Statefinder hierarchy: An extended null diagnostic for concordance cosmology

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(Received 13 November 2010; published 2 February 2011)

We show how higher derivatives of the expansion factor can be developed into a null diagnostic for concordance cosmology (ΛCDM). It is well known that the Statefinder—the third derivative of the expansion factor written in dimensionless form, $a^{(3)}/aH^3$, equals unity for ΛCDM. We generalize this result and demonstrate that the hierarchy, $a^{(n)}/aH^n$, can be converted to a form that stays pegged at unity in concordance cosmology. This remarkable property of the Statefinder hierarchy enables it to be used as an extended null diagnostic for the cosmological constant. The Statefinder hierarchy combined with the growth rate of matter perturbations defines a *composite null diagnostic* which can distinguish evolving dark energy from ΛCDM.

DOI: 10.1103/PhysRevD.83.043501 PACS numbers: 98.80.Es, 98.62.Sb, 98.65.Dx

I. INTRODUCTION

Despite its radical connotations, there is mounting observational evidence in support of a universe that is currently accelerating [1]. Theoretically there appear to be two distinct ways in which the Universe can be made to accelerate [2]: (i) through the presence of an additional component in the matter sector, which, following [3], we call *physical dark energy*. Physical DE models possess large negative pressure and lead to the violation of the strong energy condition, $\rho + 3P \geq 0$, which forms a necessary condition for achieving cosmic acceleration. Prominent examples of this class of models include the cosmological constant "Λ", Quintessence, the Chaplygin gas, etc. (ii) The Universe can also accelerate because of changes in the gravitational sector of the theory. These models (sometimes referred to as *geometrical DE* or modified gravity) include $f(R)$ theories, extra-dimensional braneworld models, etc.

Because of its elegance and simplicity the cosmological constant, with $P = -\rho$, occupies a privileged place in the burgeoning pantheon of DE models. Although the reasons behind the extremely small value of Λ remain unclear, concordance cosmology (ΛCDM) does appear to provide a very good fit to current data (although possible departures from $P = -\rho$ have also been noted [4]). Given the success and simplicity of concordance cosmology it is perhaps natural to discuss diagnostic measures which can be used to compare a given DE model with ΛCDM. "Null measures" of concordance cosmology proposed so far include the *Om* diagnostic [5,6], and the Statefinders [7]. While *Om* involves measurements of the expansion rate, $H(z)$, the Statefinders are related to the third derivative of the expansion factor. In a spatially flat ΛCDM universe, the Statefinders and *Om* remain pegged at a fixed value during our recent expansion history ($z \lesssim 10^3$).

In this paper we introduce the notion of the "*Statefinder hierarchy*" which includes higher derivatives of the expansion factor $d^n a/dt^n$, $n \geq 2$. We demonstrate that, for concordance cosmology, all members of the Statefinder hierarchy can be expressed in terms of elementary functions of the deceleration parameter q (equivalently the density parameter Ω_m). This property singles out the cosmological constant from evolving DE models and allows the Statefinder hierarchy to be used as an extended null diagnostic for ΛCDM.

II. THE STATEFINDER HIERARCHY

The expansion factor of the Universe can be Taylor expanded around the present epoch t_0 as follows:

$$(1+z)^{-1} := \frac{a(t)}{a_0} = 1 + \sum_{n=1}^{\infty} \frac{A_n(t_0)}{n!}[H_0(t-t_0)]^n \quad (1)$$

where

$$A_n := \frac{a^{(n)}}{aH^n}, \qquad n \in N; \quad (2)$$

$a^{(n)}$ is the nth derivative of the scale factor with respect to time. Historically different letters of the alphabet have been used to describe various derivatives of the scale factor. Thus $q \equiv -A_2$ is the deceleration parameter, while A_3, which was first discussed in [8], has been called the Statefinder "r"[7] as well as the jerk "j"[9]. A_4 is the snap "s" and A_5 is the lerk "l", etc. (See for instance [9–11] and references therein.)

It is quite remarkable that, in a spatially flat universe consisting of pressureless matter and a cosmological constant (henceforth referred to as concordance cosmology or ΛCDM), all the A_n parameters can be expressed as elementary functions of the deceleration parameter q, or the density parameter Ω_m. For instance[1]

[1]It is easy to see that successive differentiations of $\ddot{a} = a(\dot{H} + H^2)$ and $R = 6(\dot{H} + 2H^2)$ in conjunction with (3), can be used to express higher derivatives of the Hubble parameter, $H^{(n)}/H^{n+1}$, and the Ricci scalar, $R^{(n)}/6H^{n+2}$, as polynomial expansions in q (or Ω_m), for concordance cosmology.

$$\begin{aligned} A_2 &= 1 - \frac{3}{2}\Omega_m, \\ A_3 &= 1, \\ A_4 &= 1 - \frac{3^2}{2}\Omega_m, \\ A_5 &= 1 + 3\Omega_m + \frac{3^3}{2}\Omega_m^2, \\ A_6 &= 1 - \frac{3^3}{2}\Omega_m - 3^4\Omega_m^2 - \frac{3^4}{4}\Omega_m^3, \qquad \text{etc,} \end{aligned} \tag{3}$$

where $\Omega_m = \Omega_{0m}(1+z)^3/h^2(z)$ and $\Omega_m = \frac{2}{3}(1+q)$ in concordance cosmology. It is interesting to note that while A_3 remains pegged at unity, the remaining A_n parameters evolve with time with even A_{2n} (odd A_{2n+1}) remaining smaller (larger) than unity, see Fig. 1. All A_n approach unity in the distant future: $A_n \longrightarrow 1$ when $\Omega_m \longrightarrow 0$ and $\Omega_\Lambda \longrightarrow 1$. The above expressions allow us to define the *Statefinder hierarchy* S_n:

$$\begin{aligned} S_2 &:= A_2 + \frac{3}{2}\Omega_m, \\ S_3 &:= A_3, \\ S_4 &:= A_4 + \frac{3^2}{2}\Omega_m, \\ S_5 &:= A_5 - 3\Omega_m - \frac{3^3}{2}\Omega_m^2, \\ S_6 &:= A_6 + \frac{3^3}{2}\Omega_m + 3^4\Omega_m^2 + \frac{3^4}{4}\Omega_m^3, \qquad \text{etc.} \end{aligned} \tag{4}$$

The Statefinder stays pegged at unity for ΛCDM,

$$S_n|_{\Lambda\mathrm{CDM}} = 1, \tag{5}$$

during the entire course of cosmic expansion!

Equation (5) define a *null diagnostic* for concordance cosmology, since some of these equalities are likely to be violated by evolving DE models for which one might expect one (or more) of the Statefinders to depend upon time, see Fig. 2.

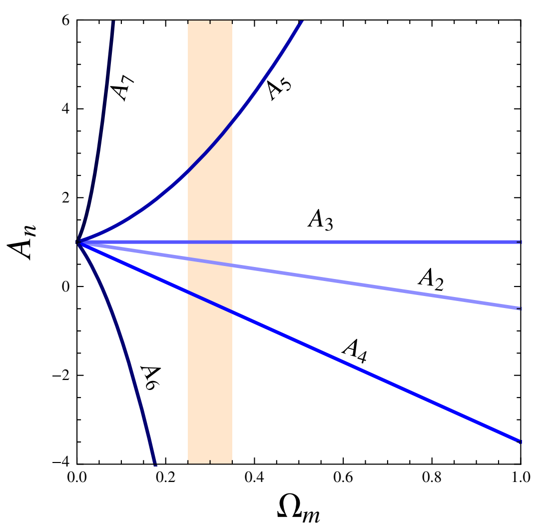

FIG. 1 (color online). A_n are plotted against $\Omega_m(z)$ for ΛCDM.

Fractional Statefinders

Interestingly, for $n > 3$ there is more than one way in which to define a null diagnostic. Using the relationship $\Omega_m = \frac{2}{3}(1+q)$, valid in ΛCDM, it is easy to see that the Statefinders can also be written in the alternate form:

$$\begin{aligned} S_4^{(1)} &:= A_4 + 3(1+q) \\ S_5^{(1)} &:= A_5 - 2(4+3q)(1+q), \qquad \text{etc.} \end{aligned} \tag{6}$$

Clearly both methods yield identical results for ΛCDM: $S_4^{(1)} := S_4 = 1$, $S_5^{(1)} := S_5 = 1$. For other DE models however, the two alternate definitions of the statefinder, S_n & $S_n^{(1)}$, are expected to give different results.

In [7] it was shown that a second Statefinder could be constructed from the Statefinder $S_3^{(1)} := S_3$, namely,

$$S_3^{(2)} = \frac{S_3^{(1)} - 1}{3(q - 1/2)}. \tag{7}$$

In concordance cosmology $S_3^{(1)} = 1$ while $S_3^{(2)}$ stays pegged at zero. Consequently the Statefinder pair $\{S_3^{(1)}, S_3^{(2)}\} = \{1, 0\}$ provides a model independent means of distinguishing evolving dark energy models from the cosmological constant [7]. In analogy with (7) we define the second member of the Statefinder hierarchy as follows:[2]

$$S_n^{(2)} = \frac{S_n^{(1)} - 1}{\alpha(q - 1/2)}, \tag{8}$$

where α is an arbitrary constant. In concordance cosmology $S_n^{(2)} = 0$ and

$$\{S_n^{(1)}, S_n^{(2)}\} = \{1, 0\}. \tag{9}$$

The second Statefinder $S_n^{(2)}$ serves the useful purpose of breaking some of the degeneracies present in $S_n^{(1)}$. For DE with a constant equation of state (EOS) w,

$$\begin{aligned} S_3^{(1)} &= 1 + \frac{9w}{2}(w+1)\Omega_{\mathrm{DE},} \qquad S_3^{(2)} = w + 1 \\ S_4^{(1)} &= 1 - \frac{27}{2}w(w+1)(w + \frac{7}{6})\Omega_{\mathrm{DE}} - \frac{27}{4}w^2(w+1)\Omega_{\mathrm{DE}}^2 \\ S_4^{(2)} &= -(w+1)(w + \frac{7}{6}) - \frac{1}{2}w(w+1)\Omega_{\mathrm{DE}} \end{aligned}$$

where $S_4^{(2)} = \frac{S_4^{(1)} - 1}{9(q - \frac{1}{2})}$ and $q - 1/2 = \frac{3w}{2}\Omega_{\mathrm{DE}}$.

[2] $\{S_3^{(1)}, S_3^{(2)}\} \equiv \{r, s\}$ in [7]. Equation (3) can be used to define still other null tests, including $Om_1 = \frac{1-\Omega_{0m}}{h^2} + \Omega_m(z)$, $\Sigma_1 := 2(1 - A_4)/9\Omega_m$, $\Sigma_2 := A_4 + 3(A_3 - A_2)$, where $Om_1 = 1$, $\Sigma_{1,2} = 1$ for ΛCDM.

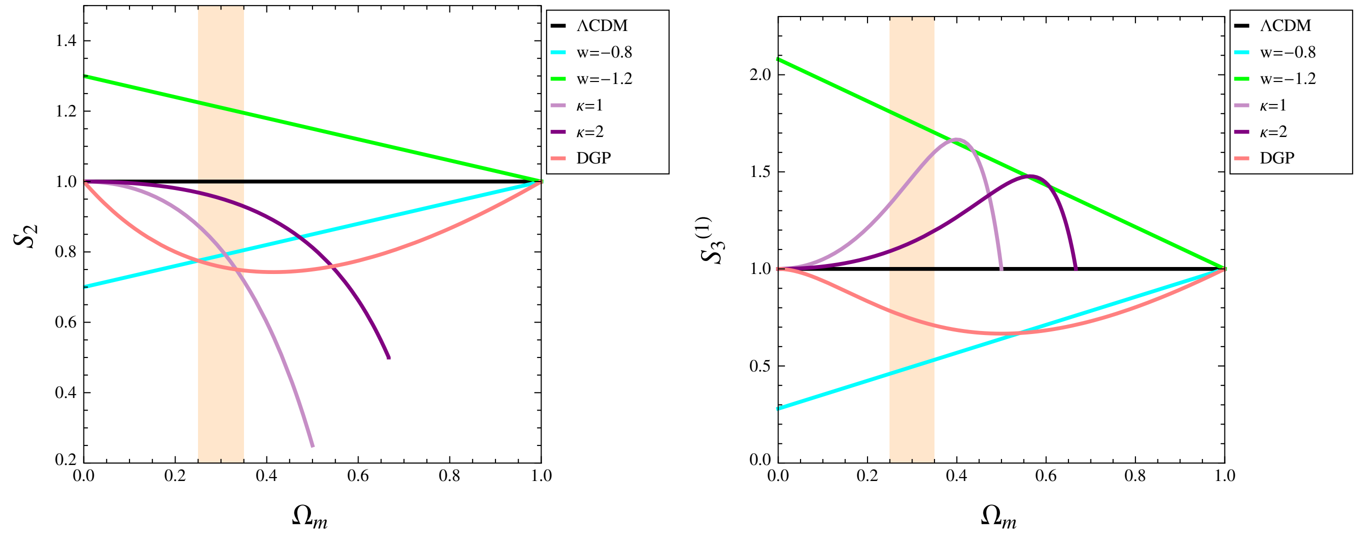

FIG. 2 (color online). The left (right) panel shows the Statefinder S_2 ($S_3^{(1)} \equiv S_3$) plotted against $\Omega_m \equiv \Omega_{0m}(1+z)^3/h^2$. Large values $\Omega_m \to 1$ correspond to the distant past ($z \gg 1$), while small values $\Omega_m \to 0$ correspond to the remote future ($z \to -1$). The models are: DE with $w = -0.8$ (blue), phantom with $w = -1.2$ (green), Chaplygin gas (purple), DGP (red). The horizontal black line shows ΛCDM. The vertical band centered at $\Omega_{0m} = 0.3$ roughly corresponds to the present epoch. Note that the near degeneracy seen in S_2 between DGP, $w = -0.8$ and $\kappa = 1$ Chaplygin gas, at $\Omega_{0m} \simeq 0.3$, is absent in $S_3^{(1)}$.

As we demonstrate in Figs. 2–5, the Statefinder hierarchy $\{S_n^{(1)}, S_n^{(2)}\}$ provide us with an excellent means of distinguishing dynamical DE models from ΛCDM. Our discussion focuses on the following models:

(1) Dark energy with a constant equation of state

$$\frac{H(z)}{H_0} = \left[\Omega_{0m}(1+z)^3 + \Omega_{DE}(1+z)^{3(1+w)}\right]^{1/2},$$

where concordance cosmology (ΛCDM) corresponds to $w = -1$. For simplicity we neglect the presence of spatial curvature and radiation.

(2) The Chaplygin gas [12] has the interesting EOS $p_c = -A/\rho_c$ while its density evolves as $\rho_c = \sqrt{A + B(1+z)^6}$. The expansion rate of a universe containing the Chaplygin gas and pressureless matter is given by

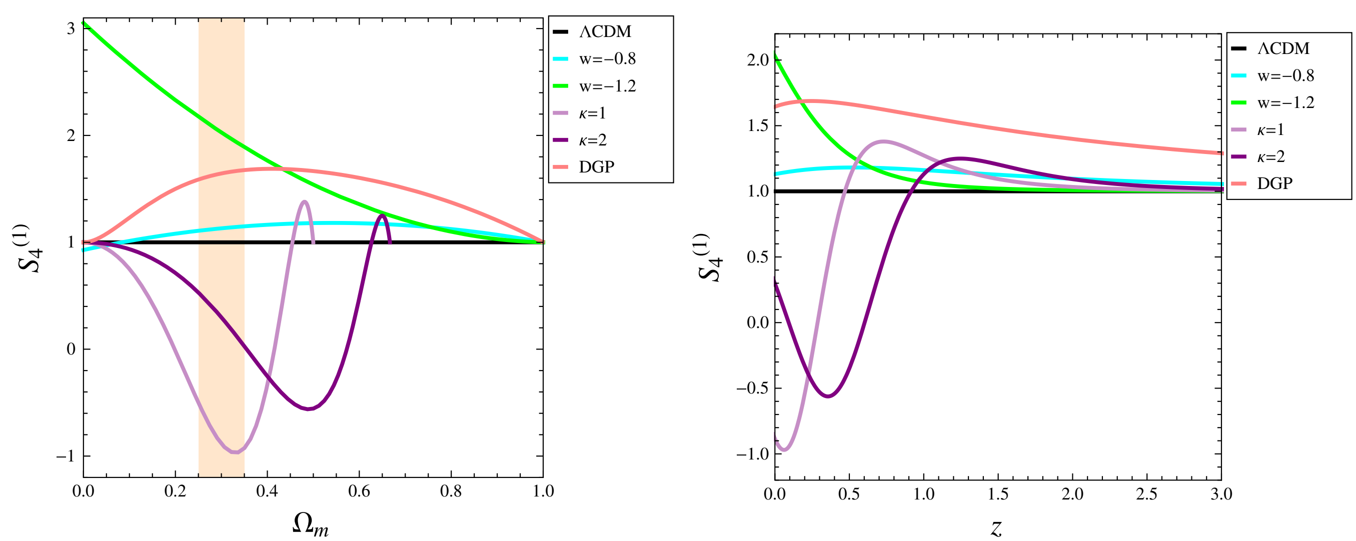

FIG. 3 (color online). The left (right) panel shows the Statefinder $S_4^{(1)}$ plotted against Ω_m (left panel) and z (right panel). The dark energy models are: DE with $w = -0.8$ (blue), phantom with $w = -1.2$ (green), Chaplygin gas (purple), DGP (red). The horizontal black line shows ΛCDM. The vertical band centered at $\Omega_{0m} = 0.3$ in the left panel roughly corresponds to the present epoch. $\Omega_{0m} = 0.3$ is assumed for DE models in the right panel.

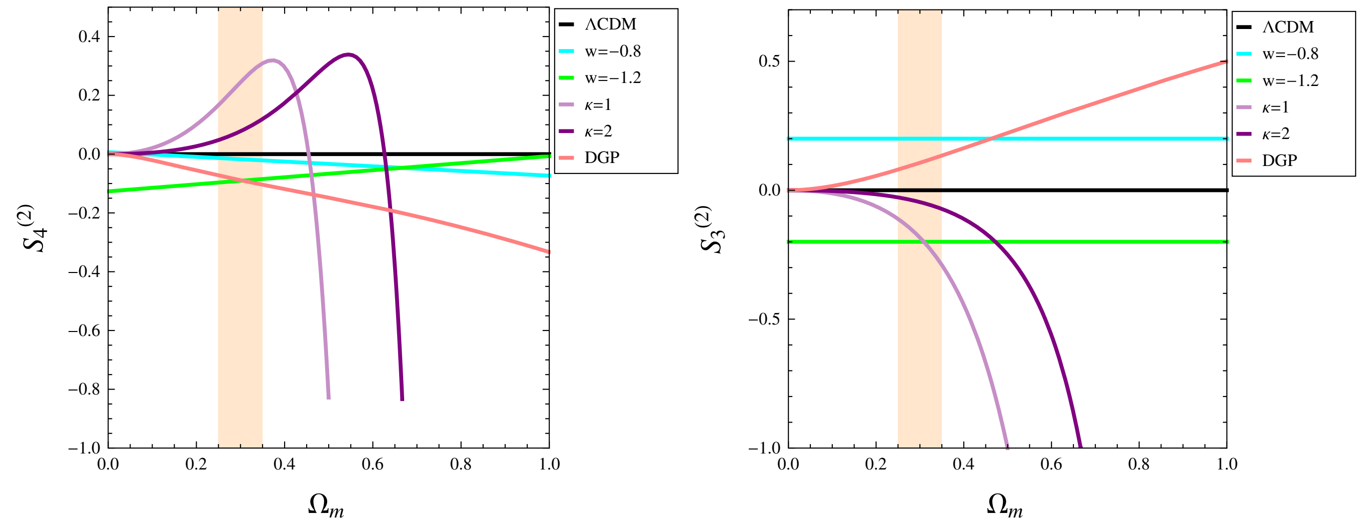

FIG. 4 (color online). The left (right) panel shows the Statefinder $S_4^{(2)}$ ($S_3^{(2)}$) plotted against Ω_m (left panel). The vertical band centered at $\Omega_{0m} = 0.3$ in the left panel corresponds to the present epoch. Comparing Fig. 4 with Fig. 3, we find that $S_4^{(2)}$ does not appear to perform as well as $S_4^{(1)}$, or even $S_3^{(2)}$, in distinguishing between the DE models considered in this paper.

$$\frac{H(z)}{H_0} = \left[\Omega_{0m}(1+z)^3 + \frac{\Omega_{0m}}{\kappa}\sqrt{\frac{A}{B} + (1+z)^6}\right]^{1/2},$$

where κ defines the ratio between the density in cold dark matter and the Chaplygin gas at the commencement of the matter-dominated stage of expansion and

$$A = B\left\{\kappa^2\left(\frac{1-\Omega_{0m}}{\Omega_{0m}}\right)^2 - 1\right\}.$$

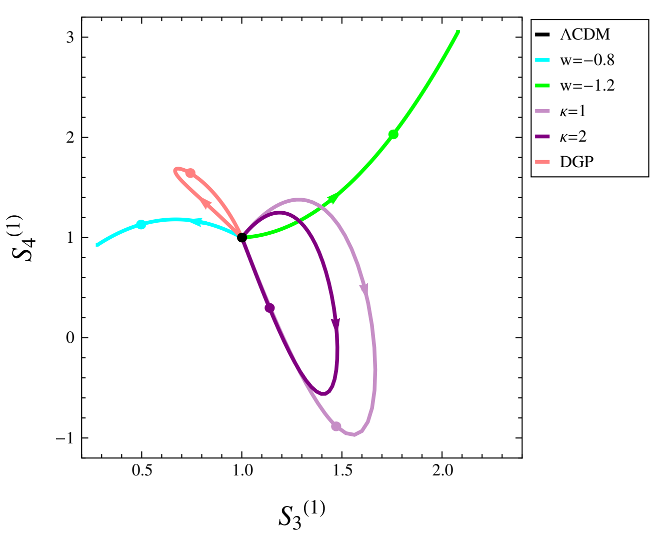

FIG. 5 (color online). The Statefinders $S_4^{(1)}$ and $S_3^{(1)} \equiv S_3$ are shown for the DE models discussed in the previous figures. The fixed point at $\{1, 1\}$ is ΛCDM. The arrows show time evolution and the present epoch in the different models is shown as a dot. $\Omega_{0m} = 0.3$ is assumed.

(3) The Braneworld model suggested by Dvali, Gabadadze and Porrati [13]:

$$\frac{H(z)}{H_0} = \left[\left(\frac{1-\Omega_{0m}}{2}\right) + \sqrt{\Omega_{0m}(1+z)^3 + \left(\frac{1-\Omega_{0m}}{2}\right)^2}\right].$$

III. GROWTH RATE OF PERTURBATIONS

The Statefinders can be usefully supplemented by the *fractional growth parameter* $\epsilon(z)$ [14]

$$\epsilon(z) := \frac{f(z)}{f_{\Lambda\mathrm{CDM}}(z)},$$

where $f(z) = d\log\delta/d\log z$ describes the growth rate of linearized density perturbations [15]

$$f(z) \simeq \Omega_m(z)^\gamma \tag{10}$$

$$\gamma(z) = \frac{3}{5 - \frac{w}{1-w}} + \frac{3}{125}\frac{(1-w)(1-\frac{3}{2}w)}{(1-\frac{6}{5}w)^3}(1-\Omega_m(z)) + \mathcal{O}[(1-\Omega_m(z))]^2. \tag{11}$$

The above approximation works reasonably well for physical DE models in which w is either a constant, or varies slowly with time. For instance $\gamma \simeq 0.55$ for ΛCDM [15,16]. It may be noted that in physical DE models such as Quintessence, the development of perturbations can be reconstructed from a knowledge of the expansion history [17]. This is not the case in modified gravity theories in which perturbation growth contains information which is complementary to that contained in the expansion history.

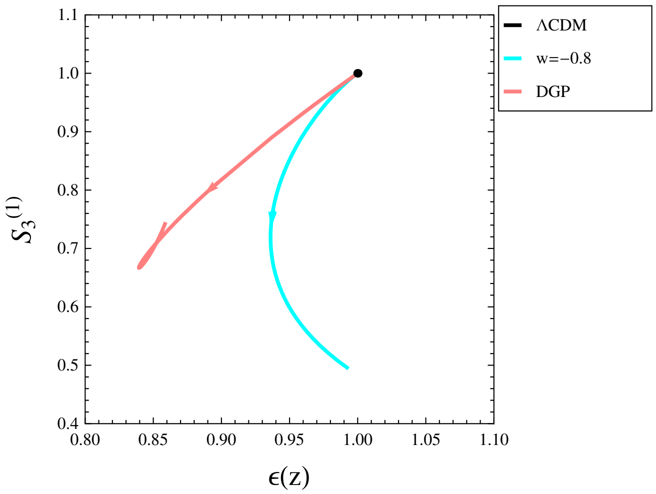

FIG. 6 (color online). The *composite null diagnostic* $\{S_3^{(1)}, \epsilon\}$ is plotted for three DE models. Arrows show time evolution which proceeds from $z = 100$ to $z = 0$.

(Eq. (10) is still a valid approximation for DGP cosmology but with $\gamma \simeq 2/3$ [18]).

For this reason, the *fractional growth parameter* $\epsilon(z)$, can be used in conjunction with the Statefinders to define a *composite null diagnostic*: CND $\equiv \{S_n, \epsilon\}$, where $\{S_n, \epsilon\} = \{1, 1\}$ for ΛCDM. Figure 6 demonstrates how DE models are distinguished by means of CND. (One can also use the *Om* diagnostic in place of the Statefinder in CND.)

IV. CONCLUSIONS

In this paper we have shown that a simple series of relationships links the Statefinder hierarchy in concordance cosmology with the deceleration/density parameters. These relationships can be used to define *null tests* for the cosmological constant. Including information pertaining to the growth rate of perturbations increases the effectiveness of this hierarchy of null diagnostics. Our results demonstrate that lower order members of the Statefinder hierarchy already differentiate quite well between concordance cosmology on the one hand, and Braneworld models and the Chaplygin gas, on the other. Since, for $n \geq 3$, the nth Statefinder S_n contains terms proportional to $w^{(n-2)}/H^{n-2}$, higher members of the hierarchy will contain progressively greater information about the evolution of the equation of state of DE. From the observational perspective, however, one might note that a determination of the S_n Statefinders involves prior knowledge of the $(n-1)$th derivative of $H(z)$. Thus only lower order Statefinders, S_n, $n \leq 4$, together with the *Om* diagnostic, may prove compatible with the quality of observational data expected in the near future.

ACKNOWLEDGMENTS

We acknowledge useful discussions with Ujjaini Alam and Alexei Starobinsky.

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