PHYSICAL REVIEW D **83,** 043522 (2011)

The big bang and inflation united by an analytic solution

Itzhak Bars[1] and Shih-Hung Chen[2]

[1]*Department of Physics and Astronomy, University of Southern California, Los Angeles, California 90089-2535, USA*

[2]*Department of Physics and School of Earth and Space Exploration, Arizona State University, Tempe, Arizona 85287-1404, USA*

(Received 1 September 2010; published 24 February 2011)

Exact analytic solutions for a class of scalar-tensor gravity theories with a hyperbolic scalar potential are presented. Using an exact solution we have successfully constructed a model of inflation that produces the spectral index, the running of the spectral index, and the amplitude of scalar perturbations within the constraints given by the WMAP 7 years data. The model simultaneously describes the big bang and inflation connected by a specific time delay between them so that these two events are regarded as dependent on each other. In solving the Friedmann equations, we have utilized an essential Weyl symmetry of our theory in 3 + 1 dimensions which is a predicted remaining symmetry of 2T-physics field theory in 4 + 2 dimensions. This led to a new method of obtaining analytic solutions in the 1T field theory which could in principle be used to solve more complicated theories with more scalar fields. Some additional distinguishing properties of the solution includes the fact that there are early periods of time when the slow-roll approximation is not valid. Furthermore, the inflaton does not decrease monotonically with time; rather, it oscillates around the potential minimum while settling down, unlike the slow-roll approximation. While the model we used for illustration purposes is realistic in most respects, it lacks a mechanism for stopping inflation. The technique of obtaining analytic solutions opens a new window for studying inflation, and other applications, more precisely than using approximations.

DOI: 10.1103/PhysRevD.83.043522 PACS numbers: 98.80.−k, 04.50.−h, 98.80.Cq

I. INTRODUCTION

Most inflation theories involve one or more scalar fields which are called inflatons [1–4]. The slow-roll approximation is a standard technique used in the study of inflation generated by different inflaton potentials. However, for the slow-roll approximation to be valid, the shape of the inflaton potential has to be shallow. This is because only with a shallow inflaton potential can the kinetic energy of the inflaton be neglected compared to the potential energy. Therefore the shape of inflaton potentials have been restricted in the past to apply the slow-roll approximation. This approximation cannot be used if we want to figure out the dynamics of inflaton fields in regions where the kinetic energy cannot be neglected. To obtain solutions in these regions, one has to solve the full second-order coupled nonlinear differential equations. In general, these kinds of equations are difficult to solve and are approached by numerical methods.

In this paper, we will analytically solve a scalar-tensor theory with a scalar field $\sigma(x)$ minimally coupled to gravity. The full action of our theory is a standard scalar-tensor theory of gravity with a single scalar

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V(\sigma)\right\}. \quad (1)$$

For the present application, σ will be the inflaton, but in other applications of our method it may have other interpretations. The potential is

$$V(\sigma) = \left(\frac{6}{\kappa^2}\right)^2\left(c\sinh^4\left(\sqrt{\frac{\kappa^2}{6}}\sigma\right) + b\cosh^4\left(\sqrt{\frac{\kappa^2}{6}}\sigma\right)\right), \quad (2)$$

where b and c are dimensionless free parameters of the potential and $\frac{1}{\kappa}$ is the reduced Planck mass $\tilde{m}_p = \frac{1}{\kappa}$, where $\tilde{m}_p = m_p/\sqrt{8\pi} = 2.43 \times 10^{18}$ GeV. For our illustrative model we will further specialize to $c = 64b$ and fix some integration constant to a specific value ($\delta = 0$; see below), for no particular reason other than plotting some graphs. With this choice of parameters we will obtain a model of inflation that matches all the current observational constraints [5–7].

Our main purpose here is to illustrate the method of analytic computation, as well as the underlying important ideas inspired by 2T physics that lead to this method and potential. The essential ingredient that permits us to obtain an exact solution is a reformulation of the above theory as a gauge fixed version of a theory with two conformally coupled scalars ϕ and s. This structure includes an essential local scaling symmetry, or Weyl symmetry, that reduces the theory to one degree of freedom while also generating the Newton constant from an initial scale invariant theory with no dimensionful constants. This symmetry structure in 3 + 1 dimensions is a direct outcome of 2T gravity [8] from which the Weyl symmetry emerges as a leftover of general coordinate reparametrizaton symmetry in 4 + 2 dimensions. The surviving Weyl symmetry can be gauge fixed in different ways. One gauge choice gives the theory defined above which appears very difficult to solve. Another gauge choice leads to fully decoupled equations for two scalars, for the special potential given above, for the case of any conformally flat metric, such as the Robertson-Walker case. These structures and the method of solution will be explained in more detail in Sec. III.

The observational constraints on inflation include the value of the scalar spectral index, the tensor to scalar ratio, and the amplitude of scalar perturbation. Following the definition of the potential-slow-roll approximation given by Liddle and Lyth [9,10], the validity of the slow-roll approximation for some potential $V(\sigma)$ requires two slow-roll parameters ε_V and η_V to be much smaller than 1:

$$\varepsilon_V \equiv \frac{1}{2\kappa^2}\left(\frac{V'(\sigma)}{V(\sigma)}\right)^2 \ll 1 \quad \text{and} \quad |\eta_V| \equiv \left|\frac{1}{\kappa^2}\frac{V''(\sigma)}{V(\sigma)}\right| \ll 1. \tag{3}$$

It turns out that our model in Eqs. (1) and (2) requires the inflaton to evolve in a period of time from the big bang to well into the period of inflation when the slow-roll approximation does not apply. During this time, there are several interesting behaviors of the exact solutions that cannot be captured by the slow-roll approximation, including the connection between the big bang and inflation, and an oscillatory behavior of the inflaton field as it settles down, which could not be discussed before. After some time well into inflation, the kinetic energy decreases monotonically, and thus the equation of state asymptotically approaches $w \rightarrow -1$ at late times.

Exact solutions for several different inflation potentials have been reported [11–16], but none of them fall into the category we discuss in this paper. There are several interesting properties of our model.

(i) First, if we trace back the dynamics of the cosmological scale factor $a(\tau)$ analytically, there is a specific time when it vanishes $a(\tau_{\rm BB}) = 0$ thereby defining the big bang. This point in conformal time $\tau = \tau_{\rm BB}$ is the definition of beginning of physical "time" $t(\tau_{\rm BB}) = 0$ as defined by a comoving observer.

(ii) Second, our exact solution simultaneously captures analytically both the big bang and inflation. In particular, inflation does not happen right after the big bang; rather it is determined by the model that it happens around 10^5 Planck times $t_I \sim 10^5 t_{\rm Planck}$ after the big bang. Furthermore, unlike usual practice in inflation models, we do not artificially insert a boundary value for the inflaton $\sigma(\tau)$; rather it is predicted to start from infinity at the big bang and then smoothly connect to the inflation period.

(iii) Third, unlike most inflation theories where the inflaton field $\sigma(\tau)$ decreases monotonically during inflation, after it drops from its infinite value at the big bang, it oscillates around the potential minimum during inflation. So, by contrast to widely used approximations in past analyses [5], in our model time cannot be exchanged with field strength, showing again that there is no substitute for our type of exact analytic solutions.

Among these properties, the first one is tentative, since it is not clear we can apply a purely classical theory to the very early moment of the Universe where quantum effects are expected to play an important role. But in any case we think that the interesting physical picture of the relationship between the big bang and inflation may still remain as a physically correct feature. On the opposite end, inflation in our model does not end by itself; rather we need to introduce other mechanisms to end inflation around 10^7 Planck times after the big bang. The mechanism of stopping inflation requires additional physical features involving reheating that may be possible to incorporate naturally in a more complete physical theory. For this reason we leave it to future investigations, while in this paper the primary focus is on introducing a technique for obtaining exact solutions for the scalar-tensor theory with the potential (2) and showing that it provides an attractive description of the big bang and inflation.

The paper is organized as follows: In Sec. II we review the phenomenological constraints of inflation theories. In Sec. III we explain in detail a technique for obtaining analytic solutions which we adopt from our study of 2T gravity. This enabled us to obtain analytic solutions for the standard gravity action in (1). In Sec. IV we construct a model of inflation that matches all the observational constraints. In Sec. V we summarize our conclusions and point out some future directions.

II. PHENOMENOLOGICAL CONSTRAINTS OF INFLATION MODELS

The latest observational constraints on inflation models from WMAP 7 years data can be found in Ref. [6]. In this section we will briefly review the zeroth-order and first-order properties of inflation theories.

First, an inflation theory must be able to generate a period of inflation which is defined as $\frac{d^2 a_E}{dt^2} > 0$, where $a_E(t)$ is the scale factor of the flat Friedmann-Robertson-Walker metric in the Einstein frame

$$\begin{aligned} ds^2 &= -dt^2 + a_E^2(t)(dx^2 + dy^2 + dz^2) \\ &= a_E^2(\tau)(-d\tau^2 + dx^2 + dy^2 + dz^2). \end{aligned} \tag{4}$$

In the second line we have pulled out a common factor $a_E^2(\tau) \equiv a_E^2(t(\tau))$ and defined the conformal time τ, with $t = t(\tau)$ given by the relation $a_E(\tau)d\tau = dt$. Using conformal time, the derivative with respect to time t can be rewritten as $\frac{d}{dt} = \frac{1}{a_E}\frac{d}{d\tau}$; therefore, the Hubble parameter $H \equiv [(\frac{da_E}{dt})/a_E]$ can be expressed as

$$H(\tau) = \frac{\dot{a}_E(\tau)}{a_E^2(\tau)}, \tag{5}$$

where the overdot denotes derivative with respect to conformal time. Second, an inflation theory should produce more than 60 e-folds of expansion in order to solve the horizon and the flatness problems, which means $\ln(\frac{a_E^{\rm end}}{a_E^{\rm begin}}) > 60$, where $a_E^{\rm end}$ and $a_E^{\rm begin}$ are the scale factors at the end and the beginning of inflation, respectively. The above two constraints are the zeroth-order properties of an inflation theory that comes from a purely classical gravity theory.

To include the fluctuations of the fields that cause the primordial perturbation, one should match the amplitude of fluctuations at the horizon crossing to the current observational anisotropy of the CMB. This is explained in detail in [17–20], while here we will outline the procedure. We consider the small oscillations for the perturbed Friedmann-Robertson-Walker metric and the inflaton

$$ds^2 = a_E^2(\tau)\{-(1+2A)d\tau^2 - 2\partial_i B dx^i d\tau + [(1+2\mathcal{R})\delta_{ij} + \partial_i\partial_j H_T]dx^i dx^j\}, \tag{6}$$

$$\sigma = \sigma_E(\tau) + \delta\sigma. \tag{7}$$

In this paper we will provide exact analytic solutions for the backgrounds $a_E(\tau)$ and $\sigma_E(\tau)$. For the perturbations we choose a coordinate reparametrization gauge in which $\delta\sigma = 0$ and concentrate on the relevant curvature perturbation R. To compute the amplitude of this perturbation, one first expands the action in Eq. (1) up to second order and then puts the background evolution on shell. The resulting action for the perturbation is

$$S_{(2)} = \frac{1}{2}\int d\tau d^3x \frac{\dot{\sigma}_E^2}{H^2}[\dot{\mathcal{R}}^2 - (\partial_i\mathcal{R})^2]. \tag{8}$$

After defining a rescaled amplitude $v \equiv zR$, using the factor

$$z(\tau) \equiv \frac{\dot{\sigma}_E(\tau)}{H(\tau)} \tag{9}$$

that depends on background evolution, the action for the perturbation $v(\tau, \vec{x})$ becomes

$$S_{(2)} = \frac{1}{2}\int d\tau d^3x\left[(\dot{v})^2 - (\partial_i v)^2 + \frac{\ddot{z}}{z}v^2\right]. \tag{10}$$

The Fourier transform of $v(\tau, \vec{x})$ defines the mode function $v_k(\tau)$ in momentum space:

$$v(\tau, \vec{x}) = \int \frac{d^3x}{(2\pi)^3}(a_k v_k(\tau)e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger v_k^*(\tau)e^{-i\vec{k}\cdot\vec{x}}). \tag{11}$$

It satisfies the Mukhanov-Sasaki equation which is obtained by minimizing $S_{(2)}$:

$$\ddot{v}_k + \left[k^2 - \frac{\ddot{z}}{z}\right]v_k = 0. \tag{12}$$

This equation is to be solved along with a physical boundary condition that corresponds to choosing a particular vacuum. Then one can compute the power spectrum $P_{\mathcal{R}}(k)$, the spectral index $n_s(k)$, and the running of the spectral index $n'(k)$ as follows:

$$P_{\mathcal{R}}(k) \equiv \frac{|v_k(\tau_*(k))|^2}{z^2(\tau_*(k))}; \qquad \tau_*(k) \text{ given by } a_E(\tau_*)H(\tau_*) = k, \tag{13}$$

$$n_s(k) - 1 \equiv \frac{d\ln(k^3 P_{\mathcal{R}}(k))}{d\ln k}, \qquad n'(k) \equiv \frac{dn_s(k)}{d\ln k}. \tag{14}$$

Notice that all the above quantities are evaluated at the horizon crossing ($k = a_E H$) which defines the time $\tau_*(k)$. This is because the curvature perturbation is freezed out when the wave length $1/k$ stretches outside the horizon. In general, $\frac{\ddot{z}}{z}$ (given below) is a complicated function of τ that renders the analytic solution to Eq. (12) difficult or impossible.

The solution of Eq. (12) has commonly been discussed for the cases in which the background $a_E(\tau)$ and $\sigma_E(\tau)$ can be approximated by the de Sitter background and the slow-roll approximations. The factor $\frac{\ddot{z}}{z}$ can then be approximated by $\frac{\alpha}{\tau^2}$ for which Eq. (12) is solvable analytically. However, in our model, the behavior of $\frac{\ddot{z}}{z}$ is far from this common fit function $\frac{\alpha}{\tau^2}$. Hence we suggest a different fit function to approximate $\frac{\ddot{z}}{z}$ in the relevant time period close to $\tau \sim \tau_*(k)$ so that this equation is solvable analytically.

The tensor perturbation h_{ij} is discussed in a similar way, starting with

$$ds^2 = a_E^2(-d\tau^2 + (\delta_{ij} + h_{ij}(\tau, \vec{x}))dx^i dx^j). \tag{15}$$

The action for this perturbation is

$$S_{(2)} = \frac{1}{\kappa^2}\int d\tau d^3x a_E^2(\tau)[\dot{h}_{ij}^2 - (\partial_k h_{ij})^2]. \tag{16}$$

The Fourier transform of $h_{ij}(\tau, \vec{x})$ is $a^s(k)\varepsilon_{ij}^s(k)h_k^s(\tau)e^{i\vec{k}\cdot\vec{x}}$ plus the Hermitian conjugate, where $\varepsilon_{ij}^s(k)$ is the spin-two polarization tensor and s denotes the polarization. The rescaled amplitude defined by $\mu_k^s \equiv \frac{1}{2}a_E h_k^s$ is then governed by the following action for the mode $\mu_k^s(\tau)$:

$$S_{(2)} = \frac{1}{2\kappa^2}\sum_s \int d\tau d^3k\left[(\dot{\mu}_k^s)^2 - \left(k^2 - \frac{\ddot{a}_E}{a_E}\right)(\mu_k^s)^2\right]. \tag{17}$$

In our model the behavior of $\frac{\ddot{a}_E}{a_E}$ is similar to the standard case and can be approximated by α/τ^2, so the Mukhanov equation for $\mu_k^s(\tau)$ can be solved in the usual way. As in the scalar case, we can then compute the power spectrum for the tensor, $P_T(k)$, at the horizon crossing. Finally, another observational quantity, the tensor to scalar ratio r, is defined as

$$r \equiv \frac{P_T(k)}{P_{\mathcal{R}}(k)}. \tag{18}$$

The phenomenologically allowed ranges for r and n and n' are plotted in [5,6]. A pure power law spectrum should predict zero running $n' \sim 0$. Finally, the amplitude of scalar perturbations is also observed to be around $P_{\mathcal{R}}(k) \sim 10^{-5}$ [7].

In this paper we construct a model of inflation that satisfies all of the above observational constraints.

III. SOLVING THE THEORY ANALYTICALLY

In this section we will solve analytically the equations of motion for the action (1) Assuming the scalar field σ is homogeneous in space and using the flat Friedmann-Robertson-Walker metric (4), there are two well known independent Einstein equations and one equation for σ [21]:

$$\frac{\dot{a}_E^2}{a_E^4} = \frac{\kappa^2}{3}\left[\frac{1}{2a_E^2}\dot{\sigma}^2 + V(\sigma)\right], \tag{19}$$

$$\frac{\ddot{a}_E}{a_E^3} - \frac{\dot{a}_E^2}{a_E^4} = -\frac{\kappa^2}{3}\left[\frac{1}{a_E^2}\dot{\sigma}^2 - V(\sigma)\right], \tag{20}$$

$$\frac{\ddot{\sigma}}{a_E^2} + 2\frac{\dot{a}_E}{a_E^3}\dot{\sigma} + V'(\sigma) = 0, \tag{21}$$

where a prime represents the derivative with respect to σ. The first two equations are the $\mu\nu = 00$ and $\mu\nu = 11$ components of the Einstein equations, and the third equation is the equation of motion for σ, all expressed in terms of the conformal time τ. These three equations are coupled second-order nonlinear differential equations. We will obtain all the exact solutions of these equations as displayed in Eqs. (53) and (54) when the potential is as given in Eq. (2). To obtain this solution we will use a technique which we developed in the context of 2T gravity that results in a theory with Weyl symmetry when reduced to a 1T shadow. For this reason we give a brief outline of how this is inspired from 2T gravity.

A. The 2T approach to ordinary gravity

Here we will not discuss 2T gravity itself [8,22], which is a theory in $d+2$ dimensions. We will only mention the crucial property of this formulation, namely, that it has the right mix of gauge symmetries to eliminate all ghosts from the 2T fields (including those extra timelike components in vector or tensor fields) and yield shadow fields in two lower dimensions that are ghost-free fields in physical interacting 1T field theories in d dimensions. Dualities relate the many possible shadow 1T field theories that emerge in the process of gauge fixing. For our purposes here we concentrate only on the so-called "conformal shadow." The conformal shadow (like other shadows as well) in d dimensions captures holographically all the gauge invariant content of the 2T-gravity parent theory in $d+2$ dimensions [23].

The action for the conformal shadow of *pure* 2T gravity yields ordinary 1T general relativity in d dimensions with some constraints imposed on it. In particular, it contains a dilaton ϕ and the full action has the form

$$S_{\text{grav}} = \int d^d x\sqrt{-g}\left\{z_d\phi^2 R(g) + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right\}, \tag{22}$$

where $\phi(x)$ and $g_{\mu\nu}(x)$ are the d-dimensional shadows of their higher-dimensional counterparts and the potential is unique $V(\phi) = \lambda\phi^{2d/(d-2)}$ up to an undetermined dimensionless constant λ. The standard curvature term with the Newton constant G is *not permitted* as a consequence of the gauge symmetries of the parent theory. So there are no dimensionful constants in this theory. In this shadow, due to the predetermined constant (fixed by the gauge symmetries in 2T gravity)

$$z_d \equiv \frac{d-2}{8(d-1)}, \tag{23}$$

there is an emergent local scaling (Weyl) symmetry in d dimensions which is a remnant of general coordinate transformations in the extra $1+1$ dimensions [22]. Since the coefficient of $R(g)$ must be positive, the dilaton must have the wrong sign kinetic energy to satisfy the Weyl symmetry, so ϕ is a ghost. Using the Weyl gauge symmetry the shadow dilaton can be gauge fixed to a constant ϕ_0 (thus eliminating the ghost which would also have been a Goldstone boson after condensation), yielding precisely Einstein's general relativity with an arbitrary cosmological constant $S_{\text{grav}} = \int d^d x\sqrt{-g}\{\phi_0^2 R(g) - \lambda\phi_0^{2d/(d-2)}\}$, where the condensate ϕ_0^2 must be interpreted in terms of Newton's constant $\phi_0^2 = (16\pi G_d)^{-1}$.

Matter fields can be added to 2T gravity, including Klein-Gordon type scalars, Dirac or Weyl spinors, and Yang-Mills type vectors, all in $d+2$ dimensions. There are special restrictions on each one of these, on the form of their kinetic energies and the forms of permitted interactions among themselves and with the gravitational fields. These restrictions emerge from the underlying gauge symmetry.

Here we are only concerned with the conformal shadow of this theory when it includes only gravity coupled to scalar fields. These scalar fields are all the elementary scalars that one would introduce in a complete theory (thus including the dilaton, inflaton, Higgs boson, supersymmetry partners, grand unified theory scalars, etc.). The emerging conformal shadow then has the following form in the language of the ordinary field theory in d dimensions with one time [8,22]:

$$S = \int d^d x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g^{\mu\nu}\sum_i\partial_\mu s_i\partial_\nu s_i + z_d\left(\phi^2 - \sum_i s_i^2\right)R(g) - \phi^{2d/(d-2)}f\left(\frac{s_i}{\phi}\right)\right), \tag{24}$$

where $f(s_i/\phi)$ is an arbitrary function of its arguments s_i/ϕ. This shadow automatically has the local Weyl scale symmetry:

$$g'_{\mu\nu} = e^{2\lambda(x)} g_{\mu\nu}, \quad \phi' = e^{(1-d/2)\lambda(x)}\phi, \quad s'_i = e^{(1-d/2)\lambda(x)} s_i, \tag{25}$$

with an arbitrary gauge parameter $\lambda(x)$, which can be verified directly in d dimensions. Indeed the predicted special value of z_d and the form of the potential $V(\phi, s_i) = \phi^{2d/(d-2)} f(\frac{s_i}{\phi})$ that scales like $V(t\phi, ts_i) = t^{2d/(d-2)} V(\phi, s_i)$ are crucial properties to realize this local symmetry. We emphasize that a Weyl symmetry was not one of the gauge symmetries of the parent 2T-gravity field theory in $d+2$ dimensions; rather it emerges in the conformal shadow in d dimensions as a remnant from the general coordinate symmetry in the extra dimensions [22]. Thus the Weyl symmetry is a signature of 2T physics.

In this coupling of gravity to matter, demanded by 2T physics, there is an interesting model-independent physics prediction [8] to be emphasized. Every physical scalar s_i in the complete field theory must be a conformal scalar that couples to the curvature term just like the dilaton. But to avoid being ghosts the s_i must have the opposite relative sign in the kinetic term. Then in the conformal shadow the curvature term is predicted to take the form $(\phi^2 - \sum_i s_i^2)R(g)$ with a required relative minus sign as shown. Hence the gravitational constant must emerge from the condensates of all the scalars, not only the dilaton's. This predicts a physical effect, that the effective gravitational constant $G \sim (\phi^2 - \sum_i s_i^2)^{-1}$ is not really a constant; rather it must increase after every phase transition of the Universe as a whole. Since the dominant part of each field is the condensate after the phase transition, ignoring the small fluctuations, the effective gravitational constant is approximately a constant in between the phase transitions. Thus the Newton constant we measure today cannot be the same as the analogous constant before the various transitions occurred, such as inflation, grand unification, supersymmetry breaking, electroweak symmetry breaking, etc. Of course the earlier phase transitions supply the dominant condensates in the sum.

There is also the curious possibility that $G \sim (\phi^2 - \sum_i s_i^2)^{-1}$ could turn negative if the other scalars dominate over the dilaton in some regions of the Universe, or in the history of the Universe, thus producing antigravity in those parts of spacetime. The effects of this idea on cosmology is one of the motivations that led us to investigate the solutions of this theory, as we will do in the rest of this paper. Interestingly, we found that the big bang is related to the vanishing of the gauge invariant $(1 - \sum_i s_i^2/\phi^2)$ at which point the effective G changes sign. The familiar portion of the Universe (with positive G) and its evolution are described by positive values of the factor $(1 - \sum_i s_i^2/\phi^2)$ starting with zero at the big bang and staying in the positive region throughout the history of the Universe. The value of the gauge invariant quantity $(1 - \sum_i s_i^2/\phi^2)$ oscillates in the positive region before it reaches its asymptotic value 1. As a result, the vanishing of this *gauge invariant* quantity determines the big bang.[1]

B. The model for the big bang and inflation

To solve the differential equations for inflation (19)–(21) and even harder ones, we will start with a theory of the type (24) that includes two scalar fields ϕ and s, with a Weyl local scaling symmetry. Our strategy is to use the Weyl symmetry to make some convenient gauge choices. In one gauge the theory will reduce to the standard inflaton theory in Eqs. (1) and (2), while in another gauge it will reduce to a completely solvable theory. Since each one is a gauge choice, the solvable theory is dual to the inflaton theory.[2] The inflaton equations (19)–(21) are then solved by transforming the solution from the fully solvable version.

The starting point is then the action

$$S = \int d^4x \sqrt{-g}\left(\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g^{\mu\nu}\partial_\mu s \partial_\nu s + \frac{1}{12}(\phi^2 - s^2)R - \phi^4 f\left(\frac{s}{\phi}\right)\right), \tag{26}$$

where we have used $z_4 = \frac{1}{12}$. Here f can be an arbitrary function of $\frac{s}{\phi}$ but will be later set to $f(\frac{s}{\phi}) = c(\frac{s}{\phi})^4 + b$ in the present application to reproduce the model of Eq. (2). Here ϕ and s are real scalars while ϕ has the wrong sign in the kinetic term. This makes ϕ a ghost degree of freedom. With the special coefficient $\frac{1}{12}$ in the coupling of the scalar fields to the Ricci curvature, this action is invariant under the local Weyl transformation in Eq. (25). Because of this local Weyl symmetry, we can eliminate the ghost degree of freedom by gauge fixing. So the theory is actually *ghost-free*. This Weyl symmetry will play a crucial roll in solving the theory (1). Note that under the Weyl transformations the quantity $(1 - s^2/\phi^2)$ is gauge invariant, so the sign or the zeros of the effective gravitational coupling $(\phi^2 - s^2) \sim G^{-1}$ are the same for all gauge choices.

[1] Meanwhile, the behavior of the gauge-dependent quantity $G \sim (\phi^2 - \sum_i s_i^2)^{-1}$ varies according to the gauge choice; for example, it can be taken to be a positive constant in the Einstein frame but only in spacetime regions where it is positive.

[2] This duality, which is in the form of a familiar gauge transformation in the present case, is a simple example of a rich set of dualities among 1T physical systems predicted by 2T physics but missed systematically in 1T physics. Choosing a Weyl gauge in the present context amounts to choosing how to embed d dimensions in $d+2$ dimensions, thus creating a perspective of how an observer in d dimensions perceives some "shadow" of phenomena that occur in $d+2$ dimensions. For a recent summary of 2T physics that includes a description of such phenomena see [23].

Varying the action (26) with respect to all its degrees of freedom, we derive the equations of motion

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = T_{\mu\nu}, \tag{27}$$

$$\nabla^2 \phi = \frac{1}{6}\phi R - 4\phi^3 f\left(\frac{s}{\phi}\right) + \frac{s}{\phi^2} f'\left(\frac{s}{\phi}\right), \tag{28}$$

$$\nabla^2 s = \frac{1}{6} s R + \phi^3 f'\left(\frac{s}{\phi}\right), \tag{29}$$

where $f'(\frac{s}{\phi})$ denotes the derivative of f with respect to its argument $\frac{s}{\phi}$. The energy momentum tensor $T_{\mu\nu}$ is

$$T_{\mu\nu} = \frac{6}{(\phi^2 - s^2)}\Big[-\partial_\mu\phi\partial_\nu\phi + \partial_\mu s\partial_\nu s - \frac{1}{6}(g_{\mu\nu}\nabla^2 - \nabla_\mu\partial_\nu)(\phi^2 - s^2) + g_{\mu\nu}\left(\frac{1}{2}\partial\phi\cdot\partial\phi - \frac{1}{2}\partial s\cdot\partial s - \phi^4 f\left(\frac{s}{\phi}\right)\right)\Big]. \tag{30}$$

The above equations are valid for all gauge choices of the Weyl symmetry.

Now we select gauges. To start with, we will choose the so-called Einstein gauge where the coefficient in front of the Ricci curvature is set to the usual gravitational constant

$$\frac{1}{12}(\phi_E^2 - s_E^2) = \frac{1}{2\kappa^2}. \tag{31}$$

The subscripts E indicate that these fields, including the metric $(g_E)_{\mu\nu}$, are in the Einstein gauge. So in this gauge, ϕ_E is related to the field s_E as

$$\phi_E = \pm\left(s_E^2 + \frac{6}{\kappa^2}\right)^{1/2}. \tag{32}$$

The $\pm$ plays no role because ϕ_E appears always quadratically. Now, inserting this into the action (24) and further using the following field redefinition:

$$s_E = \frac{\sqrt{6}}{\kappa}\sinh\left(\frac{\kappa\sigma}{\sqrt{6}}\right), \tag{33}$$

we find that the gauge fixed form of the action (26) reduces to the inflaton action (1) and relates the general inflaton potential $V(\sigma)$ to the general function $f(s/\phi)$. This result indicates that the physics of (1) is completely equivalent to the physics of (26) for any desired potential. Hence we can use the Weyl symmetric version (26) in other gauges to tackle the solution of the inflaton theory.

Now suppose we wish to discuss the case of a conformally flat spacetime defined by the line element

$$ds^2 = a^2(x)(\eta_{\mu\nu}dx^\mu dx^\nu), \tag{34}$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric while the scale factor $a(x)$ is an *arbitrary* function of spacetime. The Robertson-Walker metric (4) that we discuss later is a special case where the scale factor is a function of only time, but for now we are considering the more general spacetime dependence in $a(x)$.

The equations above (27)–(29) simplify for the conformally flat metric. They now contain three fields: $a(x)$, $\phi(x)$, and $s(x)$. Under the Weyl transformations we can form two invariants, which we can choose as (here we state the results more generally for any d, but for the application we use $d = 4$)

$$\tilde{\phi} = a^{(d/2-1)}\phi \quad \text{and} \quad \tilde{s} = a^{(d/2-1)}s. \tag{35}$$

All the equations above, when specialized to the conformal spacetime, can be written in terms of only these two gauge invariants.

To proceed, we will choose a Weyl gauge where the metric is actually flat. In this gauge we have $a_{\text{flat}}(x) = 1$, while ϕ_{flat} and s_{flat} are still arbitrary but they are equal to the gauge invariants defined above:

$$a_{\text{flat}}(x) = 1{:}\ \tilde{\phi} = \phi_{\text{flat}}, \qquad \tilde{s} = s_{\text{flat}}. \tag{36}$$

This shows that by choosing the "flat gauge" we can extract all the gauge invariant information in the equations of motion (27)–(29) for a conformally flat spacetime.

In the flat gauge, since $R(\eta) = 0$ and $\sqrt{-\eta} = 1$, the action (26) becomes

$$S = \int d^d x\left(\frac{1}{2}(\partial_\mu\phi_{\text{flat}})^2 - \frac{1}{2}(\partial_\mu s_{\text{flat}})^2 - \phi_{\text{flat}}^{2d/(d-2)} f\left(\frac{s_{\text{flat}}}{\phi_{\text{flat}}}\right)\right), \tag{37}$$

where the conformally flat metric $g_{\mu\nu}$ is actually flat $\eta_{\mu\nu}$.

Now we note that in this gauge the two scalar fields ϕ_{flat} and s_{flat} (equivalently the two gauge invariants $\tilde{\phi}$ and $\tilde{s}$) will have completely decoupled dynamics if $f(s/\phi)$ takes the special form $f(s/\phi) = c(s/\phi)^{d/(d-2)} + b$. This is when the inflaton potential takes the special form in Eq. (2) when $d = 4$. In this case the action describes two completely decoupled scalars ϕ_{flat} and s_{flat} where ϕ_{flat} has a wrong sign for its kinetic term:

$$S = \int d^d x\left(\frac{1}{2}(\partial_\mu\phi_{\text{flat}})^2 - \frac{1}{2}(\partial_\mu s_{\text{flat}})^2 - b\phi_{\text{flat}}^{2d/(d-2)} - c s_{\text{flat}}^{2d/(d-2)}\right). \tag{38}$$

The equations of motion (28) and (29) in this gauge are

$$\Box\phi_{\text{flat}} = -\frac{2d}{d-2} b\phi_{\text{flat}}^{(d+2)/(d-2)}, \qquad \Box s_{\text{flat}} = \frac{2d}{d-2} c s_{\text{flat}}^{(d+2)/(d-2)}, \tag{39}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. In addition to these equations one should also impose the constraints that follow from general coordinate symmetry (which has been gauge fixed) that imply, through Eq. (27), that the stress tensor in Eq. (30) specialized to the flat gauge must also vanish:

$$T^{\rm flat}_{\mu\nu} = 0. \tag{40}$$

There remains solving the equations in this gauge. It should be emphasized that the decoupling of $\phi_{\rm flat}$ and $s_{\rm flat}$ is valid for the general conformally flat metric, that is, for any spacetime dependence of $a(x)$, and therefore for any $a_E(x)$ in the Einstein gauge in the standard form of the theory of Eqs. (1) and (2) (i.e. not only the Robertson-Walker case).

In solving these decoupled equations we should not forget that the acceptable solutions in the flat gauge must still satisfy the gauge invariant requirement that the sign and the zeros of $(1 - s^2/\phi^2)$ must be the same in all gauge fixed versions. In particular, if we wish to relate to the Einstein gauge in which $(1 - s_E^2/\phi_E^2) = (2z_d\kappa^2\phi_E^2)^{-1}$ is always positive [see Eq. (32)], then the corresponding acceptable solutions in the flat gauge must also satisfy $(1 - s^2_{\rm flat}/\phi^2_{\rm flat}) \geq 0$. The zero in the flat gauge at $s_{\rm flat}/\phi_{\rm flat} = \pm 1$ may be attained only when in the Einstein gauge s_E and ϕ_E tend to infinity according to their relation in Eq. (32). According to the relation to the inflaton σ in Eq. (33) the zero can happen in any gauge only when the inflaton σ in the Einstein frame blows up. We will see through exact solutions that this gauge invariant zero corresponds to the big bang.

C. Relating the Einstein and flat gauges

Notice that the quantity $\frac{s}{\phi}$ is scale invariant, so it can be expressed in various Weyl gauges as follows:

$$\frac{s}{\phi} = \frac{s_{\rm flat}}{\phi_{\rm flat}} = \frac{s_E}{\phi_E}, \tag{41}$$

and by using Eq. (33) this allows us to relate σ to $s_{\rm flat}/\phi_{\rm flat}$ as follows:

$$\frac{s_{\rm flat}}{\phi_{\rm flat}} = s_E\left(s_E^2 + \frac{1}{2z_d\kappa^2}\right)^{-(1/2)} = \tanh(\sqrt{2z_d}\kappa\sigma). \tag{42}$$

Thus the inflaton field σ in the Einstein gauge is related to the flat-gauge fields as follows:

$$\sigma = \frac{1}{\sqrt{8z_d}\kappa}\ln\left|\frac{\phi_{\rm flat}+s_{\rm flat}}{\phi_{\rm flat}-s_{\rm flat}}\right| \underset{d=4}{\rightarrow} \frac{\sqrt{6}}{2\kappa}\ln\left|\frac{\phi_{\rm flat}+s_{\rm flat}}{\phi_{\rm flat}-s_{\rm flat}}\right|. \tag{43}$$

The other field in the Einstein gauge is the scale factor $a_E(\tau)$ which appears in the equations (19)–(21) we wish to solve. It is related to the flat fields by a Weyl gauge transformation involving a local parameter $\lambda(x)$ consistent with Eq. (25). Hence we can determine $\lambda(x)$ through the equation

$$\begin{aligned} s_E &= e^{(1-d/2)\lambda(x)}s_{\rm flat}, \\ \phi_E &= e^{(1-d/2)\lambda(x)}\phi_{\rm flat} = \left(s_E^2 + \frac{1}{2z_d\kappa^2}\right)^{1/2}. \end{aligned} \tag{44}$$

So, consistent with the Einstein gauge of Eq. (31), we can write

$$e^{(d-2)\lambda(x)} = 2z_d\kappa^2(\phi^2_{\rm flat} - s^2_{\rm flat}), \quad \text{for } (\phi^2_{\rm flat} - s^2_{\rm flat}) \geq 0. \tag{45}$$

Now, using Eq. (45) we can figure out the scale factor in the Einstein gauge as

$$\begin{aligned} a_E &= e^{\lambda(x)}a_{\rm flat} \\ &= [2z_d\kappa^2(\phi^2_{\rm flat} - s^2_{\rm flat})]^{1/(d-2)} \underset{d=4}{\rightarrow} \frac{\kappa}{\sqrt{6}}(\phi^2_{\rm flat} - s^2_{\rm flat})^{1/2}, \end{aligned} \tag{46}$$

where we used $a_{\rm flat} = 1$. So, Eqs. (43) and (46) provide the duality transformation for relating the Einstein-gauge fields $(a_E(x), \sigma(x))$ to the flat-gauge fields $(\phi_{\rm flat}(x), s_{\rm flat}(x))$. In these expressions we may substitute the gauge invariants $\tilde{\phi}$ and $\tilde{s}$ instead of $\phi_{\rm flat}$ and $s_{\rm flat}$.

It should be emphasized that this approach works in any dimension, not just four. By solving the simpler decoupled equations (39) for $(\phi_{\rm flat}(x), s_{\rm flat}(x))$ and then using this duality (which depends on d), we will obtain the solutions for the much more complicated coupled differential equations for the fields $(a_E(x), \sigma(x))$ for any conformally flat metric in the Einstein gauge, i.e. any $g^E_{\mu\nu} = a_E(x)\eta_{\mu\nu}$ with arbitrary x^μ dependence. Special examples of such spacetime metrics include $\mathrm{AdS}_d, \mathrm{AdS}_{d-1}\times S^1, \ldots, \mathrm{AdS}_{d-n}\times S^n$, any maximally symmetric space, any conformally flat space including singular ones, etc., and of course the Robertson-Walker expanding universe that we wish to discuss next. It is worth noticing that these special conformally flat spacetimes in d dimensions, which can all be mapped to our decoupled system, have a hidden global $SO(d, 2)$ symmetry, since they are obtained as shadows of the completely flat spacetime in $d + 2$ dimensions in the context of 2T physics [24,25].

D. Solution of inflation equations

For the problem of inflation we need to specialize to the much simpler homogeneous fields that depend only on the conformal time τ and take $d = 4$. The solution for the complicated equations (19)–(21) for $a_E(\tau)$ and $\sigma(\tau)$ can now be obtained by solving the decoupled simple equations $(\phi_{\rm flat}(\tau), s_{\rm flat}(\tau))$ plus our duality transformation (43) and (46). Then, Eqs. (39) take the form

$$-\ddot{\phi}_{\rm flat} = -4b\phi^3_{\rm flat}, \qquad -\ddot{s}_{\rm flat} = 4cs^3_{\rm flat}, \tag{47}$$

where the derivatives are with respect to τ. A first integral is obtained in the form

$$-\frac{\dot{\phi}^2_{\rm flat}}{2} + b\phi^4_{\rm flat} = -E_\phi, \qquad \frac{\dot{s}^2_{\rm flat}}{2} + cs^4_{\rm flat} = E_s. \tag{48}$$

Using the 00 component of Eq. (27), or Eq. (40), we find that the constants are related: $E_s = E_\phi = E$, where E is the energy density of $s_{\rm flat}$ and $(-E)$ is the energy density of $\phi_{\rm flat}$, so

$$\frac{\dot\phi_{\rm flat}^2}{2} - b\phi_{\rm flat}^4 = \frac{\dot s_{\rm flat}^2}{2} + cs_{\rm flat}^4 = E. \tag{49}$$

This implies that the total energy density in the flat gauge is zero which makes perfect sense. If the total energy density were not zero, then we could not have a static flat metric.

The nature of the solutions can now be ascertained intuitively because (49) looks like the problem of a nonrelativistic particle moving in a quartic potential (upside or downside, depending on the signs of b and c) at a fixed energy E (which can be positive, negative, or zero).

More precisely, Eqs. (47) are second-order nonlinear differential equations. A first integral is already given by Eq. (49), and this leads to first-order nonlinear differential equations $\dot\phi_{\rm flat} = \pm\sqrt{2E + 2b\phi_{\rm flat}^4}$ and $\dot s_{\rm flat} = \pm\sqrt{2E - 2cs_{\rm flat}^4}$ that are easily integrated. For the case $E = 0$ the solutions are simple. For $E \neq 0$ the solutions are expressed in terms of the famous Jacobi elliptic function $cn(z|\frac{1}{2})$ as seen below. The function $cn(z|m)$ with a more general label m is a doubly periodic meromorphic function [similar to the cosine function $\cos(\omega z)$], where m is the parameter which determines the period of the function (for more information see [26] and the appendix). All the possible solutions for different regions of the parameters are listed below, where we have abbreviated $cn(z) \equiv cn(z|\frac{1}{2})$ and defined the parameters

$$\zeta \equiv \left|\frac{4b}{c}\right|^{1/4}, \quad \tilde\tau \equiv 2|cE|^{1/4}\tau, \quad \delta = \text{constant}. \tag{50}$$

Here ζ is a dimensionless parameter that depends on the ratio of b and c, and $\tilde\tau$ is a dimensionless conformal time. The purpose of introducing ζ and $\tilde\tau$ is just to simplify the expressions in the tables below. However, we will still use the dimensionful τ in some of the solutions where c or E or both of them are zero. The constant δ is a relative shift in dimensionless conformal time. The origin of the conformal time can be changed arbitrarily because of translation symmetry of the equations under $\tau \to \tau + \tau_0$. Therefore τ_0 has no physical meaning but the relative time δ which corresponds to initial conditions has physical meaning.

There are four free parameters in the solutions of these two decoupled second-order differential equations, which are E_ϕ, E_s, δ, and τ_0. We have just argued that τ_0 is not a physical quantity that can be fixed arbitrarily; also Eq. (27) requires $E_\phi = E_s = E$. This reduces the number of undetermined integration parameters associated with boundary conditions to only the two parameters E and δ. Together with the two parameters b and c that determine the shape of the inflaton potential, we have totally four parameters in the theory which can be adjusted to fit phenomenological observations.

The following three tables elaborate on all possible solutions of Eqs. (47), corresponding to all possible solutions of the Einstein equations (19)–(21) in the Einstein frame

Since all solutions of $\phi_{\rm flat}$ and $s_{\rm flat}$ are known, it is straightforward to calculate σ and a_E from the duality (or gauge) transformation in the previous subsection and the results listed in Tables I, II, and III. One can check that Eqs. (43) and (46) are actually solutions to differential equations (19)–(21) by direct substitution provided $\phi_{\rm flat}$ and $s_{\rm flat}$ are one of the possible solutions in Tables I, II, and III that satisfies Eq. (49). Verifying this requires only the second derivatives supplied by Eq. (47) and the first derivatives supplied by Eq. (49). Of course, these are consistent with the properties of the Jacobi elliptic functions listed in the appendix.

We have used conformal time as the evolution parameter; however, cosmologists usually discuss time using physical time t that is measured by a comoving observer.

TABLE I. Solutions for $b > 0$.

b	E	c	$\phi_{\rm flat}$	$s_{\rm flat}$
$b > 0$	$E > 0$	$c > 0$	$\left(\frac{E}{b}\right)^{1/4}\left[\frac{1-cn(\zeta\tilde\tau)}{1+cn(\zeta\tilde\tau)}\right]^{1/2}$	$\left(\frac{E}{c}\right)^{1/4} cn(\tilde\tau + \delta)$
		$c = 0$	$\left(\frac{E}{b}\right)^{1/4}\left[\frac{1-cn(\zeta\tilde\tau)}{1+cn(\zeta\tilde\tau)}\right]^{1/2}$	$\sqrt{2E}(\tau + E^{-1/4}\delta)$
		$c < 0$	$\left(\frac{E}{b}\right)^{1/4}\left[\frac{1-cn(\zeta\tilde\tau)}{1+cn(\zeta\tilde\tau)}\right]^{1/2}$	$\left\lvert\frac{E}{c}\right\rvert^{1/4}[(1 - cn(\sqrt{2}\tilde\tau + \delta))/(1 + cn(\sqrt{2}\tilde\tau + \delta))]^{1/2}$
	$E = 0$	$c > 0$	$1/\pm\sqrt{2b}\tau$	No solution
		$c = 0$	$1/\pm\sqrt{2b}\tau$	$s_0 = \text{const}$
		$c < 0$	$1/\pm\sqrt{2b}\tau$	$1/\pm\sqrt{-2c}\tau + \kappa\delta$
	$E < 0$	$c \geqslant 0$	$\left\lvert\frac{E}{b}\right\rvert^{1/4}\frac{1}{cn((\zeta/\sqrt{2})\tilde\tau)}$	No solution
		$c < 0$	$\left\lvert\frac{E}{b}\right\rvert^{1/4}\left[1/cn\left(\frac{\zeta}{\sqrt{2}}\tilde\tau\right)\right]$	$\left\lvert\frac{E}{c}\right\rvert^{1/4}\frac{1}{cn(\tilde\tau+\delta)}$

TABLE II. Solutions for $b = 0$.

b	E	c	$\phi_{\rm flat}$	$s_{\rm flat}$
$b = 0$	$E > 0$	$c > 0$	$\pm\sqrt{2E}(\tau + E^{-1/4}\delta)$	$(\frac{E}{c})^{1/4}cn(\tilde{\tau})$
		$c = 0$	$\pm\sqrt{2E}(\tau + E^{-1/4}\delta)$	$\sqrt{E}\tau$
		$c < 0$	$\pm\sqrt{2E}(\tau + E^{-1/4}\delta)$	$(\zeta/\sqrt{2})\left\lvert\frac{E}{b}\right\rvert^{1/4}\left[\frac{1-cn(\tilde{\tau})}{1+cn(\tilde{\tau})}\right]^{1/2}$
	$E = 0$	$c > 0$	$\phi_0 = \text{const}$	No solution
		$c = 0$	$\phi_0 = \text{const}$	$s_0 = \text{const}$
		$c < 0$	$\phi_0 = \text{const}$	$1/\pm\sqrt{-2c}\tau + \kappa\delta$
	$E < 0$	All c	No solution	

TABLE III. Solutions for $b < 0$.

b	E	c	$\phi_{\rm flat}$	$s_{\rm flat}$
$b < 0$	$E > 0$	$c > 0$	$\left\lvert\frac{E}{b}\right\rvert^{1/4}cn((\zeta/\sqrt{2})\tilde{\tau})$	$\left\lvert\frac{E}{c}\right\rvert^{1/4}cn(\tilde{\tau} + \delta)$
		$c = 0$	$\left\lvert\frac{E}{b}\right\rvert^{1/4}cn((\zeta/\sqrt{2})\tilde{\tau})$	$\pm\sqrt{2E}(\tau + E^{-1/4}\delta)$
		$c < 0$	$\left\lvert\frac{E}{b}\right\rvert^{1/4}cn((\zeta/\sqrt{2})\tilde{\tau})$	$\left\lvert\frac{E}{c}\right\rvert^{1/4}\left[\frac{1-cn(\tilde{\tau}+\delta)}{1+cn(\tilde{\tau}+\delta)}\right]^{1/2}$
	$E \leq 0$	All c	No solution	

To convert the conformal time to the physical time, one needs to perform an integral:

$$t = \int dt = \int a_E(\tau)d\tau. \tag{51}$$

This follows from the definition of conformal time. In the example we are going to discuss, the physical time will diverge logarithmically as conformal time approaches a critical finite value τ_∞. This is because as conformal time approaches this critical value, our solution indicates that the scale factor in the Einstein gauge diverges as $a_E(\tau) \sim \frac{1}{\tau - \tau_\infty}$. With all the information given in this section, we can proceed to build a phenomenological inflation model.

IV. A MODEL OF INFLATION

In this section, we will construct a model of inflation that matches all of the phenomenological constraints. The specific solution we are using is the one in Table I with $b > 0$, $c > 0$, and $E > 0$. The solution for this range of parameter space is

$$\phi_{\rm flat} = \left(\frac{E}{b}\right)^{1/4}\left[\frac{1 - cn(\zeta\tilde{\tau})}{1 + cn(\zeta\tilde{\tau})}\right]^{1/2}, \qquad s_{\rm flat} = \left(\frac{E}{b}\right)^{1/4}\frac{\xi}{\sqrt{2}}cn(\tilde{\tau} + \delta). \tag{52}$$

From Eqs. (43) and (46) the scale factor in the Einstein gauge and the inflaton can be written as (recall $\zeta \equiv |\frac{4b}{c}|^{1/4}$ and $\tilde{\tau} \equiv 2|cE|^{1/4}\tau$)

$$a_E = \frac{\kappa}{\sqrt{6}}\left(\frac{E}{b}\right)^{1/4}\left\{\left[\frac{1 - cn(\zeta\tilde{\tau})}{1 + cn(\zeta\tilde{\tau})}\right] - \frac{\zeta^2}{2}[cn(\tilde{\tau} + \delta)]^2\right\}^{1/2} \tag{53}$$

and

$$\sigma_E = \frac{\sqrt{6}}{\kappa}\frac{1}{2}\ln\left(\frac{1 + \frac{\zeta}{\sqrt{2}}cn(\tilde{\tau} + \delta)\left[\frac{1+cn(\zeta\tilde{\tau})}{1-cn(\zeta\tilde{\tau})}\right]^{1/2}}{1 - \frac{\zeta}{\sqrt{2}}cn(\tilde{\tau} + \delta)\left[\frac{1+cn(\zeta\tilde{\tau})}{1-cn(\zeta\tilde{\tau})}\right]^{1/2}}\right). \tag{54}$$

It should be noted that the energy parameter E determines the scale of a_E, so E could be fixed if we want to normalize a_E in the conventional way, $a_E(\text{today}) = 1$, but since we are discussing the early Universe we will keep it as shown above.

In the remainder of the discussion we will further specialize to a specific value of the available constants by taking $\delta = 0$ and $\zeta = \frac{1}{2}$ (or $64b = c$). Although there is no reason *a priori* to prefer this model, we will take this as an example to show that our solution is compatible with a phenomenological model of inflation. A study of phenomenologically consistent more general parameter space, determined by numerical analysis, is expected in following papers.

The scale factor a_E as a function of $\tilde{\tau}$ is plotted in Fig. 1.

The interesting behavior of this particular solution is that at $\tilde{\tau} = \tilde{\tau}_{\rm BB} \approx 0.92$ the scale factor is exactly zero: $a_E(\tau_{\rm BB}) = 0$. This defines the origin for comoving time $t(\tau_{\rm BB}) = 0$ at the big bang.[3] Unlike pure exponential inflation, where the conformal time of the big bang is equal to $-\infty$, our model gives a finite conformal time for the big bang. Of course the finite value $\tilde{\tau}_{\rm BB} \approx 0.92$ is not physically significant since τ can be translated by an arbitrary amount as remarked earlier. A finite conformal time for the big bang is a general property of our scalar-tensor theory (1).

After the big bang, the scale factor increases monotonically to infinity at $\tilde{\tau}_\infty \approx 7.4$. Converting conformal time to physical time t for a comoving observer using Eq. (51) the corresponding physical time is infinite.[4] This is because the integral (51) for t diverges logarithmically when $\tilde{\tau}$ approaches $\tilde{\tau}_\infty$.

Inflation is defined as the region $\frac{d^2}{dt^2}(a_E) > 0$, or equivalently $\frac{d}{d\tau}(\frac{\dot{a}_E}{a_E}) > 0$. Using (53) with $\zeta = \frac{1}{2}$, the inflation region is $\tilde{\tau} > \tilde{\tau}_I \approx 2.87$. The conversion from the conformal time to the physical time $t(\tilde{\tau})$ depends on the magnitude of the dimensionless variable b. This value of b can be determined by requiring the scalar perturbation amplitude to be of order 10^{-5} [7]. This requires b to be around 10^{-12}. Using this small value of b we find that the time for inflation $\tilde{\tau} = \tilde{\tau}_I$ corresponds to $t_I = 1.6 \times 10^{-38}$ s or 3×10^5 Planck times after the big bang. Therefore, after fitting observation for the amplitude, the time of inflation relative to the big bang is *predicted* in our model. At the time t_I the energy density of the Universe is 7.9×10^{15} GeV, which is 3 orders of magnitude smaller than the Planck scale. This guarantees the theory can be applied at the time of the beginning of inflation.

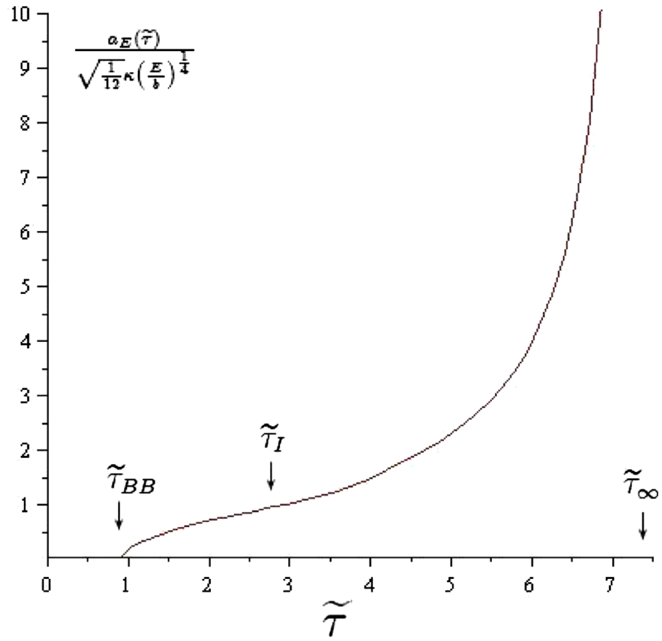

FIG. 1 (color online). The scale factor $a_E(\tilde{\tau})$ in the Einstein frame, from Eq. (53) with $\zeta = \frac{1}{2}$ and $\delta = 0$.

The analytic expression for the Hubble parameter $H = \frac{\dot{a}_E}{a_E^2}$ that indicates the expansion rate of the Universe is

$$H = \left(\frac{\kappa^2}{6}\right)^{-(1/2)} (\phi_{\rm flat}^2 - s_{\rm flat}^2)^{-3/2} (\phi_{\rm flat}\dot{\phi}_{\rm flat} - s_{\rm flat}\dot{s}_{\rm flat}), \tag{55}$$

which is plotted in Fig. 2.

From Fig. 2 we can see that the Hubble parameter decreases from infinity at the big bang monotonically to an asymptotic value. The Hubble rate does not change much after inflation starts at $\tilde{\tau}_I \approx 2.87$; thus during inflation we can approximate the conformal time as $\tau \approx -\frac{1}{a_E H}$ but not before.

The fact that $\frac{d}{d\tau}(\frac{\dot{a}_E}{a_E}) > 0$ when $\tilde{\tau} > \tilde{\tau}_I$ indicates that our theory does not have a mechanism for stopping inflation. However, like most of the inflation theories, one follows the dynamics of the inflaton to a certain point and claims that the inflaton decays into the standard bodel particles. This is called reheating [27,28]. Similarly, in our theory, we will assume after the theory produces enough inflation, another aspect of the complete theory will take over. The reheating mechanism is not the focus of this paper but certainly it is an important problem to investigate in the future.

From Eq. (54), we see that when the scale factor vanishes $a_E(\tau_{\rm BB}) = 0$ the inflaton must blow up logarithmically. Hence we find that the inflaton field σ drops from infinity at the big bang to zero in a finite conformal time and then keeps oscillating around the potential minimum eventually dropping to 0 at $\tilde{\tau} = \tilde{\tau}_\infty$. Unlike most inflation

[3]At the moment of the big bang $\tilde{\tau}_{\rm BB}$, it also happens that the gauge invariant variable $(1 - s^2/\phi^2)$ vanishes. The solution in terms of $(1 - s_{\rm flat}^2/\phi_{\rm flat}^2)$ is allowed to change sign in the flat gauge, but if we insist that the physics must be the physics described in the Einstein gauge in terms of the comoving time interval $0 \le t < \infty$, then we must confine the solution to only $(1 - s_{\rm flat}^2/\phi_{\rm flat}^2) \ge 0$. But it is interesting to note that in a more general gauge there are solutions in which the gauge invariant quantity $(1 - s^2/\phi^2)$ can go through zero and change sign, thus making a transition to a region with antigravity where the effective Newton constant $G \sim (\phi^2 - s^2)^{-1}$ is negative. Of course, applying a classical theory at the big bang, where $(1 - s^2/\phi^2)$ changes sign, is incomplete; however, this can be a guideline for a more complete quantum theory of the big bang, with possibly new physics insights.

[4]The area under this curve up to some point $\tilde{\tau}$ gives the comoving time t [see Eq. (51)]. So, it is intuitive to define $t = 0$ to correspond to $\tilde{\tau}_{\rm BB}$ while $t =$ infinity corresponds to $\tilde{\tau} = \tilde{\tau}_\infty$. The solution (53) for the scale factor a_E as a function of $\tilde{\tau}$ is actually a periodic function, so there is more to the curve than is shown in Fig. 1, but since the range for the comoving time is already infinite in one-quarter period in the space of $\tilde{\tau}$, the physics of the model in comoving time $0 \le t < \infty$ is already represented by the portion of the curve shown in Fig. 1.

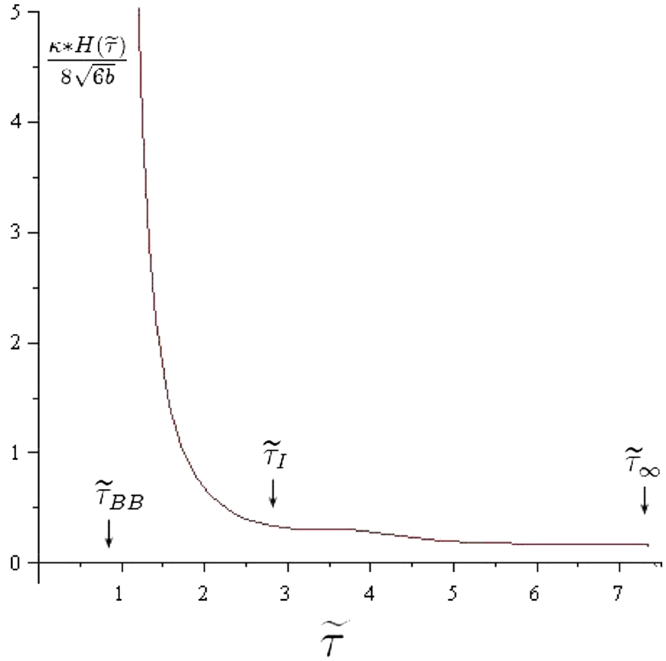

FIG. 2 (color online). The Hubble parameter $H(\tilde{\tau})$.

theories, this eliminates the need for declaring some arbitrary initial value for the inflaton. As shown in Fig. 3, this kind of inflation theory was never reported before. Furthermore, most inflation theories based on slow roll simply assume that inflation ends before the inflaton reaches its potential minimum, but this assumption is not valid as seen in the plot of our analytic solution. Hence,

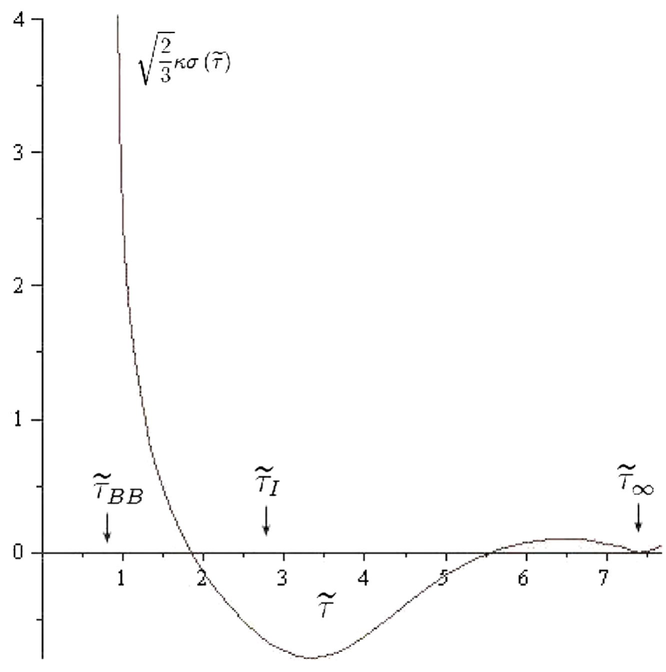

FIG. 3 (color online). The inflaton field drops from infinity at the big bang and then oscillates around the potential minimum, becoming zero at $\tilde{\tau} = \tilde{\tau}_\infty$.

within the slow-roll approximation, neither the connection to the big bang nor the oscillation around the potential minimum could be discovered.

To calculate the power spectrum $P_{\mathcal{R}}$, the spectral index n_s, n', and the ratio of tensor to scalar perturbation r, we have to solve the Mukhanov-Sasaki equation (12) with a proper boundary condition. The solvability of this equation hinges on the properties of the function $\ddot{z}(\tau)/z(\tau)$ whose analytic form in our model is given by

$$\frac{\ddot{z}}{z} = \frac{\partial_\tau^2(\dot{\sigma}_E/H)}{(\dot{\sigma}_E/H)}, \tag{56}$$

where $\dot{\sigma}_E/H$ is computed analytically by using the expressions for $\sigma_E(\tau)$ and $a_E(\tau)$ in terms of the solutions $\phi_{\rm flat}$ and $s_{\rm flat}$ given above as

$$z = \frac{\dot{\sigma}}{H} = \frac{(\phi_{\rm flat}\dot{s}_{\rm flat} - s_{\rm flat}\dot{\phi}_{\rm flat})}{(\phi_{\rm flat}\dot{\phi}_{\rm flat} - s_{\rm flat}\dot{s}_{\rm flat})}(\phi_{\rm flat}^2 - s_{\rm flat}^2)^{1/2}. \tag{57}$$

The explicit behavior of $\frac{\ddot{z}}{z}$ in our model is plotted in Fig. 4. It seen that this is very different than the usual approximation α/τ^2; therefore as explained earlier in the paper following Eq. (13), we cannot use the standard approximation to the Mukhanov-Sasaki equation.

To solve the Mukhanov-Sasaki equation in the vicinity of the horizon crossing $\tau \sim \tau_*(k)$, as defined in Eq. (13), it is appropriate to approximate the curve $\frac{\ddot{z}}{z}$ with the dotted curve shown in Fig. 4. This corresponds to the upside down dotted parabola in the figure:

$$\frac{\ddot{z}}{z} \approx -k_0^2 - \alpha^2(\tau - \tau_{\rm max})^2. \tag{58}$$

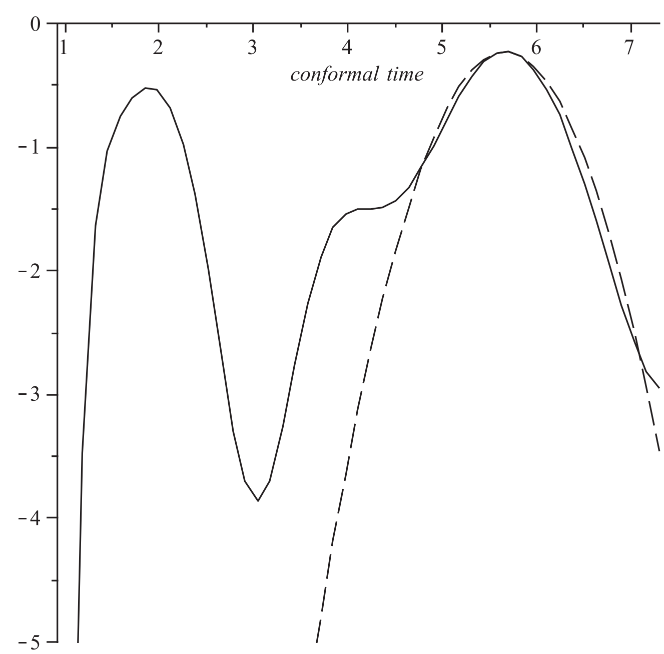

FIG. 4. Exact curve for $\frac{\ddot{z}}{z}(\tilde{\tau})$, solid line; quadratic approximation, dashed line.

We emphasize that the horizon crossing time $\tau_*(k)$ is down the tail past the peak $\tau_*(k) > \tau_{\max}$. Here the peak of the parabola is at time $\tau = \tau_{\max}$, while $-k_0^2$ is the value of $\frac{\ddot{z}}{z}$ at that time $\frac{\ddot{z}}{z}(\tau_{\max}) = -k_0^2$. Both of these quantities $(\tau_{\max,} - k_0^2)$ as well as the parameter α^2 are determined by our analytic solution for $\frac{\ddot{z}}{z}$ from Eq. (57). Our analytic solution in Eq. (52) depends on three parameters, namely, $(\frac{E}{b}, \xi, \delta)$, but $\frac{\ddot{z}}{z}$ depends only on (ξ, δ), which in turn fully determine the three parameters of the fitting parabola. The Mukhanov-Sasaki equation then takes the form

$$\ddot{v}_k + [(k^2 + k_0^2) + \alpha^2(\tau - \tau_{\max})^2]v_k = 0. \tag{59}$$

With this fitting function one can obtain analytic solutions for $v_k(\tau)$ in the desired neighborhood of $\tau < \tau_*(k)$ in terms of hypergeometric functions and choose an appropriate boundary condition at the peak of the inverted parabola at $\tau_{\max}$.

The curves shown in the various figures in this paper assume the values of $\delta = 0$ and $\xi = 1/2$ as an illustrative example. For this choice of parameters we find the fit function is given by $\frac{\ddot{z}}{z} \approx -0.23–1.2(\tau - 5.66)^2$.

The details of the computation for $v_k(\tau)$, and the corresponding $P_{\mathcal{R}}$, n_s, n', and r outlined in Eqs. (13), (14), and (18), can be found in [29]. Below, we plot the predicted spectral index $n_s(\tilde{\tau})$ and running of the spectral index $n'(\tilde{\tau})$, as well as constrain the tensor to scalar ratio $r(\tilde{\tau})$, as follows (where we recall $\tilde{\tau} = 2|cE|^{1/4}\tau$).

The observational data restrict the value of the spectral index n_s to between 0.92 and 1 at 60 e-folds before the end of inflation and the tensor to scalar ratio between 0 and 0.5. These ranges are the two-sigma region which corresponds to 95% confidence limit from the WMAP 7 years data [5]. Since in our theory inflation has to be truncated by hand, we just need to know if there is a time during inflation such that the values of n_s and r reside within this range. If such a time exists, we can let inflation develop for 60 more e-folds and truncate the theory.

From Fig. 5 we see that at $\tilde{\tau} \approx 7.05$, or equivalently $t = 3 \times 10^{-37}$ s after the big bang, the value of n_s is around 0.97.

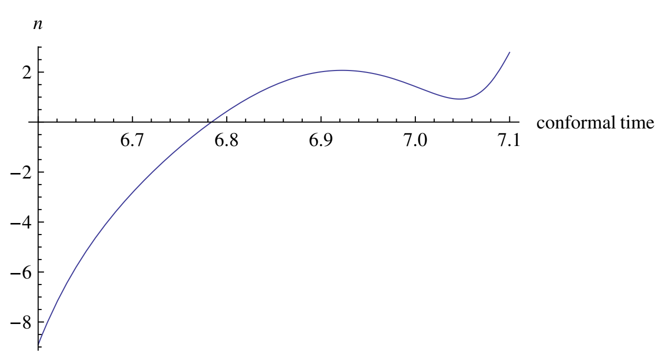

FIG. 5 (color online). The spectral index $n_s(\tilde{\tau})$. At $\tilde{\tau} = 7.05$ the value is 0.97.

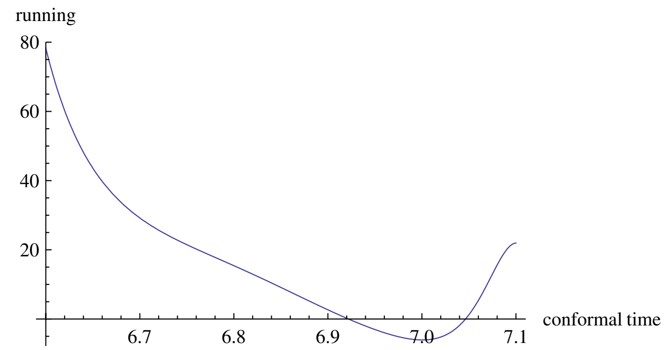

FIG. 6 (color online). The running of the spectral index $n'(\tilde{\tau})$. At $\tilde{\tau} = 7.05$ its value is close to zero.

The analytic expression for n' is not very transparent. It is plotted in Fig. 6. We can see that at $\tilde{\tau}_\infty \approx 7.05$ the running is around 0. This is within the upper bound of the WMAP 7 years data estimates of around 0.05.

Meanwhile, the observational upper bound of the tensor to scalar ratio r is satisfied in our model and merely constrains the boundary condition of the tensor mode perturbation equation at $\tau_{\max}$ [29].

V. CONCLUSION AND FUTURE DIRECTIONS

In this paper we have established a definite relation between the big bang and inflation, where neither one could be considered as separate events due to separate physical reasons. This is a new point of view for the cosmological evolution of our Universe.

We have also presented both a new method for solving exactly certain scalar-tensor theories as well as a sufficiently realistic new model of inflation. The theory has very few parameters, yet it makes phenomenologically consistent interesting predictions based on an analytic solution and simultaneously reveals several new physical features not discussed before.

The method consists of enlarging the action to a form with local Weyl symmetry, with some special potential. Both of these aspects were inspired by 2T physics. We can then solve the theory in a gauge where exact solutions are obtained.

This method should be regarded as one of the useful technical by-products of 2T gravity, while the physics content is also one of the novel predictions of 2T physics. The new features of our solution could have been obtained within 1T physics but was not, thus demonstrating that 1T physics lacks the guidance that 2T physics supplies systematically in the form of hidden dualities and hidden symmetries as explained in [23–25].

We could in principle use the same techniques to obtain exact solutions for more general inflation models with more than one scalar field and more complicated potentials. As we have seen in this paper, a theory that appears intractable may be presented as a complicated 1T-physics shadow of a 2T-physics theory, while in another shadow that corresponds to a more tractable gauge choice, the theory is handled much more easily and even solvable. As illustrated in [23–25] 2T physics offers this kind of new insights that are not available in 1T physics. In this paper we have essentially used this duality idea of 2T physics for a specific case in the context of 2T gravity, but we expect more general applications of this concept in future work.

The exact solutions obtained with our methods allowed us to analyze the theory more precisely than using the slow-roll approximation. This revealed properties of the theory that could not be found by slow-roll analyses, such as the predictions for the time delay between the big bang and inflation or the oscillatory behavior of the inflaton.

Satisfying the constraints coming from the first-order phenomenological parameters, the spectral index, the running of the spectral index, and the amplitude of the scalar power spectrum, we have constructed a phenomenological model, with the action (1) and inflaton potential

$$V(\sigma) = \left(\frac{6}{\kappa^2}\right)^2 b\left(64\sinh^4\left(\sqrt{\frac{\kappa^2}{6}}\sigma\right) + \cosh^4\left(\sqrt{\frac{\kappa^2}{6}}\sigma\right)\right), \tag{60}$$

where b is a positive dimensionless parameter of order 10^{-12}. A small parameter like b is common in general inflation theories. This makes the coefficient of the potential V of order $(\frac{6}{\kappa^2})^2 b \sim (6\times 10^{15}\ \mathrm{GeV})^4$, revealing perhaps an interesting scale since it is not too far from the grand unification scale.

This particular theory has several interesting behaviors. However, it does not predict when inflation ends, which is one of its weak points. In a complete theory, a modification around the grand unified theory scale mentioned above could provide the desired mechanism to end inflation.

It is also interesting to note that we obtain a cyclic cosmology, not unlike Ref. [30], under the following conditions:

1. A special value of the integration constant δ given by the quarter period of the elliptic δ function $cn(\tilde{\tau}+\delta)$.
2. A quantized value of the parameter $\zeta = \frac{2}{n}$, with integer n. (61)

Under these conditions our solution describes an universe that expands from a big bang, shrinks to a big crunch, and repeats this cycle indefinitely. In the meantime the equation of state w

$$w \equiv \frac{\dot{\sigma}_E^2/2a_E^2 - V(\sigma_E)}{\dot{\sigma}_E^2/2a_E^2 + V(\sigma_E)} \tag{62}$$

never goes below -1. In this case the effective gravitational "constant" $(\phi^2_{\text{flat}}(\tau) - s^2_{\text{flat}}(\tau))$ that appears in the action equation (26) never changes sign in any cycle although it becomes zero at each big bang or big crunch. This cyclic cosmological scenario will be discussed in more detail separately in a future paper [31].

The applications of our equations are not restricted to inflation theories. They might also be useful to discuss dark energy to explain the current expansion of the Universe. However, this will require a fine-tuning of our parameters. If one wants to use a similar model for dark energy, b has to be as small as 10^{-120}. This is because the current Hubble time is way too big compared to the Planck time. Including supersymmetry in our approach may play a role to suppress the value of emergent cosmological constant at late times (see [32]). A modified version of our theory might be extendible up to the current era and then be applicable also to the physics of dark energy.

ACKNOWLEDGMENTS

This work was partially supported by the U.S. Department of Energy, Grant No. DE-FG03-84ER40168. We thank Elena Pierpaoli, Loris Colombo, Nicholas Warner, Yueh-Cheng Kuo, and Guillaume Quélin at USC for helpful discussions

APPENDIX: PROPERTIES OF JACOBI ELLIPTIC FUNCTIONS

The Jacobi elliptic functions used in this paper are $sn(z|m)$, $cn(z|m)$, and $dn(z|m)$. There are two parameters

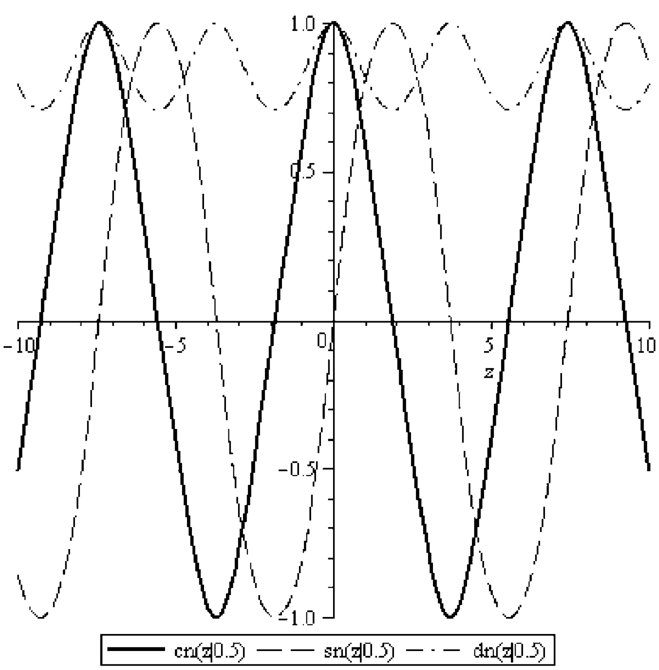

FIG. 7. Plots of $cn(z|\frac{1}{2})$ (solid line), $sn(z|\frac{1}{2})$ (dashed line), and $dn(z|\frac{1}{2})$ (dash-dotted line).

z and m for each of them. The parameter m determines the period under translations of the parameter z. The period is given by

$$\text{period } = 4 * \int_0^{\pi/2} \frac{d\theta}{(1 - m\sin^2\theta)^{1/2}}. \qquad (A1)$$

With $m = 0$ the period for all the Jacobi elliptic functions is 2π. For $m = 0$ these functions reduce to the familiar sine and cosine, $sn(z|0) = \sin(z)$, $cn(z|0) = \cos(z)$, and then $dn(z) = 1$. In our solutions we have $m = \frac{1}{2}$. This results in the period to be about 7.42 in z, but note that z is related by a factor to $\tilde{\tau}$ or it is translated by δ in the various entries in Tables I, II, and III. The functions $sn(z|\frac{1}{2})$, $cn(z|\frac{1}{2})$, and $dn(z|\frac{1}{2})$ are plotted in Fig. 7.

We can see that the behavior of sn and cn is very similar to their counterpart trigonometric functions. They also satisfy properties similar to trigonometric functions, such as

$$(sn(z|m))^2 + (cn(z|m))^2 = 1; \quad m(sn(z|m))^2 + (dn(z|m))^2 = 1. \qquad (A2)$$

The derivatives of Jacobi elliptic functions are given in terms of expressions somewhat similar to those for trigonometric functions:

$$\frac{d}{dz} sn(z|m) = cn(z|m) \times dn(z|m),$$
$$\frac{d}{dz} cn(z|m) = -sn(z|m) \times dn(z|m),$$
$$\frac{d}{dz} dn(z|m) = -m \times sn(z|m) \times cn(z|m).$$

One can look up more properties of Jacobi elliptic functions at [26].

[1] A. H. Guth, Phys. Rev. D **23**, 347 (1981).
[2] A. D. Linde, Phys. Lett. **129B**, 177 (1983).
[3] A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).
[4] D. La and P. J. Steinhardt, Phys. Rev. Lett. **62**, 376 (1989).
[5] W. H. Kinney, E. W. Kolb, A. Melchiorri, and A. Riotto, Phys. Rev. D **78**, 087302 (2008).
[6] E. Komatsu *et al.*, Astrophys. J. Suppl. Ser. **192**, 18 (2011).
[7] E. F. Bunn, A. R. Liddle, and M. White, Phys. Rev. D **54**, R5917 (1996).
[8] I. Bars, Phys. Rev. D **77**, 125027 (2008).
[9] A. R. Liddle and D. H. Lyth, Phys. Lett. B **291**, 391 (1992).
[10] A. R. Liddle, P. Parsons, and J. D. Barrow, Phys. Rev. D **50**, 7222 (1994).
[11] D. S. Salopek and J. R. Bond, Phys. Rev. D **42**, 3936 (1990).
[12] F. Lucchin and S. Matarrese, Phys. Rev. D **32**, 1316 (1985).
[13] R. Easther, Classical Quantum Gravity **10**, 2203 (1993).
[14] R. Easther, Classical Quantum Gravity **13**, 1775 (1996).
[15] J. D. Barrow, Phys. Rev. D **48**, 1585 (1993).
[16] J. D. Barrow, Phys. Rev. D **49**, 3055 (1994).
[17] D. Baumann, arXiv:0907.5424v1.
[18] E. D. Stewart and D. H. Lyth, Phys. Lett. B **302**, 171 (1993).
[19] D. H. Lyth and A. Riotto, Phys. Rep. **314**, 1 (1999).
[20] S. Dodelson, *Modern Cosmology* (Academic, New York, 2003), ISBN .
[21] A. Friedman, Gen. Relativ. Gravit. **31**, 12 (1999).
[22] I. Bars and S. H. Chen, Phys. Rev. D **79**, 085021 (2009).
[23] For a recent summary and status of 2T physics, see I. Bars, Int. J. Mod. Phys. A **25**, 5235 (2010).
[24] I. Bars, S. H. Chen, and G. Quelin, Phys. Rev. D **76**, 065016 (2007).
[25] I. Bars and G. Quelin, Phys. Rev. D **77**, 125019 (2008).
[26] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), ISBN .
[27] L. Kofman, A. D. Linde, and A. A. Starobinsky, Phys. Rev. Lett. **73**, 3195 (1994).
[28] L. Kofman, A. D. Linde, and A. A. Starobinsk, Phys. Rev. D **56**, 3258 (1997).
[29] S. H. Chen and J. B. Dent, arXiv:1012.4811.
[30] P. Steinhardt and N. Turok, New Astron. Rev. **49**, 43 (2005).
[31] I. Bars, S. H. Chen (unpublished).
[32] I. Bars, Phys. Rev. D **82**, 125025 (2010).