessary to avoid ambiguity, we will denote operators O or states $|\psi\rangle$ which live in the Hilbert space $\mathcal{H}^{(j)}$ of the j'th subsystem by a superscript (j); for example, $O^{(j)}$ and $|\psi^{(j)}\rangle$. Orders in perturbation theory will be denoted by a subscript (j), as in, e.g., $P = P_{(0)} + P_{(1)} + O(\epsilon^2)$. We work in the interaction picture.

II. VACUUM ENTANGLEMENT

Let us first briefly review vacuum entanglement with Unruh-DeWitt detectors [1,2,6]. To begin, let us denote the overall Hilbert space by $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \mathcal{H}^{(3)}$, where the first two Hilbert spaces belong to two Unruh-DeWitt detectors and where the third Hilbert space is that of a quantum massless scalar field. The total Hamiltonian H of the system with respect to the coordinate time t is

$$\begin{aligned} H &= H_F + H_D + H_{\mathrm{int}} \\ H_F &= \frac{1}{2}\int d^3x[\pi^2(x) + (\nabla\phi(x))^2] \\ H_D &= \sum_{j=1}^{2}[(E_g + \Delta E)|e^{(j)}\rangle\langle e^{(j)}| + E_g|g^{(j)}\rangle\langle g^{(j)}|]\frac{d\tau_j(t)}{dt} \\ H_{\mathrm{int}}(t) &= \sum_{j=1}^{2}\alpha_j\eta(\tau_j(t))(|e^{(j)}\rangle\langle g^{(j)}|e^{i\Delta E\tau_j(t)} \\ &\quad + |g^{(j)}\rangle\langle e^{(j)}|e^{-i\Delta E\tau_j(t)})\phi(x_j(\tau_j(t)))\frac{d\tau_j(t)}{dt}, \end{aligned} \tag{1}$$

where H_F is the Hamiltonian of a free massless scalar field, H_D is the Hamiltonian of the two detectors, $H_{\mathrm{int}}(t)$ is the interaction Hamiltonian [8] in the interaction picture, α_j is the coupling constant of the j'th detector ($j \in \{1, 2\}$), $\phi(x_j(\tau_j))$ is the field at the point of the j'th detector, and $m^{(j)}(\tau_j) := (|e^{(j)}\rangle\langle g^{(j)}|e^{i\Delta E\tau_j} + |g^{(j)}\rangle\langle e^{(j)}|e^{-i\Delta E\tau_j})$ is the monopole matrix of the j'th detector. The function $\eta(\tau_j)$ will be used to describe the continuous switching on and off of the detectors and τ_j is the proper time of the j'th detector.

Let us first consider the special case where $\tau_1(t) = \tau_2(t) = \tau(t)$ such that the evolution operator $U = Te^{-i\int dtH_{\mathrm{int}}(t)}$ acting on states takes the form

$$\begin{aligned} U = T\exp\Big\{-i\int d\tau[\alpha_1\eta(\tau)m^{(1)}(\tau)\phi(x_1(\tau)) \\ + \alpha_2\eta(\tau)m^{(2)}(\tau)\phi(x_2(\tau))]\Big\}. \end{aligned} \tag{2}$$

We assume that the initial state of the system is $|0g^{(1)}g^{(2)}\rangle$. After the unitary evolution of the total system, we trace out the field and obtain at $O(\alpha^2)$ [1,6]

$$\rho_f^{(1,2)} = \mathrm{Tr}_{(3)}(U|0g^{(1)}g^{(2)}\rangle\langle 0g^{(1)}g^{(2)}|U^\dagger) = \begin{pmatrix} 0 & 0 & 0 & X \\ 0 & P_1 & Y & 0 \\ 0 & Y^* & P_2 & 0 \\ 0 & 0 & 0 & 1-P_1-P_2 \end{pmatrix} + O(\alpha^4), \quad (3)$$

in the basis $|e^{(1)}e^{(2)}\rangle$, $|e^{(1)}g^{(2)}\rangle$, $|g^{(1)}e^{(2)}\rangle$, and $|g^{(1)}g^{(2)}\rangle$. The matrix elements P_j, X, and Y read

$$P_j = \alpha_j^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \eta(\tau)\eta(\tau') \times e^{-i\Delta E(\tau-\tau')} \times D(x_j(\tau), x_j(\tau')), \quad (4)$$

$$X = -\alpha_1\alpha_2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\tau} d\tau' \eta(\tau)\eta(\tau') \times e^{i\Delta E(\tau+\tau')} \times (D(x_1(\tau), x_2(\tau')) + D(x_2(\tau), x_1(\tau'))), \quad (5)$$

$$Y = \alpha_1\alpha_2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \eta(\tau)\eta(\tau') \times e^{-i\Delta E(\tau-\tau')} \times D(x_1(\tau), x_2(\tau')), \quad (6)$$

where $D(x, y) = \langle 0|\phi(x)\phi(y)|0\rangle$. To measure the entanglement of $\rho_f^{(1,2)}$, we use the negativity [9] which gives

$$N = \max(\sqrt{(P_1 - P_2)^2 + 4|X|^2} - P_1 - P_2, 0) + O(\alpha^4). \quad (7)$$

In order to obtain more explicit results, let us consider, for example, the case where the detectors are inertial and separated by a constant proper distance L_p in Minkowski space-time, in which case $x_j^0(\tau) = \tau$. Let us also assume that $\alpha_1 = \alpha_2 = \alpha$ such that in Minkowski space we have $P_1 = P_2 = P$ and $N = 2\max(|X| - P) + O(\alpha^4)$. For simplicity we choose the switching functions to be Gaussian $\eta(\tau) = e^{-\tau^2/(2\sigma^2)}$, where σ is the detector on-time. This also means that $1/\sigma$ is the energy uncertainty due to the detector switching and if $L_p/\sigma > 1$, the world lines of the two detectors are essentially spacelike while they are on.

In Eq. (7), following the terminology of [3], we will call X the exchange term and P_j the local noise term. This is because X can be interpreted as describing the exchange of virtual quanta between the two detectors, and P_j can be interpreted as describing the detection of virtual quanta by detector j. In order to allow the introduction of gravity (which will enter mostly through the propagator), it will be useful to view P_j and X as functions of the propagator. To this end, let us already ensure that the time ordering is respected. For X this is straightforward since the time integrations respect time ordering by construction. We can still simplify X by using the variable change $s = \tau - \tau'$ and $u = \tau + \tau'$ such that we have

$$X = -\alpha^2 e^{-\sigma^2\Delta E^2}\sigma\sqrt{\pi}\int_0^\infty ds e^{-s^2/(4\sigma^2)} \times (G(\vec{x}_1, \vec{x}_2, s) + G(\vec{x}_2, \vec{x}_1, s)), \quad (8)$$

where $G(x, y) = \langle 0|T\phi(x)\phi(y)|0\rangle = G(\vec{x}, \vec{y}, x^0 - y^0)$ is Feynman propagator [10]. For P_j, we introduce a convenient change of variables for the double integral over the (τ, τ') plane [11], making $u = \tau$, $s = \tau - \tau'$ in the lower half-plane $\tau' < \tau$ and $u = \tau'$, $s = \tau' - \tau$ in the upper half-plane $\tau < \tau'$. Then, P_j becomes

$$P_j = 2\alpha^2\sigma\sqrt{\pi}\Re\left(\int_0^\infty ds e^{-s^2/(4\sigma^2) - i\Delta E s} G(\vec{x}_j, \vec{x}_j, s)\right). \quad (9)$$

In Minkowski space-time we use the Boulware vacuum such that the propagator is given by

$$G(x, y) = \frac{-1}{4\pi^2[(x^0 - y^0)^2 - |\vec{x} - \vec{y}|^2 - i\epsilon]}, \quad (10)$$

where $\lim_{\epsilon\to 0^+}$ is implicit. Using Eq. (10) in Eqs. (9) and (8) we obtain the local noise and the exchange term in Minkowski space-time [6],

$$P = \frac{\alpha^2}{4\pi}(e^{-\Delta E^2\sigma^2} - \Delta E\sqrt{\pi}\sigma \mathrm{erfc}(\Delta E\sigma)), \quad (11)$$

$$X = \frac{\alpha^2\sigma i}{4L_p\sqrt{\pi}} e^{-\Delta E^2\sigma^2 - L_p^2/4\sigma^2} \mathrm{erfc}\left(\frac{-iL_p}{2\sigma}\right), \quad (12)$$

where $L_p = |\vec{x}_1 - \vec{x}_2|$ and $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$.

Below, we will study the behavior of these parameters and the negativity numerically, in particular, also in the case with gravity. Here, let us note that, in an interesting regime, these equations are tractable analytically. It is the regime where on one hand $\Delta E\sigma \gg 1$, i.e., where the energy gap is large compared to the energy uncertainty $1/\sigma$ that comes with a detector on-time of effective duration σ. It is also the regime where on the other hand also $L_p/\sigma \gg 1$, i.e., where the detectors are too far from another to causally communicate while on. In this regime, we have $P \approx \frac{\alpha^2 e^{-\Delta E^2\sigma^2}}{8\pi\Delta E^2\sigma^2}$ and $|X| \approx \frac{\alpha^2\sigma^2 e^{-\Delta E^2\sigma^2}}{2\pi L_p^2}$.

From these expressions we see that to get a nonvanishing negativity, $\sigma\Delta E$ must be at least as large as $\sigma\Delta E > \frac{L_p}{2\sigma}$. However, we also see that $\sigma\Delta E$ should not be chosen too large. This is because as $\sigma\Delta E \to \infty$, the negativity N vanishes—which had to be expected because as $\sigma\Delta E$ increases the Gaussian switching function becomes more and more adiabatic, and this implies that the final state returns to the ground state of the free theory that we started with, which is not entangled. To optimize the negativity, we set $\frac{\partial N}{\partial \Delta E_{\mathrm{opt}}} = 0$, which yields $\Delta E_{\mathrm{opt}} \approx \frac{L_p}{2\sigma^2}(1 + 2\sigma^2/L_p^2)$. The resulting negativity is then $N_{\mathrm{opt}} \approx \frac{4\alpha^2\sigma^4 e^{-L_p^2/4\sigma^2}}{\pi L_p^4}$.

III. NEWTONIAN LIMIT

Let us now briefly review the Newtonian limit of general relativity; see, e.g., [12]. In this limit we can write the metric as $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ where $|h_{\alpha\beta}| \ll 1$. Note that under a small change of coordinates $x^\mu \to x^\mu + \xi^\mu$ the term $h_{\alpha\beta}$ has a gauge transformation $h_{\alpha\beta} \to h_{\alpha\beta} - \partial_\beta \xi_\alpha - \partial_\alpha \xi_\beta$. Let us define the quantity $\bar{h}^{\mu\nu} := h^{\mu\nu} - \eta^{\mu\nu} h^\alpha_\alpha/2$. To simplify the Einstein equation, we choose to work in the Lorentz gauge in which $\bar{h}^{\mu\nu}{}_{,\nu} = 0$. In this gauge, the linearized Einstein equation reads $\partial_\alpha \partial^\alpha \bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}$. In the Newtonian limit the gravitational field is too weak to produce velocities near the speed of light, thus only the T^{00} component of the stress-energy tensor contributes significantly and we can make the approximation $\partial_\alpha \partial^\alpha \approx \nabla^2$. This means that the Einstein equation can be approximated as $\partial_\alpha \partial^\alpha \bar{h}^{00} \approx \nabla^2 \bar{h}^{00} \approx -16\pi\rho$. From this we conclude that the dominant component of $\bar{h}^{\mu\nu}$ is $\bar{h}^{00}$, such that in terms of $h^{\alpha\beta}$ we have $h^{00} = h^{ii} = \bar{h}^{00}/2$. Thus, the line element takes the form

$$ds^2 = -(1 - \bar{h}^{00}/2)dt^2 + (1 + \bar{h}^{00}/2)(dx^2 + dy^2 + dz^2). \tag{13}$$

Now, assume we have a compact object, say, a star of dark matter that does not interact with the quantum field and is of radius R_o and of constant density $\rho = 3M/(4\pi R_o^3)$. We solve $\nabla^2 \bar{h}^{00} \approx -16\pi\rho$ with the usual boundary conditions $\bar{h}^{00}(|\vec{r}| \to \infty) = 0$, $\frac{\partial \bar{h}^{00}(\vec{r})}{\partial r}(r = 0) = 0$ and with the continuity conditions $\bar{h}^{00}(|\vec{r}| \to R_o - \epsilon) = \bar{h}^{00}(|\vec{r}| \to R_o + \epsilon)$, $\frac{\partial \bar{h}^{00}}{\partial r}(|\vec{r}| \to R_o - \epsilon) = \frac{\partial \bar{h}^{00}}{\partial r}(|\vec{r}| \to R_o + \epsilon)$ in the limit $\epsilon \to 0$. This gives

$$\bar{h}^{00}(\vec{r}) = \begin{cases} \frac{2M}{R_o}\left(3 - \frac{|\vec{r}|^2}{R_o^2}\right) & \text{when } |\vec{r}| < R_o, \\ \frac{4M}{|\vec{r}|} & \text{when } |\vec{r}| > R_o \end{cases}, \tag{14}$$

so to have $|h_{\alpha\beta}| \ll 1$ we require $M/R_o \ll 1$.

IV. DETECTORS ON THE CURVED BACKGROUND

Let us now consider the two Unruh-DeWitt detectors on the background of the weak gravitational field. We assume that the two detectors and the center of the star are all on the same axis. Therefore, detector 1 is located at a fixed distance r_1 from the center of the star, and similarly detector 2 is located at $r_2 = r_1 + L$ from the center of the star. This means that their proper times do not coincide $\tau_1(t) \neq \tau_2(t)$, so we may write the evolution operator as

$$U = T\exp\left\{-i\int d\tau_1 \alpha\left[\eta(\tau_1)m^{(1)}(\tau_1)\phi(x_1(\tau_1)) + \eta(\tau_2(\tau_1))m^{(2)}(\tau_2(\tau_1))\phi(x_2(\tau_2(\tau_1)))\frac{d\tau_2(\tau_1)}{d\tau_1}\right]\right\}, \tag{15}$$

and using Eq. (13) we have

$$\tau_2(\tau_1) = \tau_1\sqrt{\frac{1 - \bar{h}^{00}(r_2)/2}{1 - \bar{h}^{00}(r_1)/2}} = \tau_1\left(1 - \frac{\bar{h}^{00}(r_2)}{4} + \frac{\bar{h}^{00}(r_1)}{4} + O([\bar{h}^{00}]^2)\right). \tag{16}$$

To simplify our analysis we want to avoid this blueshift effect. To do this, we assume that the two detectors are close enough such that their internal clocks have the same speed at first order in perturbation theory. This will be so if $|\bar{h}^{00}(r_2)/4 - \bar{h}^{00}(r_1)/4| \lesssim O([\bar{h}^{00}(r_2)]^2)$, which for detectors outside the star gives $L \lesssim 16M$. Under that assumption, we have $\tau_2 = \tau_1(1 + O([\bar{h}^{00}]^2))$ such that one can easily verify that Eqs. (4) and (5) still hold up to $O([\bar{h}^{00}]^2)$.

We assume that the detectors are pointlike and that their quantum position uncertainties ΔX are much smaller than their separation: $\Delta X \ll L$. This means that the detectors, possessing some mass m, have quantum momentum fluctuations obeying $m\Delta v \gg 1/L \gg 1/(16M)$. These fluctuations cause a Doppler broadening of the energy gap ΔE and in order to be able to neglect this effect we must require $\Delta v \ll 1$. We thus arrive at the requirement that the mass of the detectors obey $m \gg 1/(16M)$. In other words, we must have (Schwarzschild radius of star)/(Compton wavelength of a detector) $\gg 1$, which clearly always holds.

In Eq. (7), we are therefore left with two types of first-order contributions to the exchange term X and the local noise term P_j, namely $P_j = P_{(0)} + \tilde{P}_{j(1)} + \delta P_{j(1)}$ and $X = X_{(0)} + \tilde{X}_{(1)} + \delta X_{(1)}$. Here, the $\tilde{X}_{(1)}$ and $\tilde{P}_{j(1)}$ are due to the time dilation caused by the star. We denote by $\delta X_{(1)}$ and $\delta P_{j(1)}$ the contributions that arise from the modification of the propagator on the curved background. We will calculate the perturbative expansion of the propagator $G(x, y) = G_{(0)}(x, y) + G_{(1)}(x, y)$ in the next section. We note here already that since it is widely believed that the Boulware-type vacuum is the right vacuum for a quantum field in a Newtonian gravitational potential [13,14], we can use Eq. (10) for $G_{(0)}(x, y)$. This means that we can already evaluate the contributions $\tilde{X}_{(1)}$ and $\tilde{P}_{j(1)}$. First, we note that

$$x_j(\tau_j) = \left(\frac{\tau_j}{\sqrt{1 - \bar{h}^{00}(r_j)/2}}, \vec{r}_j\right). \tag{17}$$

Thus, when $L \lesssim 16M$ we have, using Eq. (10),

$$G_{(0)}(|\vec{x}_1(\tau) - \vec{x}_2(\tau')|, x_1^0(\tau) - x_2^0(\tau')) = (1 - \bar{h}^{00}(r_1)/2)G_{(0)}(L_p(1 - \bar{h}^{00}(r_1)/2), \tau - \tau') + O([\bar{h}^{00}]^2), \tag{18}$$

where

$$L_p := \int_{r_1}^{r_1+L} \sqrt{1 + \bar{h}^{00}(r)/2}\,dr \approx L(1 + \bar{h}^{00}(r_1)/4) \tag{19}$$

is the proper distance between the two detectors. Hence, when we put this back in Eqs. (9) and (8) we have the first-order corrections

$$\tilde{P}_{j(1)} = -\frac{\bar{h}^{00}(r_j)}{2} P_{(0)} \tag{20}$$

$$\tilde{X}_{(1)} = -\frac{\bar{h}^{00}(r_1)}{2}\left(X_{(0)} + L_p \frac{\partial X_{(0)}}{\partial L_p}\right), \tag{21}$$

where the 0th order terms are given by Eqs. (11) and (12).

V. CORRECTION TO THE PROPAGATOR

In this section we compute the first-order correction to the propagator on the perturbed background. The first steps of our calculation can be found in [13]. To focus on the correction caused by gravity, we now assume that the field is minimally coupled to curvature and to the matter that composes the star. Under these assumptions, the propagator is a Green's function of the Klein-Gordon operator

$$\Box_x G(x, y) = \frac{i\delta(x, y)}{\sqrt{-g(x)}}, \tag{22}$$

where $\Box_x f(x) = \frac{1}{\sqrt{-g}} \partial_\mu[\sqrt{-g} g^{\mu\nu} \partial_\nu f(x)]$. The first-order correction to g is $g = -1 - h^\alpha_\alpha = -1 - \bar{h}^{00}$. Using $G(x, y) = G_{(0)}(x, y) + G_{(1)}(x, y)$ we have

$$\frac{1}{\sqrt{1 + h^\alpha_\alpha}} \partial_\mu[\sqrt{1 + h^\alpha_\alpha}(\eta^{\mu\nu} - h^{\mu\nu}) \times \partial_\nu(G_{(0)}(x, y) + G_{(1)}(x, y))] = \frac{i\delta(x, y)}{\sqrt{1 + h^\alpha_\alpha}}. \tag{23}$$

Expanding everything to first order only and using the fact that $G_{(0)}(x, y)$ solves the 0th order equation, we obtain

$$\begin{aligned} &- h^{\mu\nu}\partial_\mu\partial_\nu G_{(0)}(x, y) + \Box_{(0)x} G_{(1)}(x, y) \\ &\quad - \partial_\mu h^{\mu\nu}\partial_\nu G_{(0)}(x, y) + \partial_\mu(h^\alpha_\alpha/2)\eta^{\mu\nu}\partial_\nu G_{(0)}(x, y) \\ &= -i\delta(x, y)h^\alpha_\alpha/2, \end{aligned} \tag{24}$$

where we used $\Box_{(0)x} := \eta^{\mu\nu}\partial_\mu\partial_\nu$. Using again the fact that $i\delta(x, y) = \Box_{(0)x} G_{(0)}(x, y)$ we can simplify the previous equation,

$$\begin{aligned} \Box_{(0)x} G_{(1)}(x, y) &= \partial_\mu \bar{h}^{\mu\nu}\partial_\nu G_{(0)}(x, y) + \bar{h}^{\mu\nu}\partial_\mu\partial_\nu G_{(0)}(x, y) \\ &= \bar{h}^{\mu\nu}\partial_\mu\partial_\nu G_{(0)}(x, y) \\ &= \bar{h}^{00}(x)\partial^2_{x^0} G_{(0)}(x, y), \end{aligned} \tag{25}$$

where we used the fact that we are in the Lorentz gauge and that in the Newtonian limit $\bar{h}^{00}$ is the dominant component of $\bar{h}^{\mu\nu}$. Note that since the space-time we consider is static and asymptotically flat, the propagator $G(x, y)$ can be seen as the analytic continuation of the unique Green's function on the positive definite section [13]. Since this holds order by order in perturbation theory, at first-order perturbation we can use $G_{(0)}$ as the inverse of $\Box_{(0)}$ such that

$$G_{(1)}(x, y) = -i \int d^4 z G_{(0)}(x, z)\bar{h}^{00}(z)\partial^2_{z^0} G_{(0)}(z, y). \tag{26}$$

This equation gives us explicitly the first-order correction to the propagator. It is clear from this equation that the entire space-time perturbation will modify the propagator, and the most significant contribution will come from the patch of space-time near x and y. We now insert $G_{(0)}(x, y)$ in Eq. (26) and, using the fact that $\bar{h}^{00}(x)$ is independent of time, we obtain

$$\begin{aligned} G_{(1)}(x, y) = \frac{-i}{16\pi^4} \int dz^0 d^3 z \bar{h}^{00}(\vec{z}) &\times \left[\frac{8(\tilde{z} + s)^2}{(\tilde{z} - z_1)^3(\tilde{z} - z_2)^3(\tilde{z} - z_o)(\tilde{z} + z_o)}\right. \\ &\left. - \frac{2}{(\tilde{z} - z_1)^2(\tilde{z} - z_2)^2(\tilde{z} - z_o)(\tilde{z} + z_o)}\right], \end{aligned} \tag{27}$$

where we use the definitions $Z_x := |\vec{x} - \vec{z}|$, $Z_y := |\vec{y} - \vec{z}|$, $s := x^0 - y^0$, $\tilde{z} := z^0 - x^0$, $z_o := X + i\epsilon$, $z_1 := -s + Y + i\epsilon$, and $z_2 := -s - Y - i\epsilon$. We can then perform the z^0 integration with the residue theorem. We choose a closed contour in the upper half of the complex plane and the upper part of the contour is equal to zero because the integrand vanishes sufficiently rapidly as $z^0 = \mathrm{Re}^{i\theta}|_{R\to\infty}$. We thus have

$$\begin{aligned} G_{(1)}(x, y) &= \frac{1}{8\pi^3} \int d^3 z \bar{h}^{00}(\vec{z}) \left[\frac{8(z_o + s)^2}{(z_o - z_1)^3(z_o - z_2)^3 2z_o}\right. \\ &\quad + 4\frac{d^2}{d\tilde{z}^2}\left(\frac{(\tilde{z} + s)^2}{(\tilde{z} - z_2)^2(\tilde{z} - z_o)(\tilde{z} + z_o)}\right) \\ &\quad \times \Bigg|_{\tilde{z}=z_1} - \frac{2}{(z_o - z_1)^2(z_o - z_2)^2 2z_o} \\ &\quad \left. - \frac{d}{d\tilde{z}}\left(\frac{2}{(\tilde{z} - z_2)^2(\tilde{z} - z_o)(\tilde{z} + z_o)}\right)\Bigg|_{\tilde{z}=z_1}\right] \\ &= \frac{1}{8\pi^3} \int d^3 z \bar{h}^{00}(|\vec{z}|) \\ &\quad \times \left[\frac{3(s^2 + Z_x Z_y)(Z_x + Z_y) + Z_x^3 + Z_y^3}{(Z_x Z_y + i\epsilon)(s^2 - [Z_x + Z_y + i\epsilon]^2)^3}\right]. \end{aligned} \tag{28}$$

Using the above equation we may now evaluate $\delta P_{j(1)}$ and $\delta X_{(1)}$. For $\delta P_{j(1)}$, we have $Z_x = Z_y = Z$, such that the correction to the propagator can be greatly simplified with a simple change of variables:

$$G_{(1)}(\vec{x}, \vec{x}, s) = \frac{1}{2r\pi^2} \int_0^\infty dR R \bar{h}^{00}(R) \times \int_{|r-R|}^{r+R} dv \frac{3s^2 + 4v^2}{(s^2 - 4v^2 - i\epsilon)^3}, \tag{29}$$

where r is the distance between $\vec{x}$ and the center of the star and $s = x^0 - y^0$. The v integral can be performed analytically. Note that Eqs. (9) and (8) were derived for detectors in Minkowski space-time, where $x^0(\tau) = y^0(\tau) = \tau$. The effect of the time dilation in the corrected propagator is a second-order term which we neglect since we are only interested in the first-order effect. In other words, the corrected propagator in terms of proper time can be expanded as

$$G_{(1)}(\vec{x},\vec{x},s(1+\bar{h}^{00}(r_1)/4)) = G_{(1)}(\vec{x},\vec{x},s) + \frac{s\bar{h}^{00}(r_1)}{4}\frac{\partial}{\partial s}G_{(1)}(\vec{x},\vec{x},s) + O([\bar{h}^{00}]^3), \tag{30}$$

and to be consistent with our current first-order expansion we neglect the second term. We can thus use Eqs. (9) and (8) with the first-order correction of the propagator and with no time dilation, that is, $x^0(\tau) = y^0(\tau) = \tau$. For the same reason, we can also use at this order $L_p = L$. Therefore, we can put Eq. (29) in Eq. (9) and we obtain the first-order correction to the local noise $\delta P_{j(1)}$:

$$\delta P_{j(1)} = \frac{\alpha^2\sigma\sqrt{\pi}}{\pi^2 r_j}\Re\Big\{\int_0^\infty ds e^{-s^2/(4\sigma^2)-i\Delta E s}\int_0^\infty dR \times R\bar{h}^{00}(R)\Big[\Big(\ln\Big(\frac{2(r_j+R)+(s-i\epsilon)}{2(r_j+R)-(s-i\epsilon)}\Big) - \ln\Big(\frac{2|r_j-R|+(s-i\epsilon)}{2|r_j-R|-(s-i\epsilon)}\Big)\Big)\frac{1}{4(s-i\epsilon)^3} - \frac{2(r_j+R)(2(r_j+R)^2-s^2)}{(s-i\epsilon)^2(4(r_j+R)^2-(s-i\epsilon)^2)^2} + \frac{2|r_j-R|(2(r_j-R)^2-s^2)}{(s-i\epsilon)^2(4(r_j-R)^2-(s-i\epsilon)^2)^2}\Big]\Big\}. \tag{31}$$

Similarly, for the exchange term $\delta X_{(1)}$, we put Eq. (28) in Eq. (8) and we then use a simple change of variables to obtain

$$\delta X_{(1)} = -\frac{\alpha^2\sigma\sqrt{\pi}e^{-\sigma^2\Delta E^2}}{2\pi^2 r_1}\int_0^\infty ds e^{-s^2/(4\sigma^2)} \times \int_0^\infty dR R\bar{h}^{00}(R)\int_{|r_1-R|}^{r_1+R} dv_1 v_1 \times \Big[\frac{3(s^2+v_1v_2)(v_1+v_2)+v_1^3+v_2^3}{(v_1v_2+i\epsilon)(s^2-[v_1+v_2+i\epsilon]^2)^3}\Big], \tag{32}$$

where $v_2 = \sqrt{v_1^2(1+\frac{L_p}{r_1})+L_p(r_1+L_p-\frac{R^2}{r_1})}$. The s integration can be performed analytically, such that we are left with a relatively simple expression for $\delta X_{(1)}$ which involves only two integrations:

$$\delta X_{(1)} = \frac{\alpha^2\sigma\sqrt{\pi}e^{-\sigma^2\Delta E^2}}{2\pi^2 r_1}\int_0^\infty dR R\bar{h}^{00}(R)\int_{|r_1-R|}^{r_1+R} dv_1 \times \frac{1}{16(v_2+i\epsilon)\sigma^4}\Big\{i\pi\,\mathrm{erfc}\Big(\frac{-i(v_1+v_2)}{2\sigma}\Big) \times e^{-(v_1+v_2))^2/(4\sigma^2)}[2\sigma^2-(v_1+v_2)^2] - 2\sqrt{\pi}\sigma(v_1+v_2)\Big\}. \tag{33}$$

VI. NEGATIVITY ON THE PERTURBED BACKGROUND

We now have all the tools to compute explicitly the corrected negativity. Using the $\bar{h}^{00}(|\vec{r}|)$ of Eq. (14) in Eqs. (31) and (33), we can find $\delta P_{j(1)}$ and $\delta X_{(1)}$ by numerically evaluating the remaining integrals. $\tilde{P}_{j(1)}$ and $\tilde{X}_{(1)}$ can then be evaluated exactly using Eqs. (20) and (21) such that we arrive at the full noise term $P_j = P_{(0)} + \tilde{P}_{j(1)} + \delta P_{j(1)}$ and the full exchange term $X = X_{(0)} + \tilde{X}_{(1)} + \delta X_{(1)}$ using Eqs. (11) and (12) for the 0th order terms. This allows us to compute the negativity between the two detectors using Eq. (7). Note that even if we could find the approximate behavior of X and P_j as a function of the different parameters this would not be sufficient for our analysis. Indeed, we need to know exactly which one of X or P_j is greater to evaluate the negativity and conclude whether or not gravity increased the extractable entanglement of the vacuum. It is thus for this reason that a numerical analysis is required.

Numerical evaluations indicate that $|X|$ linearly increases with the strength of the gravitational potential M/R_o of the star while P_j linearly decreases with M/R_o. Therefore, the negativity N linearly increases with the strength of the gravitational field M/R_o. Note that the linear dependence on M/R_o is simply due to the fact that we used perturbation theory and kept only the first-order term. In other words, the linear dependence did not need numerical evaluation. However, the sign of the linear dependence did need numerical evaluation. In a similar fashion, numerical evaluation of $|X|$ and P_j indicate that the correction to the negativity N decreases roughly like R_o/r_1 as $r_1/R_o \to \infty$ but remains positive, see Fig. 1. Note that

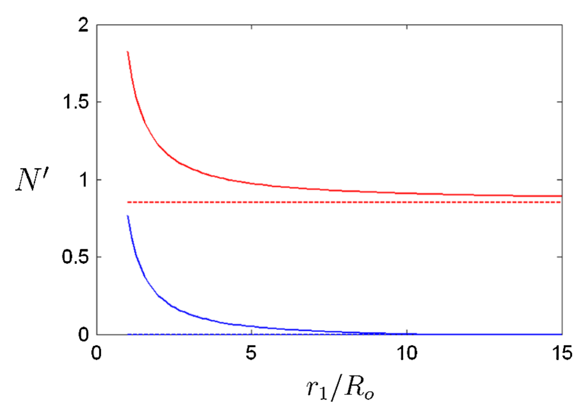

FIG. 1 (color online). $N' = 8\pi^2 N/\alpha^2$ as a function of r_1/R_o. Here $\sigma\Delta E = 0.00674$, $R_o\Delta E = 1$, and $M/R_o = 0.001$. The upper (red) curve corresponds to $L_p/\sigma = 1.409$ and the lower (blue) curve to $L_p/\sigma = 1.484$. The upper and lower dashed lines are the asymptotes $r_1/R_o \to \infty$. The parameters $\{L_p/\sigma, \sigma\Delta E, R_o\Delta E\}$ were chosen to display two regimes in which entanglement enhancement and entanglement creation occur, respectively.

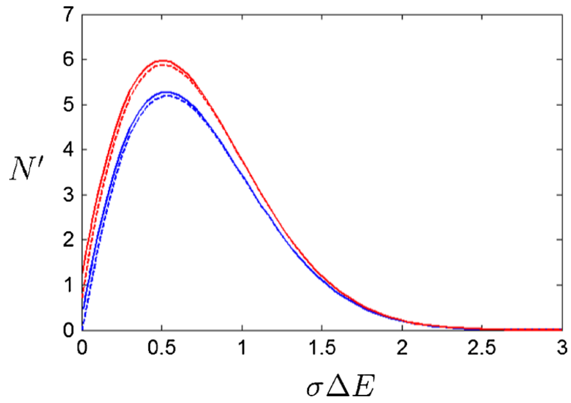

FIG. 2 (color online). $N' = 8\pi^2 N/\alpha^2$ as a function of $\sigma\Delta E$. Here $\Delta E R_o = 1$, $r_1/R_o = 1.1$, and $M/R_o = 0.001$. The upper (red) curve corresponds to $L_p/\sigma = 1.409$ and the lower (blue) curve to $L_p/\sigma = 1.484$. The upper and lower dashed lines are the asymptotes $r_1/R_o \to \infty$.

since M/R_o is the amplitude of the perturbation it is appropriate to chose its value to be as small as 0.001. Moreover, in Fig. 1 the two curves correspond to different choices of parameter L_p/σ: the choice $L_p/\sigma = 1.409$ was made to exhibit an example where the negativity N is nonzero even without the gravitational field. We therefore see entanglement enhancement by gravity but not entanglement creation. The choice $L_p/\sigma = 1.484$ was made to exhibit an example where $N = 0$ and $|X| \approx P$ in the absence of the gravitational field. We therefore see not only entanglement enhancement by gravity here but also entanglement creation by gravity. Also the choice of $R_o\Delta E = 1$ was made to allow one to display both entanglement enhancement and creation regimes. We remark that it implies the exchange of virtual particles of typical wavelength of the order of the radius of the star. Other choices of parameters are possible but generally only display entanglement enhancement. Furthermore, note that the negativity's dependence on $\Delta E\sigma$ is essentially unchanged by the gravitational field as one can see in Fig. 2.

As we previously mentioned this effect scales linearly with the strength of the gravitational field M/r_1, so for detectors with $\sigma \gg 1/\Delta E$ and $\sigma \gtrsim L_p$ we have for the Earth $N_{(1)} \lesssim 10^{-9} N_{(0)}$, while for the Sun we have $N_{(1)} \lesssim 10^{-6} N_{(0)}$. Since entanglement swapping from the entanglement $N_{(0)}$ of the vacuum has still not been observed, we expect that observing $N_{(1)}$ will be very difficult. Nevertheless, it should be interesting to see if this effect can be modeled in a quantum field analog such as a linear ion trap [15].

VII. OUTLOOK

Our calculations depended on the assumption that the vacuum of the quantum field is described by the Boulware vacuum. If we had considered two Unruh-DeWitt detectors near a black hole in an Unruh or a Kruskal vacuum [8], the Hawking temperature seen by both detectors would have increased the local noise significantly such that the entanglement between both detectors should be degraded, not enhanced. It should be interesting to investigate in detail to what extent entanglement extraction by detectors near black holes and stars is affected by the properties of the corresponding vacuum states.

In this context, it should also be interesting to investigate whether one can effectively model a black hole by using a confining potential, for instance, on a shell. Indeed, a trapping potential can have horizons, so it may be possible to have a nontrivial vacuum in which particle production occurs because of the potential. Such an analysis could show the Hawking effect and its various open questions in a new light.

Since we observed that the exchange term $|X|$ increases because of the gravitational field, it is tempting to speculate on the Casimir-Polder force near a constant density star. Indeed, the exchange term and the Casimir-Polder force have essentially the same interpretation, that is they are the result of a continuous exchange of virtual particles. We therefore conjecture that Casimir or Casimir-Polder forces can slightly increase in a weak gravitational field. Let us recall, however, that the main contribution to this effect stems from the exchange of virtual photons whose wavelength is of the order of the spatial separation of the two objects. In our calculations, we assumed that the separation of the detectors is small compared to the radius of the star. However, in the regimes considered in Fig. 1, we had that $R_o\Delta E = 1$, i.e., that the virtual photons are of the wavelength of the radius of the star. This means that we are here in the near-field regime, where the Casimir effect reduces to the van der Waal's effect.

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