Energy of string loops and thermodynamics of dark energy

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We discuss the thermodynamic aspects of a simple model of cosmic string loops, whose energy is nonlinearly related to their lengths. We obtain in a direct way an equation of state having the form $p = -(1+\alpha)\rho/3$, with ρ the energy density and $1+\alpha$ the exponent which relates the energy u_l of a loop with its length l as $u_l \sim l^{1+\alpha}$. In the linear situation ($\alpha = 0$) one has $p = -\rho/3$, in the quadratic one ($\alpha = 1$) $p = -2\rho/3$, and in the cubic case ($\alpha = 2$) $p = -\rho$. For all values of α the entropy goes as $S \sim (2-\alpha)L^{3/2}$ (L being the string length density). The expression of S is useful to explore the behavior of such string loops under adiabatic expansion of the Universe. Thermodynamic stability suggests that the gas of string loops must coexist with several long strings, longer than the horizon radius.

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I. INTRODUCTION

Dark energy is thought to be the majoritary ingredient of the Universe, but its composition is unknown [1–5]. Its main physical feature is to produce an acceleration in the expansion of the Universe. According to Einstein's equations of general relativity, this requires that $3p + \rho$, with p the pressure and ρ the energy density, should be negative [1–5]. This follows from the form of the energy-momentum tensor, which for an ideal fluid with four-velocity u^μ has the form $T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu}$, where $g^{\mu\nu}$ is the metric tensor of the space-time, ρ the energy density, and p the pressure. Introduction of this tensor into the Einstein equations leads to the evolution equation for the cosmic scale factor a

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\sum_i(\rho_i + 3p_i), \tag{1}$$

where G is the gravitational constant and the summation is over all constituents of the Universe (baryonic matter, radiation, neutrinos, dark matter, and dark energy). The Hubble ratio $\dot{a}/a$ is related to the total energy density as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\sum_i\rho_i \quad \text{(flat universe)}. \tag{2}$$

It follows from (1) that an homogeneous fluid with $p = -\rho/3$ would not produce gravitational effects. Furthermore, the energy conservation law states that

$$\dot{\rho}_i = -3\frac{\dot{a}}{a}(\rho_i + p_i). \tag{3}$$

The first term in the right-hand side is the decrease of mass per unit volume due to the expansion, and the second one is the work done by the pressure. This equation states that the relation between p and ρ also influences the rate of decrease of the energy density along the Universe expansion. If $p = -\rho$, the work of expansion would cancel the reduction of energy density, and therefore the energy density would stay constant during the expansion.

Research of systems with the property $p = -w\rho$, with w a numerical parameter in the interval $1/3 \leq w \leq 1$, is an interesting topic in cosmology as well as in thermodynamics. The value $w = 2/3$ was considered to be close to the observational data on cosmic expansion some years ago [6,7], but more recent results favor values closer to $w = 1$ [8].

Some candidates for dark energy range from an extremely tiny cosmological constant to a variety of exotic fields as scalar fields, tachyons, k-essence, topological defects in cosmic fields, and so on [1,2]. In particular, the idea of topological defects and cosmic strings has been examined, for instance, in Refs. [9–17].

The aim of this paper is to present a thermodynamic analysis of a simple model of a system of cosmic string loops whose energy is nonlinearly related to their length. By starting from three simple hypotheses, the thermodynamic model is built, giving equations of state for the energy, pressure, and entropy of the system. This is a naive analysis where relativistic effects are not taken into account but which reproduces in a simple way a set of interesting physical features of this system. Being a thermodynamic model, it does not deal with the dynamical microscopic aspects nor with the physical origin and properties of the elementary ingredients, but it relates the assumed microscopic properties with macroscopic thermodynamic behavior. Thus, whether the cosmic loops disappear rapidly or slowly because of gravitational radiation, for instance, is not the topic of our analysis, which is devoted to the thermodynamic properties of the system as far as it exists. From our thermodynamical equations, and from the hypothesis of a reversible adiabatic expansion, the energy density and the entropy are obtained in terms of the cosmic scale factor (and therefore of the volume).

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From a thermodynamic point of view, the system is characterized by a given average length of loops and by the number of loops. The use of thermodynamic methods does not necessarily imply that the system is in equilibrium, but it may be in a nonequilibrium steady state, this average length and number of loops being the result of a fast and complex dynamics of breaking and reconnections under the effects of the expansion. The condition of using these methods is that the microscopic internal dynamics of the subsystem being considered is much faster than its macroscopic dynamics; it is not necessary that the different subsystems forming the system are in mutual thermodynamic equilibrium, but thermodynamic methods can be used for each of them provided the dynamics of their mutual energy exchange is much slower than the internal microscopic dynamics of each subsystem [18]. We assume that this is the case here, and we focus our interest on the consequences of this hypothesis rather than examining *a priori* the consistency of the hypotheses, which would require a better knowledge of the true microscopic nature and the true dynamics of the internal processes of the system is being considered. Since these microscopic details are not completely known, and since the current observations of it are of a macroscopic nature, it is logical to use thermodynamics and statistical mechanics as methods to derive the macroscopic consequences of microscopic models, in order to keep only those microscopic models compatible with the macroscopic observations.

Our approach may be useful to complement other analyses of the thermodynamics of dark energy which start from the dynamical equations for the cosmic expansion (1)–(3), relating the thermodynamic equations to some features of the expansion. Our approach starts from the definition of temperature in terms of the average energy of the hypothetical cosmic loops and on two hypotheses concerning the form of the energy of the individual loops as a function of their corresponding length and a scale-invariant hypothesis relating the average string loop length to the average separation between neighboring cosmic loops.

The plan of the paper is the following. In Sec. IIm the model is presented and the thermodynamic functions are thoroughly studied. In Sec. III, a study of the thermodynamic stability is made, a critical value for the temperature of the cosmic loops is determined, and the cosmological consequences are analyzed. A microscopic statistical interpretation of these results is proposed in Sec. IV. In the concluding remarks we compare our results to those obtained by previous authors [19–26].

II. TEMPERATURE, PRESSURE, AND ENTROPY FOR A GAS OF STRING LOOPS

The first hypothesis of the model is to assume that the temperature T of the system of closed cosmic string loops is given by

$$\tilde{c}k_B T = \langle u_l \rangle, \tag{4}$$

where u_l is the energy of a loop of length l, k_B is the Boltzmann's constant, $\tilde{c}$ is a constant of the order of 1, and the angle brackets denote the average over the loops of different lengths. This hypothesis comes from assuming that the energy is related only to the length of the loops; for the sake of generality, we write the numerical parameter $\tilde{c}$. For instance, if different normal modes of the string were assumed to act as different degrees of freedom, the value of $\tilde{c}$ would be related to the total number of degrees of freedom. We will show that its particular value does not modify essentially our conclusions and, in particular, that it has no influence on the equation of state for the pressure in terms of the energy density [Eq. (14)]. The robustness of this and other results is of interest because the value of $\tilde{c}$ could depend, in principle, on the number of independent degrees of freedom. The value of this hypothesis, which is plausible but not self-evident, will be judged in terms of the interest of the conclusions achieved from them.

The second hypothesis is to assume that the average length of the loops behaves as

$$\langle l \rangle = \lambda L^{-1/2}, \tag{5}$$

L being the total length of loops per unit volume, which has dimensions of (length)$^{-2}$, and λ a dimensionless constant. This hypothesis follows from microscopic analysis [15] which will be commented on in Sec. IV and is related to the statistical properties of the system of loops. In physical terms, Eq. (5) implies that the average length and curvature radius of loops are of the order of their average separation, which is given, on purely geometrical grounds, by $L^{-1/2}$ (L being the string length per unit volume).

Note that relation (5) is scale-invariant, because if all the sizes of the system are multiplied by a numerical factor, both the average radius of the loops as their average separation would increase in the same factor and (5) would remain unchanged.

Equation (5) implies that the total number of loops per unit volume N, given by $N = L/\langle l \rangle$, will behave as $N = \lambda^{-1} L^{3/2}$. Thus, a reduction of the average size of the loops at constant L will imply a big increase of the number of loops, and vice versa.

In the case of cosmic string loops, it is usually assumed that their energy is proportional to their length [9,10,12–17]. Here, for the sake of exploring a wider range of physical possibilities, we assume that the energy u_l of a line of length l is

$$u_l = a_0 l^{1+\alpha}, \tag{6}$$

with α a constant exponent and a_0 a constant given by $a_0 = \mu_V l_0^{-\alpha}$, μ_V being a coefficient of dimensions of energy per unit length, also called the tension of the string, and l_0 a fixed reference length. The term α in the exponent of the expression (6) describes possible deviations with respect to the linear behavior in the energy-length relation.

A deviation with respect to the linear behavior may be an intrinsic property of the loops or may be attributed to the interaction between other cosmic strings or with the several parts of the string loop; for instance, a source of discrepancy with respect to the linear expression may be the presence of kinks, arising from recombinations of previous loops; if the kink is very acute, the two parts of it may have a strong interaction between themselves, thus leading to contributions which would be absent in a smooth model of lines. The coefficient α is not necessarily an integer but it could be a fraction; that is, the interactions between loops or between other parts of the same loop could contribute to fractal aspects in such a way that the contribution to the energy is neither linear nor quadratic [27].

Here, we will not discuss the physical origin of the parameter α but its influence on the macroscopic equations of state. It could be guessed, however, that a positive (negative) value of α corresponds to situations where long wavelength perturbations have a higher (lower) contribution to the energy than short wavelength perturbations. Indeed, an exponent higher than 1 in $u_l \sim l^{1+\alpha}$ gives proportionally a higher increase of energy when l increases, and the opposite follows for exponents lower than 1.

Thus, relations (4)–(6) lead to $a_0\langle l^{1+\alpha}\rangle = \tilde{c}k_BT$. In view of (5), and of $\langle l^{1+\alpha}\rangle = \beta\langle l\rangle^{1+\alpha}$, with β being a numerical constant which depends on the detailed form of the statistical distribution of the lengths of the loops, which will be discussed below, in Sec. IV, temperature scales with L as

$$\tilde{c}k_BT = a_0\beta\lambda^{1+\alpha}L^{-(1+\alpha)/2} \equiv A_TL^{-(1+\alpha)/2}, \tag{7}$$

with $A_T = a_0\beta\lambda^{1+\alpha}$ [the value of β is explicitly obtained in Eq. (27) for a potential distribution function with a lower cutoff for the loop length]. This implies that the energy density per unit volume ρ of the system of string loops depends on T as

$$\rho = \tilde{c}k_BTN = \frac{A_T}{\lambda}L^{(2-\alpha)/2} = \frac{A_T}{\lambda}\left(\frac{\tilde{c}k_B}{A_T}\right)^{-[(2-\alpha)/(1+\alpha)]}T^{-[(2-\alpha)/(1+\alpha)]}. \tag{8}$$

Since, according to the fundamental Gibbs equation of thermodynamics, the entropy per unit volume s satisfies the relation $(\partial s/\partial\rho) = T^{-1}$, we obtain

$$\frac{\partial s}{\partial L} = \frac{\partial s}{\partial \rho}\frac{\partial \rho}{\partial L} = \frac{1}{T}\frac{A_T}{\lambda}\frac{2-\alpha}{2}L^{-\alpha/2} = \tilde{c}k_B\frac{2-\alpha}{2\lambda}L^{1/2}. \tag{9}$$

Integrating this equation, we get for the entropy per unit volume $s(L)$

$$s(L) - s_0 = \tilde{c}k_B\frac{2-\alpha}{3\lambda}(L^{3/2} - L_0^{3/2}). \tag{10}$$

It is worth noting that the exponent of L in this expression does not depend on α. This implies that the value of α will not have an influence of the evolution of L as a function of time in an adiabatic expansion (i.e. at constant total entropy). Furthermore, to have a positive entropy requires α to be less than or equal to 2, which excludes the possibility of a phantom energy, i.e. of a system with $p < -\rho$ (we will come back to this point at the end of the paper in a discussion on the chemical potential of cosmic loops) [19,28–32].

Since $s(L)$ is the contribution of the loops to the entropy density of the Universe, it is logical to expect that when $L = 0$, this contribution to the entropy will be 0; therefore, one can ignore the additive factor in (10); i.e. one may set $s_0 = 0$ and $L_0 = 0$.

To find the expression of the pressure p we need the total entropy $S = S(U, V)$ as a function of the total energy U and the volume V, to apply the well known thermodynamic relation $(\partial S/\partial V)_U = p/T$ [33]. We assume that $S(U, V) \equiv s(\rho)V$, in other words, that entropy (as well as energy) are extensive quantities, namely, that for a uniform system, the total energy and total entropy are equal to their respective densities per unit volume times the total volume, i.e. $U = \rho V$ and $S = sV$. This is not the only conceivable possibility, because systems with long range interaction, such as the gravitational one, may be expected to behave in a nonadditive way. Thus, the present hypothesis is admissible provided that the mutual gravitational interaction of neighboring loops, of the order of $GM^2/L^{-1/2}$, M being the average mass of a loop, is much less than the average energy of a single loop $Mc^2 = k_BT = (1/\tilde{c})A_TL^{-(1+\alpha)/2}$, with c being the light speed.

Recall that according to (5) both the average lengths of loops as their average separation are of the order of $L^{-1/2}$. This requires that $(GA_T/c^4\tilde{c})L^{-\alpha/2} \ll 1$. In the linear case, when $\alpha = 0$, this means that $G\mu_V/c^4 \ll 1$, with μ_V being the tension of the string or energy per unit length. In fact, the usual analyses of cosmic strings take for $G\mu_V/c^4$ values ranging from 10^{-6} to 10^{-30}, in any case much smaller than 1 [9–11], and therefore the extensivity hypothesis may be safely used.

In view of the relation (8) between ρ and L, and of the extensivity hypothesis that we have just discussed, neglecting, as has been said, the constant of integration in (10), we may write the total entropy as

$$S(U, V) = \tilde{c}k_B\frac{2-\alpha}{3\lambda}L^{3/2}V = \tilde{c}k_B\frac{2-\alpha}{3\lambda}\left[\frac{\lambda}{A_T}\right]^{3/(2-\alpha)} \times \left(\frac{U}{V}\right)^{3/(2-\alpha)}V \propto U^{3/(2-\alpha)}V^{-[(1+\alpha)/(2-\alpha)]}. \tag{11}$$

Note, incidentally, that in an adiabatic expansion (i.e. constant entropy), in a universe dominated by loops, one would have, according to (11), $L^{3/2}V = \text{constant}$. If $a(t)$ is the cosmic scale factor, and therefore $V(t) \sim a^3(t)$, this would yield $L(t) \sim a^{-2}(t)$. Thus, in view of (8), in an adiabatic expansion, the energy density ρ of loops changes as

$$\rho = \frac{U}{V} \sim a^{-2(1-(\alpha/2))}(t). \tag{12}$$

This means that under cosmic expansion the energy density of cosmic loops reduces in a slower way than that of matter ($\rho_{\rm mat} \sim a^{-3}$) or that of radiation ($\rho_{\rm rad} \sim a^{-4}$). As a consequence, it will become dominant at long times, provided the loops may subsist long enough, as they may disappear by emission of gravitational waves, decaying into baryonic matter, or coalescing into long strings not being loops [15]. Furthermore, by introducing (12) into (2) it follows that in the loop-dominated universe, the scale factor will evolve as $a(t) \sim t^{2/(2-\alpha)}$; when $\alpha = 0$, $a(t) \sim t$, and when $\alpha = 1$, $a(t) \sim t^2$.

From expression (11), the pressure may be obtained from the well known thermodynamical relation $(\partial S/\partial V)_U = p/T$, leading to

$$\frac{p}{T} = -\tilde{c} k_B \frac{\alpha + 1}{3\lambda} \left[\frac{\lambda}{A_T}\right]^{3/(2-\alpha)} \rho^{3/(2-\alpha)}. \tag{13}$$

Combining this equality with expression (8) for T in terms of ρ leads to

$$p = -\frac{1+\alpha}{3}\rho. \tag{14}$$

Note that this result does not depend on the value of the coefficient $\tilde{c}$ introduced in Eq. (4), which indicates that the concrete number of degrees of freedom of the loop is not relevant for the relation between p and ρ.

The negative pressure is a consequence of the fact that—according to (11)—S decreases when V increases at constant U; i.e. at a constant value of the total length of cosmic strings $\mathcal{L} = LV$, for values of α between 2 and -1. For α higher than 2, the entropy would become negative. We will comment on this point in the final paragraphs.

An increase of V at constant $\mathcal{L}$ implies a decrease of L, thus yielding a decrease in the number of strings per unit volume (recall that $N \sim L^{3/2}$) and in the entropy density s. Because of the relation $dW = -pdV$, with dW the work done on the system in a change of volume dV, a negative pressure means that work must be done on the system to expand it, in contrast with usual materials with positive pressure. This work will increase the length of the strings during the expansion and may allow them to become the dominant constituent of the Universe, at a sufficient long time.

Note finally that the chemical potential of the loops will be zero, because the fundamental equation for S in (11) does depend only on U and on V, but not on N, and the chemical potential is related to the differential of the entropy with respect to N at U and V constant. The vanishing of the chemical potential is analogous to that for photons. In a following paper we will further explore the relation between the thermodynamics of string loops and of photons.

III. STABILITY ANALYSIS: STRING LOOPS AND OPEN STRINGS

In (14) we have found that the coefficient relating p to ρ is $-1/3$ in the linear case ($\alpha = 0$), which would lead to a cosmic expansion at a constant rate [as we have seen that in this case $a(t)$ proportional to t], $-2/3$ [for $\alpha = 1$, which implies a constant acceleration, as $a(t)$ proportional to t^2], or -1 (for $\alpha = 2$). It is known that if $p = -\rho$, the energy density remains constant in the expansion, as shown by Eq. (3). This is consistent with relation (8), which implies that for $\alpha = 2$, ρ does not depend on L: In this case, expansion could modify the total length and number of loops, but this would not imply a change in the energy density.

From a thermodynamic perspective, it must be outlined that the specific heat of this system is negative, because from (8) we obtain

$$C_{\rm loops} = \left(\frac{\partial U}{\partial T}\right)_V = -\frac{2-\alpha}{1+\alpha}\frac{A_T}{\lambda} V \left(\frac{\tilde{c} k_B}{A_T}\right)^{-[(2-\alpha)/(1+\alpha)]} T^{-[3/(1+\alpha)]}. \tag{15}$$

The feature of negative specific heat is analogous to that of black hole thermodynamics [4,34] or other gravitational systems, as a gas submitted to a central gravitational potential. This would imply that during the accelerated expansion the gas of cosmic loops is heating, which means that they will become longer or coalescing to yield longer loops and, eventually, infinite strings, which would be the situation for $T \to \infty$.

Such a kind of process has been studied from a microscopic perspective in Ref. [15]. In fact, a negative specific heat implies a thermodynamic instability of the system [33]. This would lead the system to evolve towards a gas of loops plus a number of single long strings, comparable to or longer than the horizon radius, which have a positive heat capacity. Indeed, in contrast with the gas of loops, the number of very long strings changes very slowly. This implies that in the relation $L = N\langle l\rangle$, N will be fixed and $\langle l\rangle$ will be proportional to L, instead to $L^{-1/2}$ as in (5). Since $\rho = N\tilde{c}k_B T$, it follows that the heat capacity is $C_{\rm long} = NV\tilde{c}k_B$, which is positive. The global stability requirement is

$$\frac{1}{C_{\rm long}} + \frac{1}{C_{\rm loops}} \geq 0. \tag{16}$$

Therefore, a system of NV long strings plus a gas of cosmic loops may be thermodynamically stable provided that the mentioned condition is realized. To get this condition, we take into account that according to (15) the heat capacity of a gas of loops, for example, with $\alpha = 0$, is $C_{\rm loops} = -2A_T^3\lambda^{-1}V(\tilde{c}k_B)^{-2}T^{-3}$. Then, the stability condition will be

$$\frac{1}{N\tilde{c}k_B} - \frac{\lambda k_B^2\tilde{c}^2T^3}{2A_T^3} \geq 0. \tag{17}$$

This leads to the following condition for the temperature of the gas of loops in stable equilibrium with long strings:

$$k_BT \leq k_BT_C = \frac{A_T}{\tilde{c}}\left(\frac{2}{\lambda N}\right)^{1/3}. \tag{18}$$

Thus, instead of imagining a pure gas of cosmic loops, it seems more realistic to imagine a mixture of a few very long loops and a gas of loops. The situation discussed in the previous paragraph is analogous to that of a Schwarzschild black hole inside a theoretical box [34]. The heat capacity of black holes is negative, varies as M^{-2}, M being the mass of the black hole, and cannot be stable by itself, but it arrives to a stable state by emission of Hawking radiation when

$$\frac{1}{C_{\rm blackhole}} + \frac{1}{C_{\rm rad}} \geq 0. \tag{19}$$

A natural question is whether the loop temperature should imply consequences on radiation emission and, therefore, on the cosmic background radiation. The answer is negative, however, because dark energy, as well as dark matter, lacks electromagnetic interaction and, therefore, cannot emit or absorb electromagnetic radiation. However, it may have an indirect influence on the large-angle correlation functions of the tiny anisotropies of the cosmic microwave background.

IV. MICROSCOPIC INTERPRETATION

To give a microscopic basis to the starting hypothesis (5) and to check the consistence of the results which have been found, we briefly comment on some aspects obtained from a more microscopic perspective, based on a distribution function of the length of loops [15]. Let $n(l)dl$ be the number of loops of length l comprised between l and $l + dl$, per unit volume. Following some previous authors [15] we assume that $n(l)$ has a power form

$$n(l) = Bl^{-q}(l_{\rm min})^{-p}, \tag{20}$$

where q and B are positive dimensionless constants and $l_{\rm min}$ the minimum length of vortex loops. For B to be dimensionless, the exponent p must have the form $p = 4 - q$, because $n(l)dl$ must have dimensions of $(\text{length})^{-3}$. In general, the value $q = 5/2$ is especially favored in theoretical analyses of these topics. Note that this kind of distribution is found as the steady state solution of a kinetic equation for the distribution of the length of loops in a nonexpanding universe [15]. In Ref. [16], $q = 5/2$ was obtained for an expanding universe in the matter era and $q = 2.8$ in a radiation era; in any case, $q > 2$ as required by Eq. (25). Furthermore, distribution functions having the power-law form are scale-invariant, and therefore they are consistent with the scale-invariant macroscopic hypothesis (5).

The number N of loops per unit volume, as well as the loop length density L, the loop energy per unit volume ρ, and the entropy per unit volume s, are found to be, for $q > 2 + \alpha$,

$$N = \int_{l_{\rm min}}^{\infty} B\frac{l^{-q}}{(l_{\rm min})^{4-q}}dl = \frac{B}{q-1}(l_{\rm min})^{-3}, \tag{21}$$

$$L = \int_{l_{\rm min}}^{\infty} B\frac{l^{-q+1}}{(l_{\rm min})^{4-q}}dl = \frac{B}{q-2}(l_{\rm min})^{-2}, \tag{22}$$

$$\rho = \int_{l_{\rm min}}^{\infty} B\frac{a_0 l^{-q+1+\alpha}}{(l_{\rm min})^{4-q}}dl = B\frac{a_0}{q-2-\alpha}(l_{\rm min})^{-2+\alpha}, \tag{23}$$

$$\begin{aligned} s &= -\tilde{c}k_BN\int_{l_{\rm min}}^{\infty}\frac{n(l)}{N}\log\frac{l_0n(l)}{N}dl \\ &= \tilde{c}k_B\frac{B}{q-1}l_{\rm min}^{-3}\left[\frac{q}{q-1} - \log\frac{(q-1)l_0}{l_{\rm min}}\right]. \end{aligned} \tag{24}$$

The factor l_0 inside the logarithmic term of the definition of the entropy is written to have a dimensionless combination inside the logarithm; l_0 could be replaced by some other characteristic length, such as, for instance, $l_{\rm min}$, which has a well-defined physical meaning; this term will contribute logarithmically to the factor multiplying to $l_{\rm min}^{-3}$, which is the dominant contribution which will be examined below.

From (21) and (22) it follows that the average loop length is

$$\langle l\rangle = \frac{L}{N} = \frac{q-1}{q-2}l_{\rm min} = \frac{B^{1/2}(q-1)}{(q-2)^{3/2}}L^{-1/2}, \tag{25}$$

because from (22) it follows that $l_{\rm min} = B^{1/2}(q-2)^{-1/2}L^{-1/2}$. Then, a potential distribution of the form (20) is seen to lead to relation (5) that we have taken as the starting point. It is also seen that for this relation to be valid the exponent q in (20) must be higher than 2. This condition implies a finite value of $\langle l\rangle$ and, therefore, that no infinite strings are taken in consideration here. This is indeed satisfied under the common assumption that $q = 5/2$.

The average energy and temperature are related by

$$\begin{aligned} \tilde{c}k_BT &= \frac{\rho}{N} = a_0\frac{q-1}{q-2-\alpha}l_{\rm min}^{1+\alpha} \\ &= a_0\frac{q-1}{q-2-\alpha}\left[\frac{B}{q-2}\right]^{(1+\alpha)/2}L^{-[(1+\alpha)/2]}. \end{aligned} \tag{26}$$

This shows that our proposal (4) for the temperature of the strings in terms of L is also corroborated from a microscopic reasoning. Furthermore, it allows us to find the numerical factor β introduced above Eq. (7) relating $\langle l^{1+\alpha}\rangle$ and $\langle l\rangle^{1+\alpha}$ as $\langle l^{1+\alpha}\rangle = \beta\langle l\rangle^{1+\alpha}$. It turns out that

$$\beta = \frac{(q-2)^{1+\alpha}}{(q-1)^{\alpha}(q-2-\alpha)}; \tag{27}$$

this depends on the exponent q defining the statistics, as was expected, as well as on the exponent α. Note that if an upper cutoff $l_{\max}$ for the length had been considered besides the lower cutoff $l_{\min}$, the relation between $\langle l\rangle$ and L would have been not so simple, but it would have been dependent also on the ratio $l_{\min}/l_{\max}$. It could be considered that Eq. (27) also requires that $q > 2+\alpha$; if this is so, this could be an interesting restriction of the possible couples of values of q and α. Anyway, the value $q = 5/2$ has been obtained in the literature by considering linear loops, with $\alpha = 0$; a more general analysis for nonlinear strings with $\alpha > 0$ should be carried out in the future.

V. CONCLUDING REMARKS

To summarize the main ideas, here we have analyzed a simple model to study several thermodynamical functions which could be of interest for the modelization of dark energy. In fact, this model has been inspired by analyses of vortex loops in fully developed turbulence in superfluid helium [27,35–37]. The connection between cosmological models and superfluid models has already a fruitful tradition, summarized in Ref. [38]. We have studied the microscopic requirements in the relation between the energy u_l of a loop of length l and its length, through the relation $u_l \sim l^{1+\alpha}$, and have seen that the linear, the quadratic, and the cubic cases lead to specially significant relations $p = -(1/3)\rho$, $p = -(2/3)\rho$, and $p = -\rho$, respectively.

Our paper is based on only two hypotheses (4) and (5), which are microscopically consistent with a potential distribution function for the length of the loops, used in previous work on cosmic strings [15], plus a hypothesis on the relation between the energy and the length of loops. In our opinion, the expressions (11) for the entropy and (14) for the pressure are the most salient ones.

In the former presentation it could seem that our model leads to a time-independent factor w, in contrast to models based on dynamical scalar fields [2–5], which yield a time-dependent $w(t)$. However, our model may also lead in a natural way to a time-dependent $w(t)$. Indeed, if three kinds of loops (linear, quadratic, and cubic) are present in the Universe, their contributions will change in a different form with the scale factor $a(t)$, according to (12). Assume that for a given situation, for which we take $a = 1$, the respective proportions of the energy are $\rho_i = x_i\rho$, with ρ the total energy density corresponding to the loops and ρ_i the energy density of loops with energy proportional to l^i; then the total contribution of the pressure will change as

$$p = -\frac{1}{3}\frac{x_1a^{-2} + 2x_2a^{-1} + 3x_3}{x_1a^{-2} + x_2a^{-1} + x_3}\rho. \tag{28}$$

Then, the effective pressure will change from $-(1/3)\rho$ for short times to $p = -\rho$ for long times; i.e. the coefficient relating the pressure to the density will change from $-1/3$ to -1. To these contributions one should add the radiation contribution $p_{\rm rad} = (1/3)\rho_{\rm rad}$ and the matter contribution $p_m = 0$ with the respective variations as a^{-4} and a^{-3}. Then, in this scenario one begins with a dominating radiation and $p = (1/3)\rho$, afterwards a dominating matter, and finally a dominating dark energy with p going to $p = -\rho$ in the long time.

On the other side, Eq. (14) shows that we do not assume *a priori* a relation $p = -w\rho$, with w a constant, but that this relation follows from hypotheses (4) and (5) and the microscopic hypothesis (6), and illustrate the relation between the microscopic equation (6) and the macroscopic equation (14). Note that because of (5), the entropy may be rewritten as $S \sim \langle l\rangle^{-3}V$, from which it follows that in an adiabatic cosmic expansion $\langle l\rangle$ would change as $\langle l\rangle \sim a(t)$; i.e. the loops would be stretched in the same proportion as the cosmic horizon. In terms of the temperature, we may rewrite this expression as $S \sim T^{-3/(\alpha+1)}V$, which becomes $S \sim T^{-3}V$ when $\alpha = 1$.

An equation of this form has also been proposed in Ref. [28] but derived from the dynamical properties of the cosmic expansion with dark energy rather than starting from a purely thermodynamic formalism. In particular, this relation implies that the gas of loops heats up during the expansion, as a consequence of the work done on it by the negative pressure. In this interpretation of temperature, the gas of loops will not be in equilibrium with the cosmic event horizon, which is attributed a temperature in an analogous way to the black hole temperature and which is inversely proportional to the Hubble radius. Thus, the horizon temperature decreases, whereas the string loop temperature increases. This is in contrast with some proposals [29] that the dark energy is in thermal equilibrium with the apparent horizon. To examine this point, a careful analysis of the meaning of temperature for each constituent must be carried out. Because of expansion, it is not necessary to assume that all components are at the same temperature, if the different constituents interact weakly. The multiplicity of values of temperature in nonequilibrium situations is well known (see [30] for an extensive discussion).

Another topic of discussion in the thermodynamics of dark energy concerns whether the chemical potential of dark energy is zero or not [28,31]. In our analysis, the chemical potential is zero, as the entropy (11) depends only on U and V but not on N, as in radiation thermodynamics, where the chemical potential is zero. This vanishing of the chemical potential was also assumed in Ref. [32] and other previous papers mentioned in [32]. In [28] it is argued that if the chemical potential of dark energy is zero, phantom energy (i.e. $p < -\rho$) is forbidden, because the entropy would become negative. At first sight, our formalism seems to point also to this fact; however, since the entropy (11) we have studied is the contribution of the strings, but the other components of the system (radiation and matter) have in principle a non-negative entropy, the entropy of the strings could be negative without the total entropy of the system being negative. Therefore, this aspect should be analyzed in more detail by taking into account the total entropy of the full system. However, our analysis does indeed point to a zero chemical potential for the strings and not to a negative chemical potential. But in [28] it is shown that a negative chemical potential of dark energy could permit the existence of phantom energy, whereas a zero or positive chemical potential would rule it out. However, the introduction of the chemical potential in [28] is not based on deep physical grounds but only in a formal analysis concerning the possibility of phantom energy.

A qualitative, but speculative, argument against phantom energy could be based on a topological interpretation of the coefficient α. In particular, the situation with $\alpha = 1$, corresponding to $U_l = a_0 l^2$, may be interpreted as an energy proportional to an area; one possible interpretation could be to imagine that, instead of loops of length l, the basic entities of our system are surfaces of dimension $l \times l$, whose relation to cosmic walls should be explored. Along this line of thought, one could also imagine that the situation with $\alpha = 2$ would correspond to three-dimensional entities. Thus, in a three-dimensional space this would be the maximum possible value of α, because for α higher than 2 the entities would have a dimensionality higher than 3, which is the dimensionality of the embedding space. This interpretation is only tentative and cannot be completely trusted for the moment, but it seems to give an intuitive hint of why phantom energy would not be possible in a three-dimensional space. Its possibility could not be excluded, however, in a four-dimensional space, as those arising in some brane universe models.

The interest of the model presented in this paper is therefore to yield a thermodynamic analysis wider than the more usual models focusing on the equations of state for the pressure and to be mathematically very simple—without pretending to grasp the true and definitive physics of dark energy. Topics to be analyzed in the future include, for instance, the interaction between dark energy and dark matter or with other cosmic components [22–26], in order to avoid the so-called coincidence problem.

This kind of result points out that thermodynamics, being not based on the concrete microscopic details, may provide useful insights relating very different microscopic phenomenologies in interesting macroscopic connections.

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