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Particle motion around black hole in Hořava-Lifshitz gravity

Ahmadjon Abdujabbarov,* Bobomurat Ahmedov,† and Abdullo Hakimov‡

Institute of Nuclear Physics, Ulughbek, Tashkent 100214, Uzbekistan
Ulugh Beg Astronomical Institute,
Astronomicheskaya 33, Tashkent 100052, Uzbekistan

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Analytical solutions of Maxwell equations around a black hole immersed in an external uniform magnetic field in the background of the Kehagias-Sfetsos (KS) asymptotically flat black hole solution of Hořava-Lifshitz gravity have been found. The influence of a magnetic field on the effective potential of the radial motion of a charged test particle around a black hole immersed in an external magnetic field in Hořava-Lifshitz gravity has been investigated by using the Hamilton-Jacobi method. An exact analytical solution for dependence of the minimal radius of the circular orbits $r_{\rm mc}$ from KS parameter ω for motion of a test particle around a spherical symmetric black hole in Hořava-Lifshitz gravity has been derived. The critical values of the particle's angular momentum for captured particles by a black hole in Hořava-Lifshitz gravity have been obtained numerically. The comparison of the obtained numerical results with the astrophysical observational data on the radii of the innermost stable circular orbits gives us the estimation of the parameter as $\omega \simeq 3.6 \cdot 10^{-24}\ \mathrm{cm}^{-2}$.

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I. INTRODUCTION

Recently, Petr Hořava suggested a new candidate quantum field theory of gravity with a dynamical critical exponent equal to $z = 3$ in the UV (ultraviolet). This theory is a nonrelativistic power-counting renormalizable theory in four dimensions, which admits the Lifshitz scale-invariance in time and space that reduces to Einstein's general relativity at large scales [1,2]. The Hořava theory has received a great deal of attention, and since its formulation various properties and characteristics have been extensively analyzed, ranging from formal developments [3], cosmology [4], dark energy [5], dark matter [6], and spherically symmetric or axial symmetric solutions [7].

In the paper [8], the possibility of observationally testing Hořava gravity at the scale of the Solar System, by considering the classical tests of general relativity (perihelion precession of the planet Mercury, deflection of light by the Sun, and the radar echo delay) for the Kehagias-Sfetsos (KS) asymptotically flat black hole solution of Hořava-Lifshitz gravity has been considered. The stability of the Einstein static universe by considering linear homogeneous perturbations in the context of an infrared (IR) modification of Hořava gravity has been studied in [9]. Potentially observable properties of black holes in the deformed Hořava-Lifshitz gravity with Minkowski vacuum: the gravitational lensing and quasinormal modes have been studied in [10]. The authors of the paper [11] derived the full set of equations of motion, and then obtained spherically symmetric solutions for the UV-completed theory of Einstein proposed by Hořava.

*ahmadjon@astrin.uz
†ahmedov@astrin.uz
‡abdullo@astrin.uz

Black hole solutions and the full spectrum of spherically symmetric solutions in the five-dimensional nonprojectable Hořava-Lifshitz type gravity theories have been recently studied in [12]. Geodesic stability and the spectrum of entropy/area for a black hole in Hořava-Lifshitz gravity via quasinormal modes approach are analyzed in [13]. Particle geodesics around a Kehagias-Sfetsos black hole in Hořava-Lifshitz gravity are also investigated by authors of the paper [14]. Recently observational constraints on Hořava-Lifshitz gravity have been found from the cosmological data [15]. Authors of the paper [16] have found all spherical black hole solutions for two, four, and six derivative terms in the presence of a Cotton tensor.

This paper is organized as follows. We look for exact solutions of vacuum Maxwell equations in spacetime of the black hole immersed in a uniform magnetic field in IR-modified Hořava-Lifshitz gravity in Sec. II. In our recent paper [17], an exact analytical solution for dependence of the radius of the innermost stable circular orbits (ISCO) $r_{\rm ISCO}$ from brane tension for motion of a test particle around a black hole in the braneworld has been analyzed. We extend it to the motion of charged particles around a black hole immersed in a uniform magnetic field in Hořava-Lifshitz gravity using the Hamilton-Jacobi method in Sec. III. We obtain the effective potential for a charged test particle with a specific angular momentum, orbiting around the black hole, as a function of the external magnetic field, and the IR-modified parameter in Hořava-Lifshitz gravity. In Sec. IV, we find the exact expression for dependence of a minimal radius of circular orbit from the parameter ω, which is responsible for the IR-modified term in the Hořava-Lifshitz action, for the test particle moving in the equatorial plane of the black hole (when the external magnetic field is neglected for the simplicity of calculations). Then we present the numerical results of the capture cross section of the slowly moving test particles by the black hole in Hořava-Lifshitz gravity. The concluding remarks are given in Sec. V.

We use a system of units in which $c = G = 1$, a space-like signature $(-, +, +, +)$ and a spherical coordinate system (t, r, θ, φ). Greek indices are taken to run from 0 to 3. We will indicate vectors with bold symbols (e.g., $\mathbf{B}$).

II. BLACK HOLE IMMERSED IN A UNIFORM MAGNETIC FIELD

The static and spherical symmetric spacetime metric of the black hole with mass M in Hořava-Lifshitz gravity takes form (see e.g, [8–10])

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2, \quad (1)$$

where the metric functions Φ and Λ depend on the radial coordinate r only.

We consider the Kehagias and Sfetsos asymptotically flat solution [18], given by

$$e^{2\Phi(r)} = e^{-2\Lambda(r)} = 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}. \quad (2)$$

A Killing vector ξ^μ, being an infinitesimal generator of an isometry, satisfies to the equation

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0, \quad (3)$$

which can be used in order to rewrite the equation

$$\xi_{\alpha;\beta;\gamma} - \xi_{\alpha;\gamma;\beta} = -\xi^\lambda R_{\lambda\alpha\beta\gamma}, \quad (4)$$

which defines the Riemann curvature tensor $R_{\lambda\alpha\beta\gamma}$ in the form

$$\xi^{\alpha;\beta}{}_{;\beta} = \xi^\gamma R_{\gamma\beta}{}^{\alpha\beta} = R^\alpha{}_\gamma \xi^\gamma. \quad (5)$$

For spacetime of the KS black hole, the right-hand side of Eq. (5) can be expressed as $R^\alpha{}_\gamma\xi^\gamma = \eta^\alpha$, and consequently the Maxwell equations as

$$F^{\alpha\beta}{}_{;\beta} = -2C_0(\xi^{\alpha;\beta}{}_{;\beta} - \eta^\alpha) = 0, \quad (6)$$

where $\eta^\alpha = \{0, 0, 0, 6M^2/\omega r^6\}$ is just the first order approximation in ω^{-1} of the relation $R^\alpha{}_\gamma\xi^\gamma$ (see [18]), and neglecting the time component η^t which can be explained as follows. If one considers the electrical neutrality of the source

$$4\pi Q = 0 = \frac{1}{2}\oint F^{\alpha\beta}{}_{*}dS_{\alpha\beta},$$

where ${}_{*}dS_{\alpha\beta}$ is the element of a 2-surface, and evaluate the value of the integral through the spherical surface at the asymptotic infinity ($r \to \infty$, $\omega r^2 \to \infty$), one can obtain that the time component of the potential will vanish identically (see, for more details [19]). Taking into the account the Lorentz gauge, the electromagnetic field tensor $F_{\alpha\beta}$ can be selected as

$$\begin{aligned} F_{\alpha\beta} &= C_0(\xi_{\beta;\alpha} - \xi_{\alpha;\beta} + 2f_{\alpha\beta}) \\ &= -2C_0(\xi_{\alpha;\beta} + a_{\beta,\alpha} - a_{\alpha,\beta}). \end{aligned} \quad (7)$$

Here, C_0 is constant and 4-potential a^α, being responsible for the KS parameter ω, can be found from the equation $\Box a^\alpha = \eta^\alpha$.

Finally, one can express the electromagnetic potential as a sum of two contributions

$$A^\alpha = \tilde{A}^\alpha + a^\alpha, \quad (8)$$

where $\tilde{A}^\alpha$ is the potential being proportional to the Killing vectors. To find the solution for $\tilde{A}^\alpha$, we exploit the existence in this spacetime of a timelike Killing vector $\xi^\alpha_{(t)}$ and spacelike one $\xi^\alpha_{(\varphi)}$ being responsible for stationarity and axial symmetry of geometry, such that they satisfy the Killing Eq. (3) and consequently the wavelike equations (in vacuum spacetime) $\Box\xi^\alpha = 0$, which gives a right to write the solution of vacuum Maxwell equations $\Box\tilde{A}^\alpha = 0$ for the vector potential $\tilde{A}_\alpha$ of the electromagnetic field in the Lorentz gauge in the simple form $\tilde{A}^\alpha = C_1\xi^\alpha_{(t)} + C_2\xi^\alpha_{(\varphi)}$ [20]. The constant $C_2 = B/2$, where the gravitational source is immersed in the uniform magnetic field $\mathbf{B}$ being parallel to its axis of rotation. The value of the remaining constant C_1 will vanish, which can be easily shown from the asymptotic properties of spacetime (1) at the infinity (see e.g., our preceding papers [17,19] for the details of typical calculations).

The second part a^α of the total vector potential of the electromagnetic field is produced by the presence of the KS parameter ω and has the following solution:

$$a^\alpha = \frac{B}{2}\left\{0, 0, 0, \frac{3M^2}{10\omega r^4}\right\}.$$

Finally, the 4-vector potential A_α of the electromagnetic field will take a form

$$A_0 = A_1 = A_2 = 0, \quad A_3 = \frac{1}{2}Br^2\sin^2\theta\left(1 + \frac{3M^2}{10\omega r^4}\right). \quad (9)$$

The orthonormal components of the electromagnetic fields measured by a fixed observer with the four-velocity components $(u^\alpha)_{\rm obs} \equiv \exp(-\Phi)\{1, 0, 0, 0\}$; $(u_\alpha)_{\rm obs} \equiv -\exp(\Phi)\{1, 0, 0, 0\}$ are given by expressions

$$B^{\hat{r}} = B\left(1 + \frac{3M^2}{10\omega r^4}\right)\cos\theta, \quad (10)$$

$$B^{\hat{\theta}} = e^{\Phi(r)}B\left(1 - \frac{6M^2}{10\omega r^4}\right)\sin\theta, \quad (11)$$

which depend on the lapse function of the metric (1).

In the limit of flat spacetime, i.e. for $\omega r^2 \to \infty$ and $2M/r \to 0$, expressions (10) and (11) give

$$\lim_{\omega r^2\to\infty, 2M/r\to 0} B^{\hat{r}} = B\cos\theta, \quad (12)$$

$$\lim_{\omega r^2\to\infty, 2M/r\to 0} B^{\hat{\theta}} = B\sin\theta. \quad (13)$$

As expected, expressions (12) and (13) coincide with the solutions for the homogeneous magnetic field in the Newtonian spacetime.

III. MOTION OF THE CHARGED PARTICLES AROUND A BLACK HOLE

It is very important to study in detail the motion of charged particles around a black hole in Hořava-Lifshitz gravity immersed in a uniform magnetic field given by a 4-vector potential (9), with the aim to find astrophysical evidence for the existence of the KS parameter ω.

We shall study the motion of the charged test particles around a black hole in Hořava-Lifshitz gravity using the Hamilton-Jacobi equation

$$g^{\mu\nu}\left(\frac{\partial S}{\partial x^{\mu}} + eA_{\mu}\right)\left(\frac{\partial S}{\partial x^{\nu}} + eA_{\nu}\right) = -m^2, \tag{14}$$

where e and m are the charge and the mass of a test particle, respectively. Since t and φ are the Killing variables, one can write the action in the form

$$S = -\mathcal{E}t + \mathcal{L}\varphi + S_{r\theta}(r, \theta), \tag{15}$$

where the conserved quantities $\mathcal{E}$ and $\mathcal{L}$ are the energy and the angular momentum of a test particle at infinity. Substituting it into Eq. (14), one can get the equation for the inseparable part of the action

$$-m^2 = -e^{2\Lambda(r)}\mathcal{E}^2 + e^{2\Phi(r)}\left(\frac{\partial S_{r\theta}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_{r\theta}}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left[\mathcal{L} + \frac{eB}{2}r^2\left(1 + \frac{3M^2}{10\omega r^4}\right)\sin^2\theta\right]^2. \tag{16}$$

One can easily separate variables in this equation in the equatorial plane $\theta = \pi/2$ and obtain the equation for radial motion

$$\left(\frac{dr}{ds}\right)^2 = \mathcal{E}^2 - V_{\mathrm{eff}}(\mathcal{L}, r, \epsilon, \omega), \tag{17}$$

where s is the proper time along the trajectory of a particle and

$$V_{\mathrm{eff}}(\mathcal{L}, r, \epsilon, \omega) = \left[1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}\right] \times \left\{\left[\frac{\mathcal{L}}{r} + \left(\frac{r}{2M} + \frac{3M^3}{10\tilde{\omega} r^3}\right)\epsilon\right]^2 + 1\right\} \tag{18}$$

can be interpreted as an effective potential of the radial motion, where the dimensionless parameter $\tilde{\omega} = \omega M^2$ is introduced. Here, we have changed $\mathcal{E} \rightarrow \mathcal{E}/m$ and $\mathcal{L} \rightarrow \mathcal{L}/m$. The effective potential besides the energy, the angular momentum, the KS parameter ω and the radius of the motion also depends on the dimensionless parameter $\epsilon = eBM/m$ which characterizes the relative influence of a uniform magnetic field on the motion of the charged particles. In Fig. 1, the radial dependence of the effective potential of the radial motion of the charged particle around a black hole for the different values of the dimensionless parameter $\tilde{\omega} = \omega M^2$ and magnetic parameter ϵ are shown. One can easily see that orbits of the particles become more unstable with the increasing of the parameter ϵ. (Similar results have been obtained in our previous paper [19].) The influence of the dimensionless parameter $\tilde{\omega}$ is sufficient near the black hole; as it is seen from the figure, the potential carries the repulsive character. It means that the particle coming from infinity and passing by the source will not be captured; it will be reflected and will go to infinity again. The orbits start to be only parabolic or hyperbolic and no more circular or elliptical orbits exist with decreasing the dimensionless parameter $\tilde{\omega}$, i.e. captured particles by the central object are going to leave the black hole.

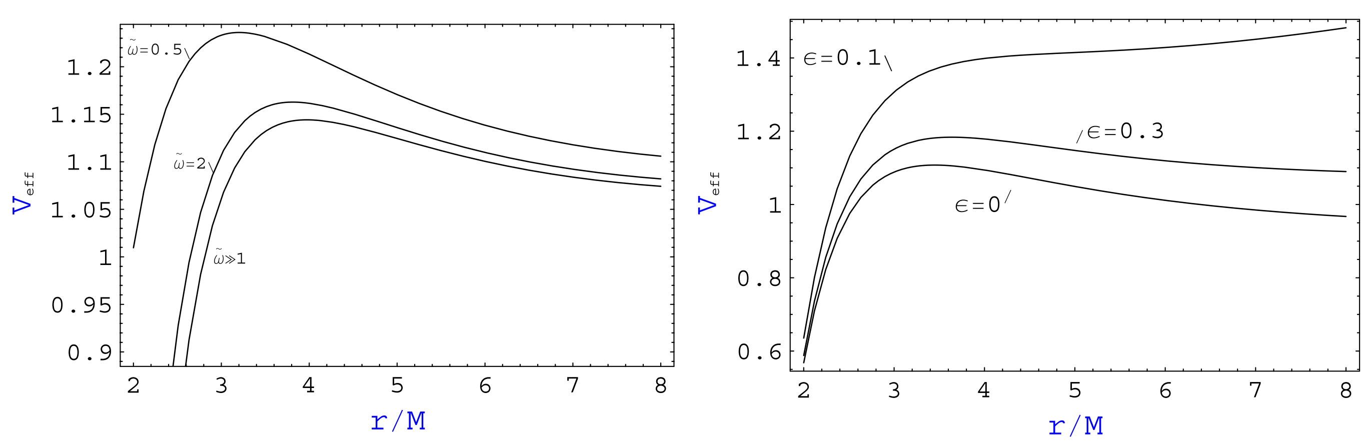

FIG. 1 (color online). Radial dependence of the effective potential of the radial motion of the charged particle around a black hole immersed in a uniform magnetic field in Hořava-Lifshitz gravity. In the left graph, the effective potential of the radial motion of the charged particle around a black hole has been shown for the different values of the dimensionless parameter $\tilde{\omega}$; the values of the momentum $\mathcal{L}/mM = 4.3$ and the magnetic parameter $\epsilon = 0.03$ are fixed. For comparison, we have also plotted this dependence in the case of the Schwarzschild black hole, corresponding to $\tilde{\omega} \gg 1$. In the right graph, the effective potential of the radial motion of the charged particle around a black hole has been shown for the different values of the magnetic parameter ϵ; the values of the momentum $\mathcal{L}/mM = 4.3$ and the dimensionless parameter $\tilde{\omega} = 1$ are fixed.

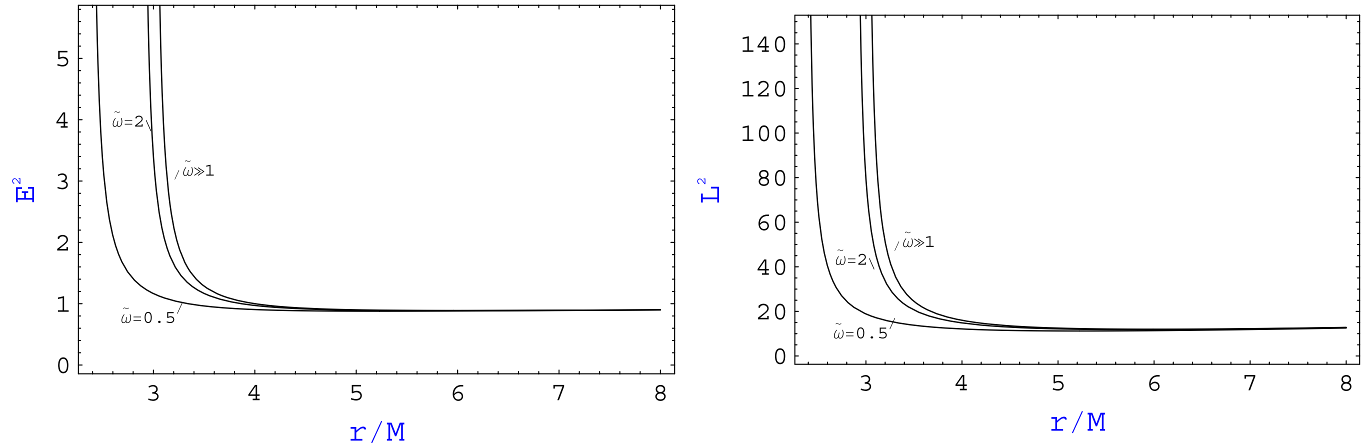

FIG. 2 (color online). Radial dependence of energy (left graph) and angular momentum (right graph) of the circular orbits around a black hole in Hořava-Lifshitz gravity for the different values of the dimensionless parameter $\tilde{\omega}$. For comparison, we have also plotted this dependence in the case of the Schwarzschild black hole, corresponding to $\tilde{\omega} \gg 1$.

IV. CIRCULAR ORBITS AROUND A BLACK HOLE IN HORíAVA-LIFSHITZ GRAVITY

In order to find solution for the ISCO radius of $r_{\rm ISCO}$, we assume that the external magnetic field is absent.

The expression (17) can now be written as

$$\left(\frac{dr}{ds}\right)^2 = f(r) = \mathcal{E}^2 - \left[1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}\right]\left(\frac{\mathcal{L}^2}{r^2} + 1\right). \tag{19}$$

Using Eq. (19) and the condition of occurrence of circular orbits $f(r) = 0$, $f'(r) = 0$, one can easily find expressions for energy $\mathcal{E}$ and angular momentum $\mathcal{L}$ of a circular orbit of radius r_c, which are given as

$$\mathcal{E}^2 = \left(1 + \frac{M - r^3(\Sigma - 1)\omega}{3M - r\Sigma}\right)(1 - r^2\omega(\Sigma - 1)), \tag{20}$$

$$\mathcal{L}^2 = r^2 \frac{M - \omega r^3(\Sigma - 1)}{3M - r\Sigma}, \tag{21}$$

where notation $\Sigma = (1 + 4M/r^3\omega)^{1/2}$ has been used. Figure 2 shows the radial dependence of both the energy

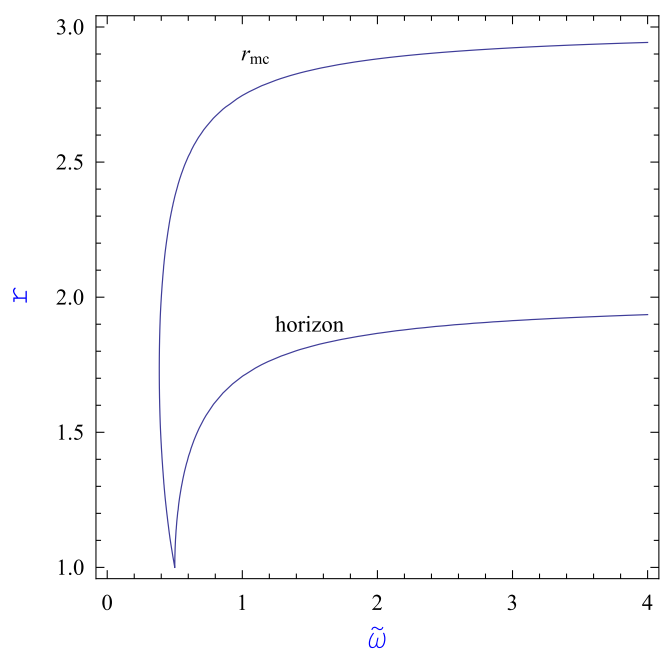

FIG. 3 (color online). Dependence of the radius of the horizon and $r_{\rm mc}$ from the $\tilde{\omega}$.

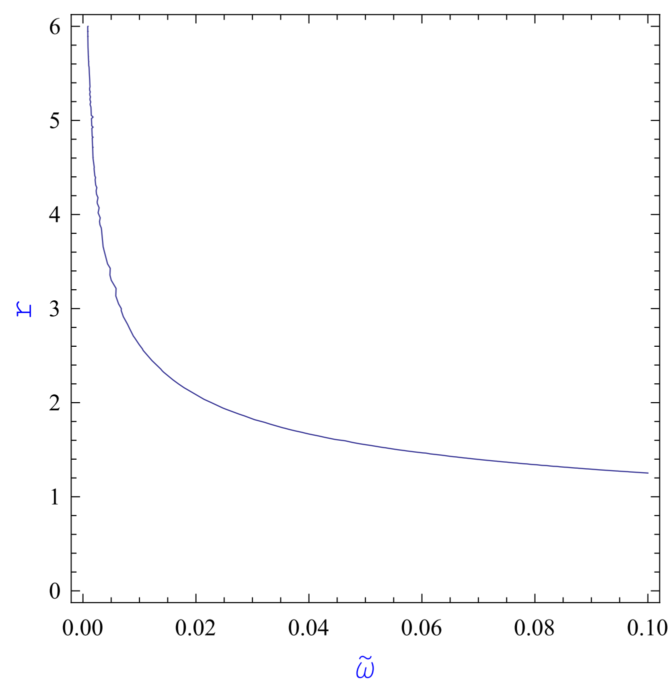

FIG. 4 (color online). Dependence of the ISCO radius from the $\tilde{\omega}$.

TABLE I. The innermost stable circular orbits around the black hole and the critical values of the momentum of the particles falling down to the central black hole in Hořava-Lifshitz gravity.

$\tilde{\omega}$	0.5	1	2	4	6	8	10	12
$r_{\rm ISCO}$	5.236 55	5.663 95	5.840 24	5.921 93	5.948 34	5.9614	5.969 18	5.974 36
$\mathcal{L}^2_{\rm cr}$	14.77	15.454	15.7395	15.8725	15.9156	15.9369	15.9496	15.9581

and the angular momenta of the test particle moving on circular orbits in the equatorial plane. One can easily see that circular orbits corresponding to constant value of the energy and momentum of the test particle shift to the central object with the decreasing of the parameter $\tilde{\omega}$.

For the existing circular orbits, the expression for angular momentum (21) of the test particle requires, in particular, that $r\Sigma - 3M \geq 0$. Consequently, one can easily find the minimum radius for circular orbits $\tilde{r}_{\rm mc} = r_{\rm mc}/M$:

$$\tilde{r}_{\rm mc} = \begin{cases} \frac{3\tilde{\omega}}{(\Delta-2\tilde{\omega}^2)^{1/3}} + \frac{(\Delta-2\tilde{\omega}^2)^{1/3}}{\tilde{\omega}}, & \tilde{\omega} < \frac{2\sqrt{3}}{9} \\ 2\sqrt{3}\cos\left[\frac{1}{3}\arccos\left(-\frac{2\sqrt{3}}{9}\frac{1}{\tilde{\omega}}\right)\right], & \tilde{\omega} \geq \frac{2\sqrt{3}}{9} \end{cases} \tag{22}$$

where $\Delta = (4\tilde{\omega}^4 - 27\tilde{\omega}^6)^{1/2}$. The obtained Eq. (22) is the original one.

As it was expected in the limiting case when $\tilde{\omega} \gg 1$, i.e. when metric (1) coincides with the well-known Schwarzschild metric, one can easily obtain the known result for the minimal radius of circular orbits around the Schwarzschild black hole as $\tilde{r}_{\rm mc} = 3$.

In Fig. 3, the dependence of both radii of the horizon and $r_{\rm mc}$ from the dimensionless parameter $\tilde{\omega}$ are shown. In the case of $\omega \geq 2\sqrt{3}/9$, the decreasing of the KS parameter $\tilde{\omega}$ forces the minimum radii of the circular orbits $r_{\rm mc}$ to shift to the central object. In the case of $\omega < 2\sqrt{3}/9$, there is no lower limit for $r_{\rm mc}$, which means that circular orbits can be present near the black hole.

The minimum radius for a stable circular orbit will occur at the point of inflexion of the function $f(r)$, or in other words where the supplement conditions $f(r) = f'(r) = 0$ with the relation $f''(r) \geq 0$ are satisfied. The numerical results for the values of ISCO radii for the different values of the parameter $\tilde{\omega}$ in the case of $\tilde{\omega} \geq 0.5$ are given in Table I (the second line). From the results, one can easily get in the limit of Schwarzschild spacetime $\omega r^2 \to \infty$ the standard value for ISCO radius as $r_{\rm ISCO} = 6M$.

Next, we will consider the capture cross section of slowly moving test particles by a black hole in Hořava-Lifshitz gravity. (Slow motion means that $\mathcal{E} \simeq 1$ at the infinity.) The critical value of the particle's angular momentum, $\mathcal{L}_{\rm cr}$, hinges upon the existence of a multipole root of the function $f(r)$ in (19) [21]. We give the numerical results for $\mathcal{L}^2_{\rm cr}$ in Table I (the third line).

In the limiting case, i.e. when $\tilde{\omega} \gg 1$ the value of the critical angular momentum is $\mathcal{L} = 4$, which coincides with critical angular momentum for particle capture cross section for the Schwarzschild black hole [22]. As a particle having the critical angular momentum travels from infinity toward the black hole in Hořava-Lifshitz gravity, it spirals into an unstable circular orbit.

V. CONCLUSION

Constraints for the KS parameter from the Solar System tests were found as $(5.660 \pm 3.1) \cdot 10^{-26}$ cm^{-2} [8]. The similar constraints for the parameter $\omega \simeq 1.25 \cdot 10^{-25}$ cm^{-2} have been found from the quantum interference effects [23]. In Fig. 4, the dependence of the $r_{\rm ISCO}$ from the dimensionless KS parameter $\tilde{\omega}$ is shown. From the figure, one can see that in the presence of the parameter $\tilde{\omega}$, ISCO shifts to the central black hole. One can easily compare the obtained numerical results with observational data for ISCO radius for some candidates of rotating black holes [24]. One can obtain the lower value for the parameter as $\omega \simeq 3.6 \cdot 10^{-24}$ cm^{-2}.

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