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Light cones in relativity: Real, complex, and virtual, with applications

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We study geometric structures associated with shear-free null geodesic congruences in Minkowski space-time and asymptotically shear-free null geodesic congruences in asymptotically flat space-times. We show how in both the flat and asymptotically flat settings, complexified future null infinity $\mathfrak{I}^+_{\mathbb{C}}$ acts as a "holographic screen," interpolating between two dual descriptions of the null geodesic congruence. One description constructs a complex null geodesic congruence in a complex space-time whose source is a complex worldline, a virtual source as viewed from the holographic screen. This complex null geodesic congruence intersects the real asymptotic boundary when its source lies on a particular open-string type structure in the complex space-time. The other description constructs a real, twisting, shear-free or asymptotically shear-free null geodesic congruence in the real space-time, whose source (at least in Minkowski space) is in general a closed-string structure: the caustic set of the congruence. Finally we show that virtually all of the interior space-time physical quantities that are identified at null infinity $\mathfrak{I}^+$ (center of mass, spin, angular momentum, linear momentum, and force) are given kinematic meaning and dynamical descriptions in terms of the complex worldline.

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I. INTRODUCTION

In this paper, we describe some interesting structures based in classical special and general relativity which bear some resemblance to dualities known as "holographic" dualities which have emerged elsewhere in theoretical physics over the past decades (cf. [1–3]). Though these holographic dualities usually involve the use of highly nonclassical machinery such as supersymmetry or string theory, most famously in the case of the AdS/CFT correspondence [4–6], we emphasize that our discussion here will use no such tools; we work entirely in the context of classical four-dimensional Lorentzian space-time. The structures we are interested in emerge naturally from the study of light-cone foliations (and their generalization to asymptotically shear-free null geodesic congruences) in space-time and have simply been overlooked in prior research. Although it would certainly be presumptuous for us to suggest that our work here has any true connection with holographic duality as it is known to most theoretical physicists, we do find a holographic screen, open and closed classical strings, and other suggestive objects, all of which can be given real physical meaning in four-dimensional space-time.

Specifically, it is the purpose of this note to first explore the properties of ordinary "run of the mill" light cones and then turn to their generalization via complex and virtual light cones in four-dimensional Lorentzian space-times. More precisely, we study the properties of light cones and their complex generalizations both in Minkowski space and in asymptotically flat (vacuum and Einstein-Maxwell) space-times in the neighborhood of future null infinity. This is followed by a discussion of physical applications of these ideas and constructs. These complex light cones are first applied to the structure of *real* Maxwell fields in *real* Minkowski space. The complex cones in flat space-time are then generalized and applied to study equations of motion in general relativity. Along the way we point out a pretty duality between the complex light cones and real shear-free, but twisting, null geodesic congruences.

The first issue raised comes from the simple question: In Minkowski space, avoiding or ignoring their apex, what are the geometric properties that a set of null geodesics must have in order to be a light cone? How can they be determined to be light cones even far from their apex? The answer is simple: First of all, the family of relevant null geodesics (the light-cone generators) must be (null) surface forming; they must lie on a null surface and thus have vanishing "twist." As a result, these surfaces must be foliated by null geodesics whose tangent vectors are determined by the gradient of the surface. Second, they must have vanishing shear and nonvanishing divergence. (The plane null surfaces can be thought of as light cones but with their apex at infinity; we ignore this case.) From this requirement, light cones posses the topology of $S^2 \times \mathbb{R}$. For us, the most relevant feature is their *vanishing shear*. Even far from their apex (i.e., even at future null infinity $\mathfrak{I}^+$) on the S^2 portion of the surface, if the shear vanishes, then it has an apex and the surface is a light cone. This case

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can be generalized from an individual light cone to a family of light cones: If Minkowski space is (partially) foliated by a null geodesic congruence (NGC), can we tell at $\mathfrak{I}^+$ that the geodesics all focus to a timelike worldline in the interior? The answer again is simple: If the congruence has vanishing twist and shear at $\mathfrak{I}^+$ and nonvanishing divergence and furthermore has no members lying tangent to $\mathfrak{I}^+$ itself (the regularity condition), then there is a one-parameter family of light cones and a timelike worldline to which the NGC converges.

The main goal of this work is to investigate and analyze how this asymptotic description of light cones can be generalized, and what applications to physics it might have. The generalization will be in two distinctly different but related directions.

First of all, in the context of Minkowski space, we define and describe *complex light cones*. They will be determined solely from the properties of specific sets of null directions at complexified null infinity ($\mathfrak{I}^+_\mathbb{C}$), the analytic continuation of Penrose's future null infinity $\mathfrak{I}^+$. These complex null directions, normal to specific slices of $\mathfrak{I}^+_\mathbb{C}$, define—by following them backwards in time—complex null geodesics (and complex light cones) which converge to points in complex Minkowski space [7,8]. In general there will be a subset of points in complex Minkowski space where one (or more) of its light-cone generators intersects real Minkowski space at real $\mathfrak{I}^+$. If instead of the complex light cone of a single point in complex Minkowski space we take a complex analytic "timelike" (to be defined) worldline parametrized by the complex parameter τ, we would have a two-real-dimensional set of complex points (from the real and imaginary parts of τ) and their light cones. We show that for any fixed value of the real part there is a one-dimensional set of points such that the envelope formed by their individual light cones intersects real $\mathfrak{I}^+$ on an S^2 slice. As the real parameter changes we obtain a one-parameter family of real slicings of $\mathfrak{I}^+$.

At this point, an interesting duality emerges. On one hand, if we start from each of these slices and move backwards into the complex space along the complex null directions, these trajectories converge to an imaginary line segment in the complex space. On the other hand, there is a dual method (described later) for following null geodesics from the slices back into the real space-time; this yields a real shear-free, but twisting, null geodesic congruence. It is precisely this twist which links the two pictures: The "distance" of the complex worldline from the real Minkowski space-time in the first picture is a measure of the twist of the real congruence in the latter picture. The caustic set of the real (dual) congruence is (in general) a closed curve moving in real time, something analogous to a classical closed string [8].

The extension of these ideas to asymptotically flat Einstein space-times initially seems to be impossible. Standard light cones from any given space-time point will undergo such distortions from the curvature of the space-time itself that little or no memory of their origin will remain when they arrive at $\mathfrak{I}^+$. Nevertheless, we can consider the possibility of using the procedure that was successful in the Minkowski space-time case by asking for null geodesic congruences in the neighborhood of $\mathfrak{I}^+$ that are shear-free and nontwisting. In the general asymptotically flat case, *shear-free* null geodesic congruences do not exist—but there are always null geodesic congruences that are *asymptotically shear-free* in the neighborhood of $\mathfrak{I}^+$. Unfortunately, to use this idea effectively again entails the analytic extension of the space-time a small distance into the complex. Working on the complexification of $\mathfrak{I}^+$ (i.e., on $\mathfrak{I}^+_\mathbb{C}$), there is a construction of complex slices or "cuts" whose complex null normals can be used to determine *asymptotically shear-free and twist-free complex null geodesics* [7,9,10]. In fact one can construct a four-complex-dimensional family of such complex cuts which define a four-complex-dimensional manifold frequently referred to as $\mathcal{H}$ space [11–14]. The immediately relevant feature for us is that these complex null geodesics from each complex cut converge or focus to a point in $\mathcal{H}$ space [15]. It will be shown later that real structures associated with $\mathcal{H}$ space can be found and that real physics can be interpreted as taking place in $\mathcal{H}$ space [7,16]. The $\mathcal{H}$ space can thus be viewed as the virtual image space seen by looking backwards along complex null directions from a sphere of points on $\mathfrak{I}^+_\mathbb{C}$. It is this property that could allow us to refer to $\mathfrak{I}^+_\mathbb{C}$ as a holographic screen.

The prior discussion of complex Minkowski space (which is a special case of $\mathcal{H}$ space) can be extended to $\mathcal{H}$ space. There is a subset of points in $\mathcal{H}$ space where one (or more) of the light-cone generators (null geodesics) coming from a complex point intersects real asymptotically flat space-time at real $\mathfrak{I}^+$. If instead of the complex light cone of a single point in complex $\mathcal{H}$ space we take a complex timelike (to be defined) worldline parametrized by the complex parameter $\tau = s + i\lambda$, we would have a two-real-dimensional set of complex points (from the real and imaginary parts of τ) and their associated complex light cones. For any fixed value of the real part, s, there is a one-dimensional set of points (a finite interval parametrized by λ) such that the envelope formed by their individual light cones intersects real $\mathfrak{I}^+$ on a S^2 slice. As the real parameter s changes we obtain in the $\mathcal{H}$ space a ribbon (the finite interval moving in "s time") which could be called a classical open string; from the null cones of points on this ribbon, we get a one-parameter family of real slicings on $\mathfrak{I}^+$. All the information about the ribbon is encoded (holographically) in the one-parameter family of real $\mathfrak{I}^+$ slicings and a null direction field on $\mathfrak{I}^+$. In other words, there is a duality between the coded information on $\mathfrak{I}^+$ and $\mathcal{H}$-space information. A further related duality is that a given complex analytic worldline in $\mathcal{H}$ space (via its associated ribbon) yields in the physical space-time (via a complex-conjugate action) a real twisting but asymptotically shear-free null geodesic congruence in the real space-time.

The question of where this beautiful mathematical structure makes contact with physical issues does have a simple answer—the details, however, are rather complicated.

The simple answer is that an (analytic) asymptotically flat Maxwell field in Minkowski space with nonvanishing total charge generates, in the complex Minkowski space, a unique complex analytic worldline: the *complex center of charge* worldline, where the real part describes the standard center of charge while the ribbon thickness encodes magnetic dipole information [17,18]. For the case of asymptotically flat space-times there are two situations: the vacuum asymptotically flat and the Einstein-Maxwell asymptotically flat space-times. For the vacuum case there is a unique complex $\mathcal{H}$-space worldline that contains, from the real part, the equations of motion for the physical center of mass and, from the ribbon thickness, the spin angular momentum, with both interpretations arising from the "view" at infinity, $\mathfrak{I}^+$ [7,16]. These are (loosely) analogues of measuring the total charge at infinity via Gauss's law or observing the Bondi energy-momentum vector at infinity.

In Sec. II, the preliminaries, we introduce our notation and results from earlier investigations that will be needed here. Specifically we first discuss conventions and notation followed by a description of flat-space null geodesic congruences. The section ends with a brief summary of properties of asymptotically flat spaces and their null geodesic congruences. Section III deals with real space-time structures that are associated with the complex worldlines, first in complex Minkowski space and then in $\mathcal{H}$ space. In Sec. IV, we apply the ideas associated with the complex worldlines to real physical ideas. In particular, we show that a real asymptotically flat Maxwell field (with nonvanishing charge) determines a complex worldline (the complex center of charge) that carries information about both the electric and magnetic dipole moments. This construction is then generalized to asymptotically flat space-times, where the complex mass dipole moment (the real mass dipole moment plus "i" times the angular momentum) determines an $\mathcal{H}$-space worldline. The Bianchi identities then yield kinematic definitions, equations of motion, angular momentum, and conservation laws. All take place in $\mathcal{H}$ space, which we interpret as a virtual image space. This information is coded into the real space-time by functions on real $\mathfrak{I}^+$. The results are very reminiscent of ordinary Newtonian dynamical laws of motion. Though partially a summary of results presented elsewhere in the literature (e.g., [7,16,18]), our presentation includes several simplifications and alterations. In Sec. V, we summarize the earlier discussion and speculate on what meaning and possible future use there might be to the observations made here. The appendix provides some background on tensorial spin-s spherical harmonics, which are used throughout this work.

We again stress that the strange results described here lie wholly in standard four-dimensional classical physics. There is no need—other than assumed analyticity—to rely on drastic modifications of space-time properties such as supersymmetry or higher dimensions. The results are here to be seen and perhaps understood. It would have been a cruel god to have laid down such a pretty scheme and not have it mean something deep.

II. FOUNDATIONS: $\mathfrak{I}^+$, NULL GEODESIC CONGRUENCES, AND ASYMPTOTIC FLATNESS

In this section we summarize several of the basic ideas and tools which are needed in our later discussions. The explanations are rather concise and extensive proofs are omitted. In large part, much of what is covered in this section should be familiar to many or even most workers in general relativity.

A. Conventions and notation

The arena for most of our discussion is the neighborhood of the "far (infinite) null future" of our space-time (intuitively the end points of future-directed null geodesics) for both Minkowski space and asymptotically flat space-times. This region, first defined and studied by Roger Penrose and referred to as future null infinity $\mathfrak{I}^+$, is constructed by the rescaling of the space-time metric by a conformal factor which approaches zero asymptotically, the zero value defining $\mathfrak{I}^+$ [19–21]. This process leads to the (future-null) boundary being a null hypersurface for the conformally rescaled metric with topology $S^2 \times \mathbb{R}$. An easy visualization of the boundary $\mathfrak{I}^+$ is as the past light cone of the point $\mathbf{I}^+$, future timelike infinity. A natural coordinatization of $\mathfrak{I}^+$ and its neighborhood is via a Bondi coordinate system: $(u, r, \zeta, \bar{\zeta})$. In this system, u, the Bondi time, labels the null surfaces of the space-time that intersect $\mathfrak{I}^+$; r is the affine parameter along the null geodesics of the constant u surfaces; and $\zeta = e^{i\phi}\cot(\theta/2)$ is the complex stereographic angle that labels the null geodesics of $\mathfrak{I}^+$, the S^2 portion of $\mathfrak{I}^+$ [9].

In Minkowski space, the Bondi coordinates $(u, \zeta, \bar{\zeta})$ of $\mathfrak{I}^+$ can be constructed from the intersection of the future-null cones of the timelike worldline at the Minkowski space spatial origin, i.e., from the line, $x^a = (t, 0, 0, 0)$. The cone has the form

$$\begin{aligned} x^a &= u_{\mathrm{ret}}\delta_0^a + r\hat{l}^a(\zeta, \bar{\zeta}), \\ \zeta &= e^{i\phi}\cot(\theta/2), \\ u_{\mathrm{ret}} &= t - r = \sqrt{2}u \\ \hat{l}^a &= \frac{\sqrt{2}}{2(1+\zeta\bar{\zeta})}(1+\zeta\bar{\zeta}, \zeta+\bar{\zeta}, -i(\zeta - i\bar{\zeta}), -1+\zeta\bar{\zeta}) \\ &= \frac{\sqrt{2}}{2}(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \end{aligned}$$

with $(\zeta, \bar{\zeta})$ labeling the sphere of null directions at the origin and $u_{\rm ret}$ the retarded time. $\mathfrak{I}^+$ is the limit as r tends to infinity.

Remark II.1.—Note that $u_{\rm ret}$, u, and t (and the variable τ introduced later) all have the dimensions of length. In Sec. IV, the velocity of light, c, will be explicitly introduced via the replacement $(u_{\rm ret}, u, t, \tau) \rightarrow (cu_{\rm ret}, cu, ct, c\tau)$ so that $u_{\rm ret}$, u, t, and τ have the dimensions of time.

Remark II.2.—We note that the round sphere metric $ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$ becomes in stereographic coordinates $ds^2 = 4P^{-2}d\zeta d\bar{\zeta}$, with $P = 1 + \zeta\bar{\zeta}$.

To reach $\mathfrak{I}^+$, we simply let $r \rightarrow \infty$, so that $\mathfrak{I}^+$ has coordinates $(u, \zeta, \bar{\zeta})$. The choice of a Bondi coordinate system is not unique, there being a variety of Bondi coordinate systems to choose from. The coordinate transformations between any two are known as Bondi-Metzner-Sachs (BMS) transformations or as the BMS group (cf. [22,23]).

Our assumption of the analyticity of the space-time then allows for the complexification of $\mathfrak{I}^+$. For this complexification (i.e., extension to $\mathfrak{I}^+_{\mathbb{C}}$), we allow u to take on complex values close to the real and free $\bar{\zeta}$ from being the complex conjugate of ζ. It is then denoted by $\tilde{\zeta} \approx \bar{\zeta}$. (Often we take this as implicitly understood and just use $\bar{\zeta}$.)

Associated with the Bondi coordinates is a (Bondi) null tetrad system, $(l^a, n^a, m^a, \bar{m}^a)$ (cf. [9,24]):

$$l^a l_a = n^a n_a = m^a m_a = \bar{m}^a \bar{m}_a = 0,$$
$$l^a n_a = -m^a \bar{m}_a = 1.$$

The first tetrad vector l^a is the tangent to the geodesics of the constant u null surfaces given by

$$l^a = \frac{dx^a}{dr} = g^{ab}\nabla_b u, \tag{2.1}$$

$$l^a \nabla_a l^b = 0, \tag{2.2}$$

$$l^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial r}. \tag{2.3}$$

The second null vector n^a is tangent to the null geodesics lying on $\mathfrak{I}^+$, normalized so that

$$l_a n^a = 1. \tag{2.4}$$

The remaining vector m^a and its complex conjugate are tangent to the S^2 slices of constant u.

An important construct is the family of past light cones from each point of $\mathfrak{I}^+$ (or $\mathfrak{I}^+_{\mathbb{C}}$). Each past cone is determined by a sphere's worth of null directions at $\mathfrak{I}^+$ with each null direction labeled by the associated sphere coordinate. These coordinates are chosen as the complex stereographic coordinates and are denoted by the complex-conjugate pair $(L, \bar{L})$. An arbitrary field of null directions on $\mathfrak{I}^+$ (and consequently an arbitrary null geodesic congruence that intersects $\mathfrak{I}^+$) can then be described by the function $L = L(u, \zeta, \bar{\zeta})$ or its analytic extension to $\mathfrak{I}^+_{\mathbb{C}}$.

Often we will use a very specific form of a null tetrad given in Minkowski coordinates and parametrized by the points on the sphere in stereographic coordinates $(\zeta, \bar{\zeta})$ and denoted by the overhat:

$$\begin{aligned}
\hat{l}^a &= \frac{\sqrt{2}}{2(1+\zeta\bar{\zeta})}(1+\zeta\bar{\zeta}, \zeta+\bar{\zeta}, -i(\zeta - i\bar{\zeta}), -1+\zeta\bar{\zeta}) \\
&= \left(\frac{\sqrt{2}}{2}, \frac{1}{2}Y^0_{1i}\right), \\
\hat{n}^a &= \frac{\sqrt{2}}{2(1+\zeta\bar{\zeta})}(1+\zeta\bar{\zeta}, -(\zeta+\bar{\zeta}), i(\zeta - i\bar{\zeta}), 1-\zeta\bar{\zeta}) \\
&= \left(\frac{\sqrt{2}}{2}, -\frac{1}{2}Y^0_{1i}\right), \\
\hat{m}^a &= \eth l^a = \frac{\sqrt{2}}{2(1+\zeta\bar{\zeta})}(0, 1-\bar{\zeta}^2, -i(1+\bar{\zeta}^2), 2\bar{\zeta}) \\
&= (0, -Y^1_{1i}), \\
\hat{\bar{m}}^a &= \bar{\eth} l^a = \frac{\sqrt{2}}{2(1+\zeta\bar{\zeta})}(0, 1-\zeta^2, i(1+\zeta^2), 2\zeta), \\
&= (0, -Y^{-1}_{1i}).
\end{aligned}$$

As $(\zeta, \bar{\zeta})$ move over the complex plane, $\hat{l}^a$ and $\hat{n}^a$ range over the light cone. The spin-s harmonics [25], $Y^s_{l,ijk\ldots}(\zeta, \bar{\zeta})$, which are frequently used, are described in the appendix.

B. Flat-space null geodesic congruences

In Minkowski space $\mathbb{M}$, the future light cones from an arbitrary timelike worldline $x^a = \xi^a(s)$ can be described by the NGC

$$x^a = \xi^a(s) + r\hat{l}^a(\zeta, \bar{\zeta}) \tag{2.5}$$

with r the affine parameter on each of the light-cone generators. This construct is easily generalized to complex Minkowski space $\mathbb{M}_{\mathbb{C}}$, where light cones from $\xi^a(\tau)$ (now an arbitrary complex analytic worldline with complex affine parameter τ), and its corresponding *complex* NGC is

$$z^a = \xi^a(\tau) + r\hat{l}^a(\zeta, \tilde{\zeta}), \tag{2.6}$$

where r is now complex and $(\zeta, \tilde{\zeta})$ are independent of each other.

The Sachs complex optical parameters for an arbitrary NGC (real or complex) are the complex divergence and shear of the congruence [24,26],

$$\rho = \tfrac{1}{2}(-\nabla_a l^a + i\,\mathrm{curl}\, l^a), \tag{2.7}$$

$$\sigma = \nabla_{(a} l_{b)} m^a m^b, \tag{2.8}$$

where

$$\mathrm{curl}\, l^a \equiv \sqrt{(\nabla_{[a} l_{b]} \nabla^a l^b)}.$$

These satisfy the flat-space optical equations:

$$D\rho = \rho^2 + \sigma\sigma,$$
$$D\sigma = (\rho + \bar{\rho})\sigma, \tag{2.9}$$
$$D = l^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial r},$$

with r the affine parameter along the geodesics. The optical parameters for the above light-cone congruence can be calculated directly from Eq. (2.5) yielding

$$\rho = -r^{-1}, \qquad \sigma = 0. \tag{2.10}$$

By reversing the statement and assuming a NGC with vanishing shear and real divergence, the optical equations become

$$D\rho = \rho^2.$$

The integral (i.e., $\rho = -r^{-1}$) is the same as Eq. (2.10), thus showing that a NGC with real divergence and vanishing shear is the light-cone congruence of a (real) timelike worldline.

An arbitrary NGC in Minkowski space can be described by the three-parameter, $(u, \zeta, \bar{\zeta})$, family of null geodesics

$$x^a = u(\hat{l}^a + \hat{n}^a) - \bar{L}\hat{m}^a - L\hat{\bar{m}}^a + (r - r_0)\hat{l}^a, \tag{2.11}$$

where r is the affine parameter, $r_0 = r_0(u, \zeta, \bar{\zeta})$ is the arbitrary origin for the affine parameter, and $L = L(u, \zeta, \bar{\zeta})$, the determining function of the congruence, is an arbitrary complex spin-weight one function of the parameters. The three parameters $(u, \zeta, \bar{\zeta})$ are the Bondi coordinates of the intersection points of the null geodesics with $\mathfrak{I}^+$. The optical parameters are determined by L, which is the stereographic angle field on $\mathfrak{I}^+$ that determines the directions of the null geodesics.

The condition for a NGC with vanishing shear is that the function L must satisfy the nonlinear partial differential equation [27]

$$\eth L + L\dot{L} = 0, \tag{2.12}$$

where $\dot{L} = L_{,u}$ and $\eth$ is the spin-weighted covariant derivative on the 2-sphere (see the appendix for details) [28]. This can be integrated by introducing an auxiliary complex variable $\tau = T(u, \zeta, \bar{\zeta})$, related to L by the so-called Cauchy-Riemann equation (related to the existence of a Cauchy-Riemann structure on $\mathfrak{I}^+$ [29])

$$\eth T + L\dot{T} = 0 \tag{2.13}$$

and then using its inversion

$$u = G(\tau, \zeta, \bar{\zeta}), \tag{2.14}$$

$$\tau = T(G(\tau, \zeta, \bar{\zeta}), \zeta, \bar{\zeta}) \equiv \tau. \tag{2.15}$$

After a process of implicit differentiation (cf. [7,10,30]), Eq. (2.12) becomes

$$\eth_\tau^2 G = 0, \tag{2.16}$$

with

$$L = \eth_\tau G|_{\tau = T(u,\zeta,\bar{\zeta})},$$

where the subscript τ indicates that the differentiation is at τ held constant. From this it follows that the regular solutions to Eq. (2.17) can be given implicitly in terms of G as

$$u = G(\tau, \zeta, \bar{\zeta}) \equiv \xi^a(\tau)\hat{l}_a \Leftrightarrow \tau = T(u, \zeta, \bar{\zeta}),$$
$$L = \eth G \equiv \xi^a(\tau)\hat{m}_a = \xi^a(T(u, \zeta, \bar{\zeta}))\hat{m}_a. \tag{2.17}$$

Several remarks must be made here:

(i) $\xi^a(\tau)$ are four arbitrary complex analytic functions of the complex parameter τ which can be interpreted as determining a complex worldline in complex Minkowski space.

(ii) τ can be regauged by the analytic function $\tau^* = F(\tau)$. Often it is useful to chose $\xi^0(\tau) \equiv \tau$.

(iii) Since τ is complex we must allow u to take complex values which requires the complexification of $\mathfrak{I}^+$, denoted $\mathfrak{I}^+_{\mathbb{C}}$.

(iv) When the $\xi^a(\tau)$ are real functions of a real variable s, Eq. (2.11) reduces to Eq. (2.5) (i.e., to the real worldline light-cone congruence).

(v) For a complex set $\xi^a(\tau)$, the NGC, Eq. (2.11), is a real shear-free NGC but with a *nonvanishing* twist. The caustic set is in general a closed curve moving in time.

An important observation that plays a major role for us is the following: From the same L, two different "conjugate" versions can be constructed. The first is obviously the complex conjugate given by $\bar{L} = \bar{\xi}^a(\bar{\tau})\hat{\bar{m}}_a$ while the second, referred to as the holomorphic conjugate, is given by $\tilde{L} = \bar{\eth}G = \xi^a(\tau)\hat{\bar{m}}_a$. Using $\tilde{L}$ in Eq. (2.11) instead of $\bar{L}$, we obtain another shear-free NGC but now it is the complex congruence, given earlier by Eq. (2.6):

$$z^a = \xi^a(\tau) + r\hat{l}^a(\zeta, \tilde{\zeta}).$$

In other words, the cut function Eq. (2.17) describes a family of null cones with an apex on the complex line, $z^a = \xi^a(\tau)$. We now have on $\mathfrak{I}^+$ and $\mathfrak{I}^+_{\mathbb{C}}$ two different tetrad systems (obtained by null rotations from the Bondi tetrad) coming from $\bar{L}$ and $\tilde{L}$, namely,

$$l^a \to l^{*a} = l^a - \frac{\bar{L}}{r}m^a - \frac{L}{r}\bar{m}^a + O(r^{-2}), \tag{2.18}$$

$$m^{*a} = m^a - \frac{\bar{L}}{r}n^a, \tag{2.19}$$

$$n^{*a} = n^a \tag{2.20}$$

and

$$l^a \to l_C^{*a} = l^a - \frac{\tilde{L}}{r} m^a - \frac{L}{r} \bar{m}^a + O(r^{-2}), \tag{2.21}$$

$$m^{*a} = m^a - \frac{\tilde{L}}{r} n^a, \tag{2.22}$$

$$n^{*a} = n^a. \tag{2.23}$$

The null geodesic congruence determined by l^{*a}, as mentioned earlier, is a real shear-free congruence with twist while the congruence determined by l_C^{*a} is a complex, shear-free, twist-free congruence and focuses on the complex curve $\xi^a(\tau)$.

Though the complex null geodesics with an apex on $\xi^a(\tau)$ spend most of their "time" in the complex Minkowski space, some do reach real Minkowski space and, in particular, some reach the real $\mathfrak{I}^+$. It turns out that the complex worldline and their associated light cones have real structures. They are discussed in Sec. III.

C. Asymptotic flatness

At a first glance it would appear as if it were not possible to duplicate the Minkowski space discussion of light-cone NGCs in asymptotically flat space-times. Aside from a few special cases (the algebraically special metrics) there are no Einstein space-times with shear-free NGCs. The family of future-directed null geodesics originating at a fixed space-time point traversing regions of curvature will, in general, be distorted and develop shear. Surprisingly it nevertheless is possible to duplicate virtually all the light-cone NGC results of flat space-time for the general case of asymptotically flat space-times by looking not for shear-free NGCs but instead *asymptotically shear-free congruences.* In fact such congruences are determined by a complex analytic curve in an auxiliary four-complex-dimensional space, referred to as $\mathcal{H}$ space.

Before describing these congruences we first review some relevant features of asymptotically flat space-times. Details, derivations, and proofs are largely omitted since they are easily found in the literature [7,9,24,26].

We begin by pointing out that with Bondi coordinates and tetrad the two optical parameters, the complex divergence and shear are given by

$$\rho = \bar{\rho} = -\frac{1}{r} + \frac{\sigma^0 \bar{\sigma}^0}{r^3} + O(r^{-5}), \qquad \sigma = \frac{\sigma^0}{r^2} + O(r^{-4}), \tag{2.24}$$

with $\sigma^0 = \sigma^0(u, \zeta, \bar{\zeta})$, the asymptotic Bondi shear of the NGC with the Bondi tangent vector, i.e., l^a. The σ^0, which is the free data determining the gravitational radiation, plays a major role in our discussion. Considering a new NGC with tangent vector l^{*a} defined at $\mathfrak{I}^+$ by the null rotation

$$l^{*a} = l^a - \frac{\bar{L}}{r} m^a - \frac{L}{r} \bar{m}^a + O(r^{-2}), \tag{2.25}$$

$$m^{*a} = m^a - \frac{\bar{L}}{r} n^a, \tag{2.26}$$

$$n^{*a} = n^a \tag{2.27}$$

with arbitrary $L = L(u, \zeta, \bar{\zeta})$, one finds that the asymptotic shear of the new congruence is given by a version of the Sachs theorem [27]:

$$\sigma^{0*} = \eth L + L\dot{L} - \sigma^0. \tag{2.28}$$

The condition for the new congruences to be asymptotically shear-free ($\sigma^{0*} = 0$) is thus that L satisfy

$$\eth L + L\dot{L} = \sigma^0, \tag{2.29}$$

which is the extension of the flat-space Eq. (2.12).

As in the Minkowski space case, Eq. (2.12), this can also be integrated by introducing the same auxiliary complex variable $\tau = T(u, \zeta, \bar{\zeta})$, related to L by the Cauchy-Riemann equation

$$\eth T + L\dot{T} = 0 \tag{2.30}$$

and then using its inversion

$$u = G(\tau, \zeta, \bar{\zeta}), \tag{2.31}$$

$$\tau = T(G(\tau, \zeta, \bar{\zeta}), \zeta, \bar{\zeta}) \equiv \tau. \tag{2.32}$$

Note that as in the flat case, τ is complex and we must allow the complexification of u and let $\bar{\zeta} \approx \tilde{\zeta}$. For each value of τ we obtain a complex cut of $\mathfrak{I}^+_{\mathbb{C}}$.

Again after manipulating several implicit derivatives, Eqs. (2.29) and (2.30) become

$$\eth_\tau^2 G = \sigma^0(G, \zeta, \bar{\zeta}), \tag{2.33}$$

$$L(u, \zeta, \bar{\zeta}) = \eth_\tau G|_{\tau=T(u,\zeta,\bar{\zeta})} \tag{2.34}$$

with again the subscript τ indicating that the derivatives are at τ held constant. Equation (2.33), the "good-cut equation," has been shown to depend on four complex parameters, z^a, (the $\mathcal{H}$-space coordinates), so that we can write $u = X(z^a, \zeta, \bar{\zeta})$, where we distinguish X from G by its explicit dependence on the four solution parameters. By the coordinate freedom

$$z^a \to z^{*a} = f^a(z^a),$$

the first four spherical harmonic coefficients can be chosen as the z^a (coordinate conditions on $\mathcal{H}$ space) so that we have

$$u = X(z^a, \zeta, \bar{\zeta}) = z^a \hat{l}_a(\zeta, \bar{\zeta}) + H_{l\geq 2}(z^a, \zeta, \bar{\zeta}), \tag{2.35}$$

where $H_{l\geq 2}$ are spherical harmonic contributions with $l \geq 2$. Finally, by taking an arbitrary worldline in the $\mathcal{H}$ space, $z^a = \xi^a(\tau)$, we find the general regular solution to Eq. (2.29) is given implicitly by

$$L(u, \zeta, \bar{\zeta}) = \eth_\tau G|_{\tau=T(u,\zeta,\bar{\zeta})}, \tag{2.36}$$

$$G(\tau,\zeta,\bar\zeta)\equiv X(\xi^a(\tau),\zeta,\bar\zeta), \tag{2.37}$$

$$\begin{aligned} u = G(\tau,\zeta,\bar\zeta) &= \xi^a(\tau)\hat l_a(\zeta,\bar\zeta) + H_{l\geq 2}(\xi^a(\tau),\zeta,\bar\zeta) \\ &= \xi^a(\tau)\hat l_a(\zeta,\bar\zeta) + \tilde H_{l\geq 2}(\tau,\zeta,\bar\zeta). \end{aligned} \tag{2.38}$$

As in the flat-space case, the regular asymptotically shear-free NGCs are determined by the arbitrary choice of a complex worldline in an auxiliary complex space, $\mathcal{H}$ space.

In complete analogy with the complex Minkowski space case, it turns out that if we use the complex NGC determined by the stereographic angle field, (2.36), and the associated holomorphic field, $\tilde L(u,\zeta,\bar\zeta) = \bar\eth_\tau G|_{\tau=T(u,\zeta,\bar\zeta)}$ [not $\bar L(u,\zeta,\bar\zeta) = \bar\eth_\tau \bar G|_{\bar\tau=\bar T(u,\zeta,\bar\zeta)}$] as initial directions at $\mathfrak{I}^+_{\mathbb{C}}$

$$l^{*a}_C = l^a - \frac{\tilde L}{r}m^a - \frac{L}{r}\bar m^a + O(r^{-2}), \tag{2.39}$$

the complex geodesics converge on the $\mathcal{H}$-space worldline $z^a = \xi^a(\tau)$ [15]. By pointing into complex null directions from the complexified $\mathfrak{I}^+_{\mathbb{C}}$, we have complex *virtual cones* and complex *virtual worldlines*. There will always be points on the complex worldline whose null cones partially intersect real $\mathfrak{I}^+$. We will see later that unique worldlines can be determined so that real meaning or significance can be given to them, as complex centers of charge and complex centers of mass—which include both asymptotic magnetic dipoles and angular momentum.

The basic idea that will be pursued later (in Secs. III and IV) is to identify certain terms in the asymptotic behavior of the Maxwell and Weyl tensor tetrad components with physical quantities and then see how they change when they are computed with the (rotated) complex null directions pointing towards a complex worldline, Eq. (2.39). By choosing the worldline appropriately, so that these quantities vanish, we identify the virtual complex centers of charge and mass.

As an interim step, we need the behavior of both the tetrad components of the Weyl and Maxwell tensors:

$$\begin{aligned} \psi_0 &= -C_{abcd}l^a m^b l^c m^d, \\ \psi_1 &= -C_{abcd}l^a n^b l^c m^d, \\ \psi_2 &= -\tfrac{1}{2}(C_{abcd}l^a n^b l^c n^d - C_{abcd}l^a n^b m^c \bar m^d), \\ \psi_3 &= C_{abcd}l^a n^b n^c \bar m^d, \\ \psi_4 &= C_{abcd}n^a \bar m^b n^c \bar m^d, \end{aligned} \tag{2.40}$$

and

$$\phi_0 = F_{ab}l^a m^b, \qquad \phi_1 = \tfrac{1}{2}F_{ab}(l^a n^b + m^a\bar m^b), \qquad \phi_2 = F_{ab}n^a\bar m^b. \tag{2.41}$$

Integrating both the Weyl tensor and Maxwell spin-coefficient equations leads to the peeling behavior:

$$\begin{aligned} &\psi_0 = \psi_0^0 r^{-5} + O(r^{-6}), \quad \psi_1 = \psi_1^0 r^{-4} + O(r^{-5}), \\ &\psi_2 = \psi_2^0 r^{-3} + O(r^{-4}), \quad \psi_3 = \psi_3^0 r^{-2} + O(r^{-3}), \\ &\psi_4 = \psi_4^0 r^{-1} + O(r^{-2}), \end{aligned} \tag{2.42}$$

and

$$\phi_0 = \phi_0^0 r^{-3} + O(r^{-4}), \qquad \phi_1 = \phi_1^0 r^{-2} + O(r^{-3}), \qquad \phi_2 = \phi_2^0 r^{-1} + O(r^{-2}). \tag{2.43}$$

With the coefficients satisfying the asymptotic Bianchi identities

$$\dot\psi_2^0 = -\eth\psi_3^0 + \sigma^0\psi_4^0, \tag{2.44}$$

$$\dot\psi_1^0 = -\eth\psi_2^0 + 2\sigma^0\psi_3^0, \tag{2.45}$$

$$\dot\psi_0^0 = -\eth\psi_1^0 + 3\sigma^0\psi_2^0, \tag{2.46}$$

where

$$\psi_4^0 = -\ddot{\bar\sigma}^0, \tag{2.47}$$

$$\psi_3^0 = \eth\dot{\bar\sigma}^0, \tag{2.48}$$

and asymptotic Maxwell equations

$$\dot\phi_1^0 = -\eth\phi_2^0, \tag{2.49}$$

$$\dot\phi_0^0 = -\eth\phi_1^0 + \sigma^0\phi_2^0. \tag{2.50}$$

All the r^{-n} coefficients, which are functions on $\mathfrak{I}^+$, i.e., functions of $(u,\zeta,\bar\zeta)$, have physical meaning, e.g., multipole moments, etc. The four quantities $\{\psi_1^0, \psi_2^0, \phi_0^0, \phi_1^0\}$ are the most important to us, due to the following properties:

(i) The $l = 0$ harmonic of ϕ_1^0 is the Coulomb charge q. It is assumed to be nonvanishing whenever a Maxwell field is being considered.
(ii) The $l = 1$ harmonic of ϕ_0^0 is the complex electromagnetic dipole moment: $D^i_{E\&M} = D^i_E + iD^i_M$.
(iii) The $l = 0, 1$ harmonics of ψ_2^0, slightly modified by the shear σ^0, is the Bondi energy-momentum four-vector.
(iv) The $l = 1$ harmonic of ψ_1^0, also slightly modified by the shearσ^0, encodes the center of mass dipole and angular momentum: $D^i_{\mathbb{C}(\mathrm{grav})} = D^i_{\mathrm{mass}} + ic^{-1}J^i$.

By using the tetrad transformation generated by Eq. (2.39) [see Eq. (2.21)], one finds the transformation law of the leading terms of the Weyl and Maxwell tensors:

$$\psi_0^{*0} = \psi_0^0 - 4L\psi_1^{*0} + 6L^2\psi_2^{*0} - 4L^3\psi_3^{*0} + L^4\psi_4^0, \tag{2.51}$$

$$\psi_1^{*0} = \psi_1^0 - 3L\psi_2^0 + 3L^2\psi_3^0 - L^3\psi_4^0, \tag{2.52}$$

$$\psi_2^{*0} = \psi_2^0 - 2L\psi_3^0 + L^2\psi_4^0, \tag{2.53}$$

$$\psi_3^{*0} = \psi_3^0 - L\psi_4^0, \tag{2.54}$$

$$\psi_4^0 = \psi_4^{*0}, \tag{2.55}$$

$$\phi_0^{*0} = \phi_0^0 - 2L\phi_1^0 + L^2\phi_2^0, \tag{2.56}$$

$$\phi_1^{*0} = \phi_1^0 - L\phi_2^0, \tag{2.57}$$

$$\phi_2^{*0} = \phi_2^0. \tag{2.58}$$

Later, setting to zero the $l = 1$ parts of ψ_1^{*0} and ϕ_0^{*0}, we can determine two different worldlines (when a Maxwell field is present) that can be referred to, respectively, as the complex centers of mass and charge.

III. REAL STRUCTURES FROM THE COMPLEX WORLDLINE

Our task in this section is to find the real structures that are lying in the complex worldlines and their complex light cones.

A. Flat-space real structure

We first examine the case of flat space-time with a complex Minkowski worldline, $z^a = \xi^a(\tau)$, and associated light-cone cut of $\mathfrak{I}^+_{\mathbb{C}}$,

$$u = \xi^a(\tau)\hat{l}_a(\zeta, \bar{\zeta}). \tag{3.1}$$

To answer our question, what values of τ allow real values of u, we first write $\tau = s + i\lambda$ (s and λ real), decompose the right-hand side of Eq. (3.1) into its real and imaginary parts, and set the imaginary part to zero [8]:

$$u = \tfrac{1}{2}(\xi^a(s+i\lambda)\hat{l}_a(\zeta,\bar{\zeta}) + \bar{\xi}^a(s-i\lambda)\hat{l}_a(\zeta,\bar{\zeta})) + \tfrac{1}{2}(\xi^a(s+i\lambda)\hat{l}_a(\zeta,\bar{\zeta}) - \bar{\xi}^a(s-i\lambda)\hat{l}_a(\zeta,\bar{\zeta}), \tag{3.2}$$

$$0 = [\xi^a(s+i\lambda) - \bar{\xi}^a(s-i\lambda)]\hat{l}_a(\zeta,\bar{\zeta}). \tag{3.3}$$

Considering Eq. (3.3) as an implicit equation defining

$$\lambda = \Lambda(s, \zeta, \bar{\zeta}) \tag{3.4}$$

we have that the allowed values of τ are given by

$$\tau = s + i\Lambda(s, \zeta, \bar{\zeta}). \tag{3.5}$$

The real values of u are thus given by the one-parameter (s) family of slicings

$$u = \xi^a(s + i\Lambda(s,\zeta,\bar{\zeta}))l_a(\zeta,\bar{\zeta}). \tag{3.6}$$

Assuming small values for the imaginary part of $\xi^a(\tau) = \xi^a_R(\tau) + i\xi^a_I(\tau)$, $[(\xi^a_R(\tau), \xi^a_I(\tau))$ both real analytic functions] and hence small $\Lambda(s,\zeta,\bar{\zeta})$, it has been shown that $\Lambda(s,\zeta,\bar{\zeta})$ (for a fixed value of s) is a bounded smooth function on the $(\zeta,\bar{\zeta})$ sphere, with maximum and minimum values $\lambda_{\max} = \Lambda(s,\zeta_{\max},\bar{\zeta}_{\max})$ and $\lambda_{\min} = \Lambda(s,\zeta_{\min},\bar{\zeta}_{\min})$, respectively. Furthermore on the sphere, there are a circle's (S^1) worth of curves between $(\zeta_{\min},\bar{\zeta}_{\min})$ and $(\zeta_{\max},\bar{\zeta}_{\max})$ such that $\Lambda(s,\zeta,\bar{\zeta})$ is a monotonically increasing function on each curve. Hence there will be a family of circles on the $(\zeta,\bar{\zeta})$ sphere where the value of λ is a constant, ranging between $\lambda_{\max}$ and $\lambda_{\min}$.

Summarizing, we have the result that in the complex τ plane there is a ribbon or strip given by all values of s and a line segment parameterized by λ between $\lambda_{\min}$ and $\lambda_{\max}$ such that the complex light cones from each of the associated points, $\xi^a(s+i\lambda)$, all have *some null geodesics* that intersect real $\mathfrak{I}^+$. More specifically, for each allowed value of $\tau = s + i\Lambda$ there will be a circle's worth of complex null geodesics leaving the point $\xi^a(s+i\lambda)$ reaching real $\mathfrak{I}^+$. It is the union of these null geodesics, corresponding to the circles on the $(\zeta,\bar{\zeta})$ sphere from the line segment, that produces the real family of cuts, Eq. (3.6).

The real structure associated with a complex worldline is then the one-parameter family of slices (cuts) Eq. (3.6) and angle field $L(u,\zeta,\bar{\zeta})$ on each point of the cuts.

The dual point of view, as previously mentioned, is to start with the same L as used earlier:

$$u \equiv \xi^a(\tau)\hat{l}_a \Leftrightarrow \tau = T(u,\zeta,\bar{\zeta}),$$
$$L = \bar{\eth}_\tau G \equiv \xi^a(\tau)\hat{m}_a = \xi^a(T(u,\zeta,\bar{\zeta}))\hat{m}_a, \tag{3.7}$$

which was used with the holomorphic $\tilde{L}$,

$$\tilde{L} = \eth_\tau G \equiv \xi^a(\tau)\bar{\hat{m}}_a = \xi^a(T(u,\zeta,\bar{\zeta}))\bar{\hat{m}}_a,$$

but now, instead, use the complex conjugate of L:

$$\bar{L} = \bar{\eth}_\tau \bar{G} = \bar{\xi}^a(\bar{\tau})\bar{\hat{m}}_a$$

for the null directions pointing inward. In this case one obtains again a real shear-free NGC but now with twist $\Sigma(u,\zeta,\bar{\zeta})$ which comes from the complex divergence:

$$\rho = -\frac{1}{r + i\Sigma}, \tag{3.8}$$

$$2i\Sigma = \eth\bar{L} + L(\bar{L})^{\cdot} - \bar{\eth}L - \bar{L}\dot{L}.$$
$$= (\xi^a(\tau) - \bar{\xi}^a(\bar{\tau}))(n_a - l_a). \tag{3.9}$$

As was claimed earlier, the twist is proportional to the imaginary part of the complex worldline and consequently we have the real structure coming from two (dual) places.

B. Asymptotically flat-space real structure

The extension of the above argument to the case of asymptotically flat space-times is relatively simple. Again assuming that the Bondi shear is sufficiently small and the $\mathcal{H}$-space complex worldline is not too far from the "real," the solution to the good-cut equation (2.33),

$$u = \xi^a(\tau)\hat{l}_a(\zeta,\bar{\zeta}) + \tilde{H}_{l\geq 2}(\tau,\zeta,\bar{\zeta}) \equiv G(\tau,\zeta,\bar{\zeta}), \tag{3.10}$$

with $\tau = s + i\lambda$, is decomposed into real and imaginary parts:

$$G(\tau,\zeta,\bar{\zeta}) = \tfrac{1}{2}(G(s+i\lambda,\zeta,\bar{\zeta}) + \bar{G}(s-i\lambda,\zeta,\bar{\zeta})) + \tfrac{1}{2}(G(s+i\lambda,\zeta,\bar{\zeta}) - \bar{G}(s-i\lambda,\zeta,\bar{\zeta})). \quad (3.11)$$

Setting the imaginary part to zero and solving for λ we obtain an expression of the form

$$\lambda = \Lambda(s,\zeta,\bar{\zeta}).$$

As in the flat case, for fixed $s = s_0$, Λ has values on a line segment bounded between some $\lambda_{\min}$ and $\lambda_{\max}$. The allowed values of τ are again on a ribbon in the τ plane, all values of s and values on the λ-line segments.

Each level curve of the function $\lambda = \Lambda(s_0,\zeta,\bar{\zeta}) =$ constant on the $(\zeta,\bar{\zeta})$ sphere (closed curves or isolated points) determines a specific subset of the null directions and associated null geodesics on the light cone of the complex point $\xi^a(s_0 + i\Lambda(s_0,\zeta,\bar{\zeta}))$ that intersect the real $\mathfrak{I}^+$. These geodesics will be referred to as *real geodesics*. As λ moves over all allowed values of its segment, we obtain the set of $\mathcal{H}$-space points $\xi^a(s_0 + i\Lambda(s_0,\zeta,\bar{\zeta}))$ and their collection of real geodesics. From Eq. (3.11), these real geodesics intersect $\mathfrak{I}^+$ on the cut

$$u = G(s_0 + i\Lambda(s_0,\zeta,\bar{\zeta}),\zeta,\bar{\zeta}).$$

As s varies we obtain a one-parameter family of cuts. If these cuts do not intersect with each other, we say that the complex worldline $\xi^a(\tau)$ is by definition a timelike line. This occurs when the time component of the real part of the complex velocity vector, $v^a(\tau) = d\xi^a(\tau)/d\tau$, is sufficiently large.

C. Summary of real structures

To put the ideas of this section into perspective we collect the claims.

(i) In Minkowski space, the future-directed light cones emanating from a real timelike worldline, $x^a = \xi^a(s)$, intersect future null infinity $\mathfrak{I}^+$ on a one-parameter family of spherical nonintersecting cuts.

(ii) The complex light cones emanating from a timelike complex analytic curve in complex Minkowski space, $z^a = \xi^a(\tau)$ parametrized by the complex parameter $\tau = s + i\lambda$, have for each fixed value of s and λ a limited set null geodesics that reach real $\mathfrak{I}^+$. However, for a ribbon in the complex τ plane (i.e., a region topologically $\mathbb{R}\times I$, with $s \in \mathbb{R}$ and $\lambda \in I = [\lambda_{\min},\lambda_{\max}]$), there will be many null geodesics intersecting $\mathfrak{I}^+$. Such null geodesics were referred to as real geodesics. More specifically, for a fixed s, there is a limited range of λ such that all the real null geodesics intersect $\mathfrak{I}^+$ in a full cut, leading to a one-parameter family of real (distorted sphere) slicings. The ribbon is the generalization of the real worldline and the slicings are the analogues of the spherical slicings. When the ribbon shrinks to a line it degenerates to the real case. We can consider the ribbon as a generalized worldline and the real null geodesics from constant s portion of the ribbon as a generalized light cone.

(iii) For the case of asymptotically flat space-times, the real light cones from interior points are replaced by the virtual light cones generated by the asymptotically shear-free NGCs. These cones emanate from a complex virtual worldline $z^a = \xi^a(\tau)$ in the associated $\mathcal{H}$ space. As in the case of complex Minkowski space, there is a ribbon in the τ plane where the real null geodesics emanate from. The real null geodesics coming from a cross section of the strip at fixed s (as in the complex Minkowski case) intersect $\mathfrak{I}^+$ in a cut, the collection of cuts yielding a one-parameter family of cuts. The situation is exactly the same as in the complex Minkowski space case except that the spherical harmonic decomposition of these cuts is in general more complicated.

Example: The (charged) Kerr metric

Considering the Kerr or the charged Kerr metrics (or even more generally any asymptotically flat stationary metric), we have immediately that the Bondi shear σ^0 vanishes and hence the associated $\mathcal{H}$ space is complex Minkowski space [31,32]. From the stationarity and a real origin shift and rotation, the complex worldline can be put into the form

$$\xi^a(\tau) = (\tau, 0, 0, ia), \quad (3.12)$$

with a being the Kerr parameter. The complex cut function is then

$$u = \xi^a(\tau)\hat{l}_a(\zeta,\bar{\zeta}) = \frac{\tau}{\sqrt{2}} - \frac{i}{2}aY^0_{1,3}(\zeta,\bar{\zeta}), \quad Y^0_{1,3}(\zeta,\bar{\zeta}) = -\sqrt{2}\frac{1-\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}}, \quad (3.13)$$

so that the angle fields are

$$L = \sqrt{2}ia\frac{\bar{\zeta}}{1+\zeta\bar{\zeta}},$$
$$\bar{L} = -\sqrt{2}ia\frac{\zeta}{1+\zeta\bar{\zeta}},$$
$$\tilde{L} = \sqrt{2}ia\frac{\zeta}{1+\zeta\bar{\zeta}}.$$

Using $\tau = s + i\lambda$ in Eq. (3.13), the reality condition $u = \bar{u}$ on the cut function is that

$$\lambda = \Lambda(s,\zeta,\bar{\zeta}) = \frac{\sqrt{2}}{2}aY^0_{1,3}(\zeta,\bar{\zeta}),$$

so that on the τ ribbon, λ ranges between $\pm\sqrt{2}$ and the real slices from the ribbon become simply $u = s/\sqrt{2}$. □

Though we are certainly not making the claim that one can in reality ''observe'' these complex worldlines that arise from (asymptotically) shear-free congruences, we nevertheless claim that they can be observed in a different sense. In the following section we will show that there are simple physical measurements that do determine these complex worldlines.

IV. APPLICATIONS

We can now explore uses of our observations concerning light cones and their generalizations. The first issue addressed is the application, in Minkowski space, to the Maxwell equations and, in particular, to asymptotically vanishing Maxwell fields with nonvanishing charge q. Specifically, we show that such solutions naturally define a complex worldline that can be identified or referred to as the complex center of charge. It is determined from the complex Minkowski space points where the (suitably defined) *complex electromagnetic dipole* (a combination of the electric dipole moment plus the i magnetic dipole moment) vanishes.

The analogous problem for asymptotically flat space-times (either vacuum or Einstein-Maxwell) is addressed with a unique worldline again arising, this time from the gravitational part with its identification as the complex center of mass. These are the $\mathcal{H}$-space points where the *complex gravitational dipole* (identified as the mass dipole plus i angular momentum) vanishes. For the Einstein-Maxwell case there will be, in addition, a complex center of charge line.

A few words of explanation in a much simpler situation might be of use. In Minkowski space, in a given Lorentz frame and coordinate origin, with given charge and current distributions (or given mass and spin distribution), one defines the electric dipole moment (mass dipole) on any time slice by an space integral over the charge density (or mass density) times the position. By shifting the spatial origin, the dipole moment becomes a *space-time field* depending on the origin shift:

$$\vec{D}^* = \vec{D} - q\vec{R}.$$

The zero values of this field determine the center of charge (or center of mass); $\vec{R} = q^{-1}\vec{D}$.

By extending this idea to include the magnetic dipole moment

$$\vec{D}_{E\&M} = \vec{D}_E + i\vec{D}_M$$

and allowing the position $\vec{R}$ to take on complex values, we find the space dependence of the complex dipole moments given by

$$\vec{D}^*_{E\&M} = \vec{D}_{E\&M} - qR_{\mathbb{C}}, \tag{4.1}$$

so that the complex center of charge is given by $\vec{D}^*_{E\&M} = 0$ or

$$\vec{R}_{\mathbb{C}} = q^{-1}\vec{D}_{E\&M}. \tag{4.2}$$

The difficulty with this construction is that it is not Lorentz invariant: The transformations of the dipoles from one Lorentz frame to another is nonlocal and one does not obtain (in any obvious manner) a unique center of charge or mass worldline.

We use an alternate procedure to find the different ''centers of motion.'' Namely, the complex dipoles are first identified from the asymptotic solutions with interior sources: They are identified from the $l = 1$ harmonics in the tetrad components (spin-coefficient components) of the asymptotic Maxwell field and the asymptotic Weyl tensor, Eqs. (2.40) and (2.41). [See the discussion immediately after Eq. (2.43).] These quantities *depend on the choice of the tetrad vectors* at $\mathfrak{I}^+$. If we choose the tetrad so that the null vector $l = l^*_C$ determines a shear-free (or asymptotically shear-free) null geodesic congruence that focuses on points in complex Minkowski space (or in the general relativity case, on points in the virtual $\mathcal{H}$ space), we see that the associated dipole is a function (three complex components) on the complex Minkowski space (or $\mathcal{H}$ space). The vanishing set of this function (generically) determines the complex worldline that is referred to as the complex center of charge or mass. The idea is then to express the moments in terms of the complex worldline—or, as an alternative, find the complex worldline in terms of the complex dipole. To implement this (in principle straightforward) procedure is in practice rather involved, requiring severe approximations and Clebsch-Gordon expansions of spherical harmonic products. We illustrate the procedure in detail with the Maxwell field in flat space and then report the results (obtained earlier) for the Einstein and Einstein-Maxwell cases with a minimum of detail.

A. Maxwell fields in Minkowski space

Beginning with a complex worldline in $\mathbb{M}_{\mathbb{C}}$, $z^a = \xi^a(\tau)$, its family of cuts of $\mathfrak{I}^+_{\mathbb{C}}$ is, as discussed earlier,

$$u \equiv u_B = c^{-1}\xi^a(\tau)\hat{l}_a(\zeta,\bar{\zeta}) \equiv \frac{\sqrt{2}\tau}{2} - \frac{1}{2}c^{-1}\xi^i Y^0_{1i},$$
$$L(u,\zeta,\bar{\zeta}) = c^{-1}\xi^a(\tau)\hat{m}_a(\zeta,\bar{\zeta}) = c^{-1}\xi^i(\tau)Y^1_{1i} \tag{4.3}$$

with L the angle field of its null normals. Note that c has been explicitly reintroduced so that the cut function u, with u_{ret} and τ, has the dimensions of time. This has the annoying effect of causing the frequent appearance of c.

Remark IV.1.—To avoid a plethora of terms involving $\sqrt{2}$ we switch from the Bondi time $u \equiv u_B$ to the retarded time $u_{\mathrm{ret}} = \sqrt{2}u_B$ so that

$$u_{\mathrm{ret}} = \tau - \frac{\sqrt{2}}{2}c^{-1}\xi^i Y^0_{1i}. \tag{4.4}$$

Derivatives with respect to $u_{\rm ret}$ are denoted by a prime: $\partial_{u_{\rm ret}}F = F'$.

We now illustrate how an asymptotically flat Maxwell field with nonvanishing charge determines a unique complex center of charge worldline, $\xi^a(\tau)$.

We have, first, the asymptotic solution

$$\phi_0 = \frac{\phi_0^0}{r^3} + O(r^{-4}), \qquad \phi_1 = \frac{\phi_1^0}{r^2} + O(r^{-3}), \qquad \phi_2 = \frac{\phi_2^0}{r} + O(r^{-2}) \tag{4.5}$$

with the spherical harmonic decomposition

$$\phi_0^0 = \phi_{0i}^0 Y_{1i}^1 + \phi_{0ij}^0 Y_{2ij}^1 + \cdots, \tag{4.6}$$

$$\phi_1^0 = q + \phi_{1i}^0 Y_{1i}^0 + \phi_{1ij}^0 Y_{2ij}^0 + \cdots, \tag{4.7}$$

$$\phi_2^0 = \phi_{2i}^0 Y_{1i}^{-1} + \phi_{2ij}^0 Y_{2ij}^{-1} + \cdots \tag{4.8}$$

and physical identifications

$$\begin{aligned}
\phi_0^0 &= 2q\eta^i(u_{\rm ret})Y_{1i}^1 + c^{-1}Q_{\mathbb{C}}^{ij\prime}(u_{\rm ret})Y_{2ij}^1 + \cdots,\\
\phi_1^0 &= q + \sqrt{2}qc^{-1}\eta^{i\prime}(u_{\rm ret})Y_{1i}^0 + \frac{\sqrt{2}}{6}c^{-2}Q_{\mathbb{C}}^{ij\prime\prime}(u_{\rm ret})Y_{2ij}^0 + \cdots,\\
\phi_2^0 &= -2qc^{-2}\eta^{i\prime\prime}(u_{\rm ret})Y_{1i}^{-1} - \frac{1}{3}c^{-3}Q_{\mathbb{C}}^{ij\prime\prime\prime}(u_{\rm ret})Y_{2ij}^{-1} + \cdots.
\end{aligned} \tag{4.9}$$

The quantities $q\eta^i = D_{E\&M}^i = D_E^i + iD_M^i$ and $Q_{\mathbb{C}}^{ij}$ are, respectively, the complex (electric and magnetic) dipole and complex quadrupole.

Under the null tetrad rotation, Eq. (2.21)

$$l^{\mathbf{a}} \rightarrow l_C^{*a} = l^a - c\frac{\tilde{L}}{r}m^a - c\frac{L}{r}\bar{m}^a + O(r^{-2}), \tag{4.10}$$

$$m^{*a} = m^a - c\frac{\tilde{L}}{r}n^a, \tag{4.11}$$

$$n^{*a} = n^a, \tag{4.12}$$

the leading Maxwell field terms transform as

$$\begin{aligned}
\phi_0^{*0} &= \phi_0^0 - 2cL\phi_1^0 + c^2L^2\phi_2^0,\\
\phi_1^{*0} &= \phi_1^0 - cL\phi_2^0,\\
\phi_2^{*0} &= \phi_2^0.
\end{aligned} \tag{4.13}$$

The procedure to determine $\xi^a(\tau)$ is the following:

In the first equation of Eq. (4.13), written as

$$\phi_0^0 = \phi_0^{*0} + 2cL\phi_1^0 - c^2L^2\phi_2^0, \tag{4.14}$$

replace the $u_{\rm ret}$ (appearing in ϕ_0^0, ϕ_1^0, and ϕ_2^0) by $u_{\rm ret} = \tau - (\sqrt{2}/2)c^{-1}\xi^iY_{1i}^0$ from (4.3), and for fixed τ, assume that the $l = 1$ terms in ϕ_0^{*0} vanish.

Formally, by extracting the remaining $l = 1$ terms in Eq. (4.14) via the integral at constant τ,

$$\oint_{S^2}\phi_0^0 Y_{1i}^{-1}dS = \oint_{S^2}(2cL\phi_1^0 - c^2L^2\phi_2^0)Y_{1i}^{-1}dS, \tag{4.15}$$

we have the exact functional relationship between the dipole $q\eta^i$ and the worldline $\xi^a(\tau)$.

Unfortunately, it is extremely difficult to get explicit relations from Eq. (4.15) and approximations applied to Eq. (4.14) must be used. Our basic approximation is to consider the $\xi^a(\tau)$ to be of the form $\xi^a(\tau) = (\tau, \xi^i(\tau))$ [using a τ rescaling of the form $\tau \rightarrow F(\tau)$] with both ξ^i and the η^i to be "small." We retain only terms up to second order and harmonic expansions up to $l = 2$.

Writing out Eq. (4.14),

$$\phi_0^0 = \phi_0^{*0} + 2cL\phi_1^0 - c^2L^2\phi_2^0,$$

using Eqs. (4.9), with $u_{\rm ret} = \tau - (\sqrt{2}/2)c^{-1}\xi^i(\tau)Y_{1i}^0$ and $L = c^{-1}\xi^i(\tau)Y_{1i}^1$, by omitting cubic terms including $L^2\phi_2^0$, and then using the first two terms of the Taylor series

$$F\left(\tau - \frac{\sqrt{2}}{2}c^{-1}\xi^iY_{1i}^0\right) = F(\tau) - \frac{\sqrt{2}}{2}c^{-1}\xi^iY_{1i}^0F'(\tau) \tag{4.16}$$

and the Clebsch-Gordon expansions of the products of the spherical harmonics (see the appendix) we finally have (after simplification) for just the $l = 1$ harmonic terms

$$q\eta^k = q\xi^k - i\frac{q}{2}\xi^lc^{-1}\eta^{i\prime}\epsilon_{ilk} + \frac{\sqrt{2}}{10}c^{-2}Q_{\mathbb{C}}^{ik\prime\prime}\xi^i. \tag{4.17}$$

The first thing we notice is the linear relation:

$$\eta^j(\tau) = \xi^j(\tau). \tag{4.18}$$

This can be fed back into Eq. (4.17) in either of two ways resulting in either of the relations:

$$\begin{aligned}
\eta^k &= \xi^k - i\frac{1}{2}c^{-1}\xi^l\xi^{i\prime}\epsilon_{ilk} + \frac{\sqrt{2}}{10}q^{-1}c^{-2}Q_{\mathbb{C}}^{ik\prime\prime}\xi^i,\\
\xi^k &= \eta^k + i\frac{1}{2}c^{-1}\eta^l\eta^{i\prime}\epsilon_{ilk} - \frac{\sqrt{2}}{10}q^{-1}c^{-2}Q_{\mathbb{C}}^{ik\prime\prime}\xi^i,
\end{aligned} \tag{4.19}$$

which determine the complex dipole in terms of the complex worldline or the worldline in terms of the complex dipole.

Note that, though all the expressions are functions of τ, the τ can be replaced by $u_{\rm ret}$ with no other changes needed due to our approximation scheme.

B. Asymptotically flat space-times

Turning now to the Einstein (or Einstein-Maxwell) case, we basically repeat the procedure used in the Minkowski space Maxwell field example.

We begin with an unknown complex worldline in $\mathcal{H}$ space, $z^a = \xi^a(\tau)$, to be determined by the existing asymptotically flat space-time. Its family of cuts and null normal angle field of $\mathfrak{I}_{\mathbb{C}}^+$ is, as discussed earlier [cf. (2.36) and (2.33), etc.], given by

$$\begin{aligned} u &= c^{-1}G(\tau,\zeta,\bar{\zeta}) \\ &= c^{-1}\xi^a(\tau)\hat{l}_a(\zeta,\bar{\zeta}) + c^{-1}H_{l\geq 2}(\xi^a(\tau),\zeta,\bar{\zeta}) \\ &= c^{-1}\xi^a(\tau)\hat{l}_a(\zeta,\bar{\zeta}) + c^{-1}\tilde{H}_{l\geq 2}(\tau,\zeta,\bar{\zeta}), \\ &= \frac{\sqrt{2}\tau}{2} - \frac{1}{2}c^{-1}\xi^i(\tau)Y^0_{1i} + c^{-1}\xi^{ij}(\tau)Y^0_{2ij} + \cdots, \end{aligned} \tag{4.20}$$

$$\begin{aligned} L(u,\zeta,\bar{\zeta}) &= c^{-1}\eth_\tau G(\tau,\zeta,\bar{\zeta})|_{\tau=T(u,\zeta,\bar{\zeta})} \\ &= c^{-1}\xi^i(\tau)Y^1_{1i} - 6c^{-1}\xi^{ij}(\tau)Y^1_{1ij} + \cdots, \end{aligned} \tag{4.21}$$

$$\sigma^0(\tau,\zeta,\bar{\zeta}) = \eth^2_{(\tau)}G(\tau,\zeta,\bar{\zeta}) = 24\xi^{ij}(\tau)Y^2_{2ij} + \cdots. \tag{4.22}$$

We now show how a given asymptotically flat spacetime determines the complex center of charge worldline $\xi^a(\tau)$.

Returning to the "peeling" theorem:

$$\begin{aligned} &\psi_0 = \psi_0^0 r^{-5} + O(r^{-6}), \qquad \psi_1 = \psi_1^0 r^{-4} + O(r^{-5}), \\ &\psi_2 = \psi_2^0 r^{-3} + O(r^{-4}), \qquad \psi_3 = \psi_3^0 r^{-2} + O(r^{-3}), \\ &\psi_4 = \psi_4^0 r^{-1} + O(r^{-2}), \end{aligned}$$

with the transformation law of the leading terms under a null rotation,

$$\begin{aligned} \psi_0^{*0} = \psi_0^0 - 4cL\psi_1^{*0} + 6c^2L^2\psi_2^{*0} - 4c^3L^3\psi_3^{*0} \\ + c^4L^4\psi_4^0, \end{aligned} \tag{4.23}$$

$$\psi_1^{*0} = \psi_1^0 - 3cL\psi_2^0 + 3c^2L^2\psi_3^0 - c^3L^3\psi_4^0, \tag{4.24}$$

$$\psi_2^{*0} = \psi_2^0 - 2cL\psi_3^0 + c^2L^2\psi_4^0, \tag{4.25}$$

$$\psi_3^{*0} = \psi_3^0 - cL\psi_4^0, \tag{4.26}$$

$$\psi_4^{*0} = \psi_4^0, \tag{4.27}$$

we can then determine the transformation law for the physical quantities that are identified in the following harmonic components.

The (truncated) harmonic expansions with their (approximate) physical identifications are

$$\begin{aligned} \psi_0^0 &= \psi_0^{0ij}Y^2_{2ij} + \cdots, \\ \psi_1^0 &= \psi_1^{0i}Y^1_{1i} + \cdots, \\ \psi_2^0 &= \Psi - \eth^2\bar{\sigma}^0 - c^{-1}\sigma^0(\bar{\sigma}^0)', \\ \Psi &= \bar{\Psi} = \Psi^0 + \Psi^i Y^0_{1i} + \cdots, \\ \psi_3^0 &= c^{-1}\eth\bar{\sigma}^{0\prime}, \\ \psi_4^0 &= -c^{-2}\bar{\sigma}^{0\prime\prime}, \end{aligned} \tag{4.28}$$

and

$$\psi_0^{0ij} = \text{approximately, the quadrupole,} \tag{4.29}$$

$$D^i_{\mathbb{C}(\mathrm{grav})} = D^i_{(\mathrm{mass})} + ic^{-1}J^i = -\frac{c^2\sqrt{2}}{12G}\psi_1^{0i}, \tag{4.30}$$

$$\psi_1^{0i} = -\frac{6\sqrt{2}G}{c^2}(D^i_{(\mathrm{mass})} + ic^{-1}J^i), \tag{4.31}$$

$$\Psi \equiv \psi_2^0 + \eth^2\bar{\sigma}^0 + c^{-1}\sigma^0\dot{\bar{\sigma}}^0, \tag{4.32}$$

$$\begin{aligned} \Psi &= \bar{\Psi} = \Psi^0 + \Psi^i Y^0_{1i} + \cdots \\ &= -M_B\frac{2\sqrt{2}G}{c^2} - \frac{6G}{c^3}P^iY^0_{1i} + \cdots, \quad \text{mass aspect,} \end{aligned} \tag{4.33}$$

$$M_B = -\frac{c^2}{2\sqrt{2}G}\Psi^0, \qquad \text{Bondi mass,} \tag{4.34}$$

$$P^i = -\frac{c^3}{6G}\Psi^i, \qquad \text{Bondi linear momentum,} \tag{4.35}$$

$$\sigma^0 = 24\xi^{ij}Y^2_{2ij} + \cdots \qquad \text{Bondi asymptotic shear,} \tag{4.36}$$

$$\xi^{ij} = (\xi^{ij}_R + i\xi^{ij}_I) = \frac{G}{12\sqrt{2}c^4}(Q^{ij\prime\prime}_{\mathrm{Mass}} + iQ^{ij\prime\prime}_{\mathrm{Spin}}). \tag{4.37}$$

Remark IV.2.—The relationship between ψ_1^{0i} and the mass dipole $D^i_{(\mathrm{mass})}$ and angular momentum J^i is usually considered to be more complicated than stated here, often involving quadratic terms in the Bondi shear [33]. The trouble is that there are disagreements in these quadratic terms in the different versions. We have simply left them out here and note that in our approximations the disputed terms do not appear.

Concentrating on the transformation of the dipole, among Eqs. (4.23), (4.24), (4.25), (4.26), and (4.27), our focus is only on Eq. (4.24), which can be rewritten as

$$\psi_1^0 = \psi_1^{*0} + 3cL\psi_2^0 - 3c^2L^2\psi_3^0 + c^3L^3\psi_4^0. \tag{4.38}$$

We first note that the $l = 1$ term in ψ_1^0 is proportional to the complex gravitational dipole $D^i_{\mathbb{C}(\mathrm{grav})}(u)$. Our procedure is now to replace all the u's that appear in Eq. (4.38) by $u_{\mathrm{ret}} = (\sqrt{2}c)^{-1}\xi^a(\tau)\hat{l}_a(\zeta,\bar{\zeta}) + (\sqrt{2})^{-1}H_{l\geq 2}(\xi^a(\tau),\zeta,\bar{\zeta})$ [i.e., Eq. (4.20)], and remember that L also depends on $\xi^a(\tau)$. Then, from our basic assumption, we take the $l = 1$ term in ψ_1^{*0} to vanish and finally extract the $l = 1$ harmonic coefficients from Eq. (4.20). Formally, this is done via the integral expression

$$\oint_{S^2}\psi_1^0 Y^{-1}_{1i}dS = \oint_{S^2}(3cL\psi_2^0 - 3c^2L^2\psi_3^0 + c^3L^3\psi_4^0)Y^{-1}_{1i}dS, \tag{4.39}$$

which on the left side contains τ and the dipole $D^i_{\mathbb{C}(\text{grav})}$ while the right side contains τ, the unknown worldline $\xi^a(\tau)$, and the Bondi shear σ^0.

Though in principle this equation should allow us to establish the relationship between $D^i_{\mathbb{C}(\text{grav})}$ and $\xi^a(\tau)$, in practice this is not possible: We must to return to Eq. (4.38) and use harmonic and Clebsch-Gordon expansions with severe approximations and finally collect the $l = 1$ terms directly.

1. A poor approximation: Results

We first describe a preliminary procedure for extracting the complex worldline from an asymptotically flat space-time rather explicitly. The following approximations are used: The Bondi mass is taken as zeroth order while all other variables are first-order with the calculations done keeping terms up to second order. In this preliminary version, the harmonic expansions keep only the $l = (0, 1)$ terms. This implies the severe condition that the Bondi shear (σ^0) be taken to be zero. Later this condition is relaxed.

Via these approximations Eq. (4.38) becomes

$$\psi_1^0(u, \zeta, \bar{\zeta}) = \psi_1^{*0} + 3cL(\tau, \zeta, \bar{\zeta})\Psi(u, \zeta, \bar{\zeta}). \tag{4.40}$$

Using the retarded time $u_{\text{ret}} = \sqrt{2}u$ instead of the Bondi time, then replacing all the u_{ret}'s by $u_{\text{ret}} = \tau - (\sqrt{2}/2)\times c^{-1}\xi^i(\tau)Y^0_{1i}$, and finally Taylor expanding with (4.16), Eq. (4.40) becomes (with physical identifications inserted):

$$\psi_1^{0i}(\tau)Y^1_{1i} - \frac{\sqrt{2}}{2}c^{-1}\psi_1^{0i\prime}(\tau)\xi^j Y^0_{1j}Y^1_{1i} = -\frac{6\sqrt{2}G}{c^2}M_B\xi^i(\tau)Y^1_{1i} - \frac{18G}{c^3}P^i\xi^j(\tau)Y^0_{1i}Y^1_{1j}. \tag{4.41}$$

Finally after Clebsch-Gordon expansions and the use of the (linearized) Bianchi identity, Eq. (2.44),

$$\sqrt{2}\psi_1^{0\prime} = 2c\Psi^i Y^1_{1i} \Rightarrow \psi_1^{0i\prime} = \sqrt{2}c\Psi^i = -\frac{6\sqrt{2}G}{c^2}P^i, \tag{4.42}$$

the three complex $l = 1$ coefficients of (4.41) (with $\xi^k = \xi^k_R + i\xi^k_I$) yield

$$(D^k_{(\text{mass})} + ic^{-1}J^k) = M_B(\xi^k_R + i\xi^k_I) - \epsilon_{ijk}c^{-1}P^i(i\xi^j_R - \xi^j_I), \tag{4.43}$$

or the pair of real equations

$$D^k_{(\text{mass})}(\tau) = M_B\xi^k_R(\tau) + c^{-1}\epsilon_{ijk}P^i\xi^j_I, \tag{4.44}$$

$$J^k(\tau) = cM_B\xi^k_I(\tau) + \epsilon_{ijk}P^j\xi^i_R(\tau). \tag{4.45}$$

From Eq. (4.42), we immediately get the kinematic definition of the Bondi momentum in terms of the complex worldline. In addition, we have the conservation of angular momentum which arises from the reality of the mass aspect Ψ, Eq. (4.33):

$$P^i = D^{i\prime}_{(\text{mass})} = M_B\xi^{k\prime}_R + c^{-1}\epsilon_{ijk}M_B(\xi^{i\prime}_R\xi^j_I)^\prime, \tag{4.46}$$

$$J^{k\prime} = 0. \tag{4.47}$$

There are several things of significance that should be pointed out here.

(i) If the higher gravitational moments and electromagnetic terms were included, these results, Eqs. (4.44), (4.45), (4.46), and (4.47), would all be augmented by further terms. In particular, there would be a non-vanishing angular-momentum flux. See below.

(ii) In the expression for the angular momentum there are two terms, the second being the conventional orbital angular momentum while the first has been identified, via the Kerr metric and the charged Kerr metric [34], as the intrinsic spin angular momentum.

(iii) The mass dipole contains the conventional $M\vec{R}$ plus a momentum-spin interaction term that creates a spin-velocity coupling contribution to the linear momentum.

These results—basically kinematic, aside from the conservation of angular momentum—have been derived, with severe approximations, by associating the idea of a complex center of mass curve with a complex curve in $\mathcal{H}$ space.

In the same vein (with the same approximations), we obtain the dynamic law for the motion of the real part of ξ^k (i.e., ξ^k_R). From the second Bianchi identity, Eq. (2.45)

$$\sqrt{2}\psi_2^{0\prime} = -c\eth\psi_3^0 + c\sigma^0\psi_4^0 \Rightarrow \psi_2^{0\prime} = 0, \tag{4.48}$$

we have that M_B and P^i are constant, i.e., conservation of energy and momentum:

$$M_B^\prime = P^{i\prime} = 0. \tag{4.49}$$

2. A better approximation (with Bondi shear): Results

For a more accurate description and determination of the complex worldline associated with a given asymptotically flat Einstein (or in the following subsection, Einstein-Maxwell) space-time we restore, in the calculations, the Bondi shear and include the effects of the Einstein-Maxwell equations. Rather than redoing the calculations from the beginning, using the same procedures as in the previous section, we simply give the final results. The approximations are basically the same: The Bondi mass is zero-order, while all other variables are first-order; in the calculations only quadratic terms are retained. However the harmonic expansions now include the $l = (0, 1, 2)$ harmonics. In addition, since a Maxwell field (with non-vanishing total charge) is allowed, we have not only the complex center of mass line, $z^a = \xi^a(\tau)$, but as well the complex center of charge line. It is denoted by $z^a = \eta^a(\tau)$.

In general the two lines are different, though in special circumstance they can coincide.

The idea is to start with Eq. (4.40), use the known expression for $\psi_1^0(u_{\rm ret}, \zeta, \bar{\zeta})$ and $\Psi(u_{\rm ret}, \zeta, \bar{\zeta})$, in terms of the physical gravitational moments, then replace every $u_{\rm ret}$ by

$$u_{\rm ret} = \frac{\sqrt{2}}{2} c^{-1} X(\xi^a(\tau), \zeta, \bar{\zeta}),$$
$$= \tau - \frac{\sqrt{2}}{2} c^{-1} \xi^i(\tau) Y^0_{1i} + \sqrt{2} c^{-1} \xi^{ij}(\tau) Y^0_{1ij} + \cdots, \quad (4.50)$$

set the $l = 1$ coefficients of ψ_1^{*0} to zero, and finally extract the $l = 1$ coefficients from the entire equation—a long process involving repeated Clebsch-Gordon expansions (cf. [7,16]). This process leads to

$$\psi_1^{0i} = -6\sqrt{2} G c^{-2} (D^i_{\rm (mass)} + i c^{-1} J^i),$$
$$(D^i_{\rm (mass)} + i c^{-1} J^i) = M_B \xi^i + i c^{-1} P^j \xi^l \epsilon_{lji} - \frac{4G}{5c^5} P^i Q^{ij\prime\prime}_{\rm Grav} - \frac{3}{5c^2} \bar{Q}^{il\prime\prime\prime}_{\rm Grav} \xi^l - i \frac{3G}{5c^6} Q^{lk\prime\prime}_{\rm Grav} \bar{Q}^{kj\prime\prime}_{\rm Grav} \epsilon_{lji}, \quad (4.51)$$

or from the real and imaginary parts,

$$D^i_{\rm (mass)} = M_B \xi^i_R + c^{-1} \xi^l_I P^j \epsilon_{ijl} - \frac{4G}{5c^5} P^i Q^{ij\prime\prime}_{\rm Mass} - \frac{3}{5c^2} (Q^{il\prime\prime\prime}_{\rm Mass} \xi^l_R + Q^{il\prime\prime\prime}_{\rm Spin} \xi^l_I) - \frac{6G}{5c^6} Q^{kl\prime\prime}_{\rm Mass} Q^{kj\prime\prime}_{\rm Spin} \epsilon_{lji},$$
$$J^i = M_B c \xi^i_I + P^j \xi^l_R \epsilon_{lji} - \frac{4G}{5c^5} P^i Q^{ij\prime\prime}_{\rm Spin} - \frac{3}{5c} (Q^{il\prime\prime\prime}_{\rm Mass} \xi^l_I - Q^{il\prime\prime\prime}_{\rm Spin} \xi^l_R),$$

the definition of the mass dipole and angular momentum in terms of the complex worldline

The kinematic definition of the linear momentum and the angular-momentum conservation law are then found by extracting the $l = 1$ harmonics from the Bianchi identity, Eq. (2.44) (the evolution equation for ψ_1^0)

$$\dot{\psi}^{\,0}_1 = -\eth \psi^0_2 + 2\sigma^0 \psi^0_3,$$

or

$$\psi_1^{0\prime} = -\frac{\sqrt{2}}{2} c \eth \Psi + \frac{\sqrt{2}}{2} c \eth^3 \bar{\sigma} + 3\sigma^0 \eth(\bar{\sigma}'), \quad (4.52)$$

leading to

$$(D^{i\prime}_{\rm (mass)} + i c^{-1} J^{i\prime}) = P^i + i \frac{12G}{5c^6} Q^{kl\prime\prime\prime}_{\rm Grav} \bar{Q}^{lj\prime\prime\prime}_{\rm Grav} \epsilon_{jki}.$$

Then inserting the expressions for $D^i_{\rm (mass)}$ and J^i we obtain, from the real part, an expression for the linear momentum

$$P^i = D^{i\prime}_{\rm (mass)} - \frac{12G}{5c^6} (Q^{kl\prime\prime}_{\rm Mass} Q^{lj\prime\prime}_{\rm Spin})' \epsilon_{jki}, \quad (4.53)$$

$$P^i = M_B \xi^{i\prime}_R + \mathfrak{P}^i,$$
$$\mathfrak{P}^i = c^{-1} (\xi^l_I P^j)' \epsilon_{jli} - \frac{4G}{5c^5} (P^i Q^{ij\prime\prime}_{\rm Mass})' - \frac{3}{5c^2} (Q^{il\prime\prime\prime}_{\rm Mass} \xi^l_R + Q^{il\prime\prime\prime}_{\rm Spin} \xi^l_I)' - \frac{3G}{c^6} (Q^{kl\prime\prime}_{\rm Mass} Q^{lj\prime\prime}_{\rm Spin})' \epsilon_{jki}, \quad (4.54)$$

and, from the imaginary part, the angular-momentum conservation law:

$$J^{i\prime} = ({\rm Flux})^i, \quad (4.55)$$

$$J^i = M_B c \xi^i_I + P^j \xi^l_R \epsilon_{lji} - \frac{4G}{5c^5} P^i Q^{ij\prime\prime}_{\rm Spin} - \frac{3}{5c} (Q^{il\prime\prime\prime}_{\rm Mass} \xi^l_I - Q^{il\prime\prime\prime}_{\rm Spin} \xi^l_R), \quad (4.56)$$

$$({\rm Flux})^i = \frac{12G}{5c^5} (Q^{lk\prime\prime}_{\rm Mass} Q^{jl\prime\prime\prime}_{\rm Mass} + Q^{kl\prime\prime}_{\rm Spin} Q^{jl\prime\prime\prime}_{\rm Spin}) \epsilon_{ijk}. \quad (4.57)$$

Finally, from the $l = 0, 1$ parts of the Bianchi identity, Eq. (4.45), the evolution equation for the mass aspect

$$\dot{\psi}^{\,0}_2 = -\eth \psi^0_3 + \sigma^0 \psi^0_4,$$

or

$$\Psi' = \frac{\sqrt{2}}{c} \sigma^{0\prime} \bar{\sigma}^{0\prime}, \quad (4.58)$$

we obtain both the energy loss expression and the evolution of the momentum, i.e., the equations of motion.

The energy (mass) loss equation, from the $l = 0$ part, is

$$M_B' = -\frac{G}{5c^7} (Q^{ij\prime\prime}_{\rm Mass} Q^{ij\prime\prime\prime}_{\rm Mass} + Q^{ij\prime\prime}_{\rm Spin} Q^{ij\prime\prime\prime}_{\rm Spin}),$$

the known quadrupole expression, while the momentum loss equation, from the $l = 1$ part of (4.58), becomes a version of Newton's second law:

$$P^{k\prime} = F^k_{\rm recoil}, \qquad F^k_{\rm recoil} \equiv \frac{2G}{15c^6} (Q^{lj\prime\prime}_{\rm Spin} Q^{ij\prime\prime\prime}_{\rm Mass} - Q^{lj\prime\prime}_{\rm Mass} Q^{ij\prime\prime\prime}_{\rm Spin}) \epsilon_{ilk}. \quad (4.59)$$

Finally substituting the P^i from Eq. (4.53), we have Newton's second law of motion:

$$M_B \xi^{i\prime\prime}_R = F^i_{\rm recoil} - M_B' \xi^{i\prime}_R - \mathfrak{P}^{i\prime} \equiv F^i. \quad (4.60)$$

3. Results for Einstein-Maxwell space-times

The calculations that were performed earlier for the vacuum general relativity case can be extended to the Einstein-Maxwell case with considerably more effort. Rather than going into the details we will simply present the main results. The Maxwell field considered has only charge and dipole terms: With a bit of effort quadrupole terms could be included. The main change needed is the modification of the asymptotic Bianchi identities to include the Maxwell field:

$$\dot{\psi}_2^0 = -\eth\psi_3^0 + \sigma^0\psi_4^0 + k\phi_2^0\bar{\phi}_2^0, \tag{4.61}$$

$$\dot{\psi}_1^0 = -\eth\psi_2^0 + 2\sigma^0\psi_3^0 + 2k\phi_1^0\bar{\phi}_2^0, \tag{4.62}$$

$$k = 2Gc^{-4}, \tag{4.63}$$

where the fields ϕ_1^0 and ϕ_2^0 are given by Eqs. (4.9).

The result of the calculations are

$$\psi_1^{0i} = -\frac{6\sqrt{2}G}{c^2}(D^i_{(\mathrm{mass})} + ic^{-1}J^i), \tag{4.64}$$

$$\begin{aligned}(D^k_{(\mathrm{mass})} + ic^{-1}J^k) = M_B\xi^k &+ \frac{i\epsilon_{mik}}{c}\xi^m P^i \\ &- i\frac{q^2}{3c^2}\epsilon_{mik}\xi^m\bar{\eta}^{i\prime\prime\prime} - \frac{4G}{5c^5}P^iQ^{ik\prime\prime}_{\mathrm{Grav}} \\ &+ \frac{\sqrt{2}Gq^2}{15c^6}\bar{\eta}^{j\prime\prime}Q^{kj}_{\mathrm{Grav}} - \frac{3}{5c^2}\xi^j\bar{Q}^{kj\prime\prime}_{\mathrm{Grav}} \\ &- i\frac{3G}{5c^6}\epsilon_{mjk}Q^{im\prime\prime}_{\mathrm{Grav}}\bar{Q}^{ij\prime\prime}_{\mathrm{Grav}}\end{aligned} \tag{4.65}$$

or

$$\begin{aligned}D^i_{(\mathrm{mass})} = M_B\xi^i_R &- c^{-1}P^j\xi^k_I\varepsilon_{kji} - \frac{3}{5c^2}(\xi^j_R Q^{ij\prime\prime}_{\mathrm{Mass}} + \xi^j_I Q^{ij\prime\prime}_{\mathrm{Spin}}) \\ &+ \frac{q^2}{3c^2}(\xi^j_I\eta^{k\prime\prime}_R - \xi^j_R\eta^{k\prime\prime}_I)\epsilon_{jki} \\ &- \frac{4G}{5c^5}P^jQ^{ij\prime\prime}_{\mathrm{Mass}} - \frac{3G}{5c^6}Q^{kl\prime\prime\prime}_{\mathrm{Mass}}Q^{kj\prime\prime}_{\mathrm{Spin}}\epsilon_{lji} \\ &+ \frac{\sqrt{2}Gq^2}{15c^6}(\eta^{j\prime\prime}_R Q^{ij\prime\prime}_{\mathrm{Mass}} + \eta^{j\prime\prime}_I Q^{ij\prime\prime}_{\mathrm{Spin}})\end{aligned} \tag{4.66}$$

and

$$\begin{aligned}J^i = cM_B\xi^i_I &+ \xi^k_R P^j\epsilon_{kji} - \frac{3}{5c}(Q^{ij\prime\prime}_{\mathrm{Mass}}\xi^j_I - Q^{ij\prime\prime}_{\mathrm{Spin}}\xi^j_R) \\ &+ \frac{q^2}{3c}(\xi^k_R\eta^{j\prime\prime}_R + \xi^k_I\eta^{j\prime\prime}_I)\epsilon_{kji} - \frac{4G}{5c^4}P^jQ^{ij\prime\prime}_{\mathrm{Spin}} \\ &+ \frac{\sqrt{2}Gq^2}{15c^5}(\eta^{j\prime\prime}_R Q^{ij\prime\prime}_{\mathrm{Spin}} - \eta^{j\prime\prime}_I Q^{ij\prime\prime}_{\mathrm{Mass}}).\end{aligned} \tag{4.67}$$

From the Bianchi identity, Eq. (4.62), we have

$$\begin{aligned}D^{i\prime}_{(\mathrm{mass})} + ic^{-1}J^{i\prime} = P^i &+ \frac{2q^2}{3c^3}\bar{\eta}^{i\prime\prime} + \frac{12Gi}{5c^6}Q^{kl\prime\prime}_{\mathrm{Grav}}\bar{Q}^{lj\prime\prime\prime}_{\mathrm{Grav}}\epsilon_{jki} \\ &+ \frac{2\sqrt{2}iq^2}{3c^4}\eta^{k\prime}\bar{\eta}^{j\prime\prime}\epsilon_{jki}\end{aligned}$$

or, from the real part,

$$\begin{aligned}P^i = M_B\xi^{i\prime}_R &- \frac{2q^2}{3c^3}\eta^{i\prime\prime}_R - c^{-1}(P^j\xi^k_I)'\epsilon_{kji} \\ &+ \frac{q^2}{3c^2}(\xi^j_I\eta^{k\prime\prime}_R - \xi^j_R\eta^{k\prime\prime}_I)'\epsilon_{jki} + \frac{2\sqrt{2}q^2}{3c^4}(\eta^{k\prime}_I\eta^{j\prime}_R)'\epsilon_{jki} \\ &- \frac{4G}{5c^5}(P^jQ^{ij\prime\prime}_{\mathrm{Mass}})' + \frac{\sqrt{2}Gq^2}{15c^6}(\eta^{j\prime\prime}_R Q^{ij\prime\prime}_{\mathrm{Mass}} + \eta^{j\prime\prime}_I Q^{ij\prime\prime}_{\mathrm{Spin}})' \\ &- \frac{3}{5c^2}(\xi^j_R Q^{ij\prime\prime}_{\mathrm{Mass}} + \xi^j_I Q^{ij\prime\prime}_{\mathrm{Spin}})' + \frac{3G}{c^6}(Q^{lk\prime\prime}_{\mathrm{Spin}}Q^{lj\prime\prime}_{\mathrm{Mass}})'\epsilon_{jki},\end{aligned} \tag{4.68}$$

and imaginary part

$$\begin{aligned}J^{i\prime} = (\mathrm{Flux})^i = \frac{2\sqrt{2}q^2}{3c^3}&(\eta^{k\prime}_R\eta^{j\prime\prime}_R + \eta^{k\prime}_I\eta^{j\prime\prime}_I)\epsilon_{jki} \\ &+ \frac{12G}{5c^5}(Q^{lk\prime\prime}_{\mathrm{Mass}}Q^{lj\prime\prime\prime}_{\mathrm{Mass}} + Q^{lk\prime\prime}_{\mathrm{Spin}}Q^{lj\prime\prime\prime}_{\mathrm{Spin}})\epsilon_{kji} - \frac{2q^2}{3c^2}\eta^{i\prime\prime}_I.\end{aligned} \tag{4.69}$$

Alternatively, we can consider the term $\frac{2q^2}{3c^2}\eta^{i\prime\prime}_I$ as a contribution to the total angular momentum rather than the flux. Using this alternative definition of angular momentum,

$$J^i_T = J^i + \frac{2q^2}{3c^2}\eta^{i\prime}_I,$$

where J^i is given by Eq. (4.67), leads to a modified flux law:

$$\begin{aligned}J^{i\prime}_T = (\mathrm{Flux})^i_T = \frac{2\sqrt{2}q^2}{3c^3}&(\eta^{k\prime}_R\eta^{j\prime\prime}_R + \eta^{k\prime}_I\eta^{j\prime\prime}_I)\epsilon_{jki} \\ &+ \frac{12G}{5c^5}(Q^{lk\prime\prime}_{\mathrm{Mass}}Q^{lj\prime\prime\prime}_{\mathrm{Mass}} + Q^{lk\prime\prime}_{\mathrm{Spin}}Q^{lj\prime\prime\prime}_{\mathrm{Spin}})\epsilon_{kji}.\end{aligned} \tag{4.70}$$

Finally from the Bianchi identity, Eq. (4.61), we have the mass loss and momentum loss equations:

$$\begin{aligned}M_B' = &-\frac{G}{5c^7}(Q^{ij\prime\prime\prime}_{\mathrm{Mass}}Q^{ij\prime\prime\prime}_{\mathrm{Mass}} + Q^{ij\prime\prime\prime}_{\mathrm{Spin}}Q^{ij\prime\prime\prime}_{\mathrm{Spin}}) \\ &- \frac{2q^2}{3c^5}(\eta^{i\prime\prime}_R\eta^{i\prime\prime}_R + \eta^{i\prime\prime}_I\eta^{i\prime\prime}_I),\end{aligned} \tag{4.71}$$

$$\begin{aligned}P^{i\prime} = &\frac{2G}{15c^6}(Q^{kj\prime\prime\prime}_{\mathrm{Spin}}Q^{lj\prime\prime\prime\prime}_{\mathrm{Mass}} - Q^{kj\prime\prime\prime}_{\mathrm{Mass}}Q^{lj\prime\prime\prime\prime}_{\mathrm{Spin}})\epsilon_{lki} \\ &+ \frac{q^2}{3c^4}(\eta^{k\prime\prime}_I\eta^{j\prime\prime}_R - \eta^{k\prime\prime}_R\eta^{j\prime\prime}_I)\epsilon_{jki}.\end{aligned} \tag{4.72}$$

C. Interpretations

There are a variety of comments to be made about the physical content contained in the above kinematic and dynamical relations:

(i) A subtle (but not essential) comment for completeness should be made. We have used the symbol $q\eta^i$ for the complex electromagnetic dipole and η^a for the complex center of charge. The two η's are related by a nonlinear term, given explicitly by (4.19). Rather than go into a detailed explanation of our usage which could be confusing, we note that the effect is that a small quadratic term in J_T^i is missing.

(ii) The first term of P^i, Eq. (4.68), is the standard Newtonian kinematic expression for the linear momentum, $M_B\xi_R^{k\prime}$. This is followed by a spin-momentum coupling term of the form $(\vec{S}\times\vec{P})'$.

(iii) The further term $-\frac{2}{3}c^{-3}q^2\eta_R^{i\prime\prime}$, which is a contribution to the linear momentum from the second derivative of the electric dipole moment, $q\eta_R^i$, plays a special role for the case when the complex center of mass *coincides* with the complex center of charge, $\eta^a = \xi^a$. In this case, the second term is exactly the contribution to the momentum that yields the classical radiation reaction force of classical electrodynamics [35]

$$\frac{2}{3}c^{-3}q\xi_R^{i\prime\prime\prime}. \tag{4.73}$$

In this special case we have a rather attractive identification: Since now the magnetic dipole moment is given by $D_M^i = q\xi_I^i$ and the spin by $S^i = M_Bc\xi_I^i$, we have that the gyromagnetic ratio is

$$\gamma = \frac{|S^i|}{|D_M^i|} = \frac{M_Bc}{q},$$

leading to the Dirac value of the g factor, i.e., $g = 2$.

(iv) Many of the remaining terms in P^i, though apparently second order, are really of higher order when the dynamics are considered. Others involve quadrupole interactions, which contain high powers of c^{-1}.

(v) The complex electromagnetic dipole moment, given in general by

$$D_{E\&M}^i = q(\eta_R^i + i\eta_I^i) = D_E^i + iD_M^i,$$

becomes, when the worldlines coincide, $D_{E\&M}^i = q(\xi_R^i + i\xi_I^i)$.

(vi) In the expression for J^i (4.67) we have already identified, in the earlier discussion, the first two terms $S^i = M_Bc\xi_I^j$ and $M_B\xi_R^{k\prime}\xi_R^i\epsilon_{ikj}$ as the intrinsic spin angular momentum and the orbital angular momentum (i.e., $\vec{r}\times\vec{P}$), respectively. An interesting contribution to the total angular momentum comes from the term $\frac{2}{3}c^{-2}q^2\eta_I^{i\prime} = \frac{2}{3}c^{-2}qD_M^{i\prime}$, i.e., a contribution to the total angular momentum from a time-varying magnetic dipole. A question arises: Is this an observable prediction?

(vii) Our identification of J^i as the total angular momentum in the absence of a Maxwell field agrees with most other identifications (assuming our approximations) [33]. Very strong support of this view, with the Maxwell terms added in, comes from the flux law. In Eq. (4.70) we see that there are four flux terms (more arise if we included electromagnetic quadrupole radiation): The first and second come from the Maxwell dipole flux, while the third and fourth are the gravitational quadrupole flux terms. The Maxwell dipole part of the flux is identical to that derived from pure Maxwell theory [35]. We emphasize that this angular-momentum flux law has little to do directly with the chosen definition of angular momentum. The imaginary part of the Bianchi identity, Eq. (4.45), *is* the conservation law. How to identify the different terms (i.e., identifying the time derivative of the angular momentum and the flux terms) comes from different arguments. The identification of the Maxwell contribution to total angular momentum and the flux contain certain arbitrary assignments: Some terms on the left-hand side of the equation, i.e., terms with a time derivative, could have been moved onto the right-hand side and been called "flux" terms. Our assignments were governed by the question of what terms appeared most naturally to be explicit time derivatives (thereby being assigned to the time derivative of the angular momentum) or which terms appeared to be physically more likely to be an angular-momentum term.

(viii) The angular-momentum conservation law can be considered as the evolution equation for the imaginary part of the complex worldline, i.e., $\xi_I^i(u_{\rm ret})$. The evolution for the real part is found from the Bondi energy-momentum loss equation.

(ix) The Bondi mass $M_B = -(c^2/2\sqrt{2}G)\Psi^0$ and the original mass of the Reissner-Nordström (Schwarzschild) unperturbed metric $M_{\rm RN} = -(c^2/2\sqrt{2}G)\psi_2^{00}$ (i.e., the $l=0$ harmonic of ψ_2^0) differ by a quadratic term in the shear, the $l=0$ part of $\sigma^0\dot{\bar{\sigma}}^0$. This suggests that the observed mass of an object is partially determined by its time-dependent quadrupole moment—if it exists.

(x) In the discussion of the Bondi energy loss theorem (4.71), we saw that we can relate ξ^{ij} (i.e., the $l=2$ shear term) to the gravitational quadrupole by

$$\xi^{ij} = (\xi_R^{ij} + i\xi_I^{ij}) = \frac{\sqrt{2}G}{24c^4}(Q_{\rm Mass}^{ij\prime\prime} + iQ_{\rm Spin}^{ij\prime\prime}) = \frac{\sqrt{2}GQ_{\rm Grav}^{ij\prime\prime}}{24c^4} \tag{4.74}$$

to obtain the standard quadrupole energy loss.

(xi) The Bondi mass loss theorem with electromagnetic dipole and quadrupole radiation becomes

$$M'_B = -\frac{G}{5c^7}(Q^{ij'''}_{\text{Mass}}Q^{ij'''}_{\text{Mass}} + Q^{ij'''}_{\text{Spin}}Q^{ij'''}_{\text{Spin}}) - \frac{2}{3c^5}(D^{i''}_E D^{i''}_E + D^{i''}_M D^{i''}_M) - \frac{2}{45c^7}(Q^{ij'''}_E Q^{ij'''}_E + Q^{ij'''}_M Q^{ij'''}_M) \quad (4.75)$$

with the first term the conventional gravitational radiation and with the second and third terms the electromagnetic dipole and quadrupole radiation loss. The momentum loss with electromagnetic quadrupole contributions becomes

$$P^{i'} = \frac{2G}{15c^6}(Q^{kj'''}_{\text{Spin}}Q^{lj'''}_{\text{Mass}} - Q^{kj'''}_{\text{Mass}}Q^{lj'''}_{\text{Spin}})\epsilon_{lki} + \frac{q^2}{3c^4}(\eta^{k''}_I\eta^{j''}_R - \eta^{k''}_R\eta^{j''}_I)\epsilon_{jki} + \frac{2}{135c^6}(Q^{lj'''}_M Q^{kj'''}_E - Q^{lj'''}_E Q^{kj'''}_M)\epsilon_{lki}.$$

(xii) There are several things to observe and comment on concerning Eqs. (4.59) and (4.60): Returning to the case when the complex center of mass and center of charge coincide the resulting equations of motion for the worldline, namely,

$$M_B\xi^{i''}_R + \tfrac{2}{3}c^{-3}q\xi^{i'''}_R. = F^i_{\text{recoil}} - M'_B\xi^{i'}_R - \mathfrak{P}^{i'} \equiv F^i,$$

we *observe and stress* that, aside from several extra terms, these equations coincide with the Lorentz-Dirac equations of motion—this includes the hyperacceleration term and the mass loss term $M'_B\xi^{i'}_R$. This result follows directly from the Einstein-Maxwell equations. There *was no model building* other than requiring that the two complex worldlines coincide—a strong condition, perhaps equivalent to a point particle assumption. Furthermore, there was no mass renormalization; the mass was simply the conventional Bondi mass as seen at infinity. Further structures (e.g., spin) could remain. The problem of the runaway solutions, though not solved here, is converted to the stability of the Einstein-Maxwell equations with the "coinciding" condition on the two worldlines. If the two worldlines do not coincide, i.e., the Maxwell worldline is formed by independent data, then there is no problem of unstable behavior. This suggests a resolution to the problem of the unstable solutions: One should treat the source as a structured object, not a point, and centers of mass and charge as independent quantities. Alternatively, it might be possible that the extra terms in the equations might stabilize the equations. It is, however, hard to see how this could be demonstrated.

(xiii) The F^i_{recoil} is the recoil force from momentum radiation; other force terms could be considered as gravitational radiation reaction.

(xiv) There are alternative perturbation schemes that use variations of the procedures used here. An example is the determination of the gravitational radiation in a Schwarzschild or Reissner-Nordström spacetime induced by a time-dependent Maxwell dipole radiation field. The physical identifications agree with those found in the present work. For instance, the perturbations induced by a Coulomb charge and general electromagnetic dipole Maxwell field in a Schwarzschild background lead to energy, momentum, and angular-momentum flux relations [36]:

$$M'_B = -\frac{2}{3c^5}(D^{i''}_E D^{i''}_E + D^{i''}_M D^{i''}_M), \quad P^{i'} = \frac{1}{3c^4}D^{k''}_E D^{j''}_M\epsilon_{kji}, \quad J^{k'} = \frac{2}{3c^3}(D^{i''}_E D^{j'}_E + D^{i''}_M D^{j'}_M)\epsilon_{ijk}, \quad (4.76)$$

all of which agree exactly with predictions from classical field theory [35].

The familiarity of these results and those in the full text act as an exhibit in favor of the physical identification methods described in this work. That is, they act as a confirmation of the consistency of the identification scheme.

V. DISCUSSION AND CONCLUSION

We have studied and described a variety of geometric structures, all occurring within the confines of classical special and general relativity, which strongly resemble ideas in other areas of physics. Our complex-conjugate method for describing a real, twisting shear-free, or asymptotically shear-free NGC places the congruence's caustic set (in Minkowski space, interpreted as its source) on a closed curve propagating in real time, or, in more suggestive parlance, a classical string or world tube. The dual description, or holomorphic method, constructs a complex NGC (a complex light-cone congruence) whose apex is on a complex worldline in either complex Minkowski space (the shear-free case) or $\mathcal{H}$ space (the asymptotically shear-free case). When one imposes a reality structure on the null geodesics of this congruence (i.e., asks that they intersect the real asymptotic boundary), the source becomes a real two-dimension (complex one-dimensional) open world sheet, also oddly reminiscent of string theory.

Furthermore, the role of $\mathfrak{I}^+_{\mathbb{C}}$ as the object which interpolates between these two dual descriptions is reminiscent of the conjectured holographic principle [2,3]. In particular, we can think of a real, twisting asymptotically shear-free NGC as determining data on $\mathfrak{I}^+_{\mathbb{C}}$ (i.e., the complex-conjugate construction), which in turn acts as a "lens" into $\mathcal{H}$ space, which serves as a virtual image space for the real space-time, where the real twisting NGC becomes a complex, twist-free NGC. Hence, one is tempted to refer to $\mathfrak{I}^+_{\mathbb{C}}$ as the holographic screen for some application of the holographic principle to classical general relativity. This should be contrasted against the most famous application of the holographic principle: the AdS/CFT correspondence [4,5]. Here, the AdS boundary acts as a holographic screen interpolating between type IIB string theory in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super-Yang-Mills theory in Minkowski space-time. It is interesting that we have found structures so closely resembling the underlying holographic principle but which involve nothing like extra dimensions or supersymmetry.

Our observations also raise a series of potentially interesting questions related to fleshing out the alluded-to connections between general relativity and more ambitious theories. For instance, is it possible to write down a sigma model for the embedding of the open world sheet (the holomorphic method) into $\mathcal{H}$ space which yields the real cuts of $\mathfrak{I}^+$? Also, can the role of the future asymptotic boundary as a holographic screen be made any more precise? Furthermore, we have seen (with several examples) how the geometry of the virtual image space ($\mathcal{H}$ space) allows us to make physical identifications from data on the holographic screen. This indicates that our virtual image space represents the "physical information" side of some holographic principle, in the same way as the conformal field theory side of the AdS/CFT correspondence.

It should be noted that in 't Hooft's original work connecting gauge theory and string theory in the planar limit, no supersymmetry was introduced [1]; an extra dimension for string propagation does enter for anomaly cancellation, though. It could be possible that the analytic continuation of $\mathfrak{I}^+$ to the holographic screen $\mathfrak{I}^+_{\mathbb{C}}$ in our investigation serves an analogous purpose, adding the degrees of freedom necessary to solve the good-cut equation and construct the virtual light-cone congruence. However, it is unclear whether further degrees of freedom (in the form of additional dimensions) would be needed if our notion of the source of real null geodesics as an open string in $\mathcal{H}$ space were taken seriously (i.e., if one attempted to quantize the theory of such "strings").

Other (perhaps less abstract) questions immediately arise. Since we obtain many of the standard classical mechanics kinematic and dynamic equations, is it possible that the standard quantization could shed light on the difficult issues of quantum gravity? Since the material described here is closely related to Penrose's twistor theory (via different versions of the Kerr theorem relating shear-freeness and twistors; cf. [7,32,37]) is there a twistorial version of the present findings? Though the material presented here deals with the apparent or "virtual" motion of compact sources as viewed from a great distance, can it be modified or generalized to treat interacting bodies? Can or should these complex worldlines with dynamic evolution and spin structure be taken at all seriously—and in what context?

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APPENDIX: TENSORIAL SPIN-*s* SPHERICAL HARMONICS

Throughout the text, we have made use of tensorial spin-s spherical harmonics, $Y^s_{li\ldots j}(\zeta, \bar{\zeta})$, to expand analytic functions on the 2-sphere. Although these are closely related to the usual spherical harmonics familiar from other arenas of physics, the incorporation of spin may be new to some readers. This appendix reviews the basic properties of the spin-s harmonics and includes some of the Clebsch-Gordon expansions of their products used in our calculations. For the original treatment of this material, see [25].

Before introducing these functions though, we recall the definition of the ð operator (also used throughout the text), which acts as a spin-weighted covariant derivative on S^2 [28]. Suppose f: $S^2 \to \mathbb{C}$ has spin weight s; then

$$\eth f = P^{1-s}\frac{\partial(P^s f)}{\partial\zeta}, \qquad \bar{\eth} f = P^{1+s}\frac{\partial(P^{-s} f)}{\partial\bar{\zeta}}, \qquad \text{(A1)}$$

where P is taken to be any conformal factor on the 2-sphere (in the text, we always take $P \equiv 1 + \zeta\bar{\zeta}$). We see that an application of the ð operator raises spin weight by 1, while $\bar{\eth}$ decreases it by one. As we will see, this property allows us to move among the spin-weighted tensorial spherical harmonics simply by applying differential operators.

1. Spherical harmonics

Some time ago, the generalization of ordinary spherical harmonics $Y_{lm}(\zeta, \bar{\zeta})$ to spin-weighted functions ${}_{(s)}Y_{lm}(\zeta, \bar{\zeta})$ (e.g., [28,38]) was developed to allow for harmonic expansions of spin-weighted functions on the sphere. We have used instead the tensorial form of these spin-weighted harmonics, the so-called tensorial spin-s spherical harmonics, which are formed by taking linear combinations of the ${}_{(s)}Y_{lm}(\zeta, \bar{\zeta})$ [25]:

$$Y^s_{li\ldots k} = \sum K^{sm}_{li\ldots k(s)}Y_{lm},$$

where the indices obey $|s| \leq l$ and the number of spatial indices (i.e., $i \ldots k$) is equal to l. Explicitly, these new spin-weighted harmonics can be constructed from first defining $Y^l_{li\ldots k}$.

We impose our special 2-sphere Bondi null tetrad:

$$\hat{l}^a = \frac{1}{\sqrt{2}P}(P, \zeta + \bar{\zeta}, -i(\zeta - \bar{\zeta}), -1 + \zeta\bar{\zeta}),$$
$$\hat{n}^a = \frac{1}{\sqrt{2}P}(P, -(\zeta + \bar{\zeta}), i(\zeta - \bar{\zeta}), 1 - \zeta\bar{\zeta}), \tag{A2}$$
$$\hat{m}^a = \frac{1}{\sqrt{2}P}(0, 1 - \bar{\zeta}^2, -i(1 + \bar{\zeta}^2), 2\bar{\zeta}), \quad P \equiv 1 + \zeta\bar{\zeta}.$$

We then project these to their covariant duals and take the spatial parts to obtain the 1-form components, along with a useful additional form $c_i = l_i - n_i$:

$$l_i = \frac{-1}{\sqrt{2}P}(\zeta + \bar{\zeta}, -i(\zeta - \bar{\zeta}), -1 + \zeta\bar{\zeta}),$$
$$n_i = \frac{1}{\sqrt{2}P}(\zeta + \bar{\zeta}, -i(\zeta + \bar{\zeta}), -1 + \zeta\bar{\zeta}),$$
$$m_i = \frac{-1}{\sqrt{2}P}(1 - \bar{\zeta}^2, -i(1 + \bar{\zeta}^2), 2\bar{\zeta}), \tag{A3}$$
$$c_i = -\sqrt{2}i\epsilon_{ijk}m_j\bar{m}_k.$$

From this, we define $Y^l_{li\ldots k}$ as [25]

$$Y^l_{li\ldots k} = m_i m_j \ldots m_k, \qquad Y^{-l}_{li\ldots k} = \bar{m}_i\bar{m}_j\ldots\bar{m}_k. \tag{A4}$$

The other harmonics are determined by the action of the ð operator on the harmonics we have defined in (A4). In particular, it can be shown that

$$Y^s_{li\ldots k} = \bar{ð}^{l-s}(Y^l_{li\ldots k}), \qquad Y^{-|s|}_{li\ldots k} = ð^{l-s}(Y^{-l}_{li\ldots k}). \tag{A5}$$

We now present a table of the tensorial spherical harmonics up to $l = 2$, where we truncate all of our expansions in this paper. Higher harmonics can be found in [25].

(i) $l = 0$:

$$Y^0_0 = 1.$$

(ii) $l = 1$:

$$Y^1_{1i} = m_i, \qquad Y^0_{1i} = -c_i, \qquad Y^{-1}_{1i} = \bar{m}_i.$$

(iii) $l = 2$:

$$Y^2_{2ij} = m_i m_j, \qquad Y^1_{2ij} = -(c_i m_j + m_i c_j),$$
$$Y^0_{2ij} = 3c_i c_j - 2\delta_{ij}, \qquad Y^{-2}_{2ij} = \bar{m}_i\bar{m}_j,$$
$$Y^{-1}_{2ij} = -(c_i\bar{m}_j + \bar{m}_i c_j).$$

In addition, it is useful to give the explicit relations between these different harmonics in terms of the ð operator and its conjugate. Indeed, we can see generally that applying ð once raises the spin index by one, and applying $\bar{ð}$ lowers the index by one. This in turn means that

$$ðY^l_{li\ldots k} = 0, \qquad \bar{ð}Y^{-l}_{li\ldots k} = 0.$$

Other relations for $l \leq 2$ are given by

$$\bar{ð}Y^1_{1i} = Y^0_{1i} = ðY^{-1}_{1i}ð, \quad Y^0_{1i} = -2Y^1_{1i}, \quad \bar{ð}Y^0_{1i} = -2Y^{-1}_{1i},$$
$$\bar{ð}Y^2_{2ij} = Y^1_{2ij}, \qquad \bar{ð}^2Y^2_{2ij} = Y^0_{2ij}, \qquad ðY^0_{2ij} = -6Y^1_{2ij},$$
$$ðY^1_{2ij} = -4Y^2_{2ij}.$$

Finally, due to the nonlinearity of the theory, throughout this review we have been forced to consider products of the tensorial spin-s spherical harmonics while expanding nonlinear expressions. These products can be expanded as a linear combination of individual harmonics using Clebsch-Gordon expansions. The explicit expansions for products of harmonics with $l = 1$ or $l = 2$ are given below (we omit higher products due to the complexity of the expansion expressions). Further products can be found in [25].

2. Clebsch-Gordon expansions

(i) $l = 1$ with $l = 1$:

$$Y^1_{1i}Y^0_{1j} = \frac{i}{\sqrt{2}}\epsilon_{ijk}Y^1_{1k} + \frac{1}{2}Y^1_{2ij},$$
$$Y^1_{1i}Y^{-1}_{1j} = \frac{1}{3}\delta_{ij} - \frac{i\sqrt{2}}{4}\epsilon_{ijk}Y^0_{1k} - \frac{1}{12}Y^0_{2ij},$$
$$Y^0_{1i}Y^0_{1j} = \frac{2}{3}\delta_{ij} + \frac{1}{3}Y^0_{2ij}.$$

(ii) $l = 1$ with $l = 2$:

$$Y^1_{1i}Y^2_{2ij} = Y^3_{3ijk},$$
$$Y^0_{1i}Y^0_{2jk} = -\frac{4}{5}\delta_{kj}Y^0_{1i} + \frac{6}{5}(\delta_{ij}Y^0_{1k} + \delta_{ik}Y^0_{1j}) + \frac{1}{5}Y^0_{3ijk},$$
$$Y^1_{1i}Y^0_{2jk} = \frac{2}{5}Y^1_{1i}\delta_{jk} - \frac{3}{5}Y^1_{1j}\delta_{ik} - \frac{3}{5}Y^1_{1k}\delta_{ij} + \frac{i}{\sqrt{2}}(\epsilon_{ikl}Y^1_{2jl} + \epsilon_{ijl}Y^1_{2kl}) + \frac{2}{5}Y^1_{3ijk},$$
$$Y^1_{1i}Y^1_{2jk} = -\frac{1}{6}ð(Y^1_{1i}Y^0_{2jk}),$$
$$Y^{-1}_{2ij}Y^1_{1k} = \frac{3}{10}Y^0_{1i}\delta_{jk} + \frac{3}{10}Y^0_{1j}\delta_{ik} - \frac{1}{5}Y^0_{1k}\delta_{ij} + \frac{i\sqrt{2}}{12}(\epsilon_{jkl}Y^0_{2il} + \epsilon_{ikl}Y^0_{2lj}) - \frac{1}{30}Y^0_{3ijk},$$
$$Y^0_{1i}Y^1_{2jk} = -\frac{2}{5}Y^1_{1i}\delta_{jk} + \frac{3}{5}Y^1_{1j}\delta_{ik} + \frac{3}{5}Y^1_{1k}\delta_{ij} - \frac{i}{3\sqrt{2}}(\epsilon_{ikl}Y^1_{2jl} + \epsilon_{ijl}Y^1_{2kl}) + \frac{4}{15}Y^1_{3ijk},$$
$$Y^1_{2ij}Y^{-1}_{1k} = \frac{3}{10}Y^0_{1i}\delta_{jk} + \frac{3}{10}Y^0_{1j}\delta_{ik} - \frac{1}{5}Y^0_{1k}\delta_{ij} - \frac{i\sqrt{2}}{12}(\epsilon_{jkl}Y^0_{2il} + \epsilon_{ikl}Y^0_{2lj}) - \frac{1}{30}Y^0_{3ijk},$$
$$Y^2_{2ij}Y^0_{1k} = ð(Y^2_{2ij}Y^{-1}_{1k}).$$

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