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Existence of diproton-like particles in 3 + 1 lattice QCD with two flavors and strong coupling

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Starting from quarks, gluons, and their dynamics, we consider the existence of two-baryon bound states of total isospin $I = 1$ in an imaginary-time formulation of a strongly coupled 3 + 1-dimensional $\mathrm{SU}(3)_c$ lattice QCD with two flavors and 4×4 spin matrices, defined using the Wilson action. For a small hopping parameter $\kappa > 0$ and a much smaller gauge coupling $0 < \beta \ll \kappa \ll 1$ (heavy quarks and large glueball mass), using a ladder approximation to a lattice Bethe-Salpeter equation, diproton-like bound states are found in the $I = 1$ isospin sector, with asymptotic masses $-6\ln\kappa$ and binding energies of order κ^2. By isospin symmetry, for each diproton there is also a dineutron bound state with the same mass and binding energy. The dominant two-baryon interaction is an energy-independent spatial range-one potential with an $\mathcal{O}(\kappa^2)$ strength. There is also an attraction arising from gauge field correlations associated with six overlapping bonds, but it is subdominant. The overall range-one potential results from a quark-antiquark exchange with no meson exchange interpretation (wrong spin indices). The repulsive or attractive nature of the interaction does depend on the isospin and spin of the two-baryon states. A novel representation in term of permanents is obtained for the spin, isospin interaction between the baryons, which is valid for any isospin sector.

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I. INTRODUCTION AND RESULTS

Our aim in this paper is to analyze the low-lying energy-momentum (EM) spectrum of an $\mathrm{SU}(3)_c$ lattice quantum chromodynamics (QCD) model. More precisely, we consider the question of the existence of two-baryon bound states in the total isospin $I = 1$ sector, where diprotons may be found. The existence of diprotons and dineutrons was hypothesized long ago, and has important consequences for the short time after the big bang nucleosynthesis [1,2], the resulting hydrogen concentration in the Universe, and the creation of matter and life [3–5]. Also, the existence of diprotons and dineutrons, and their stability or instability, is at the heart of a deep understanding of nuclear physics. Although some announcements have been made for the indication of the experimental evidence of diprotons and dineutrons, their existence was never confirmed [6]. To our knowledge, on the same footing is the status of their theoretical confirmation. Although protons and neutrons are part of the eightfold way particles theoretically corroborated by QCD, there is no reliable theoretical argument, based on dynamics, that supports the existence of this kind of two-baryon bound states or any other hypothesized multibaryon bound states, such as the tetraneutron [7] or more general protonium, neutronium, or other mixed states.

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The lack of knowledge on this type of fundamental question justifies making an appeal to lattice QCD in trying to reach a satisfactory answer to the problem. Here, we consider an imaginary-time version of the $\mathrm{SU}(3)_c$ Wilson lattice QCD model in 3 + 1 dimensions, with 4×4 Dirac spin matrices and with only two quark flavors (up and down). In order to be able to attack the problem of the existence of diprotons and dineutrons using reliable analytical methods, we work in the strong coupling regime. In this regime, the quarks and glueballs are heavy. If, in the Wilson action [8–11], we denote the hopping parameter by κ and the plaquette coupling parameter by β, this means we are assuming they verify the inequalities $0 < \kappa \ll 1$, $0 \leq \beta \ll \kappa$. The condition $\beta \ll \kappa$ guarantees that the lowest-lying EM spectrum is comprised of mesons and baryons. On the other hand, for $\beta \gg \kappa$, the low-lying EM spectrum consists only of glueballs and their excitations. Besides the local $\mathrm{SU}(3)_c$ gauge symmetry, our model has a global $\mathrm{SU}(2)_f$ flavor (isospin) symmetry.

Although any lattice and the strong coupling regime are physical limitations, their use still corresponds to the only procedure and domain of parameters where the QCD low-lying particle spectrum can be reached using reliable analytical techniques and for which the particle spectrum can be obtained from dynamical first principles, i.e. directly from the QCD quark-gluon dynamics. Moreover, regarding the EM particle spectrum of the model, we also point out that, up to now, the main qualitative spectral features that are supposed to hold for QCD in the Minkowski continuum spacetime were shown to be preserved in a type of lattice model like the one we treat here. Thus, if diprotons, dineutrons, and other more complex bound states are detected in strongly coupled lattice QCD, the underlying compound field structure describing these lattice bound states may give an indication of how the existence of these particles can be searched experimentally or theoretically, employing numerical simulations or eventually using other methods.

The search for diproton and dineutron-like particles in the above lattice QCD model, with strong coupling, represents one additional natural step in our long-standing program to unveil the low-lying lattice QCD EM spectrum in the strong coupling regime. In Refs. [12,13], we have rigorously validated the $\mathrm{SU}(2)_f$ part of the Gell-Mann and Ne'eman eightfold way picture for this model [14], and obtained the baryon and the meson mass splittings. The existence of the baryon and meson particles is obtained by showing the spectral properties of the lower mass gap and the upper mass gap. Hence, the baryon and meson particles are genuinely determined by making an identification with isolated dispersion curves in the model EM spectrum. The 56 baryon states, and their antiparticles, have masses $m = -3\ln\kappa - (3/4)\kappa^3 + \mathcal{O}(\kappa^6)$ and, for $\beta = 0$, there is a mass splitting of order $\mathcal{O}(\kappa^6)$ between the usual eightfold way octet and the decuplet (i.e. between the total spin $J = 1/2$ and $J = 3/2$, respectively). The 32 mesons have masses $m = -2\ln\kappa + \kappa^2 + \mathcal{O}(\kappa^4)$ and, for $\beta = 0$ there is a pseudoscalar, vector meson mass splitting (between $J = 0$ and $J = 1$) given by $2\kappa^4 + \mathcal{O}(\kappa^6)$. Mesons and their antiparticles coincide. The nonsingular part of the eightfold way particle masses is shown to be jointly analytic in κ and β, so that the above $\beta = 0$ mass splittings persist for $\beta \neq 0$.

Concerning the model bound state spectrum, we were able to extend the one-particle analysis to a lattice version of the Bethe-Salpeter (B-S) equation to analyze the two-particle spectrum. First, in [15], we analyzed the even sector $\mathcal{H}_e$, of states with an even number of fermions, of the model quantum mechanical physical Hilbert space $\mathcal{H}$ and inspected it for the existence of two-baryon bound states for the total isospin $I = 0$ and $I = 3$ sectors, below the free two-baryon threshold, which is given by twice the smallest of the baryon masses. We showed that there are diverse two-baryon bound states. Their asymptotic masses are $-6\ln\kappa$ and their binding energies are of order κ^2. In the $I = 0$ sector, the most strongly bound, two-baryon bound states correspond to a superposition of $p - n$ and $\Delta - \Delta$ total spin $S = 1$ states, like the deuteron. The more weakly bound, bound states are associated with a superposition of $p - n$ and $\Delta - \Delta$ total spin $S = 0$ states and, in addition, $\Delta - \Delta$, $S = 2$ states. There are also $\Delta - \Delta$, $S = 3$ states. In contrast to the $I = 0$ states, we find that for the maximum isospin $I = 3$ sector, there are also strongly bound and weakly bound, bound states in the lowest spin sectors $S = 0$, 1 and *no* bound states if $S = 2$, 3. Second, in [16], we analyzed the odd sector $\mathcal{H}_o$ of the Hilbert space $\mathcal{H}$ to probe the existence of pentaquarks as meson-baryon bound states. Although our methods are suitable to treat the complete model, in the above cases we solved the Bethe-Salpeter equation in the lowest nonvanishing order, which we call the ladder approximation. Under this approximation, no pentaquark was found. Here, we further the analysis of [15] and inspect for the existence of diprotons in $\mathcal{H}_e$.

Here, we extend the analysis of two-baryon bound states in [15] to the isospin $I = 1$, $I_3 = 1$ sector, restricting our treatment to states generated by the product of one-particle states. In this sector, we have $p - p$, $\Delta - \Delta$ as well as $p - \Delta$ and $n - \Delta$ states. By isospin symmetry, the $I = 1$, $I_3 = 0$, and $I_3 = -1$ sectors have the same spectrum. In particular, the $I_3 = 0$ sector has $p - n$ states and the $I_3 = -1$ sector has $n - n$ states.

As in [15,17], we use a lattice version of the B-S equation and, in lattice relative coordinates, we establish a correspondence between the partially Fourier transformed B-S equation and the resolvent equation of a one-particle lattice Schrödinger Hamiltonian. The Hamiltonian is a sum of a kinetic energy term of order κ^3 and a potential energy operator. It is found that the potential energy dominates. It is of space-range one and order κ^2, and arises from a $q - \bar{q}$ exchange. Moreover, the potential has a representation in terms of a permanent of a matrix (the permanent of a square matrix has the same terms as a determinant, but with only plus signs). This permanent describes the interaction between the quark spins of two-baryon states. The attractive or repulsive nature of the potential depends on the total spin of the two-baryon particles. A long-range bound on the B-S kernel, obtained as in Refs. [15,17], guarantees that the short-range contributions we keep are indeed the dominant ones describing the interaction between the two baryons. We find various bound states with binding energies of order κ^2. In particular, there is a $J = 0$ diproton-like bound state with approximate binding energy $\kappa^2/4$. It is described as a superposition of $p - p$ and $\Delta - \Delta$ states. By isospin symmetry, corresponding to this bound state, there is also an $I = 1$, $I_3 = -1$, $J = 0$ dineutron-like bound state and an $I = 1$, $I_3 = 0$, $J = 0$ $p - n$-like bound state, with the same binding energy. We list some of the other bound states. We give the binding energy, the spin state, and the field that best describes it: (1) $\epsilon = \kappa^2/4$, $J = 2$ (superposition of $\Delta - \Delta$, $\Delta - p$, and $\Delta - n$ states), (2) $\epsilon = \kappa^2/12$, $J = 3$ (superposition of $\Delta - \Delta$ states), and (3) $\epsilon = \kappa^2/12$, $J = 2$ (superposition of $\Delta - n$ and $\Delta - p$ states). We note that the bound states in (2) and (3) above are more weakly bound than the diproton-like bound state. Also, we emphasize that we take into account the *entire* two-baryon subspace in our determination of the bound state spectrum. This is natural from the point of view of degenerate perturbation theory, as the masses of the one-baryon states are the same, up to order κ^6. If, by chance, we erroneously consider only the $p-p$ subspace rather than the entire $I=1$, $I_3=1$ two-baryon subspace, we find a $J=0$ bound state, but the binding energy is only $\kappa^2/36$, which is greatly reduced (by a factor of 1/9). Thus, the matrix elements between $p-p$ and $\Delta-\Delta$ states are important in determining the binding energy.

Although this type of effect is expected as a general rule in the spectral analysis, we note that, by trying to make an *a priori* approximation by guessing a leading subspace of states, many analyses in the literature cannot be justified. It is also noteworthy to mention that, if we consider the $I=0$ sector and only take into account the $p-n$ subspace, then there are $J=1$, $J_z=1, 0, -1$ deuteron-like bound states with binding energies $\kappa^2/36$, which is the same as for the $J=0$ diproton, dineutron, and the $I=1$, $I_3=0$ $p-n$ bound states.

In our method, a basic and fundamental ingredient is given by obtaining spectral representations for the two- and four-baryon correlations. These representations are essential to relate analyticity properties of these correlations to the low-lying one and two-baryon spectra.

The paper is organized as follows. In Sec. II, we present a brief description of the model. In Sec. III, we give the one-particle states and their spectral properties. The two-baryon states are given in Sec. IV. Section V deals with the four-baryon functions, their spectral representations, and the associated Bethe-Salpeter equation. Section VI is devoted to the approximation to the B-S kernel we use to search for bound states. In Sec. VII, we solve the B-S equation and obtain the bound states. Concluding remarks are presented in Sec. VIII. As the essence of our methods we already presented in previous articles, and in order to avoid obscuring the flow of the text, in the body of this paper we show our results in a very direct way and relegate several technical ingredients to five appendices.

II. THE MODEL, PHYSICAL HILBERT SPACE, AND EM OPERATORS

Here, we use the two-flavor version of the model of Refs. [12,13]. It is an $\mathrm{SU}(2)_f$ lattice QCD model, with gauge group $\mathrm{SU}(3)_c$ and a partition function given formally by

$$Z = \int e^{-\mathcal{S}(\psi,\bar\psi,g)} d\psi d\bar\psi d\mu(g).$$

For a function $F(\psi, \bar\psi, g)$, the normalized statistical correlations are

$$\langle F\rangle = \frac{1}{Z}\int F(\psi,\bar\psi,g) e^{-\mathcal{S}(\psi,\bar\psi,g)} d\psi d\bar\psi d\mu(g).$$

The model action $\mathcal{S}(\psi,\bar\psi,g)$ is Wilson's improved action of Refs. [8–10],

$$\mathcal{S} = \frac{\kappa}{2}\sum \bar\psi_{a,\alpha,f}(u)\Gamma^{\epsilon e^\rho}_{\alpha\beta} U(g_{u,v})_{ab}\psi_{b,\beta,f}(u+\sigma e^\rho) + \sum_{u\in\mathbf{Z}^4_o}\bar\psi_{a,\alpha,f}(u) M_{\alpha\beta}\psi_{a,\beta,f}(u) - \frac{1}{g_0^2}\sum_p \chi(g_p), \quad (1)$$

where the first sum is over $u\in\mathbb{Z}^4_o$, $\sigma=\pm1$, $\rho=0,1,2,3$ and over repeated indices. Calling 0 the temporal direction, the lattice is given by $\mathbb{Z}^4_o$, where $u=(u^0,\vec u)=(u^0,u^1,u^2,u^3)$, $\mathbb{Z}^4_o\in\mathbb{Z}_{1/2}\times\mathbb{Z}^3$, where $\mathbb{Z}$ is the set of integers and $\mathbb{Z}_{1/2}=\pm1/2,\pm3/2,\ldots$. We use $\mathbb{Z}_{1/2}$ for technical reasons (see [12,18,19]). For each site $u\in\mathbb{Z}^4_o$, there are fermionic fields, represented by Grassmann variables $\psi_{a,\alpha,f}(u)$ and $\bar\psi_{a,\alpha,f}(u)$, which carry color $a=1,2,3$, spin $\alpha=1,2,3,4$, and flavor $f=u$, $d=1,2$ indices. For e^ρ being the unit vector for the ρ lattice direction, $\Gamma^{\pm e^\rho}=-1\pm\gamma^\rho$, where the γ^ρ are the Euclidean anticommuting Dirac matrices

$$\gamma^0=\begin{pmatrix}\mathbb{I}_2 & 0\\ 0 & -\mathbb{I}_2\end{pmatrix}$$

and

$$\gamma^j=\begin{pmatrix}0 & i\sigma^j\\ -i\sigma^j & 0\end{pmatrix};$$

σ^j, $j=1,2,3$, denotes the Hermitian traceless anticommuting Pauli matrices. For each oriented lattice bond, we associate a gauge group element $U(g_{u+e^\rho,u})\equiv U(g_{u,u+e^\rho})^{-1}$, and the last term in $\mathcal{S}$ is the usual plaquette action. $d\mu(g)$ is a product of normalized $\mathrm{SU}(3)_c$ Haar measures, and the Grassmann integrals are Berezin integrals (see [18]). Associated with the model is a physical quantum mechanical Hilbert $\mathcal{H}$ and self-adjoint energy and momentum operators $H\geq0$ and $-\pi<P^{i=1,2,3}<\pi$.

A Feynman-Kac (F-K) formula relates inner products in $\mathcal{H}$ to correlations, and points in the EM spectrum are detected as complex momentum singularities in the Fourier transform of correlations. For the usual spacetime and charge conjugation symmetries of this model, see [12,13,19]. (A new time reflection symmetry is also presented.) At strong coupling, polymer expansion methods (see Ref. [18]) ensure the thermodynamic limit of correlations exists and truncated correlations have exponential tree decay. The limiting correlations are translational invariant and extend to analytic functions both in κ and β.

III. ONE-BARYON SPECTRUM

The gauge-invariant, local composite, one-particle, baryon-creating fields are expressed as linear combinations of the unnormalized composite fields,

$$\bar B^{\,u}_{\vec\beta_b\vec g} = \epsilon_{abc}\bar\psi_{a\beta^\ell_1 g_1}\bar\psi_{b\beta^\ell_2 g_2}\bar\psi_{c\beta^\ell_3 g_3}, \quad (2)$$

where ϵ_{abc} denotes the Levi-Civita symbol, $\vec g=(\vec g_1,\vec g_2,\vec g_3)$, with $g_i, h_j=1,2=u,d$ and $\vec\beta_b=(\beta^\ell_1,\beta^\ell_2,\beta^\ell_3)$. Only *lower* spin indices ($\beta^\ell_i=3,4$) enter in the baryon fields. The normalized baryon fields of total isospin I and third component I_3 are Clebsch-Gordan linear combinations of the $\bar{B}^u$'s, and we denote them by $\bar{\chi}_{I,I_3,\vec{\beta}}$, where $\vec{\beta}$ specifies the spin state which will be explained below. The normalization is such that, at $\kappa = 0 = \beta$, $\langle \bar{\chi}(x)\chi(x)\rangle$ is the identity.

We now list the baryon fields. The $\bar{\chi}_{I,I_3,\vec{\beta}}$ are taken among the fields below,

$$\begin{aligned}
p_{\pm} &= \frac{\epsilon_{abc}}{3\sqrt{2}}(\bar{\psi}_{a+u}\bar{\psi}_{b-d} - \bar{\psi}_{a+d}\bar{\psi}_{b-u})\bar{\psi}_{c\pm u},\\
n_{\pm} &= \frac{\epsilon_{abc}}{3\sqrt{2}}(\bar{\psi}_{a+u}\bar{\psi}_{b-d} - \bar{\psi}_{a+d}\bar{\psi}_{b-u})\bar{\psi}_{c\pm d},\\
\Delta^{+}_{\pm(1/2)} &= \frac{\epsilon_{abc}}{6}(\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm u}\bar{\psi}_{c\mp d} + 2\bar{\psi}_{a\pm u}\bar{\psi}_{b\mp u}\bar{\psi}_{c\pm d}),\\
\Delta^{+}_{\pm(3/2)} &= \frac{\epsilon_{abc}}{2\sqrt{3}}\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm u}\bar{\psi}_{c\pm d}, \qquad \Delta^{-}_{\pm(3/2)} = \frac{\epsilon_{abc}}{6}\bar{\psi}_{a\pm d}\bar{\psi}_{b\pm d}\bar{\psi}_{c\pm d},\\
\Delta^{0}_{\pm(1/2)} &= \frac{\epsilon_{abc}}{6}(2\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm d}\bar{\psi}_{c\mp d} + \bar{\psi}_{a\mp u}\bar{\psi}_{b\pm d}\bar{\psi}_{c\pm d}),\\
\Delta^{0}_{\pm(3/2)} &= \frac{\epsilon_{abc}}{2\sqrt{3}}\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm d}\bar{\psi}_{c\pm d}, \qquad \Delta^{-}_{\pm(1/2)} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar{\psi}_{a\pm d}\bar{\psi}_{b\pm d}\bar{\psi}_{c\mp d},\\
\Delta^{++}_{\pm(1/2)} &= \frac{\epsilon_{abc}}{2\sqrt{3}}\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm u}\bar{\psi}_{c\mp u}, \qquad \Delta^{++}_{\pm(3/2)} = \frac{\epsilon_{abc}}{6}\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm u}\bar{\psi}_{c\pm u}.
\end{aligned} \tag{3}$$

Spin operators are defined in the Grassmann field algebra as in Ref. [12], and here, the subindexes $\pm$ denote the z component of total spin $J_z = \pm\frac{1}{2}$. n and p have total isospin (spin) $I = 1/2$ $(J = 1/2)$ and $I = 3/2$ $(J = 3/2)$ for Δ^-, Δ^0, Δ^+, and Δ^{++}. In $\bar{\chi}_{I,I_3,\vec{\beta}}$, we make the following correspondence between $\vec{\beta}$ and $\vec{J}_z$: $I = 1/2$, $\vec{\beta} = (+ - \pm) \leftrightarrow J_z = \pm 1/2$; $\quad I = 3/2$, $\quad \vec{\beta} = (\pm\pm\pm) \leftrightarrow J_z = \pm 3/2$, $\vec{\beta}(+ \pm -) \leftrightarrow J_z = \pm 1/2$. The associated physical Hilbert space states are shown to have the usual isospin and hypercharge quantum numbers. Note also that, for p, n, the first two isospins are coupled to give isospin zero. By the action of charge conjugation $\mathcal{C}$, baryons are mapped to the corresponding antibaryons, and the symmetry of the action of Eq. (1) under this transformation implies that baryons and antibaryons have identical spectral properties.

In Refs. [12,13], the lattice translational invariant and baryon two-point correlations $G(u, v)$ are defined and shown to have spectral representations for $x^0 \neq 0$, $x = u - v$. The baryonic two-point function for the basic excitation field $\bar{B}_\ell = \bar{\chi}_{I,I_3,J_z}$ is defined by, writing $G_{\ell\ell'}(u, v) \equiv G_{\ell\ell'}(x = u - v \in \mathbb{Z}^4)$,

$$\begin{aligned}
G_{\ell_1\ell_2}(u, v) &= \langle B_{\ell_1}(u)\bar{B}_{\ell_2}(v)\rangle\chi_{u^0\leq v^0}\\
&\quad - \langle \bar{B}_{\ell_1}(u)B_{\ell_2}(v)\rangle^*\chi_{u^0>v^0},
\end{aligned} \tag{4}$$

where the subscripts $\ell = (\vec{\alpha}, \vec{f})$ and $\ell' = (\vec{\beta}, \vec{h})$ are collective spin-isospin indices and $\chi_{\mathcal{S}}$ is the characteristic function or indicator of the set $\mathcal{S}$. Since a baryon field is made of three quarks and the average of an odd number of fields is zero, this correlation is already truncated. An important property concerning the baryon correlations which adjusts the calculated normalization factors in Eq. (4) is that, for coincident points,

$$\langle B^u_{\vec{\alpha}\vec{f}}\bar{B}^u_{\vec{\beta}\vec{h}}\rangle^{(0)} = -6\,\mathrm{per}(\delta_{\vec{\alpha}\vec{\beta}}\delta_{\vec{f}\vec{h}}), \tag{5}$$

where here (0) means $\kappa = 0 = \beta$ and, employing the ordinary Cayley notation for determinants, the 3×3 matrix $\delta_{\vec{\alpha}\vec{\beta}}\delta_{\vec{f}\vec{h}}$ has elements $\delta_{\alpha_i\beta_j}\delta_{f_ih_j}$, and where per$(A)$ is the permanent of the square matrix A. This result shows that the determinant and the Levi-Civita symbol ϵ conspire to give the permanent.

We briefly describe how the one-hadron spectrum is obtained in Refs. [12,13]. Concerning the two-point function (4), using the spectral representations for the operators $\check{T}_0^{x^0}$ and $\check{T}^{\vec{x}}$, we obtain the spectral representation, for $x^0 \neq 0$, with $\bar{B}_\ell \equiv \bar{B}_\ell(1/2, \vec{0})$ and $x = v - u$,

$$\begin{aligned}
G_{\ell_1\ell_2}(x) &= -(\bar{B}_{\ell_1}, \check{T}^{|x^0|}\check{T}^{\vec{x}}\bar{B}_{\ell_2})_{\mathcal{H}}\\
&= -\int_{-1}^{1}\int_{\mathbb{T}^3}(\lambda^0)^{|x^0|-1}e^{-i\vec{\lambda}.\vec{x}}d(\bar{B}_{\ell_1}, \mathcal{E}(\lambda^0, \vec{\lambda})\bar{B}_{\ell_2})_{\mathcal{H}},
\end{aligned} \tag{6}$$

for $x \in \mathbb{Z}^4$, $x^0 \neq 0$, and $G_{\ell_1\ell_2}(x)$ is an even function of $\vec{x}$ by parity symmetry.

Points in the EM spectrum are detected as singularities of $\tilde{G}(p) = \tilde{G}(p^0, \vec{p})$, the Fourier transform of $G(x)$, on the p^0 imaginary axis. $\Gamma(x)$, the convolution inverse of $G(x)$, has a faster temporal decay than $G(x)$ so that $\Gamma(p^0, \vec{p})$, the inverse of $\tilde{G}(p)$, $\tilde{\Gamma}(p)\tilde{G}(p) = 1$, has a larger strip of p^0 analyticity than G, namely, $|\mathrm{Im}\, p^0| < -(5 - \epsilon)\ln\kappa$, as compared to $-(3 - \epsilon)\ln\kappa$, $\epsilon > 0$. Thus,

$$\tilde{\Gamma}^{-1}(p) = [\mathrm{cof}\tilde{\Gamma}(p)]^t / \det\tilde{\Gamma}(p)$$

provides a meromorphic extension of $\tilde{G}(p)$. The singularities of $\tilde{G}(p)$ are given by

$$\det\tilde{\Gamma}(p^0 = iw(\vec{p}), \vec{p}) = 0,$$

and the EM spectrum is comprised of particles with dispersion curves $w(\vec{p})$, isolated from the rest of the spectrum, where $w(\vec{p}) = -3\ln\kappa - (3/4)\kappa^3 + \vec{p}_\ell^2\kappa^3/8 + \mathcal{O}(\kappa^6)$, $\vec{p}_\ell^2 = 2\sum_{j=1,2,3}(1-\cos p^j)$. The baryon masses $m = w(\vec{p}=\vec{0})$ are determined exactly and the nonsingular part $m - (-3\ln\kappa)$ is *jointly* analytic in κ and β. Also, there is an $\mathcal{O}(\kappa^6)$ mass splitting showing that the $I = 3/2$ delta baryons (decuplet) are heavier than the $I = 1/2$ p, n (octet) baryons. Separating out the p^0-independent contribution,

$$\tilde{G}(\vec{p}) = \sum_{\vec{x}\in\mathbb{Z}^3} e^{-i\vec{p}.\vec{x}} G(x^0 = 0, \vec{x}),$$

and the one-particle contributions, $\tilde{G}(p)$ admits the representation; suppressing I, I_3, J_z, and with $f(x,y) = (e^{ix}-y)^{-1} + (e^{-ix}-y)^{-1}$,

$$\tilde{G}(p) = \tilde{G}(\vec{p}) + Z(\vec{p})f(p^0, e^{-w(\vec{p})}) + \tilde{\mathcal{G}}(p), \qquad (7)$$

where $Z(\vec{p})^{-1} = -(2\pi)^3 e^{w(\vec{p})}\frac{\partial\tilde{\Gamma}}{\partial\chi}(p^0 = i\chi, \vec{p})|_{\chi=w(\vec{p})}$ and, for $\int' \equiv \int_{|\lambda^0|<\exp[-(5-\epsilon)\ln\kappa]}$, we have

$$\tilde{\mathcal{G}}_{\ell_1\ell_2}(p) = (2\pi)^3 \int' f(p^0, \lambda^0) d_{\lambda^0} d\alpha_{\vec{p},\ell_1\ell_2}(\lambda^0), \qquad (8)$$

for

$$d\alpha_{\vec{p},\ell_1\ell_2}(\lambda^0) = \int_{\mathbb{T}^3} \delta(\vec{p}-\vec{\lambda}) d_{\vec{\lambda}}(\bar{B}_{\ell_1}, \mathcal{E}(\lambda^0, \vec{\lambda})\bar{B}_{\ell_2})_{\mathcal{H}}, \qquad (9)$$

with $\vec{p} = (p^1, p^2, p^3) \in \mathbb{T}^3$ and $\mathbb{T}^n \equiv (-\pi, \pi]^n$, $n \in \mathbb{N}$. It follows that $Z^k(\vec{p}) \simeq (2\pi)^{-3}\exp[-w_k(\vec{p})]$, and $\tilde{G}(p)$ is analytic in p^0 in the strip $|\mathrm{Im}\, p^0| < -(5-\epsilon)\ln\kappa$. The last term $\tilde{\mathcal{G}}(p)$ in the right-hand side above includes only contributions to $\tilde{G}(p)$ in $\mathcal{H}$ with two or more particles.

IV. TWO-BARYON STATES

Since we are considering a QCD model, and no electromagnetic interaction is present, we treat the total isospin $I = 1 = I_3$ sector, where the proton-proton states lie, rather than the $I = 1$, $I_3 = -1$ sector, where neutron-neutron states lie. We search for $p-p$ bound states by restricting our analysis to the subspace of $\mathcal{H}$ generated by two-baryon states. Specifically, denoting here the normalized baryon-creating fields by $\bar{\chi}_{I,I_3}$, we consider the Clebsch-Gordan linear combination of fields $\bar{\Lambda}_{1,2,3,4}(x,y)$ with total isospin $I = 1$, $I_3 = 1$ given by, suppressing spin indices,

$$\begin{aligned}\bar{\Lambda}_1(x,y) = {}& \sqrt{3/10}\,\bar{\chi}_{(3/2),(3/2)}(x)\bar{\chi}_{(3/2),-(1/2)}(y) \\ & - \sqrt{2/5}\,\bar{\chi}_{(3/2),(1/2)}(x)\bar{\chi}_{(3/2),(1/2)}(y) \\ & + \sqrt{3/10}\,\bar{\chi}_{(3/2),-(1/2)}(x)\bar{\chi}_{(3/2),(3/2)}(y),\end{aligned} \qquad (10)$$

$$\begin{aligned}\bar{\Lambda}_2(x,y) = {}& \sqrt{3/4}\,\bar{\chi}_{(3/2),(3/2)}(x)\bar{\chi}_{(1/2),-(1/2)}(y) \\ & - \sqrt{1/4}\,\bar{\chi}_{(3/2),(1/2)}(x)\bar{\chi}_{(1/2),(1/2)}(y),\end{aligned} \qquad (11)$$

$$\bar{\Lambda}_3(x,y) = \bar{\chi}_{(1/2),(1/2)}(x)\bar{\chi}_{(1/2),(1/2)}(y), \qquad (12)$$

$$\begin{aligned}\bar{\Lambda}_4(x,y) = {}& \sqrt{3/4}\,\bar{\chi}_{(1/2),-(1/2)}(x)\bar{\chi}_{(3/2),(3/2)}(y) \\ & - \sqrt{1/4}\,\bar{\chi}_{(1/2),(1/2)}(x)\bar{\chi}_{(3/2),(1/2)}(y),\end{aligned} \qquad (13)$$

associated with the couplings of $\Delta-\Delta$, $\Delta-p$, $\Delta-n$, and $p-p$, and incorporating 16, 8, 4, and 8 spin states, respectively. Similarly, auxiliary fields $\Lambda_{1,2,3,4}$, with all fields unbarred, are defined. We now give the ordering of the spin basis. The $\bar{\Lambda} = (\bar{\Lambda}_1, \bar{\Lambda}_2, \bar{\Lambda}_3, \bar{\Lambda}_4)$ are linear combinations of products of the basic unnormalized composite quark fields of Eq. (2), where the individual spin indices α_i take the values $3(+)$ or $4(-)$ and the individual isospin indices take the values u ($+$ or 1) and d ($-$ or 2).

The spin state of the first (second) baryon is specified by $\vec{\sigma}(\vec{\beta})$ for which we have the following orderings:

- For the first 16 states: $(+++)(+++)$, $(+++)(++-)$, ..., $(+++)(---)$, $(++-)(+++)$, ..., $(++-)(---)$, $(+--)(+++)$, ..., $(---)(---)$;
- For the next eight states: $(+++)(+-+)$, $(+++)(+--)$, ..., $(---)(+-+)$, $(---)(+--)$ [here, the first (second) triple is the spin for isospin 3/2 (1/2)];
- For the next four states: $(+-+)(+-+)$, $(+-+)(+--)$, $(+--)(+-+)$, $(+--)(+--)$;
- For the last eight states: $(+-+)(+++)$, $(+--)(+++)$, $(+-+)(++-)$, $(+--)(++-)$, $(+-+)(+--)$, $(+--)(+--)$, $(+-+)(---)$, $(+--)(---)$ [here, the first (second) triple is the spin for isospin 1/2 (3/2)].

It is convenient to take out the normalization constants and the Clebsch-Gordan coefficients and express the $\bar{\Lambda}$ fields in terms of the $\bar{B}^u$ fields and similarly for Λ. Displaying the spin indices explicitly we write, for $i = 1, 2, \ldots, 36$,

$$\bar{\Lambda}_{\vec{\sigma}_i\vec{\beta}_i}(x,y) = \sum_{\vec{k}\,\vec{g}} F_i^{\vec{k}\,\vec{g}}\bar{B}^u_{\vec{\sigma}_i\vec{k}}(x)\bar{B}^u_{\vec{\beta}_i\vec{g}}(y), \qquad (14)$$

for $F_i^{\vec{k}\,\vec{g}} \equiv F^{\vec{\sigma}_i\vec{k}\vec{\beta}_i\vec{g}}$, and where $\bar{\Lambda}_{\vec{\sigma}_i\vec{\beta}_i}(x,y)$ means $\bar{\Lambda}_{1,2,3,4\vec{\sigma}_i\vec{\beta}_i}(x,y)$, for $1 \le i \le 16$, $17 \le i \le 24$, $25 \le i \le 28$, $29 \le i \le 36$. The $F_i^{\vec{k}\,\vec{g}}$'s are given in detail in Appendix A.

Because of the simplicity of the block structure of the four-point correlation, instead of the $\bar{\Lambda}$, the fields that are most convenient to use are

$$\bar{\Omega}_{1t_1t_2}(x,y) = \bar{\Lambda}_{1t_1t_2}(x,y),$$

$$\bar{\Omega}_{2t_1s_2}(x,y) = \frac{1}{\sqrt{2}}(\bar{\Lambda}_{2t_1s_2}(x,y) - \bar{\Lambda}_{2t_1s_2}(y,x)),$$

$$\bar{\Omega}_{3s_1s_2}(x,y) = \bar{\Lambda}_{3s_1s_2}(x,y) = p_{s_1}(x)p_{s_2}(y),$$

$$\bar{\Omega}_{4t_1s_2}(x,y) = \frac{1}{\sqrt{2}}(\bar{\Lambda}_{2t_1s_2}(x,y) + \bar{\Lambda}_{2t_1s_2}(y,x)).$$

In terms of spin components, the $\bar{\Omega}_{1,2,3}$ are ordered as before, for $\bar{\Lambda}_{1,2,3}$, and $\bar{\Omega}_4$ is ordered as $\bar{\Omega}_2$. The $\bar{\Omega}_i$ have the symmetry properties $\bar{\Omega}_{1t_1t_2}(x,y) = -\bar{\Omega}_{1t_2t_1}(y,x)$, $\bar{\Omega}_{2t_1s_2}(x,y) = -\bar{\Omega}_{2t_1s_2}(y,x)$, $\bar{\Omega}_{3s_1s_2}(x,y) = -\bar{\Omega}_{3s_2s_1}(y,x)$, and $\bar{\Omega}_{4t_1s_2}(x,y) = \bar{\Omega}_{4t_1s_2}(y,x)$.

The above two-baryon states are used to define the four-point correlations that we analyze in the next section.

V. FOUR-POINT CORRELATION AND THE B-S EQUATION

In this section, we follow the treatment of Ref. [20] for the four-point correlation. A temporal half-integer shift is made so that the components of the site coordinates are integers. We also assume the equal-time conditions $x_1^0 = x_2^0$ and $x_3^0 = x_4^0$ throughout the section.

Letting $\mathbf{x} \equiv (x_1, x_2, x_3, x_4)$, the four-point function $D_{\ell_1\ell_2\ell_3\ell_4}(\mathbf{x})$ is defined so that the ij block, $i, j = 1, 2, 3, 4$, is given by, for $x_1^0 \le x_3^0$,

$$\langle \Lambda_i(x_1x_2)\bar{\Lambda}_j(x_3x_4)\rangle \quad \text{or} \quad \langle \Omega_i(x_1x_2)\bar{\Omega}_j(x_3x_4)\rangle,$$

or, for $x_1^0 > x_3^0$, by

$$\langle \bar{\Lambda}_i(x_1x_2)\Lambda_j(x_3x_4)\rangle^* \quad \text{or} \quad \langle \bar{\Omega}_i(x_1x_2)\Omega_j(x_3x_4)\rangle^*.$$

Through the F's, $D_{\ell_1\ell_2\ell_3\ell_4}$ is expressed in terms of linear combinations of the unnormalized four-point function

$$D^u_{\ell_1\ell_2\ell_3\ell_4}(\mathbf{x}) = \langle B^u_{\ell_1}(x_1)B^u_{\ell_2}(x_2)\bar{B}^u_{\ell_3}(x_3)\bar{B}^u_{\ell_4}(x_4)\rangle, \tag{15}$$

for $x_1^0 \le x_3^0$, and

$$\langle \bar{B}^u_{\ell_1}(x_1)\bar{B}^u_{\ell_2}(x_2)B^u_{\ell_3}(x_3)B^u_{\ell_4}(x_4)\rangle^*,$$

for $x_1^0 > x_3^0$, where $\ell_{1,2,3,4}$ are collective indices $\ell_1 = \vec{\rho}\,\vec{h}$, $\ell_2 = \vec{\alpha}\,\vec{f}$, $\ell_3 = \vec{\sigma}\,\vec{k}$, and $\ell_4 = \vec{\beta}\,\vec{g}$, respectively. Hence, the $i, j = 1, \ldots, 36$ matrix element of D is given by

$$\sum_{\vec{h}\vec{f}\vec{k}\vec{g}} F_i^{\vec{h}\vec{f}} D^u_{\ell_1\ell_2\ell_3\ell_4}(\mathbf{x}) F_j^{\vec{k}\vec{g}} \equiv F_i D^u(\mathbf{x}) F_j,$$

and we write $F^\gamma D^u_{\gamma\gamma'} F^{\gamma'}$ for the matrix. The *Wickified* four-point function (unnormalized) D_0^u, defined by erroneously applying Wick's theorem to composite baryon fields, is given by

$$D^u_{0\ell_1\ell_2\ell_3\ell_4}(\mathbf{x}) = -\langle B^u_{\ell_1}(x_1)\bar{B}^u_{\ell_3}(x_3)\rangle\langle B^u_{\ell_2}(x_2)\bar{B}^u_{\ell_4}(x_4)\rangle + \langle B^u_{\ell_1}(x_1)\bar{B}^u_{\ell_4}(x_4)\rangle\langle B^u_{\ell_2}(x_2)\bar{B}^u_{\ell_3}(x_3)\rangle, \tag{16}$$

for $x_1^0 \le x_3^0$, or by

$$-\langle \bar{B}^u_{\ell_1}(x_1)B^u_{\ell_3}(x_3)\rangle^*\langle \bar{B}^u_{\ell_2}(x_2)B^u_{\ell_4}(x_4)\rangle^* + \langle \bar{B}^u_{\ell_1}(x_1)B^u_{\ell_4}(x_4)\rangle^* \times \langle \bar{B}^u_{\ell_2}(x_2)B^u_{\ell_3}(x_3)\rangle^*,$$

for $x_1^0 > x_3^0$. By taking linear combinations of these functions, we obtain D_0.

As shown in Ref. [20], via the Feynman-Kac formula, the Fourier transform $\tilde{D}(k)$, $k = (k^0, \vec{k}) \in \mathbb{T}^4$, of $D(x) = D(x_1, x_2, x_3 + x_0 + \vec{x}, x_4 + x_0 + \vec{x})$ in $x = (x_0, \vec{x}) \in \mathbb{Z}^4$, admits the spectral representation

$$\tilde{D}(k) = \tilde{D}(\vec{k}) - (2\pi)^3 \int_{-1}^{1} \int_{\mathbb{T}^3} f(k^0, \lambda^0)\delta(\vec{k} - \vec{\lambda})d\mu(\lambda_0, \vec{\lambda}), \tag{17}$$

where $\tilde{D}(\vec{k}) = \sum_{\vec{x}\in\mathbb{T}^3} e^{-i\vec{k}.\vec{x}} D(x^0 = 0, \vec{x})$ is the $x^0 = 0$ contribution to $\tilde{D}(k)$ and $d\mu(\lambda_0, \vec{\lambda}) \equiv d_{\lambda_0} d_{\vec{\lambda}}(\bar{\Lambda}((\frac{1}{2}, \vec{x}_1), (\frac{1}{2}, \vec{x}_2)), \mathcal{E}(\lambda^0, \vec{\lambda})\bar{\Lambda}((\frac{1}{2}, \vec{x}_3), (\frac{1}{2}, \vec{x}_4)))_{\mathcal{H}}$. Here, the contribution $\tilde{D}(\vec{k})$ is separated out, as D does not have a spectral representation for temporal coincident points. Setting $\vec{k} = \vec{0}$ corresponds to putting the system's spatial momentum to zero. We extend k^0 to complex values, and singularities of $\tilde{D}(k)$, for $k = (k^0 = i\chi, \vec{k} = \vec{0})$ and $e^{\pm\chi} \le 1$, are seen to be points in the mass spectrum. To detect two-baryon bound states below the lowest two-baryon threshold, given by twice the lightest baryon mass, we write $k^0 = i(2\bar{m} - \epsilon)$, where $\bar{m}$ is the smallest baryon mass, and we determine the value of $\epsilon > 0$ that gives the singularity of $\tilde{D}(k^0 = i\chi, \vec{k} = \vec{0})$, which we interpret as the bound state binding energy.

We write an equal-time representation B-S equation for D (see Refs. [15,17]) as, in matrix operator form,

$$D = D_0 + D_0 K D, \tag{18}$$

where the B-S kernel is formally given by $K = D_0^{-1} - D^{-1}$. Here, the inverses are formal, as D and D_0 have null spaces. After restricting D and D_0 to a proper space, the inverses will be shown to be well defined as matrix operators. The inverses are defined by the Neumann series as a perturbation about the $\kappa = 0 = \beta$ values of D and D_0. We denote these values by $D^{(0)}, D_0^{(0)}$. Furthermore, K is expanded, using the Neumann series, as

$$K = \sum_{n=0}^{\infty}(-1)^n[((D_0^{(0)})^{-1}\delta D_0)^n (D_0^{(0)})^{-1} - [(D^{(0)})^{-1}\delta D]^n (D^{(0)})^{-1}], \tag{19}$$

where $\delta D = D - D^{(0)}$ and $\delta D_0 = D_0 - D_0^{(0)}$.

We now write $D(0) = FCF$, with the argument 0 denoting coincident points, and determine a formula for $C^{\vec{\sigma}\,\vec{k}\,\vec{\beta}\,\vec{g}}_{\vec{\rho}\,\vec{h}\,\vec{\alpha}\,\vec{f}} = D^{u(0)}_{\ell_1\ell_2\ell_3\ell_4}(0)$. Using the Laplace expansion, we get

$$C^{\vec\sigma\,\vec k\,\vec\beta\,\vec g}_{\vec\rho\,\vec h\,\vec\alpha\,\vec f} = \sum_{\vec a\,\vec b\,\vec c\,\vec d} \epsilon_{\vec a}\epsilon_{\vec b}\epsilon_{\vec c}\epsilon_{\vec d}(-1) \times \det\begin{pmatrix} \delta_{\vec a\,\vec c}\delta_{\vec\rho\,\vec\sigma}\delta_{\vec h\,\vec k} & \delta_{\vec a\,\vec d}\delta_{\vec\rho\,\vec\beta}\delta_{\vec h\,\vec g} \\ \delta_{\vec b\,\vec c}\delta_{\vec\alpha\,\vec\sigma}\delta_{\vec f\,\vec k} & \delta_{\vec b\,\vec d}\delta_{\vec\alpha\,\vec\beta}\delta_{\vec f\,\vec g} \end{pmatrix}. \tag{20}$$

Here, the determinant is over a 6×6 matrix and $\delta_{\vec a\,\vec c}\delta_{\vec\rho\,\vec\sigma}\delta_{\vec h\,\vec k},\ldots$ denote 3×3 matrices with ij elements given by $\delta_{a_ic_j}\delta_{\rho_i\sigma_j}\delta_{h_ik_j},\ldots$. However, as already explained before, it turns out that the ϵ's and det conspire to give a permanent representation. The representation is derived in Appendix B and is given by

$$C^{\vec\sigma\,\vec k\,\vec\beta\,\vec g}_{\vec\rho\,\vec h\,\vec\alpha\,\vec f} = C^{\vec\sigma\,\vec k\,\vec\beta\,\vec g}_{1\vec\rho\,\vec h\,\vec\alpha\,\vec f} - C^{\vec\beta\,\vec g\,\sigma\vec k}_{1\vec\rho\,\vec h\,\vec\alpha\,\vec f} + C^{\vec\sigma\,\vec k\,\vec\beta\,\vec g}_{3\vec\rho\,\vec h\,\vec\alpha\,\vec f} - C^{\vec\beta\,\vec g\,\vec\sigma\,\vec k}_{3\vec\rho\,\vec h\,\vec\alpha\,\vec f}, \tag{21}$$

where

$$\begin{aligned}
C^{\vec\sigma\,\vec k\,\vec\beta\,\vec g}_{1\vec\rho\,\vec h\,\vec\alpha\,\vec f} &= -36\,\mathrm{perm}(\delta_{\vec\rho\,\vec\sigma}\delta_{\vec h\,\vec k})\,\mathrm{perm}(\delta_{\vec\alpha\,\vec\beta}\delta_{\vec f\,\vec g}),\\
C^{\vec\sigma\,\vec k\,\vec\beta\,\vec g}_{3\vec\rho\,\vec h\,\vec\alpha\,\vec f} &= 12[\mathrm{perm}(\delta_{(f_1h_2h_3)\vec k})\,\mathrm{perm}(\delta_{(h_1f_2f_3)\vec g})\\
&\quad + \mathrm{perm}(\delta_{(f_2h_2h_3)\vec k})\,\mathrm{perm}(\delta_{(f_1h_1f_3)\vec g})\\
&\quad + \mathrm{perm}(\delta_{(f_3h_2h_3)\vec k})\,\mathrm{perm}(\delta_{(f_1f_2h_1)\vec g})\\
&\quad + \mathrm{perm}(\delta_{(h_1f_1h_3)\vec k})\,\mathrm{perm}(\delta_{(h_2f_2f_3)\vec g})\\
&\quad + \mathrm{perm}(\delta_{(h_1f_2h_3)\vec k})\,\mathrm{perm}(\delta_{(f_1h_2f_3)\vec g})\\
&\quad + \mathrm{perm}(\delta_{(h_1f_3h_3)\vec k})\,\mathrm{perm}(\delta_{(f_1f_2h_2)\vec g})\\
&\quad + \mathrm{perm}(\delta_{(h_1h_2f_1)\vec k})\,\mathrm{perm}(\delta_{(h_3f_2f_3)\vec g})\\
&\quad + \mathrm{perm}(\delta_{(h_1h_2f_2)\vec k})\,\mathrm{perm}(\delta_{(f_1h_3f_3)\vec g})\\
&\quad + \mathrm{perm}(\delta_{(h_1h_2f_3)\vec k})\,\mathrm{perm}(\delta_{(f_1f_2h_3)\vec g})].
\end{aligned}$$

The first two terms are the Wickified four-point correlation, and the last two are the deviation from Wick's theorem applied to the composite fields. The matrix $D^{(0)}(0) = FCF$ is given explicitly in Appendix B, and its eigenvalues (multiplicities) are -4 (6), 0 (30). The determination of the null space of $D^{(0)}(0)$ is considerably simplified (obtained by inspection) in the case of only two distinct eigenvalues which we encounter here. For this, we recall the general formula for the orthogonal projection E_j on the eigenspace of the eigenvalue μ_j of a self-adjoint $n\times n$ matrix A. Letting $\{\mu_j\}$ denote the distinct eigenvalues, E_j is given by

$$E_j = \prod_{i\neq j}\frac{A-\mu_i}{\mu_j-\mu_i}.$$

Thus, in our case, the orthogonal projection P on the null space and $Q = 1-P$ are given by

$$P = (D^{(0)}(0)/4 + 1), \qquad Q = -D^{(0)}(0)/4. \tag{22}$$

The null space (its complement) is given by $R(P)$ $[R(Q)]$, and thus the null space (its complement) eigenvectors are simply the columns of $P(Q)$. For the determination of the space in which D, D_0 and the B-S equation act as matrix operators, see Appendix C.

VI. LADDER APPROXIMATION TO THE B-S KERNEL $\hat K$

Here, we derive the leading κ order of $\hat K$, at $\beta = 0$, which we call the ladder approximation to $\hat K$. From a systematic analysis of the short-distance behavior of K coupled with the long-range bound on its decay, we single out the leading contributions to $\hat K$ described below. At $\beta = 0$, the long-distance behavior of K is controlled using the decoupling of the hyperplane method (see [15,17,21]), yielding the explicit bounds

$$|K(x_1,x_2,x_3,x_4)| \le \begin{cases} c\kappa^{6|x_3^0-x_1^0|+2|\vec x_1+\vec x_2-\vec x_3-\vec x_4|+2|\vec x_2-\vec x_1|+2|\vec x_4-\vec x_3|} & \text{for } |x_3^0-x_2^0|\le 1\\ c\kappa^6\kappa^{8(|x_3^0-x_1^0|-1)+2|\vec x_1+\vec x_2-\vec x_3-\vec x_4|+2|\vec x_2-\vec x_1|+2|\vec x_4-\vec x_3|} & \text{for } |x_3^0-x_2^0|>1. \end{cases} \tag{23}$$

In order to analyze the B-S equation for D in (18), we use the translational invariance and pass to the lattice relative coordinates

$$\vec\xi = x_2 - x_1, \qquad \vec\eta = x_4 - x_3, \qquad \tau = x_3 - x_2.$$

These coordinates are the lattice substitute for the continuum center of mass and relative coordinates. In terms of relative coordinates the bound of Eq. (23) reads

$$|\hat K(\vec\xi,\vec\eta,\vec\tau)| \le \begin{cases} c\kappa^{6|\tau^0|+2|2\vec\tau+\vec\xi+\vec\eta|+2|\vec\xi|+2|\vec\eta|} & |\tau^0|\le 1\\ c\kappa^6\kappa^{8(|\tau^0|-1)+2|2\vec\tau+\vec\xi+\vec\eta|+2|\vec\xi|+2|\vec\eta|} & |\tau^0|>1. \end{cases} \tag{24}$$

To obtain the relative coordinate B-S equation of D of Eq. (18), we take the Fourier transform in the τ variable only, with the conjugate variable $k = (k^0,\vec k)$. We set the system spatial momentum to zero by putting $\vec k = \vec 0$. The resulting B-S equation is

$$\hat D(\vec\xi,\vec\eta,k^0) = \hat D_0(\vec\xi,\vec\eta,k^0) + \int \hat D_0(\vec\xi,\vec\xi',k^0)\hat K(\vec\xi',\vec\eta',k^0) \times \hat D(\vec\eta',\vec\eta,k^0)d\vec\xi' d\vec\eta'. \tag{25}$$

Bound states are detected as singularities in k_0 of $\hat D(\vec\xi,\vec\eta,k^0)$ on the imaginary axis below $2\bar m$, $\bar m = -3\ln\kappa - (3/4)\kappa^3 + \mathcal{O}(\kappa^6)$, twice the smallest baryon mass; $\hat D_0(\vec\xi,\vec\eta,k^0)$ is analytic up to this threshold. Equation (25) is analogous to an operator resolvent equation,

$$(H - z)^{-1} = (H_0 - z)^{-1} - (H_0 - z)^{-1} V (H - z)^{-1},$$

$H = H_0 + V$, for a one-particle lattice Schrödinger Hamiltonian. With our sign conventions, $\hat{K}$ can be interpreted as a generalized potential that can be nonlocal and energy dependent. k^0 plays the role of a spectral parameter. $\hat{D}_0$ is like minus the free resolvent. This correspondence will be made more precise below.

The $\beta = 0$ dominant contributions to $\hat{K}$ are described as follows.

(1) A space-range-one local potential of order κ^2 due to $q\bar{q}$ exchange. The spin interaction is independent of the direction, and there are spin states where the potential is attractive and repulsive.

(2) A space-range zero contact potential of order κ^0 arising from the coincident-point four-point function; it is spin independent.

(3) A space-range-zero, energy-dependent (k^0-dependent) potential of order κ^0 arising from contributions to K of space-range zero and temporal-distance one. This originates from the gauge field correlation effects involving six overlapping gauge bonds with the same orientation. The contribution is also spin independent.

As already explained above, the order κ^0 part of the potential of (3) cancels the contact potential of (2) and the resulting zero-range potential is of order κ^2, the same order as the exchange potential of (1). In solving the B-S equation, it turns out that the $q\bar{q}$ exchange dominates and gives rise to bound states for some spin states.

The above contributions are now obtained in the following:

(I) From the $n = 0$ term of K,

$$K^{(0)} = (D_0^{(0)})^{-1} - (D^{(0)})^{-1}.$$

The spectrum of $D^{(0)}(0)$ is -4 (6), 0 (30). On $R(Q)$, $D^{(0)}(0)$ acts as -4 times the identity and $D_0^{(0)}$ as -2 times the identity so that

$$K^{(0)}(0) = -(1/2) - (-1/4) = -1/4,$$

which is an attractive spin-independent potential. The contribution to $\hat{K}(\vec{\xi}, \vec{\eta}, k^0)$ is

$$-(Q/4)\delta(\vec{\xi})\delta(\vec{\xi} - \vec{\eta}). \tag{26}$$

(II) We have the temporal-one contribution to K given by

$$K^{(6)} = -(D_0^{(0)})^{-1} D_0^{(6)} (D_0^{(0)})^{-1} + (D^{(0)})^{-1} D^{(6)} (D^{(0)})^{-1},$$

which comes from the $n = 1$ term of K. Hence,

$$\begin{aligned} K^{(6)}(0, 0, e^0, e^0) = {} & -(D_0^{(0)})^{-1}(0) D_0^{(6)}(0, 0, e^0, e^0) \\ & \times (D_0^{(0)})^{-1}(e^0) + (D^{(0)})^{-1}(0) \\ & \times D^{(6)}(0, 0, e^0, e^0)(D^{(0)})^{-1}(e^0). \end{aligned}$$

On the other hand, from Eqs. (D1) and (D4) of Appendix D of Ref. [15] for the sixth hyperplane derivative, we have

$$\begin{aligned} D^{(6)} &= -\tfrac{1}{2} D^{(0)} \circ D^{(0)} - \tfrac{1}{4} D^{(0)} \odot D^{(0)}, \\ D_0^{(6)} &= -\tfrac{1}{2} D_0^{(0)} \circ D_0^{(0)}, \end{aligned}$$

so that $K^{(6)} = -(-1/2) + (-1/4) = 1/4$. Writing $k^0 = i(2\bar{m}(\kappa) - \epsilon)$, where $\epsilon \geq 0$ is the two-baryon binding energy, the contribution to $\hat{K}(\vec{\xi}, \vec{\eta}, k^0)$ is

$$(Q/4)\kappa^6 e^{2\bar{m}(\kappa) - \epsilon}\delta(\vec{\xi})\delta(\vec{\xi} - \vec{\eta}).$$

Since $\bar{m}(\kappa) = -3\ln\kappa - (3/4)\kappa^3 + \mathcal{O}(\kappa^6)$ the above is

$$(Q/4)e^{-(3/2)\kappa^3} e^{-\epsilon}\delta(\vec{\xi})\delta(\vec{\xi} - \vec{\eta}). \tag{27}$$

The sum of contributions (26) and (27) is

$$-(Q/4)(1 - e^{-(3/2)\kappa^3} e^{-\epsilon})\delta(\vec{\xi})\delta(\vec{\xi} - \vec{\eta}), \tag{28}$$

an attractive, space-range-zero, spin-independent, energy-dependent potential.

(III) The space-range-one $q\bar{q}$ exchange potential comes from the $n = 1$ term of K. We fix a nonoriented link or bond with end points $x = 0$ and $x = e^1$. We have the contributions to $K^{(2)}_{\ell_1\ell_2\ell_3\ell_4}(x_1, x_2, x_3, x_4)$ given as follows:

$$\begin{aligned} & (D^{(0)})^{-1}_{\ell_1\ell_2\ell'_1\ell'_2}(0, e^1, 0, e^1) D^{(2)}_{\ell'_1\ell'_2\ell'_3\ell'_4}(0, e^1, 0, e^1) \\ & \quad \times (D^{(0)})^{-1}_{\ell'_3\ell'_4\ell_3\ell_4}(0, e^1, 0, e^1), \end{aligned} \tag{29}$$

$$\begin{aligned} & (D^{(0)})^{-1}_{\ell_1\ell_2\ell'_1\ell'_2}(0, e^1, 0, e^1) D^{(2)}_{\ell'_1\ell'_2\ell'_3\ell'_4}(0, e^1, e^1, 0) \\ & \quad \times (D^{(0)})^{-1}_{\ell'_3\ell'_4\ell_3\ell_4}(e^1, 0, e^1, 0), \end{aligned} \tag{30}$$

and, similarly, for $-D_0$ replacing D.

We refer to Eqs. (29) and (30), respectively, as the first (second) term. However, since D_0 is of order κ^3 for the points occurring above, it is subdominant and will be dropped. Recalling that $D = FD^uF$, we obtain $D^{(2)}(0, e^1, 0, e^1)$ by expanding the numerator of $D^u(0, e^1, 0, e^1)$ in κ. After performing a gauge integration over oppositely oriented gauge bonds in the interval $(0, e^1)$, which gives a factor of $1/3$, we get

$$\begin{aligned} D^{u(2)} &= \frac{1}{12}\langle B^u_{\ell_1}(0)B^u_{\ell_2}(e^1)\bar\psi_{a\alpha_1 r_1}(0)\psi_{b\beta_1 r_1}(e^1)\Gamma^{e^1}_{\alpha_1\beta_1}\Gamma^{-e^1}_{\alpha_2\beta_2}\bar\psi_{b\alpha_2 r_2}(e^1)\psi_{a\beta_2 r_2}(0)\bar B^u_{\ell_3}(0)\bar B^u_{\ell_4}(e^1)\rangle^{(0)}_t \\ &= \frac{1}{12}\langle \psi_{a\beta_2 r_2}(0)B^u_{\ell_1}(0)\bar\psi_{a\alpha_1 r_1}(0)\bar B^u_{\ell_3}(0)\rangle^{(0)}_t\langle\psi_{b\beta_1 r_1}(e^1)B^u_{\ell_2}(e^1)\bar\psi_{b\alpha_2 r_2}(e^1)\bar B^u_{\ell_4}(e^1)\rangle^{(0)}_t\Gamma^{e^1}_{\alpha_1\beta_1}\Gamma^{-e^1}_{\alpha_2\beta_2} \\ &= \frac{1}{12}\langle \psi_{a\alpha_2 r_2}(0)B^u_{\ell_1}(0)\bar\psi_{a\alpha_1 r_1}(0)\bar B^u_{\ell_3}(0)\rangle^{(0)}_t\langle\psi_{b\alpha_1 r_1}(e^1)B^u_{\ell_2}(e^1)\bar\psi_{b\alpha_2 r_2}(e^1)\bar B^u_{\ell_4}(e^1)\rangle^{(0)}_t. \end{aligned} \tag{31}$$

Using the "come and go" property $\Gamma^{e^j}_{\alpha\beta}\Gamma^{-e^j}_{\gamma\alpha} = 0 = \Gamma^{e^j}_{\rho\gamma}\Gamma^{-e^j}_{\gamma\sigma}$, the $\psi\bar\psi$ contractions in the two factors give zero. For this reason, we have introduced the t index, meaning that we do not take into account contraction with external fields due to the come and go property. Furthermore, since the $\tilde B$ only have fermion fields with lower spin indices, the same holds for $\tilde\psi$. This implies that only the identity term of $\Gamma^{\pm e^1} = -1 \pm \gamma^1$ contributes. The above can be evaluated using the determinant formula for the averages, but it is shown in Appendix E, as for the case of coincident points, that again the ϵ's and det conspire to give a permanent representation. Denoting the site argument $(0, e^1, 0, e^1)$ of $D^u_{\vec\rho\,\vec h\,\vec\alpha\,\vec f\,\vec\sigma\,\vec k\,\vec\beta\,\vec g}(0, e^1, 0, e^1)$ by 1, we obtain

$$\begin{aligned} D^{u(2)} = 3\times\Big[&\mathrm{perm}\big(\delta_{\vec\rho(\beta_1\sigma_2\sigma_3)}\delta_{\vec h(g_1k_2k_3)}\big)\mathrm{perm}\big(\delta_{\vec\alpha(\sigma_1\beta_2\beta_3)}\delta_{\vec f(k_1g_2g_3)}\big) + \mathrm{perm}\big(\delta_{\vec\rho(\beta_2\sigma_2\sigma_3)}\delta_{\vec h(g_2k_2k_3)}\big)\mathrm{perm}\big(\delta_{\vec\alpha(\beta_1\sigma_1\beta_3)}\delta_{\vec f(g_1k_1g_3)}\big) \\ &+ \mathrm{perm}\big(\delta_{\vec\rho(\beta_3\sigma_2\sigma_3)}\delta_{\vec h(g_3k_2k_3)}\big)\mathrm{perm}\big(\delta_{\vec\alpha(\beta_1\beta_2\sigma_1)}\delta_{\vec f(g_1g_2k_1)}\big) + \mathrm{perm}\big(\delta_{\vec\rho(\sigma_1\beta_1\sigma_3)}\delta_{\vec h(k_1g_1k_3)}\big)\mathrm{perm}\big(\delta_{\vec\alpha(\sigma_2\beta_2\beta_3)}\delta_{\vec f(k_2g_2g_3)}\big) \\ &+ \mathrm{perm}\big(\delta_{\vec\rho(\sigma_1\beta_2\sigma_3)}\delta_{\vec h(k_1g_2k_3)}\big)\mathrm{perm}\big(\delta_{\vec\alpha(\beta_1\sigma_2\beta_3)}\delta_{\vec f(g_1k_2g_3)}\big) + \mathrm{perm}\big(\delta_{\vec\rho(\sigma_1\beta_3\sigma_3)}\delta_{\vec h(k_1g_3k_3)}\big)\mathrm{perm}\big(\delta_{\vec\alpha(\beta_1\beta_2\sigma_2)}\delta_{\vec f(g_1g_2k_2)}\big) \\ &+ \mathrm{perm}\big(\delta_{\vec\rho(\sigma_1\sigma_2\beta_1)}\delta_{\vec h(k_1k_2g_1)}\big)\mathrm{perm}\big(\delta_{\vec\alpha(\sigma_3\beta_2\beta_3)}\delta_{\vec f(k_3g_2g_3)}\big) + \mathrm{perm}\big(\delta_{\vec\rho(\sigma_1\sigma_2\beta_2)}\delta_{\vec h(k_1k_2g_2)}\big)\mathrm{perm}\big(\delta_{\vec\alpha(\beta_1\sigma_3\beta_3)}\delta_{\vec f(g_1k_3g_3)}\big) \\ &+ \mathrm{perm}\big(\delta_{\vec\rho(\sigma_1\sigma_2\beta_3)}\delta_{\vec h(k_1k_2g_3)}\big)\mathrm{perm}\big(\delta_{\vec\alpha(\beta_1\beta_2\sigma_3)}\delta_{\vec f(g_1g_2k_3)}\big)\Big] \equiv W^{\vec\sigma\,\vec k\,\vec\beta\,\vec g}_{\vec\rho\,\vec h\,\vec\alpha\,\vec f}/12. \end{aligned} \tag{32}$$

The first (second) permanent describes the interaction between $\vec\rho\,\vec h(\vec\alpha\,\vec f)$ and $\vec\sigma\,\vec k$, $\vec\beta\,\vec g$ spins and isospins. Thus, for i, $j = 1, 2, 3, \ldots, 36$, we write the matrix elements of Eq. (29) in the following way. First, we set

$$D^{(2)}_{ij}(0e^10e^1) = \frac{1}{12}F^{\vec\rho_i\vec h\vec\alpha_i\vec f}W^{\vec\sigma_j\vec k\vec\beta_j\vec g}_{\vec\rho_i\vec h\vec\alpha_i\vec f}F^{\vec\sigma_j\vec k\vec\beta_j\vec g} = \frac{1}{12}(u_1)_{ij}. \tag{33}$$

Next, similar to Eq. (30), we obtain

$$D^{(2)}_{ij}(0e^1e^10) = \frac{1}{12}F^{\vec\rho_i\vec h\vec\alpha_i\vec f}(-W^{\vec\beta_j\vec g\vec\sigma_j\vec k}_{\vec\rho_i\vec h\vec\alpha_i\vec f})F^{\vec\sigma_j\vec k\vec\beta_j\vec g} = \frac{1}{12}(u_2)_{ij}, \tag{34}$$

where we note the interchange $\vec\sigma_j\vec k \leftrightarrow \vec\beta_j\vec g$ and minus sign in going from (I) to (II). The above holds for the four-point function correlation for the $\tilde\Lambda$ field.

We now consider the $\tilde\Omega$ fields. It turns out that in the solution to the B-S equation the *sum* $u_+ \equiv u_1 + u_2$ and *difference* $u_- \equiv u_1 - u_2$ appear in the bound state equations [see Eqs. (43) and (44)]. The sum and difference will be seen to generate a decomposition of solutions into even and odd parity solutions [see Eq. (47)], respectively. For this reason, we will develop formulas for $u_\pm$, and we refer to u_+ and u_- as the sum and difference potentials.

To this end, we find the $\tilde\Omega$ field correlations that enter in Eqs. (33) and (34) in terms of $\tilde\Lambda$ field correlations. The symmetries of time reversal and parity show that $D_0(0e^10e^1)$, $D_0(0e^1e^10)$, and thus $D_0^{(2)}(0e^10e^1)$, $D_0^{(2)}(0e^1e^10)$, are self-adjoint. Let $\Lambda_{j\pm}(0, e^1) = \Lambda_j(0, e^1) \pm \Lambda_j(e^1, 0)$. For $j = 1$, 2, 3, 4 and the first term [see Eq. (33)],

$$\langle\Omega_j(0, e^1)\bar\Omega_2(0, e^1)\rangle^{(2)} = \langle\Omega_j(0, e^1)\frac{1}{\sqrt2}[\bar\Lambda_2(0, e^1) - \bar\Lambda_2(e^1, 0)]\rangle^{(2)}. \tag{35}$$

For the second term [see Eq. (34)],

$$\langle\Omega_j(0, e^1)\frac{1}{\sqrt2}[\bar\Lambda_2(e^1, 0) - \bar\Lambda_2(0, e^1)]\rangle^{(2)}, \tag{36}$$

so that the sum is zero.

For $j = 1$, 3, the first term gives $\langle\Omega_j(0, e^1)\bar\Omega_4(0, e^1)\rangle^{(2)} = \langle\Lambda_j(0, e^1)\frac{1}{\sqrt2}\bar\Lambda_{2+}(0, e^1)\rangle^{(2)}$, and the second term gives the same as the first term so that the sum is

$$\sqrt2\langle\Lambda_j(0, e^1)\bar\Lambda_{2+}(0, e^1)\rangle^{(2)}.$$

For the first term,

$$\langle\Omega_4(0, e^1)\bar\Omega_4(0, e^1)\rangle^{(2)} = \tfrac{1}{2}\langle[\Lambda_2(0, e^1) + \Lambda_2(e^1, 0)] \times[\bar\Lambda_2(0, e^1) + \bar\Lambda_2(e^1, 0)]\rangle^{(2)}, \tag{37}$$

and the second term is the same. Write the above as

$$\tfrac{1}{2}\langle \Lambda_2(0,e^1)\bar{\Lambda}_{2+}(0,e^1)\rangle^{(2)} + \tfrac{1}{2}\langle \Lambda_2(e^1,0)\bar{\Lambda}_{2+}(0,e^1)\rangle^{(2)},$$

and in the second $\langle\rangle^{(2)}$ translate by $-e^1$ and use parity to get equality with the first $\langle\rangle^{(2)}$. Altogether, for the sum, we get

$$2\langle \Lambda_2(0,e^1)\bar{\Lambda}_{2+}(0,e^1)\rangle^{(2)}.$$

For $j = 1, 3$, $i \le j$, the first term is $\langle \Omega_j(0,e^1)\bar{\Omega}_j(0,e^1)\rangle^{(2)} = \langle \Lambda_j(0,e^1)\bar{\Lambda}_j(0,e^1)\rangle^{(2)}$, and the second term is $\langle \Lambda_j(0,e^1)\bar{\Lambda}_j(e^1,0)\rangle^{(2)}$ so that the sum is

$$\langle \Lambda_j(0,e^1)\bar{\Lambda}_{j+}(0,e^1)\rangle^{(2)}.$$

In this way, we obtain the following block structure for $\langle \Omega\bar{\Omega}\rangle^{(2)}$, for the sum of the first and second terms

$$\begin{pmatrix} V_{11} & 0 & V_{13} & V_{14} \\ 0 & 0 & 0 & 0 \\ V_{31} & 0 & V_{33} & V_{34} \\ V_{41} & 0 & V_{43} & V_{44} \end{pmatrix}, \tag{38}$$

where $V_{11} = \langle \Lambda_1\bar{\Lambda}_{1+}\rangle^{(2)}$, $V_{13} = \langle \Lambda_1\bar{\Lambda}_{3+}\rangle^{(2)}$, $V_{14} = \sqrt{2}\langle \Lambda_1\bar{\Lambda}_{2+}\rangle^{(2)}$, $V_{33} = \langle \Lambda_3\bar{\Lambda}_{3+}\rangle^{(2)}$, $V_{34} = \sqrt{2}\langle \Lambda_2\bar{\Lambda}_{3+}\rangle^{(2)}$ (by the identity used previously), and $V_{44} = 2\langle \Lambda_2\bar{\Lambda}_{2+}\rangle^{(2)}$. Note that V_{11}, V_{13}, $V_{31} = V_{33}$ have the same values as for the $\tilde{\Lambda}$ fields. Collapsing down to a 3×3 block matrix gives

$$\begin{pmatrix} \langle \Lambda_1\bar{\Lambda}_{1+}\rangle^{(2)} & \langle \Lambda_1\bar{\Lambda}_{3+}\rangle^{(2)} & \sqrt{2}\langle \Lambda_1\bar{\Lambda}_{2+}\rangle^{(2)} \\ * & \langle \Lambda_3\bar{\Lambda}_{3+}\rangle^{(2)} & 0 \\ * & * & 2\langle \Lambda_2\bar{\Lambda}_{2+}\rangle^{(2)} \end{pmatrix}, \tag{39}$$

where we recall that the block is symmetric and we use the sign $*$ to avoid having to write the lower diagonal elements. In the second line, the zero element is $\sqrt{2}\langle \Lambda_2\bar{\Lambda}_{3+}\rangle^{(2)}$ and is found to be zero by numerical calculations.

For the sum potential reduced matrix, the spin ordering is that of $\tilde{\Lambda}_1$, $\tilde{\Lambda}_3$, $\tilde{\Lambda}_2$ for the first, second, and third blocks, respectively (such as $1, 2, \ldots, 16$, $25, \ldots, 28$, and $17, \ldots, 24$, respectively.

Similarly, for the difference potential and for $\tilde{\Omega}$ fields, we have

$$\begin{pmatrix} V_{11} & V_{12} & V_{13} & 0 \\ V_{21} & V_{22} & V_{23} & 0 \\ V_{31} & V_{32} & V_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{40}$$

In terms of $\tilde{\Lambda}$ fields, collapsing to a 3×3 block gives

$$\begin{pmatrix} \langle \Lambda_1\bar{\Lambda}_{1-}\rangle^{(2)} & \sqrt{2}\langle \Lambda_1\bar{\Lambda}_{2-}\rangle^{(2)} & \langle \Lambda_1\bar{\Lambda}_{3-}\rangle^{(2)} \\ * & 2\langle \Lambda_2\bar{\Lambda}_{2-}\rangle^{(2)} & \sqrt{2}\langle \Lambda_2\bar{\Lambda}_{3-}\rangle^{(2)} \\ * & * & \langle \Lambda_3\bar{\Lambda}_{3-}\rangle^{(2)} \end{pmatrix}. \tag{41}$$

For the difference potential reduced matrix, the spin ordering for the first, second, and third blocks is that of $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$, and $\tilde{\Lambda}_3$, respectively.

The sum u_+ and difference u_- potential matrices are given in Appendix D for the $\tilde{\Lambda}$ fields. Using these quantities, the sum and difference potential matrices for the $\tilde{\Omega}$ fields are obtained as above. We also see from the above that, for the sum and difference potentials in the $\tilde{\Omega}$ fields, we only need the first three sub-blocks of the sum and difference potentials in the $\tilde{\Lambda}$ fields. The spectral properties of these matrices will be determined in Secs. VII D and VII E.

VII. BOUND STATE ANALYSIS

This section is devoted to the analysis of two-baryon bound states. In the first subsection, making use of the previously obtained ladder approximation to the partially Fourier transformed B-S kernel, after employing the lattice relative coordinates, we obtain the bound state equations to be analyzed. These equations turn out to be a pair of decoupled equations, one involving only the sum and another for the difference potentials described in the previous section. An approximation for $\hat{D}_0(\vec{\xi},\vec{\eta},k^0)$ is suitable to analyze the bound state equations (see Ref. [20]). In the next subsection, by restricting our analysis to the $p - p$ subspace, we show that bound states occur. The spectral analysis for the sum and difference potentials is carried out separately in the next two subsections. Then, by using the spectral results for the sum and difference potentials obtained above, we determine the solutions of the bound states equations. Finally, we establish a correspondence between the relative coordinate, the partially Fourier transformed B-S equation, and a one-particle lattice Schrödinger operator resolvent equation. In this correspondence, two-baryon bound states correspond to negative energy states of a one-particle lattice Schrödinger Hamiltonian.

A. Bound state solution of the B-S equation

We will see that, in our approximation to the solution of the B-S equation, there is a decoupling over directions and the zero site. From Eqs. (33) and (34), we write the exchange contribution for $\hat{K}$ as

$$\hat{L}_{\rm ex}(\vec{\xi},\vec{\eta},k^0) = \frac{1}{48}\kappa^2 \sum_{\sigma=\pm, j=1,2,3} [u_1\delta(\vec{\xi}-\sigma e^j)\delta(\vec{\xi}-\vec{\eta}) + u_2\delta(\vec{\xi}-\sigma e^j)\delta(\vec{\xi}+\vec{\eta})], \tag{42}$$

where $1/12$ times u_1 (u_2) is the contribution from the first (second term). Here, $k^0 = i(2m_b - \epsilon)$, $\epsilon > 0$, and we solve for $\hat{D}(\vec{\xi},\vec{\eta})$, considering only $j = 1$ in Eq. (42). Doing so, we obtain

$$\hat{D}(\vec{\xi},\vec{\eta}) = \hat{D}_0(\vec{\xi},\vec{\eta}) + \hat{D}_0(\vec{\xi},\vec{\xi}')\hat{L}_{\rm ex}(\vec{\xi}',\vec{\eta}')\hat{D}(\vec{\eta}',\vec{\eta}).$$

Letting $\vec{\xi} = e^1, -e^1$, we get for $(\hat{D}(e^1,\vec{\eta}), \hat{D}(-e^1,\vec{\eta}))^T$,

$$\begin{pmatrix} 1-A & B \\ B & 1-A \end{pmatrix}^{-1} \begin{pmatrix} \hat{D}_0(e^1,\vec{\eta}) \\ \hat{D}_0(-e^1,\vec{\eta}) \end{pmatrix},$$

where $A = \hat{D}_0(e^1, e^1)u_1 + \hat{D}_0(e^1, -e^1)u_2$, $B = -\hat{D}_0(e^1, e^1)u_2 - \hat{D}_0(e^1, -e^1)u_1$, and we suppress the $\kappa^2/48$ factor from u_1 and u_2.

The structure of the above matrix is (recall the zeros of det are the bound state solutions)

$$\begin{pmatrix} A' & B \\ B & A' \end{pmatrix}.$$

For computing the det, perform the block row operation $r_1 \to r_1 - r_2$, then the block column operation $c_2 \to c_2 + c_1$ to get

$$\det\begin{pmatrix} A' & B \\ B & A' \end{pmatrix} = \det\begin{pmatrix} A' - B & 0 \\ B & A' + B \end{pmatrix} = \det(A' - B)\det(A' + B).$$

Here,

$$A' - B = 1 - \hat{D}_{0-}u_-, \tag{43}$$

$$A' + B = 1 - \hat{D}_{0+}u_+, \tag{44}$$

where we have defined $\hat{D}_{0\pm} \equiv \hat{D}_0(e^1, e^1) \pm \hat{D}_0(e^1, -e^1)$. The condition

$$\det(1 \pm \hat{D}_{0\pm}u_\pm) = 0$$

gives the bound state equations, and we see here the appearance of the sum (difference) potentials u_+ (u_-). To proceed, we approximate $\hat{D}_0(\vec{\xi}, \vec{\eta})$, essentially by neglecting the contributions to the two-point function involving more than a single particle. In our approach, we only need to take into account the exchange potential in each direction separately. Considering all directions gives rise to $\hat{D}_0(\sigma e^i, \sigma' e^j)$, $i \neq j$, which is zero in our approximation. Also, $\hat{D}_0(\sigma e^i, 0)$ is zero.

B. Approximate $\hat{D}_0(\vec{\xi}, \vec{\eta}, k^0)$

From [12], for $\tilde{\Omega}$ fields and the 1-1 block, suppressing the k^0 dependence in $\hat{D}_0$, we have

$$(\hat{D}_0)_{s_1s_2s_3s_4}(\vec{\xi}, \vec{\eta}, k^0) = (2\pi)^{-3}\int_{\mathbb{T}^3}[\tilde{G}(\vec{p})]^2[-\delta_{s_1s_3}\delta_{s_2s_4}\Xi_+(\vec{p}) + \delta_{s_1s_4}\delta_{s_2s_3}\Xi_-(\vec{p})]d\vec{p} + (2\pi)^{-3}\int_{-1}^{1}\int_{-1}^{1}\int_{\mathbb{T}^3} f(k^0, \lambda^0\lambda'^0)[-\delta_{s_1s_3}\delta_{s_2s_4}\Xi_+(\vec{p}) + \delta_{s_1s_4}\delta_{s_2s_3}\Xi_-(\vec{p})]d\lambda^0\alpha_{\vec{p}}(\lambda^0)d\lambda'^0\alpha_{\vec{p}}(\lambda'^0)d\vec{p},$$

where $\Xi_\pm(\vec{p}) = \cos\vec{p}.\vec{\xi}\cos\vec{p}.\vec{\eta} \pm \sin\vec{p}.\vec{\eta}\sin\vec{p}.\vec{\eta}$. Making the same approximation as in Ref. [12], we obtain for the 22 44 block,

$$\hat{D}_{022,44} \approx \int\frac{1}{2\epsilon}(-\delta_{s_1s_3}\delta_{s_2s_4})[\Xi_+(\vec{p},\vec{\xi},\vec{\eta}) - \Xi_+(\vec{p},\vec{\xi},-\vec{\eta}) - \Xi_+(\vec{p},-\vec{\xi},\vec{\eta}) + \Xi_+(\vec{p},-\vec{\xi},-\vec{\eta}), \Xi_+(\vec{p},\vec{\xi},\vec{\eta}) + \Xi_+(\vec{p},\vec{\xi},-\vec{\eta}) + \Xi_+(\vec{p},-\vec{\xi},\vec{\eta}) + \Xi_+(\vec{p},-\vec{\xi},-\vec{\eta})] = \int\frac{2}{\epsilon}(-\delta_{s_1s_3}\delta_{s_2s_4})[\sin\vec{p}.\vec{\xi}\sin\vec{p}.\vec{\eta}, \cos\vec{p}.\vec{\xi}\cos\vec{p}.\vec{\eta}],$$

where $\int$ is the normalized integral $(2\pi)^{-3}\int_{\mathbb{T}^3}d\vec{p}$. Also,

$$\hat{D}_{024} = \int\frac{1}{2\epsilon}(-\delta_{s_1s_3}\delta_{s_2s_4})[\Xi_+(\vec{p},\vec{\xi},\vec{\eta}) + \Xi_+(\vec{p},\vec{\xi},-\vec{\eta}) - \Xi_+(\vec{p},-\vec{\xi},\vec{\eta}) - \Xi_+(\vec{p},-\vec{\xi},-\vec{\eta})] = 0.$$

Here, we neglect the k^0-independent term, as it is finite, and the k^0-dependent term blows up as $\epsilon \to 0$. Similarly, using the isospin orthogonality relations for the two-point function, we have $(\hat{D}_0)_{12,14,13,34,23} = 0$ so that $\hat{D}_0$ is block diagonal. Also, $(\hat{D}_0)_{11,33}$ is the same as for the $\tilde{\Lambda}$ fields. Thus, we have our final approximation

$$(\hat{D}_0)_{11,33} \sim \frac{1}{\epsilon}\int[-\delta_{s_1s_3}\delta_{s_2s_4}\Xi_+(\vec{p}) + \delta_{s_1s_4}\delta_{s_2s_3}\Xi_-(\vec{p})], (\hat{D}_0)_{22,44} \sim -\tfrac{2}{\epsilon}(\delta_{s_1s_3}\delta_{s_2s_4}) \times \int[\sin\vec{p}.\vec{\xi}\sin\vec{p}.\vec{\eta}, \cos\vec{p}.\vec{\xi}\cos\vec{p}.\vec{\eta}].$$

From the solution of the B-S equation, we need $(\hat{D}_0)_0(e^1, e^1)$ and $\hat{D}_0(-e^1, e^1)$, which are related to $\hat{D}_{0\mp}$, as they appear in Eqs. (43) and (44). We have

$$\hat{D}_{0-} \approx \begin{pmatrix} \frac{1}{\epsilon}[-\delta_{s_1s_3}\delta_{s_2s_4} - \delta_{s_1s_4}\delta_{s_2s_3}] \\ \frac{1}{\epsilon}[-2\delta_{s_1s_3}\delta_{s_2s_4}] \\ \frac{1}{\epsilon}[-\delta_{s_1s_3}\delta_{s_2s_4} - \delta_{s_1s_4}\delta_{s_2s_3}] \\ 0 \end{pmatrix}, \tag{45}$$

$$\hat{D}_{0+} \approx \begin{pmatrix} \frac{1}{\epsilon}[-\delta_{s_1s_3}\delta_{s_2s_4} + \delta_{s_1s_4}\delta_{s_2s_3}] \\ 0 \\ \frac{1}{\epsilon}[-\delta_{s_1s_3}\delta_{s_2s_4} + \delta_{s_1s_4}\delta_{s_2s_3}] \\ \frac{1}{\epsilon}[-2\delta_{s_1s_3}\delta_{s_2s_4}] \end{pmatrix}. \tag{46}$$

We will see that the bound state equation associated with Eq. (44) leads to bound state solutions, even restricting to the 33 or $p - p$ block. However, the binding energy is enhanced appreciably, taking into account the full space of degenerate two-particle states, according to what degenerate perturbation theory tells us to do. We also see that

$\hat{D}_0(\sigma e^i, \sigma' e^j)$, $i \neq j$, is zero as well as $\hat{D}_0(\sigma e^i, 0)$. Thus, the bound state det equation factorizes over the 1, 2, and 3 directions, each factor as in Eqs. (43) and (44).

C. Exchange potential restricted to the $p - p$ subspace and bound states

Here, we restrict to the $p - p$ subspace (sub-block 33) and consider the two factors of the bound state equations (43) and (44). We will see that, even with this restriction, a $p - p$ bound state always occurs. We have

$$\hat{D}_{0-} \approx -\frac{1}{\epsilon}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$\hat{D}_{0+} \approx -\frac{1}{\epsilon}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For u_- and u_+, we have, suppressing the factor $1/12$,

$$u_- \approx \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$u_+ \approx -\frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the factor given in Eq. (43),

$$1 - \hat{D}_{0-}u_- = \begin{pmatrix} 1 + \frac{4a}{\epsilon} & 0 & 0 & 0 \\ 0 & 1 + \frac{2a}{\epsilon} & -\frac{2a}{\epsilon} & 0 \\ 0 & -\frac{2a}{\epsilon} & 1 + \frac{2a}{\epsilon} & 0 \\ 0 & 0 & 0 & 1 + \frac{4a}{\epsilon} \end{pmatrix},$$

and setting the determinant to zero gives $1 + 2a/\epsilon = 0$, with no solution. For the other factor in Eq. (44), we obtain

$$1 - \hat{D}_{0+}u_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{2}{3\epsilon} & \frac{2}{3\epsilon} & 0 \\ 0 & \frac{2}{3\epsilon} & 1 - \frac{2}{3\epsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Setting the determinant to zero gives $1 - 4/3\epsilon = 0$ or $\epsilon = 4/3$. So, we have a bound state solution with $J = J_z = 0$. Reinstating the factor $\kappa^2/48$, we have $\epsilon = \kappa^2/36$.

In the correspondence with the lattice Schrödinger Hamiltonian operator (see Sec. VII G below), we can consider the above as a result of a variational calculation to give an upper bound for the bound state energy.

D. Exchange sum potential u_+

Here, we consider the sum potential in the entire two-baryon subspace and determine the spectrum. It will be seen that negative eigenvalues lead to bound states [see Eq. (44)]. For the sum potential reduced matrix of Eq. (39), we have the invariant sub-blocks 2 5 17, 3 9 18 19, 4 7 10 13 26 27 20 21. We give these sub-blocks and their spectra (the multiplicities of eigenvalues are shown in the parentheses).

$$2\,5\,17\colon \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{4}{3}\sqrt{10} \\ * & -\frac{1}{3} & -\frac{4}{3}\sqrt{10} \\ * & * & \frac{2}{3} \end{pmatrix},$$

with σ: 0 (1), 6 (1), -6 (1).

$$3\,9\,18\,19\colon \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{2}{3}\sqrt{10} & \frac{2}{3}\sqrt{30} \\ * & -\frac{1}{3} & -\frac{2}{3}\sqrt{10} & -\frac{2}{3}\sqrt{30} \\ * & * & \frac{14}{3} & -\frac{4}{3}\sqrt{3} \\ * & * & * & 2 \end{pmatrix},$$

with σ: 0 (1), 6 (2), -6 (1). The -6 eigenvalue eigenvectors are

$$\begin{pmatrix} -\frac{\sqrt{10}}{4} \\ \frac{\sqrt{10}}{4} \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} -\frac{\sqrt{10}}{6} \\ \frac{\sqrt{10}}{6} \\ \frac{1}{3} \\ \frac{1}{\sqrt{3}} \end{pmatrix},$$

with $J = 2 = J_z$ and $J = 2$, $J_z = 1$, respectively.

$$4\,7\,10\,13\,26\,27\,20\,21\colon \begin{pmatrix} 0 & -1 & 1 & 0 & 2\sqrt{10} & -2\sqrt{10} & 2\sqrt{10} & 2\sqrt{10} \\ * & 0 & 0 & 1 & -2\sqrt{10} & 2\sqrt{10} & 2\sqrt{10} & 2\sqrt{10} \\ * & * & 0 & -1 & 2\sqrt{10} & -2\sqrt{10} & -2\sqrt{10} & -2\sqrt{10} \\ * & * & * & 0 & -2\sqrt{10} & 2\sqrt{10} & -2\sqrt{10} & -2\sqrt{10} \\ * & * & * & * & -1 & 1 & 0 & 0 \\ * & * & * & * & * & -1 & 0 & 0 \\ * & * & * & * & * & * & 10 & -8 \\ * & * & * & * & * & * & * & 10 \end{pmatrix}.$$

σ: $-6\,(2)$, $0\ (3)$, $6\ (3)$. The -6 eigenvalue eigenvectors are

$$\lambda = -6: \begin{pmatrix} -\frac{\sqrt{10}}{4} \\ -\frac{\sqrt{10}}{4} \\ \frac{\sqrt{10}}{4} \\ \frac{\sqrt{10}}{4} \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} \frac{\sqrt{10}}{5} \\ -\frac{\sqrt{10}}{5} \\ \frac{\sqrt{10}}{5} \\ -\frac{\sqrt{10}}{5} \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and have $J = 2$, $J_z = 0$ and $J = 0$, $J_z = 0$, respectively. In addition, we have the invariant sub-block 12 15 24 which has $J_z = -2$ and is the spin flip of 2 5 17; we also have 8 14 22 23 which has $J_z = -1$ and is the spin flip of 3 9 18 19. The blocks 1, 6, 11, 16, 25, 28 are zero. We single out the -6 eigenvalue eigenvectors, as they will be seen to be associated with bound states.

The second eigenvector only has components in $p - p$ and $\Delta - \Delta$ and is the same as the restriction to the $\Delta - \Delta$ and $p - p$ two-baryon subspace.

In summary, the only negative eigenvalue is $-6\,(6)$. Also, there are eigenvalues 0 (13), 6 (9).

E. Exchange difference potential u_-

For the difference reduced matrix of Eq. (41), the invariant blocks are 1, 16, 25, 12 15, 8 11 14 22 23 28, 3 6 9 18 19 25, and 4 7 10 13 20 21 26 27, 17, 24. We now determine the spectrum for each of these blocks.

1: $\sigma = -2\,(1)$ has $J_z = 3$; 16: $\sigma = -2\,(1)$, spin flip related to 1, has $J_z = -3$,

$$25: \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

$\sigma = -2\,(1)$, $0\ (1)$ has $J_z = 2$.

12 15: same as the 25 spin flip related to $J_z = -2$.

17: $\sigma = -2\,(1)$; 24: $\sigma = -2\,(1)$, spin flip related to 17, has $J_z = -2$.

8 11 14 22 23 28: spin flip related to 3 6 9 18 19 25, which has $J_z = 1$,

$$3\,6\,9\,18\,19\,25: \begin{pmatrix} -\frac{1}{3} & -\frac{4\sqrt{3}}{9} & -1 & \frac{\sqrt{40}}{3} & -\frac{2\sqrt{30}}{9} & -\frac{4\sqrt{30}}{9} \\ * & -\frac{10}{9} & -\frac{4\sqrt{3}}{9} & -\frac{4\sqrt{30}}{9} & \frac{4\sqrt{10}}{9} & -\frac{8\sqrt{10}}{9} \\ * & * & -\frac{1}{3} & \frac{2\sqrt{10}}{3} & -\frac{2\sqrt{30}}{9} & \frac{4\sqrt{30}}{9} \\ * & * & * & \frac{14}{3} & -\frac{20\sqrt{3}}{9} & \frac{40\sqrt{3}}{9} \\ * & * & * & * & \frac{2}{9} & -\frac{40}{9} \\ * & * & * & * & * & \frac{62}{9} \end{pmatrix}.$$

The spectrum is $-2\,(4)$, $0\ (1)$, $18\ (1)$. The -2 eigenvalue eigenvectors for the 3 6 9 18 19 25 sub-blocks are, for $\lambda = -2$,

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \frac{\sqrt{30}}{5} \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \frac{\sqrt{10}}{5} \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \sqrt{3} \\ 0 \end{pmatrix}.$$

The other invariant sub-block is

$$4\,7\,10\,13\,20\,21\,26\,27: \begin{pmatrix} 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{\sqrt{40}}{3} & -\frac{\sqrt{40}}{3} & \frac{\sqrt{40}}{3} & \frac{\sqrt{40}}{3} \\ * & -\frac{8}{9} & -\frac{8}{9} & -\frac{1}{3} & -\frac{\sqrt{40}}{9} & \frac{\sqrt{40}}{9} & -\frac{\sqrt{40}}{9} & -\frac{\sqrt{40}}{9} \\ * & * & -\frac{8}{9} & -\frac{1}{3} & -\frac{\sqrt{40}}{9} & \frac{\sqrt{40}}{9} & -\frac{\sqrt{40}}{9} & -\frac{\sqrt{40}}{9} \\ * & * & * & 0 & \frac{\sqrt{40}}{3} & -\frac{\sqrt{40}}{3} & \frac{\sqrt{40}}{3} & \frac{\sqrt{40}}{3} \\ * & * & * & * & \frac{22}{9} & -\frac{40}{9} & \frac{40}{9} & \frac{40}{9} \\ * & * & * & * & * & \frac{22}{9} & -\frac{40}{9} & -\frac{40}{9} \\ * & * & * & * & * & * & \frac{31}{9} & \frac{31}{9} \\ * & * & * & * & * & * & * & \frac{31}{9} \end{pmatrix}$$

and has $J_z = 0$, with the spectrum σ: $-2\,(4)$, $0\ (3)$, 18.

The eigenvectors are, for $\lambda = -2$,

$$\begin{pmatrix} 0 \\ -\sqrt{10} \\ -\sqrt{10} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ \sqrt{10} \\ \sqrt{10} \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 2\sqrt{10} \\ 2\sqrt{10} \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In summary, the only negative eigenvalue is -2, which has multiplicity (18). Also, there are eigenvalues 0 (7) and 18 (3).

F. Exchange potentials and bound states

We determine the bound states by determining the zeros of the determinant of the matrices given in Eqs. (43) and (44). First, consider Eq. (44). We already have the matrices $\hat{D}_{0+}$ and u_+, and we need their product. From Eq. (46), we see that the first matrix acts as $-2/\epsilon$ times the identity on the first three sub-blocks if they are antisymmetric under row spin interchange, it acts as $-2/\epsilon$ times the identity on the fourth sub-block. The row spin interchange is $1 \leftrightarrow 1$, $2 \leftrightarrow 5$, $3 \leftrightarrow 9$, $4 \leftrightarrow 13$, $6 \leftrightarrow 6$, $7 \leftrightarrow 10$, $8 \leftrightarrow 14$, $11 \leftrightarrow 11$, $12 \leftrightarrow 15$, $16 \leftrightarrow 16$ for the 1-1 sub-block, and for the 3 sub-block, it is $25 \leftrightarrow 25$, $26 \leftrightarrow 27$, $28 \leftrightarrow 28$. By inspection of the sum potential matrices presented in Sec. VII D, it is verified that the first three sub-block rows do indeed satisfy this property. So, the matrix of Eq. (44) becomes $1-(-2/\epsilon)u_+$. The determinant factorizes over the eigenvalues of u_+ and, for each negative eigenvalue λ, we have a bound state with binding energy $\epsilon = 2|\lambda|$ upon reinstating the factor $\kappa^2/48$, $\epsilon = 2|\lambda|\kappa^2/48$. From the spectral results for u_+, we give the negative eigenvalue λ, z component of total spin J_z, total spin J and multiplicity. We have $(-6, \pm 2, 2, 1)$, $(-6, \pm 1, 2, 1)$, $(-6, 0, 2, 1)$, and $(-6, 0, 0, 1)$.

Similarly, considering the matrix $\hat{D}_{0-}$ of Eq. (45), it acts as $-2/\epsilon$ times the identity on the first and third sub-blocks if they are symmetric under row spin interchange; it acts as the identity on the second and fourth sub-blocks. Taking into account the symmetry of u_-, the matrix of Eq. (43) becomes $1-(-2/\epsilon)(u_-)$. For each negative eigenvalue λ of u_-, we have a bound state with binding energy $\epsilon = 2|\lambda|$. From the spectrum of u_-, we have bound states $(-2, \pm 3, 3, 1)$, $(-2, \pm 2, 3, 1)$, $(-2, \pm 2, 2, 1)$, $(-2, \pm 1, , 4)$, and $(-2, 0, , 4)$.

Next, we explain our restriction to the 1 direction only, in obtaining bound states. In solving the B-S equation, taking into account all directions for the exchange potential and the zero site contact potential, the resulting determinant has off-diagonal blocks in the directions which involve $\hat{D}_0(\sigma e^i, \sigma' e^j)$, $i \neq j$, as well as $\hat{D}_0(0, \sigma e^i)$. In our approximation, these $\hat{D}_0$'s are zero, as the numerators integrate to zero. Thus the determinant factorizes over the zero site and the 1, 2, 3 directions. The determinant for directions 2 and 3 is the same as that of the 1 direction and results in additional bound states, as given above for the 1 direction. The zero site contribution to the determinant gives the approximate factor $\det[1-(Q/4)(1-3\kappa^3/2)(1-\epsilon)]$, and the zeros do not give rise to order κ^2 binding energies.

G. One-particle lattice Schrödinger operator and bound states

Here, we make a correspondence between the relative coordinate, the partially Fourier transformed B-S equation in our ladder approximation, and the $\hat{D}_0$ approximation with a one-particle lattice Schrödinger Hamiltonian resolvent equation. Letting $\hat{D} = -(H-z)^{-1}$, $\hat{D}_0 = -(H_0 - z)^{-1}$, $\hat{V} = \hat{L}_{\rm ex}$ with $z = -\epsilon/2$, $H_0 = (\kappa^3/8)p_\ell^2$, $p_\ell^2 = 2\sum_{j=1}^3 (1-\cos p_j)$, the B-S equation becomes

$$(H-z)^{-1} = (H_0 - z)^{-1} - (H_0 - z)^{-1} V (H-z)^{-1},$$

like the operator resolvent equation for the one-particle lattice Hamiltonian H, where

$$H = H_0 + V = (\kappa^3/8)p_\ell^2 + \hat{L}_{\rm ex},$$

$$\hat{L}_{\rm ex}(\vec{\xi}, \vec{\eta}, k^0) = \frac{\kappa^2}{48} \sum_{\sigma=\pm 1, j=1,2,3} [u_1 \delta(\vec{\xi} - \sigma e^j)\delta(\vec{\xi} - \vec{\eta}) + u_2 \delta(\vec{\xi} - \sigma e^j)\delta(\vec{\xi} + \vec{\eta})].$$

The negative eigenvalues λ of H are related to the two-baryon bound state binding energies ϵ by $|\lambda| = \epsilon/2$.

We determine the negative energy bound states of H. We drop the kinetic energy, as it is of order κ^3 and the exchange potential $\hat{L}_{\rm ex}$ is of order κ^2. Thus, we look for eigenfunctions $\psi_\pm$ of $\hat{L}_{\rm ex}$ or of $\hat{V}_r = 48\hat{L}_{\rm ex}/\kappa^2$. If we take

$$\psi_\pm(\vec{\xi}) = v_\pm(\delta(\vec{\xi} - e^1) \pm \delta(\vec{\xi} + e^1)),$$

which are functions of $\pm$ parity, then a calculation gives

$$V_r \psi_\pm(\vec{\xi}) = u_\pm v_\pm (\delta(\vec{\xi} - e^1) \pm \delta(\vec{\xi} + e^1)).$$

Hence, if $v_\pm$ is an eigenvector of $u_\pm$ with eigenvalue $\lambda_\pm$, then $\psi_\pm$ is an eigenfunction of V_r with eigenvalue $\lambda_\pm$. Alternatively, the parity decomposition of $\hat{L}_{\rm ex}$ can be seen by the identity (suppressing k^0)

$$\hat{L}_{\rm ex}(\vec{\xi}, \vec{\eta}) = \frac{\kappa^2}{48} \sum_{\sigma=\pm 1; j=1,2,3} \left[u_+ \frac{1}{2} \delta(\vec{\xi} - \sigma e^j) \phi_+(\vec{\xi}, \vec{\eta}) + u_- \frac{1}{2} \delta(\vec{\xi} - \sigma e^j) \phi_-(\vec{\xi}, \vec{\eta}) \right]. \tag{47}$$

We note that $\phi_\pm(\vec{\xi}, \vec{\eta}) \equiv [\delta(\vec{\xi} - \vec{\eta}) \pm \delta(\vec{\xi} + \vec{\eta})]/2$ is the kernel of the $\pm$ parity operator. For $\lambda_\pm < 0$, which are negative eigenvalues, there are two-baryon bound states with binding energies $\epsilon_\pm = 2|\lambda_\pm|\kappa^2/48$.

VIII. CONCLUDING REMARKS

We determine the two-baryon bound states of a two-flavor strongly coupled lattice QCD model with the improved Wilson action, in the total isospin $I = 1$, $I_3 = 1$ sector and in the subspace of the Hilbert space generated by the product of one-particle states. For zero plaquette coupling β and at the leading order in the hopping parameter κ, we use a lattice version of the B-S equation and find, in lattice relative coordinates, a correspondence between two-baryon bound states with the bound states of a one-particle lattice Schrödinger operator, where the spin states are those of the two-baryon particles. It is found that a space-range-one, spin-dependent and energy-independent potential of order κ^2 is the dominant interaction. The potential arises from a $q - \bar{q}$ nearest neighbor exchange in the contribution to the kernel K of the B-S equation. Other contributions to K coming from chains of $q - \bar{q}$ give rise to a Yukawa potential, but the nearest neighbor contributions we consider are dominant (the contribution of a $q - \bar{q}$ chain of length ℓ is of order $\kappa^{2\ell}$). Other subdominant contributions are multiple $q - \bar{q}$ exchange and chains of plaquettes and mixed plaquette-quark chains. The attractive or repulsive nature of the interaction depends on the individual quark spins of the baryons. Also, a representation in terms of a permanent is obtained for this potential. Similar permanent representations also hold for the $I = 0$ and $I = 3$ isospin sectors treated in a previous paper, as well as for the $I = 2$ sector. It is found that there are diverse bound states with binding energy of order κ^2. In particular, there is a $J = 0$ diproton-like bound state with approximate binding energy $\epsilon = \kappa^2/4$. It is described by a superposition of $p - p$ and $\Delta - \Delta$ states. By isospin symmetry, corresponding to this bound state, there is also an $I = 1$, $I_3 = 0$, $J = 0$ $p - n$ bound state and an $I = 1$, $I_3 = -1$, $J = 0$ dineutron-like bound state with the same binding energy. If we erroneously consider only the subspace of states generated by product states of protons, a bound state is found but with the binding energy reduced by a factor of $1/9$. Thus, the nondiagonal $p - p$, $\Delta - \Delta$ matrix elements are important for determining the correct bound state binding energy and wave function. It is certainly interesting to determine the bound state spectrum for the three or more flavor model to see, for example, the effect of the strange quark. Also, and more importantly, we would like to know what happens to these bound states as we leave the strong coupling regime and approach the scaling limit (physical region in the κ, β plane).

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APPENDIX A: COEFFICIENTS FOR THE $\bar{\Lambda}_i$ FIELDS

Taking into account the definition of the $\tilde{\chi}_{I,I_3}$ fields, we give the $F_i^{\vec{k}\,\vec{g}}$ coefficients for the $\bar{\Lambda}_{1234}$ fields of Eqs. (10)–(13). We use the totally symmetric property of the $\bar{B}^u_{\vec{\alpha}\,\vec{f}}$'s, namely, that the field $\bar{B}^u_{\vec{\alpha}\,\vec{f}}$ is invariant under the simultaneous permutation of the components of $\vec{\alpha}$ and $\vec{f}$. We write

$$
\begin{aligned}
\bar{\Lambda}_1 = {} & \sqrt{\frac{3}{10}}\left\{\frac{1}{6}\bar{B}^u_{(\pm\pm\pm)(uuu)}\frac{1}{2\sqrt{3}}\bar{B}^u_{(+\pm-)(uuu)}\right\}\left\{\frac{1}{2\sqrt{3}}\bar{B}^u_{(\pm\pm\pm)(udd)}\frac{1}{6}(2\bar{B}^u_{(\pm\pm\mp)(udd)} + \bar{B}^u_{(\mp\pm\pm)(udd)})\right\} \\
& - \sqrt{\frac{2}{5}}\left\{\frac{1}{2\sqrt{3}}\bar{B}^u_{(\pm\pm\pm)(uud)}\frac{1}{6}(\bar{B}^u_{(\pm\pm\mp)(uud)} + 2\bar{B}^u_{(\pm\mp\pm)(uud)})\right\}\left\{\frac{1}{2\sqrt{3}}\bar{B}^u_{(\pm\pm\pm)(uud)}\frac{1}{6}(2\bar{B}^u_{(\pm\pm\mp)(uud)} + 2\bar{B}^u_{(\pm\mp\pm)(uud)})\right\} \\
& \times \sqrt{\frac{3}{10}}\left\{\frac{1}{2\sqrt{3}}\bar{B}^u_{(\pm\pm\pm)(udd)}\frac{1}{6}(2\bar{B}^u_{(\pm\pm\mp)(udd)} + \bar{B}^u_{(\mp\pm\pm)(udd)})\right\}\left\{\frac{1}{6}\bar{B}^u_{(\pm\pm\pm)(uuu)}\frac{1}{2\sqrt{3}}\bar{B}^u_{(+\pm-)(uuu)}\right\} \\
& = \sum_{\vec{\sigma}\,\vec{k}\,\vec{\beta}\,\vec{g}}\Bigg\{\sqrt{\frac{3}{10}}\left[\frac{1}{6}\delta_{\vec{k}(+++)}(\delta_{\vec{\sigma}(+++)} + \delta_{\vec{\sigma}(---)}) + \frac{1}{2\sqrt{3}}\delta_{\vec{k}(+++)}(\delta_{\vec{\sigma}(++-)} + \delta_{\vec{\sigma}(+--)})\right]\Bigg[\frac{1}{2\sqrt{3}}\delta_{\vec{g}(+--)}(\delta_{\vec{\beta}(+++)} \\
& + \delta_{\vec{\beta}(---)}) + \frac{1}{6}2(\delta_{\vec{g}(+--)}\delta_{\vec{\beta}(++-)} + \delta_{\vec{g}(-+-)}\delta_{\vec{\beta}(+--)}) + \frac{1}{6}(\delta_{\vec{g}(--+)}\delta_{\vec{\beta}(++-)} + \delta_{\vec{g}(+--)}\delta_{\vec{\beta}(+--)})\Bigg] \\
& - \sqrt{\frac{2}{5}}\Bigg[\frac{1}{2\sqrt{3}}\delta_{\vec{k}(+++)}(\delta_{\vec{\sigma}(+++)} + \delta_{\vec{\sigma}(---)}) + \frac{1}{6}(\delta_{\vec{k}(++-)}\delta_{\vec{\sigma}(++-)} + \delta_{\vec{k}(-++)}\delta_{\vec{\sigma}(+--)}) + \frac{1}{6}2(\delta_{\vec{k}(+-+)}\delta_{\vec{\sigma}(++-)} \\
& + \delta_{\vec{k}(++-)}\delta_{\vec{\sigma}(+--)})\Bigg]\Bigg[\frac{1}{2\sqrt{3}}\delta_{\vec{g}(++-)}(\delta_{\vec{\beta}(+++)} + \delta_{\vec{\beta}(---)}) + \frac{1}{6}(\delta_{\vec{g}(++-)}\delta_{\vec{\beta}(++-)} + \delta_{\vec{g}(-++)}\delta_{\vec{\beta}(+--)}) \\
& + \frac{1}{6}2(\delta_{\vec{g}(+-+)}\delta_{\vec{\beta}(++-)} + \delta_{\vec{g}(++-)}\delta_{\vec{\beta}(+--)})\sqrt{\frac{3}{10}}\Bigg[\frac{1}{2\sqrt{3}}\delta_{\vec{k}(+--)}(\delta_{\vec{\sigma}(+++)} + \delta_{\vec{\sigma}(---)}) + \frac{1}{6}2(\delta_{\vec{k}(+--)}\delta_{\vec{\sigma}(++-)} \\
& + \delta_{\vec{k}(--+)}\delta_{\vec{\sigma}(+--)}) + \frac{1}{6}(\delta_{\vec{k}(--+)}\delta_{\vec{\sigma}(++-)} + \delta_{\vec{k}(+--)}\delta_{\vec{\sigma}(+--)})\Bigg[\frac{1}{6}\delta_{\vec{g}(+++)}(\delta_{\vec{\beta}(+++)} + \delta_{\vec{\beta}(---)}) \\
& + \frac{1}{2\sqrt{3}}\delta_{\vec{g}(+++)}(\delta_{\vec{\beta}(++-)} + \delta_{\vec{\beta}(+--)})\Bigg]\Bigg]\Bigg\}\bar{B}^u_{\vec{\sigma}\,\vec{k}}\bar{B}^u_{\vec{\beta}\,\vec{g}} = \sum_{\vec{\sigma}\,\vec{k}\,\vec{\beta}\,\vec{g}} F_1^{\vec{\sigma}\,\vec{k}\,\vec{\beta}\,\vec{g}}\bar{B}^u_{\vec{\sigma}\,\vec{k}}\bar{B}^u_{\vec{\beta}\,\vec{g}}.
\end{aligned}
$$

Similar expressions hold for $\bar{\Lambda}_{2,3,4}$. From the above, suppressing the superscript $\vec{k}\,\vec{g}$, for the $F_i^{\vec{k}\,\vec{g}}$'s of Eq. (14), we get

$$F_1 = \sqrt{\frac{3}{10}}\frac{1}{6}\,\delta_{\vec{k}(+++)}\frac{1}{2\sqrt{3}}\,\delta_{\vec{g}(+--)} - \sqrt{\frac{2}{5}}\frac{1}{2\sqrt{3}}\,\delta_{\vec{k}(++-)}\frac{1}{2\sqrt{3}}\,\delta_{\vec{g}(++-)} + \sqrt{\frac{3}{10}}\frac{1}{2\sqrt{3}}\,\delta_{\vec{k}(+--)}\frac{1}{6}\,\delta_{\vec{g}(+++)} = F_4 = F_{13}, \tag{A1}$$

$$F_2 = \sqrt{\frac{3}{10}}\frac{1}{6}\,\delta_{\vec{k}(+++)}\frac{2}{6}\,\delta_{\vec{g}(+--)} + \sqrt{\frac{3}{10}}\frac{1}{6}\,\delta_{\vec{k}(+++)}\frac{1}{6}\,\delta_{\vec{g}(--+)} - \sqrt{\frac{2}{5}}\frac{1}{2\sqrt{3}}\,\delta_{\vec{k}(++-)}\frac{1}{6}\,\delta_{\vec{g}(++-)} - \sqrt{\frac{2}{5}}\frac{1}{2\sqrt{3}}\,\delta_{\vec{k}(++-)}\frac{2}{6}\,\delta_{\vec{g}(+-+)} + \sqrt{\frac{3}{10}}\frac{1}{2\sqrt{3}}\,\delta_{\vec{k}(+--)}\frac{1}{2\sqrt{3}}\,\delta_{\vec{g}(+++)} = F_{14}, \tag{A2}$$

$$F_3 = \sqrt{\frac{3}{10}}\frac{1}{6}\,\delta_{\vec{k}(+++)}\frac{2}{6}\,\delta_{\vec{g}(-+-)} + \sqrt{\frac{3}{10}}\frac{1}{6}\,\delta_{\vec{k}(+++)}\frac{1}{6}\,\delta_{\vec{g}(+--)} - \sqrt{\frac{2}{5}}\frac{1}{2\sqrt{3}}\,\delta_{\vec{k}(++-)}\frac{1}{6}\,\delta_{\vec{g}(-++)} - \sqrt{\frac{2}{5}}\frac{1}{2\sqrt{3}}\,\delta_{\vec{k}(++-)}\frac{2}{6}\,\delta_{\vec{g}(++-)} + \sqrt{\frac{3}{10}}\frac{1}{2\sqrt{3}}\,\delta_{\vec{k}(+--)}\frac{1}{2\sqrt{3}}\,\delta_{\vec{g}(+++)} = F_{15}, \tag{A3}$$

$$F_5 = \sqrt{\frac{3}{10}}\frac{1}{2\sqrt{3}}\,\delta_{\vec{k}(+++)}\frac{2}{\sqrt{3}}\,\delta_{\vec{g}(+--)} - \sqrt{\frac{2}{5}}\frac{1}{6}\,\delta_{\vec{k}(++-)}\frac{1}{2\sqrt{3}}\,\delta_{\vec{g}(++-)} + \sqrt{\frac{3}{10}}\frac{2}{6}\,\delta_{\vec{k}(+--)}\frac{1}{6}\,\delta_{\vec{g}(+++)} + \sqrt{\frac{3}{10}}\frac{1}{6}\,\delta_{\vec{k}(--+)}\frac{1}{6}\,\delta_{\vec{g}(+++)} - \sqrt{\frac{2}{5}}\frac{2}{6}\,\delta_{\vec{k}(+-+)}\frac{1}{2\sqrt{3}}\,\delta_{\vec{g}(++-)} = F_8, \tag{A4}$$

$$F_6 = \sqrt{\frac{3}{10}}\frac{1}{2\sqrt{3}}\delta_{\vec{k}(+++)}\frac{2}{6}\delta_{\vec{g}(+--)} + \sqrt{\frac{3}{10}}\frac{1}{2\sqrt{3}}\delta_{\vec{k}(+++)}\frac{1}{6}\delta_{\vec{g}(--+)} + \sqrt{\frac{2}{5}}\frac{1}{6}\delta_{\vec{k}(++-)}\frac{1}{6}\delta_{\vec{g}(++-)} - \sqrt{\frac{2}{5}}\frac{1}{6}\delta_{\vec{k}(++-)}\frac{2}{6}\delta_{\vec{g}(+-+)} - \sqrt{\frac{2}{5}}\frac{2}{6}\delta_{\vec{k}(+-+)}\frac{1}{6}\delta_{\vec{g}(++-)} - \sqrt{\frac{2}{5}}\frac{2}{6}\delta_{\vec{k}(+-+)}\delta_{\vec{g}(+-+)} + \sqrt{\frac{3}{10}}\frac{2}{6}\delta_{\vec{k}(+--)}\frac{1}{2\sqrt{3}}\delta_{\vec{g}(+++)} + \sqrt{\frac{3}{10}}\frac{1}{6}\delta_{\vec{k}(--+)}\frac{1}{2\sqrt{3}}\delta_{\vec{g}(+++)}, \tag{A5}$$

$$F_7 = \sqrt{\frac{3}{10}}\frac{1}{2\sqrt{3}}\delta_{\vec{k}(+++)}\frac{2}{6}\delta_{\vec{g}(--+)} + \sqrt{\frac{3}{10}}\frac{1}{2\sqrt{3}}\delta_{\vec{k}(+++)}\frac{1}{6}\delta_{\vec{g}(+--)} - \sqrt{\frac{2}{5}}\delta_{\vec{k}(++-)}\frac{1}{6}\delta_{\vec{g}(-++)} - \sqrt{\frac{2}{5}}\frac{1}{6}\delta_{\vec{k}(++-)}\frac{2}{6}\delta_{\vec{g}(++-)} - \sqrt{\frac{2}{5}}\frac{2}{6}\delta_{\vec{k}(+-+)}\frac{1}{6}\delta_{\vec{g}(-++)} - \sqrt{\frac{2}{5}}\frac{2}{6}\delta_{\vec{k}(+-+)}\frac{2}{6}\delta_{\vec{g}(++-)} + \sqrt{\frac{3}{10}}\frac{2}{6}\delta_{\vec{k}(+--)}\frac{1}{2\sqrt{3}}\delta_{\vec{g}(+++)} + \sqrt{\frac{3}{10}}\frac{1}{6}\delta_{\vec{k}(--+)}\frac{1}{2\sqrt{3}}\delta_{\vec{g}(+++)}, \tag{A6}$$

$$F_9 = \sqrt{\frac{3}{10}}\frac{1}{2\sqrt{3}}\,\delta_{\vec{k}(+++)}\frac{2}{\sqrt{3}}\,\delta_{\vec{g}(+--)} - \sqrt{\frac{2}{5}}\frac{1}{6}\,\delta_{\vec{k}(-++)}\frac{1}{2\sqrt{3}}\,\delta_{\vec{g}(++-)} - \sqrt{\frac{2}{5}}\frac{2}{6}\,\delta_{\vec{k}(++-)}\frac{1}{2\sqrt{3}}\,\delta_{\vec{g}(++-)} + \sqrt{\frac{3}{10}}\frac{2}{6}\,\delta_{\vec{k}(--+)}\frac{1}{6}\,\delta_{\vec{g}(+++)} + \sqrt{\frac{3}{10}}\frac{1}{6}\,\delta_{\vec{k}(+--)}\frac{1}{6}\,\delta_{\vec{g}(+++)} = F_{12}, \tag{A7}$$

$$F_{10} = \sqrt{\frac{3}{10}}\frac{1}{2\sqrt{3}}\,\delta_{\vec{k}(+++)}\frac{2}{6}\,\delta_{\vec{g}(+--)} + \sqrt{\frac{3}{10}}\frac{1}{2\sqrt{3}}\,\delta_{\vec{k}(+++)}\frac{1}{6}\,\delta_{\vec{g}(--+)} - \sqrt{\frac{2}{5}}\frac{1}{6}\,\delta_{\vec{k}(-++)}\frac{1}{6}\,\delta_{\vec{g}(++-)} - \sqrt{\frac{2}{5}}\frac{1}{6}\,\delta_{\vec{k}(-++)}\frac{2}{6}\,\delta_{\vec{g}(+-+)} - \sqrt{\frac{2}{5}}\frac{2}{6}\,\delta_{\vec{k}(++-)}\frac{1}{6}\,\delta_{\vec{g}(++-)} - \sqrt{\frac{2}{5}}\frac{2}{6}\,\delta_{\vec{k}(++-)}\frac{2}{6}\,\delta_{\vec{g}(+-+)} + \sqrt{\frac{3}{10}}\frac{2}{6}\,\delta_{\vec{k}(--+)}\frac{1}{2\sqrt{3}}\,\delta_{\vec{g}(+++)} + \sqrt{\frac{3}{10}}\frac{1}{6}\,\delta_{\vec{k}(+--)}\frac{1}{2\sqrt{3}}\,\delta_{\vec{g}(+++)}, \tag{A8}$$

$$F_{11} = \sqrt{\frac{3}{10}}\frac{1}{2\sqrt{3}}\delta_{\vec{k}(+++)}\frac{2}{\sqrt{3}}\delta_{\vec{g}(+--)} - \sqrt{\frac{2}{5}}\frac{1}{6}\delta_{\vec{k}(++-)}\frac{1}{2\sqrt{3}}\delta_{\vec{g}(++-)} + \sqrt{\frac{3}{10}}\frac{2}{6}\delta_{\vec{k}(+--)}\frac{1}{6}\delta_{\vec{g}(+++)} + \sqrt{\frac{3}{10}}\frac{1}{6}\delta_{\vec{k}(--+)}\frac{1}{6}\delta_{\vec{g}(+++)} - \sqrt{\frac{2}{5}}\frac{2}{6}\delta_{\vec{k}(+-+)}\frac{1}{2\sqrt{3}}\delta_{\vec{g}(++-)}, \tag{A9}$$

$$\begin{aligned} F_{16} &= \sqrt{\frac{3}{10}}\frac{1}{6}\delta_{\vec{k}(+++)}\frac{2}{\sqrt{3}}\delta_{\vec{g}(+--)} - \sqrt{\frac{2}{5}}\frac{1}{2\sqrt{3}}\delta_{\vec{k}(++-)}\frac{1}{2\sqrt{3}}\delta_{\vec{g}(++-)} + \sqrt{\frac{3}{10}}\frac{1}{2\sqrt{3}}\delta_{\vec{k}(+--)}\frac{1}{6}\delta_{\vec{g}(+++)}, \\ F_{17} &= \sqrt{\frac{3}{4}}\frac{1}{6}\delta_{\vec{k}(+++)}\frac{1}{3\sqrt{2}}\delta_{\vec{g}(+--)} - \sqrt{\frac{1}{4}}\frac{1}{2\sqrt{3}}\delta_{\vec{k}(++-)}\frac{1}{3\sqrt{2}}\delta_{\vec{g}(+-+)} + \sqrt{\frac{3}{4}}\frac{1}{6}\delta_{\vec{k}(+++)}\Big(-\frac{1}{3\sqrt{2}}\Big)\delta_{\vec{g}(-+-)} \\ &\quad - \sqrt{\frac{1}{4}}\frac{1}{2\sqrt{3}}\times\delta_{\vec{k}(++-)}\Big(-\frac{1}{3\sqrt{2}}\Big)\delta_{\vec{g}(-++)} = F_{18} = F_{23} = F_{24}, \end{aligned} \tag{A10}$$

$$\begin{aligned} F_{19} &= \sqrt{\frac{3}{4}}\frac{1}{2\sqrt{3}}\delta_{\vec{k}(+++)}\frac{1}{3\sqrt{2}}\delta_{\vec{g}(+--)} + \sqrt{\frac{3}{4}}\frac{1}{2\sqrt{3}}\delta_{\vec{k}(+++)}\left(-\frac{1}{3\sqrt{2}}\right)\delta_{\vec{g}(-+-)} - \sqrt{\frac{1}{4}}\frac{1}{6}\delta_{\vec{k}(++-)}\frac{1}{3\sqrt{2}}\delta_{\vec{g}(+-+)} \\ &\quad - \sqrt{\frac{1}{4}}\frac{1}{6}\delta_{\vec{k}(++-)}\left(-\frac{1}{3\sqrt{2}}\right)\delta_{\vec{g}(-++)} - \sqrt{\frac{1}{4}}\frac{2}{6}\delta_{\vec{k}(+-+)}\frac{1}{3\sqrt{2}}\delta_{\vec{g}(+-+)} - \sqrt{\frac{1}{4}}\frac{2}{6}\delta_{\vec{k}(+-+)}\left(-\frac{1}{3\sqrt{2}}\right)\delta_{\vec{g}(-++)}, \end{aligned} \tag{A11}$$

$$\begin{aligned} F_{21} &= \sqrt{\frac{3}{4}}\frac{1}{2\sqrt{3}}\delta_{\vec{k}(+++)}\left[\frac{1}{3\sqrt{2}}(\delta_{\vec{g}(+--)} - \delta_{\vec{g}(-+-)})\right] - \sqrt{\frac{1}{4}}\frac{1}{6}\delta_{\vec{k}(-++)}\left[\frac{1}{3\sqrt{2}}(\delta_{\vec{g}(+-+)} - \delta_{\vec{g}(-++)})\right] \\ &\quad - \sqrt{\frac{1}{4}}\frac{2}{6}\delta_{\vec{k}(++-)}\left[\frac{1}{3\sqrt{2}}(\delta_{\vec{g}(+-+)} - \delta_{\vec{g}(-++)})\right] = F_{22}, \end{aligned} \tag{A12}$$

$$\begin{aligned} F_{25} &= \frac{1}{3\sqrt{2}}\delta_{\vec{k}(+-+)}\frac{1}{3\sqrt{2}}\delta_{\vec{g}(+-+)} + \frac{1}{3\sqrt{2}}\delta_{\vec{k}(+-+)}\left(-\frac{1}{3\sqrt{2}}\right)\delta_{\vec{g}(-++)} + \left(-\frac{1}{3\sqrt{2}}\right)\delta_{\vec{k}(-++)}\frac{1}{3\sqrt{2}}\delta_{\vec{g}(+-+)} \\ &\quad + \left(-\frac{1}{3\sqrt{2}}\right)\times\delta_{\vec{k}(-++)}\left(-\frac{1}{3\sqrt{2}}\right)\delta_{\vec{g}(-++)} = F_{26} = F_{27} = F_{28}, \end{aligned} \tag{A13}$$

and for $i = 29, 30, \ldots, 36$, $F_i^{\vec{k}\,\vec{g}} = F_{i-12}^{\vec{g}\,\vec{k}}$, where we note the $\vec{k}\,\vec{g} \to \vec{g}\,\vec{k}$ replacement on the right-hand side. For row values of $F_i^{\vec{\rho}\,\vec{h}\,\vec{\alpha}\,\vec{f}}$, one can make the replacement $\vec{\sigma}\,\vec{k}\,\vec{\beta}\,\vec{g} \to \vec{\rho}\,\vec{h}\,\vec{\alpha}\,\vec{f}$ in the above.

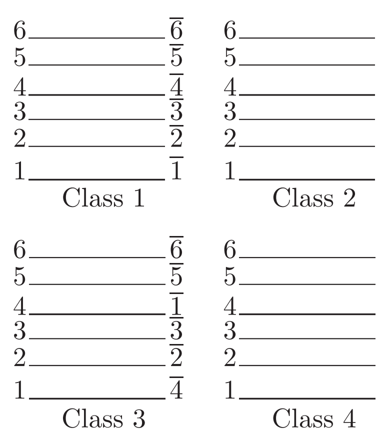
APPENDIX B: PERMANENT REPRESENTATION FOR THE COINCIDENT FOUR-POINT FUNCTION

In this appendix, we derive the permanent representation of Eq. (21), for the coincident-point correlation of four unnormalized baryon fields. We also give the $\tilde{\Lambda}$ field coincident-point matrix $D^{(0)}(0) = FCF$ of Sec. V explicitly.

As seen, the coincident-point correlation is given by

$$C_{\vec{h}\,\vec{f}}^{\vec{k}\,\vec{g}} = \langle \epsilon_{\vec{a}}\psi^3_{\vec{a}\,\vec{h}}\epsilon_{\vec{b}}\psi^3_{\vec{b}\,\vec{f}}\epsilon_{\vec{c}}\bar{\psi}^3_{\vec{c}\,\vec{k}}\epsilon_{\vec{d}}\bar{\psi}^3_{\vec{d}\,\vec{g}}\rangle,$$

where we order the $(\tilde{\psi})$ fields as $1\,2\,3\,4\,5\,6\,\bar{1}\,\bar{2}\,\bar{3}\,\bar{4}\,\bar{5}\,\bar{6}$, and for now we set all spins equal. The above is given by a sum over all the 6! pairings *connecting* $(1, 2, \ldots, 6)$ and $(\bar{1}, \bar{2}, \ldots, \bar{6})$. Considering the two triples of fields, 123 and $\bar{4}\,\bar{5}\,\bar{6}$, we classify the possible pairings according to the number of *crossovers* or contractions between fields in these triples: zero, one, two, and three. Representations of these four classes are depicted below.



We now obtain a representation in terms of permanents for each class.

Class 1: We get

$$\langle 123456\bar{1}\,\bar{2}\,\bar{3}\,\bar{4}\,\bar{5}\,\bar{6}\rangle = -\langle 123\bar{1}\,\bar{2}\,\bar{3}\rangle\langle 456\bar{4}\,\bar{5}\,\bar{6}\rangle.$$

Summing over all contractions gives

$$\begin{aligned} C^{\vec{k}\,\vec{g}}_{1\vec{h}\,\vec{f}} &= -\langle \epsilon_{\vec{a}}\psi^3_{\vec{a}\,\vec{h}}\epsilon_{\vec{c}}\bar{\psi}^3_{\vec{c}\,\vec{k}}\rangle\langle \epsilon_{\vec{b}}\psi^3_{\vec{b}\,\vec{f}}\epsilon_{\vec{d}}\bar{\psi}^3_{\vec{d}\,\vec{g}}\rangle \\ &= -36\,\mathrm{perm}_3(\delta_{\vec{f}\,\vec{k}})\,\mathrm{perm}_3(\delta_{\vec{h}\,\vec{g}}), \end{aligned}$$

where perm_n here is of an $n\times n$ matrix.

Class 2: Here, we obtain

$$\langle 123456\bar{1}\,\bar{2}\,\bar{3}\,\bar{4}\,\bar{5}\,\bar{6}\rangle = \langle 123\bar{4}\,\bar{5}\,\bar{6}\rangle\langle 456\bar{1}\,\bar{2}\,\bar{3}\rangle.$$

Summing over all contractions gives

$$\begin{aligned} C^{\vec{k}\,\vec{g}}_{2\vec{h}\,\vec{f}} &= \langle \epsilon_{\vec{a}}\psi^3_{\vec{a}\,\vec{h}}\epsilon_{\vec{d}}\bar{\psi}^3_{\vec{d}\,\vec{g}}\rangle\langle \epsilon_{\vec{b}}\psi^3_{\vec{b}\,\vec{f}}\epsilon_{\vec{c}}\bar{\psi}^3_{\vec{c}\,\vec{k}}\rangle \\ &= 36\,\mathrm{perm}_3(\delta_{\vec{h}\,\vec{g}})\,\mathrm{perm}_3(\delta_{\vec{p}\,\vec{k}}) = -C^{\vec{g}\,\vec{k}}_{1\vec{h}\,\vec{f}}. \end{aligned}$$

Class 3: Computing $\langle 123456\bar{1}\,\bar{2}\,\bar{3}\,\bar{4}\,\bar{5}\,\bar{6}\rangle$, with the $1\bar{4}$, $4\bar{1}$ pairings, we have

$$\begin{aligned} C^{\vec{k}\,\vec{g}}_{3\vec{h}\,\vec{f}} &= \langle 1\bar{4}\rangle\langle 4\bar{1}\rangle\langle 2356\bar{2}\,\bar{3}\,\bar{5}\,\bar{6}\rangle = \epsilon_{\vec{a}}\epsilon_{\vec{b}}\epsilon_{\vec{c}}\epsilon_{\vec{d}}\delta_{a_1d_1}\delta_{h_1g_1}\delta_{b_1c_1} \\ &\quad\times\delta_{f_1k_1}\langle\psi^2_{a_2a_3h_2h_3}\bar{\psi}^2_{c_2c_3k_2k_3}\rangle\langle\psi^2_{b_2b_3f_2f_3}\bar{\psi}^2_{d_2d_3g_2g_3}\rangle \\ &= \epsilon_{\vec{a}}\epsilon_{\vec{b}}\epsilon_{\vec{c}}\epsilon_{\vec{d}}\delta_{a_1d_1}\delta_{h_1g_1}\delta_{b_1c_1}\delta_{f_1k_1}\mathrm{det}_2\delta_{\vec{a}'\vec{c}'}\delta_{\vec{h}'\vec{k}'} \\ &\quad\times\mathrm{det}_2\delta_{\vec{b}'\vec{d}'}\delta_{\vec{f}'\vec{g}'}, \end{aligned}$$

where the superscript $i=1$ means we omit the ith component. Carrying out the sums on a_2, a_3, b_2, b_3 using $\epsilon_{a_1a_2a_3}\epsilon_{b_1a_2a_3}=2\delta_{a_1b_1}$, etc., and the sum over a_1b_1 of $2\delta_{a_1b_1}2\delta_{a_1b_1}$, gives 12. Thus we get, for the graph depicted above,

$$\begin{aligned} &12\delta_{h_1g_1}\delta_{f_1k_1}\Big[\delta_{h_2k_2}\delta_{h_3k_3}\delta_{f_2g_2}\delta_{f_3g_3}+\delta_{h_2k_2}\delta_{h_3k_3}\delta_{f_2g_3}\delta_{f_3g_2} \\ &\quad+\delta_{h_2k_3}\delta_{h_3k_2}\delta_{f_2g_2}\delta_{f_3g_3}+\delta_{h_2k_3}\delta_{h_3k_2}\delta_{f_2g_3}\delta_{f_3g_2}\Big] \\ &\quad=12\,\mathrm{perm}_2(\delta_{\vec{h}^1\vec{k}^1})\,\mathrm{perm}_2(\delta_{\vec{f}^1\vec{g}^1})\delta_{h_1g_1}\delta_{f_1k_1}. \end{aligned}$$

Instead of the above crossover, more generally, we have the crossover for Class 3, given by the contractions h_i (f_j) with g_m (k_ℓ) and i, j, ℓ, $m=1,2,3$. Thus, summing over all possibilities, we get

$$\sum_{ij\ell m}12\delta_{h_ig_m}\delta_{f_jk_\ell}\,\mathrm{perm}_2(\delta_{\vec{h}^i\vec{k}^\ell})\,\mathrm{perm}_2(\delta_{\vec{f}^j\vec{g}^m}).$$

Now, fix i, j and sum on ℓ, m. Also, make use of the Laplace expansion for the three-dimensional permanent in terms of the perm_2's. Consider the first perm_2 factor. We have, for $\ell=1$, $\delta_{f_jk_1}\mathrm{perm}_2(\delta_{\vec{h}^i}(k_2k_3))$; for $\ell=2$, $\delta_{f_jk_2}\mathrm{perm}_2(\delta_{\vec{h}^i}(k_1k_3))$; and for $\ell=3$, $\delta_{f_jk_3}\mathrm{perm}_2(\delta_{\vec{h}^i}(k_1k_2))$. Summing over ℓ gives, for $i=1$,

$$\mathrm{perm}\begin{pmatrix}\delta_{f_jk_1} & \delta_{f_jk_2} & \delta_{f_jk_3}\\ \delta_{h_2k_1} & \delta_{h_2k_2} & \delta_{h_2k_3}\\ \delta_{h_3k_1} & \delta_{h_3k_2} & \delta_{h_3k_3}\end{pmatrix};$$

for $i=2$, $\mathrm{perm}(\delta_{(h_1f_jh_3)\vec{k}})$; and, for $i=3$, $\mathrm{perm}(\delta_{(h_1h_2f_j)\vec{k}})$.

For the second perm_2 factor we have, for $m=1$, $\delta_{h_ig_1}\,\mathrm{perm}_2(\delta_{\vec{f}^j}(g_2g_3))$; for $m=2$, $\delta_{h_ig_2}\,\mathrm{perm}_2(\delta_{\vec{f}^j}(g_1g_3))$; and for $m=3$, $\delta_{h_ig_3}\,\mathrm{perm}_2(\delta_{\vec{f}^j}(g_1g_2))$. Summing over m, the second factor becomes, for $j=1$, $\delta_{h_ig_1}\,\mathrm{perm}(\delta_{(h_if_2f_3)\vec{g}})$; for $j=2$, $\delta_{h_ig_2}\,\mathrm{perm}(\delta_{(f_1h_if_3)\vec{g}})$; and for $j=3$, $\delta_{h_ig_3}\,\mathrm{perm}(\delta_{(f_1f_2khi)\vec{g}})$. Summing the product of the factors over i, j gives the third term of Eq. (21), which is the final formula for Class 3.

Class 4: $C^{\vec{k}\,\vec{g}}_{4\vec{h}\,\vec{f}}=-C^{\vec{g}\,\vec{k}}_{3\vec{h}\,\vec{f}}$, which is the interchange $\vec{k}$ and $\vec{g}$ in Class 3 with the addition of a minus sign.

Altogether, we get the representation

$$C^{\vec{k}\,\vec{g}}_{\vec{h}\,\vec{f}} = C^{\vec{k}\,\vec{g}}_{1\vec{h}\,\vec{f}} - C^{\vec{g}\,\vec{k}}_{1\vec{h}\,\vec{f}} + C^{\vec{k}\,\vec{g}}_{3\vec{h}\,\vec{f}} - C^{\vec{g}\,\vec{k}}_{3\vec{h}\,\vec{f}}.$$

Recall that the above calculation was done for a fixed spin. To include spin (lower indices only), we denote the coincident-point function by $C^{\vec{\sigma}\,\vec{k}\,\vec{\beta}\,\vec{g}}_{\vec{\rho}\,\vec{h}\,\vec{\alpha}\,\vec{f}}$, where the composite fields are now

$$\epsilon_{\vec{a}}\psi^3_{\vec{a}\,\vec{\rho}\,\vec{h}},\ \epsilon_{\vec{b}}\psi^3_{\vec{b}\,\vec{\alpha}\,\vec{f}},\ \epsilon_{\vec{c}}\bar{\psi}^3_{\vec{c}\,\vec{\sigma}\,\vec{k}},\ \epsilon_{\vec{d}}\bar{\psi}^3_{\vec{d}\,\vec{\beta}\,\vec{g}}$$

and the formula above holds upon including an extra δ spin function with the δ isospin function, i.e. $\delta_{(f_jh_2h_3)\vec{k}}\longrightarrow\delta_{(f_jh_2h_3)\vec{k}}\delta_{(\alpha_j\rho_2\rho_3)\vec{\sigma}}$.

Using the Laplace expansion, $C^{\vec{k}\,\vec{g}}_{\vec{h}\,\vec{f}}$ can be written as

$$C^{\vec{k}\,\vec{g}}_{\vec{h}\,\vec{f}} = \epsilon_{\vec{a}}\epsilon_{\vec{b}}\epsilon_{\vec{c}}\epsilon_{\vec{d}}\left[\langle 123456\bar{1}\,\bar{2}\,\bar{3}\,\bar{4}\,\bar{5}\,\bar{6}\rangle = (-1)\det\begin{pmatrix}\delta_{1\bar{1}} & \delta_{1\bar{2}} & \dots & \delta_{1\bar{6}}\\ \delta_{2\bar{1}} & & & \\ \vdots & & A_6 & \vdots\\ \delta_{6\bar{1}} & & & \delta_{6\bar{6}}\end{pmatrix}\right],$$

making the proper identifications of $n\bar{m}$ in $\delta_{n\bar{m}}$. For the prefactors of det_mA_m in $\langle 12\ldots m\bar{1}\,\bar{2}\ldots\bar{m}\rangle = s_m\mathrm{det}_mA_m$, we find $s_m=-1$, $m=4n-2, 4n-1$, $n=1,2,\ldots$; and $s_m=1$, $m=4n, 4n+1$, $n=0,1,\ldots$.

We now give the coincident-point matrix for the $\tilde{\Lambda}$ fields. It has the same structure as for the exchange potentials of Appendix D. Each 36×36 symmetric matrix has the upper triangular block structure $[V_{11}, V_{12}, V_{13}, V_{14}]$, $[V_{22}, V_{23}, V_{24}]$, $[V_{33}, V_{34}]$, $[V_{44}]$, and each V_{ij} sub-block has the 16 part structure $[A, C, N, R]$, $[M, B, D, P]$, $[S, T, H, J]$, $[U, X, L, K]$. We have the following:

$V_{11}(16 \times 16)$: $A = K^f$, $B = H^f$, $C = J^f$, $N = P^f$, D, R, $A = \mathrm{diag}(0, -10/9, -10/9, -1)$, $B = \mathrm{diag}(-10/9, 0, -1, -10/9)$, $C = [0_4], [10/9, 0_3], [0_4], [0_2, -1/9, 0]$, $N = [0_4], [0_4], [10/9, 0_3], [0, 1/9, 0_2]$, $D = [0_4], [0_4], [0, 1, 0_2], [0_4]$, $R = [0_4], [0_4], [0_4], [1, 0_3]$.

$V_{22}(8 \times 8)$: $A = K^f$, $B = H^f$, $C = J = J^f$, $N = P = O = R$, D, $A = \mathrm{diag}(-8/9, -2/9)$, $B = \mathrm{diag}(-2/3, -4/9)$, $C = [0_2], [2\sqrt{3}/9, 0]$, $D = [0_2], [-4/9, 0]$.

$V_{33}(4 \times 4)$: $A = K = 0$, $B = H$, D, $b = -10/9$, $D = 10/9$.

$V_{12}(16 \times 8)$: $A = -K^f$, $B = -H^f$, C, J, D, $N = P$, $S = -K^f$, $L = -C^f$, $N = -X^f$, $M = -J^f$, $T = -D^f$, $A = [0_2]$, $[4\sqrt{5}/9, 0]$, $[0, 2\sqrt{5}/9]$, $[0_2]$, $B = [0_2]$, $[0_2]$, $[0, 2\sqrt{5}/9]$, $[0_2]$, $C = [0_2], [0_2], [2\sqrt{15}/9, 0], [0, 2\sqrt{5}/9]$, $N = [0_2]$, $[0_2]$, $[0_2]$, $[2\sqrt{5}/9, 0]$, $D = [0_2]$, $[0_2]$, $[2\sqrt{5}/9, 0], [0, 2\sqrt{15}/9]$.

$V_{13}(16 \times 4)$: $A = K = 0, B = H^f, C, D, N = -C, T = D^f$, $X = N^f$, $L = C^f$, $B = [0]$, $[0]$, $[-2\sqrt{10}/9, 0]$, $[0]$ $C = [0], [0], [0], [2\sqrt{10}/9]$, $D = [0], [0], [2\sqrt{10}/9], [0]$.

$V_{23}(8 \times 4)$: 0.

$V_{14}(16 \times 8) = -V_{12}$, $V_{24} = -V_{22}$, $V_{34} = -V_{32}$, $V_{44} = V_{22}$.

The zero rows and columns are 1, 6, 11, 16, 25, 28. The invariant blocks are 2 5 17 29, 3 9 18 19 30 31, 4 7 10 13 20 21 26 27 32 33, 8 14 22 23 34 35, 12 15 24 36.

The spectrum (multiplicity) is $-4\,(6)$, $0\,(30)$.

APPENDIX C: *C*, *D*, AND D_0 CORRELATIONS FOR THE $\tilde{\Omega}$ FIELD

We now determine the coincident-point null space and the space in which D and D_0 act. For the $\tilde{\Omega}$ fields, and coincident points, we have

$$\langle \Omega_1 \bar{\Omega}_4 \rangle = \sqrt{2} \langle \Lambda_1 \bar{\Lambda}_2 \rangle, \qquad \langle \Omega_3 \bar{\Omega}_4 \rangle = \sqrt{2} \langle \Lambda_3 \bar{\Lambda}_2 \rangle,$$
$$\langle \Omega_4 \bar{\Omega}_4 \rangle = 2 \langle \Lambda_2 \bar{\Lambda}_2 \rangle,$$

and $\langle \Omega_2 \bar{\Omega}_j \rangle = 0$, since $\tilde{\Omega}_2 = 0$. The $\langle \Omega_i \bar{\Omega}_j \rangle$ matrix has the block structure

$$\begin{pmatrix} \langle \Lambda_1 \bar{\Lambda}_1 \rangle & 0 & \langle \Lambda_1 \bar{\Lambda}_3 \rangle & \sqrt{2} \langle \Lambda_1 \bar{\Lambda}_2 \rangle \\ 0 & 0 & 0 & 0 \\ * & 0 & \langle \Lambda_3 \bar{\Lambda}_3 \rangle & \sqrt{2} \langle \Lambda_3 \bar{\Lambda}_2 \rangle \\ * & 0 & * & 2 \langle \Lambda_2 \bar{\Lambda}_2 \rangle \end{pmatrix}. \tag{C1}$$

From the calculation of the matrix $\langle \Lambda_i \bar{\Lambda}_j \rangle^{(0)}$ in Appendix B, we have the invariant block matrix for $\langle \Omega \bar{\Omega} \rangle^{(0)}$:

$$4\,7\,10\,13\,26\,27\,32\,33\colon \begin{pmatrix} -1 & -\frac{1}{9} & \frac{1}{9} & 1 & \frac{2\sqrt{10}}{9} & -\frac{2\sqrt{10}}{9} & \frac{2\sqrt{10}}{9} & \frac{2\sqrt{10}}{9} \\ * & -1 & 1 & \frac{1}{9} & -\frac{2\sqrt{10}}{9} & \frac{2\sqrt{10}}{9} & \frac{2\sqrt{10}}{9} & \frac{2\sqrt{10}}{9} \\ * & * & -1 & -\frac{1}{9} & \frac{2\sqrt{10}}{9} & -\frac{2\sqrt{10}}{9} & -\frac{2\sqrt{10}}{9} & -\frac{2\sqrt{10}}{9} \\ * & * & * & -1 & -\frac{2\sqrt{10}}{9} & \frac{2\sqrt{10}}{9} & -\frac{2\sqrt{10}}{9} & -\frac{2\sqrt{10}}{9} \\ * & * & * & * & -\frac{10}{9} & \frac{10}{9} & 0 & 0 \\ * & * & * & * & * & -\frac{10}{9} & 0 & 0 \\ * & * & * & * & * & * & -\frac{8}{9} & -\frac{8}{9} \\ * & * & * & * & * & * & * & -\frac{8}{9} \end{pmatrix},$$

with the spectrum $-4\,(2)$, $0\,(6)$. Two orthogonal -4 eigenvalue eigenvectors are, with $J_z = 0$,

$$v_{-4}^{(1)} = (\sqrt{10}, \sqrt{10}, -\sqrt{10}, -\sqrt{10}, 0, 0, -4, -4),$$
$$v_{-4}^{(2)} = (-\sqrt{10}, \sqrt{10}, -\sqrt{10}, \sqrt{10}, 5, -5, 0, 0).$$

Of course, any nonzero vector with zero components except for 17 through 24 is an eigenvalue zero eigenvector. Another invariant block is

$$2\,5\,29\colon \begin{pmatrix} -\frac{10}{9} & \frac{10}{9} & 4\frac{\sqrt{10}}{9} \\ * & -\frac{10}{9} & -4\frac{\sqrt{10}}{9} \\ * & * & -\frac{16}{9} \end{pmatrix},$$

with the spectrum $0\,(2)$, $-4\,(1)$ and -4 eigenvalue eigenvector, with $J_z = 2$,

$$v_{-4} = (\sqrt{10}, -\sqrt{10}, -4).$$

Yet, another invariant block is

$$3\,9\,30\,31\colon \begin{pmatrix} -\frac{10}{9} & \frac{10}{9} & 2\frac{\sqrt{10}}{9} & 2\frac{\sqrt{30}}{9} \\ * & -\frac{10}{9} & -\frac{2}{9}\sqrt{10} & -\frac{2}{9}\sqrt{30} \\ * & * & -\frac{4}{9} & -\frac{4}{9}\sqrt{3} \\ * & * & * & -\frac{4}{3} \end{pmatrix},$$

with the spectrum $0\,(3)$, $-4\,(1)$ and -4 eigenvalue eigenvector, with $J_z = 1$,

$$v_{-4} = (\sqrt{30}, -\sqrt{30}, -2\sqrt{3}, -6).$$

There are two more invariant sub-blocks related by spin flip to the last two above, namely, 12 15 36, with $J_z = -2$ and 8 14 34 35, with $J_z = -1$. They are, respectively,

$$\begin{pmatrix} -\frac{10}{9} & \frac{10}{9} & \frac{4}{9}\sqrt{10} \\ * & -\frac{10}{9} & -\frac{4}{9}\sqrt{10} \\ * & * & -\frac{16}{9} \end{pmatrix}$$

and

$$\begin{pmatrix} -\frac{10}{9} & \frac{10}{9} & \frac{2}{9}\sqrt{30} & \frac{2}{9}\sqrt{10} \\ * & -\frac{10}{9} & -\frac{2}{9}\sqrt{30} & -\frac{2}{9}\sqrt{10} \\ * & * & -\frac{4}{3} & -\frac{4}{9}\sqrt{3} \\ * & * & * & -\frac{4}{9} \end{pmatrix}.$$

The last sub-block is related to the 3 9 30 31 invariant block by a unitary transformation generated by

$$U = \begin{pmatrix} I_2 & 0 \\ 0 & \sigma_x \end{pmatrix},$$

so it has the same spectrum.

Summarizing, $D^{(0)}(0)$ has the spectrum $-4\,(6)$, $0\,(30)$, and we denote the orthogonal projection on the null space (its complement) by $P(Q)$.

Next, we briefly discuss the origin of the null space. By Pauli exclusion in the composite fields, the rows and columns 1, 6, 11, 16, 25, 28 are zero. In these cases, $\vec{\sigma} = \vec{\beta} = (+++), (++-), (+--), (---)$ in the $\Delta - \Delta$ fields and $\vec{\sigma} = \vec{\beta} = (+-+), (+--)$ in the $p - p$ fields. However, there are other nontrivial pointwise linear relations between the Λ or $\bar{\Lambda}$ fields which give additional contributions to the null space. These relations are suggested by the null space eigenvector equation

$$D^{(0)}_{t_1t_2t_3t_4}(0) w_{t_3t_4} = 0,$$

where $w_{t_3t_4}$ are the components of the null space eigenvector w. The suggested pointwise relations in the Grassmann algebra are $\bar{\Lambda}_{t_3t_4} w_{t_3t_4} = 0$, and indeed these relations are verified.

We now consider noncoincident points. At $\kappa = \beta = 0$, D and D_0 coincide for noncoincident points. We find, for all $x_1x_2x_3x_4$,

$$\begin{aligned} D_{024t_1s_2t_3s_4}(x_1x_2x_3x_4) = \tfrac{1}{2}[& \langle \Lambda_{2t_1s_2}(x_1x_2)\Lambda_{2t_3s_4}(x_3x_4)\rangle_w \\ & \times \langle \Lambda_{2t_1s_2}(x_1x_2)\Lambda_{2t_3s_4}(x_4x_3)\rangle_w \\ & - \langle \Lambda_{2t_1s_2}(x_1x_2)\Lambda_{2t_3s_4}(x_3x_4)\rangle_w \\ & - \langle \Lambda_{2t_1s_2}(x_2x_1)\Lambda_{2t_3s_4}(x_4x_3)\rangle_w], \end{aligned}$$

where the subscript w means Wickified. At $\kappa = \beta = 0$, and for the 24 sub-block,

$$\begin{aligned} D_{024t_1s_2t_3s_4}(x_1x_2x_3x_4)^{(0)} = \tfrac{1}{2}[& -\delta_{t_1t_3}\delta_{s_2s_4}\delta(x_1-x_3)\delta(x_2-x_4) + (-1)\delta_{t_1t_3}\delta_{s_2s_4}\delta(x_1-x_4)\delta(x_2-x_3) \\ & - (-1)\delta_{t_1t_3}\delta_{s_2s_4}\delta(x_2-x_3)\delta(x_1-x_4) - (-1)\delta_{t_1t_3}\delta_{s_2s_4}\delta(x_2-x_4)\delta(x_1-x_3)] = 0. \end{aligned}$$

Similarly, we get

$$\begin{aligned} D_{022t_1s_2t_3s_4}(x_1x_2x_3x_4)^{(0)} &= -\delta_{t_1t_3}\delta_{s_2s_4}\delta(x_1-x_3)\delta(x_2-x_4) + \delta_{t_1t_3}\delta_{s_2s_4}\delta(x_1-x_4)\delta(x_2-x_3), \\ D_{033s_1s_2s_3s_4}(x_1x_2x_3x_4)^{(0)} &= -\delta_{s_1s_3}\delta_{s_2s_4}\delta(x_1-x_3)\delta(x_2-x_4) + \delta_{s_1s_4}\delta_{s_2s_3}\delta(x_1-x_4)\delta(x_2-x_3), \\ D_{044t_1s_2t_3s_4}(x_1x_2x_3x_4)^{(0)} &= -\delta_{t_1t_3}\delta_{s_2s_4}\delta(x_1-x_3)\delta(x_2-x_4) - \delta_{t_1t_3}\delta_{s_2s_4}\delta(x_1-x_4)\delta(x_2-x_3), \end{aligned}$$

and the same for D_{011}. The other sub-blocks, not related by the symmetry of $D_0^{(0)}$ to the sub-blocks above, are zero.

From the above considerations, we take D and D_0 to be defined on the space $\ell_2' = \ell_2^c + \ell_2^n$, where $\ell_2^{c(n)}$ is the coincident (noncoincident) point space. For coincident points, we take $\ell_2^c(A_c)$, where

$$A_c = \{(x_1x_2j) \in \mathbb{Z}^4 \times \mathbb{Z}^4 \times (1, 2, \ldots 6) | x_1^0 = x_2^0, \vec{x}_1 = \vec{x}_2\},$$

and j labels the six-dimensional basis of $R(Q)$. For noncoincident points, we decompose ℓ_2^n as

$$\ell_2^n = \ell_2^a(A_1^n) + \ell_2^a(A_2^n) + \ell_2^a(A_3^n) + \ell_2^s(A_4^n).$$

Let $\mathbb{N}_p$ denote the set of integers $1, 2, \ldots, p$. The A_j^n are defined as follows:

$$A_1^n = \{(x_1r_1, x_2r_2) \in (\mathbb{Z}^4 \times \mathbb{N}_4) \times (\mathbb{Z}^4 \times \mathbb{N}_4) | x_1^0 = x_2^0, \vec{x}_1 \neq \vec{x}_2\},$$

and we identify $r_{i(1,2,,3,4)}$ with the spin triples $(+++)$, $(++-)$, $(+--)$, $(---)$, respectively, of the delta fields. Similarly,

$$A_2^n = \{(x_1, x_2, j) \in \mathbb{Z}^4 \times \mathbb{Z}^4 \times (17, \ldots 24) | x_1^0 = x_2^0, \vec{x}_1 \neq \vec{x}_2\},$$

with $(17, \ldots, 24)$ corresponding to the ordered spin basis for Λ_2:

$$A_3^n = \{(x_1r_1, x_2r_2) \in (\mathbb{Z}^4 \times \mathbb{N}_2) \times (\mathbb{Z}^4 \times \mathbb{N}_2) | x_1^0 = x_2^0, \vec{x}_1 \neq \vec{x}_2\},$$

and we identify $r_{i(1,2)}$ with the spin triples $(+-+)$, $(+--)$, respectively, of the proton fields. Last, we have

$$A_4^n = \{(x_1, x_2, j) \in \mathbb{Z}^4 \times \mathbb{Z}^4 \times (29, 30, \ldots 36) | x_1^0 = x_2^0, \vec{x}_1 \neq \vec{x}_2\},$$

with $(29, \ldots, 36)$ corresponding to the ordered spin basis of Λ_2.

The superscript a (s) above means the restriction to the antisymmetric (symmetric) subspace. On $\ell_2^{\prime}$, D and D_0 are well defined using their decay bounds. Also, $(D^{(0)})^{-1}$ and $(D_0^{(0)})^{-1}$, as well as D^{-1} and D_0^{-1}, are well defined. Decomposing $f \in \ell_2^n$ as $f = f_1 + f_2 + f_3 + f_4$, we see that $D_0^{(0)} f_j = -2 f_j$ for $j = 1, 2, 3, 4$, such that it acts as a multiple of the identity.

APPENDIX D: SUM AND DIFFERENCE EXCHANGE POTENTIALS FOR $\tilde{\Lambda}$ FIELDS

In this appendix, we give the sum $F^{\rho h \alpha f} W^{\sigma k \beta g}_{\rho h \alpha f} F^{\sigma k \beta g} \pm \ldots (-W^{\beta g \sigma k}_{\rho h \alpha f}) \ldots$ and difference potential matrices for $\tilde{\Lambda}$ fields. They are used to calculate the sum and difference exchange potentials for the $\tilde{\Omega}$ fields of Eqs. (38)–(41) which are conveniently used in the B-S bound state, Eqs. (43) and (44). They are related to the sum and difference potentials which correspond to the $+$ and $-$ parity sectors in the lattice Schrödinger operators, as seen in Eq. (47). Each 36×36 symmetric matrix has the upper triangular block structure $[V_{11}, V_{12}, V_{13}, V_{14}]$, $[V_{22}, V_{23}, V_{24}]$, $[V_{33}, V_{34}]$, $[V_{44}]$, and each V_{ij} sub-block has the 16 part structure $[A, C, N, R]$, $[M, B, D, P]$, $[S, T, H, J]$, $[U, X, L, K]$. We use the shorthand notation $F^\gamma W^{\gamma'}_\gamma F^{\gamma'} \pm (F^\gamma (-W^{t\gamma'}_\gamma) F^{\gamma'}) \ldots$, and omitted sub-blocks are zero. The exact values are given below.

Sum—$V_{11}(16 \times 16)$: $A = K^f = K$, $B = H^f = H$, $C = J^f$, $J = -C$, $N = P^f = P$, $A = \mathrm{diag}(0, -1/3, -1/3, 0)$; $B = \mathrm{diag}(-1/3, 0, 0, -1/3)$; $C = [0_4]$, $[1/3, 0_3]$, $[0_4]$, $[0_2, -1/3, 0]$; $N = [0_4]$, $[0_4]$, $[1/3, 0_3]$, $[0, 1/3, 0_2]$; $V_{22}(8 \times 8)$: $A = K^f$, $B = H^f$, $C = J^f = J$, D, $A = \mathrm{diag}(1/3, 7/3)$; $B = \mathrm{diag}(1, 5/3)$; $C = [0_2]$, $[-2\sqrt{3}/3, 0]$; $D = [0_2]$, $[-4/3, 0]$; $V_{33}(4 \times 4)$: $A = K = 0$, $B = H^f = H$, D, $B = -1/3$, $D = 1/3$; $V_{12}(16 \times 8)$: $A = -K^f$, $B = -H^f$, $C = -L^f$, $N = -X^f = P$, $D = -T^f$, $M = -J^f$, $A = [0_2]$, $[4\sqrt{5}/3, 0]$, $[0, 2\sqrt{5}/3]$, $[0_2]$; $B = [0_2]$, $[0_2]$, $[0, 2\sqrt{5}/3]$, $[0_2]$; $C = [0_2]$, $[0_2]$, $[2\sqrt{15}/3, 0]$, $[0, 2\sqrt{5}/3]$; $N = [0_2]$, $[0_2]$, $[0_2]$, $[2\sqrt{5}/3, 0]$; $D = [0_2]$, $[0_2]$, $[2\sqrt{5}/3, 0]$, $[0, 2\sqrt{15}/3]$; $J = [0_2]$, $[0_2]$, $[0_2]$, $[0, 4\sqrt{5}/3]$; $V_{13}(16 \times 8)$: $A = K = 0$, $B = H^f$, $C = L^f$, $N = -X^f$, $D = T^f$, $N = -C$, $D = -B$, $B = [0]$, $[0]$, $[-2\sqrt{10}/3]$, $[0]$; $C = [0]$, $[0]$, $[0]$, $[2\sqrt{10}/3]$; $V_{23}(8 \times 4)$: 0 and $V_{14}(16 \times 8) = -V_{12}$, $V_{24}(8 \times 8) = -V_{22}$, $V_{34}(4 \times 8) = -V_{32}$, $V_{44}(8 \times 8) = -V_{22}$.

The zero rows and columns are 1, 6, 11, 16, 25, 28. The other invariant blocks are

2 5 17 29	$J_z = 2$
3 9 18 19 30 31	$J_z = 1$
4 7 10 13 20 21 26 27	$J_z = 0$
8 14 22 23 34 35	$J_z = -1$
12 15 24 36	$J_z = -2.$

The spectrum (multiplicity) is $-6\,(6)$, $0\,(21)$, $6\,(9)$.

Difference—$V_{11}(16 \times 16)$: $A = K^f$, $B = H^f$, $C = J^f$, $N = P$, D, $A = \mathrm{diag}(-2, -1, -1/3, 0)$, $B = \mathrm{diag}(-1, -10/9, -8/9, -1/3)$, $C = [0_4]$, $[-1, 0_3]$, $[0, -4\sqrt{3}/9, 0_2]$, $[0_2, -1/3, 0]$, $N = [0_4]$, $[0_4]$, $[-1/3, 0_3]$, $[0, -1/3, 0_2]$, $D = [0_4]$, $[-4\sqrt{3}/9, 0_3]$, $[0, -8/9, 0_2]$, $[0_2, -4\sqrt{3}/9]$; $V_{22}(8 \times 8)$: $A = K^f$, $B = H^f$, $C = J$, D, $A = \mathrm{diag}(-1, 7/3)$, $B = \mathrm{diag}(1/9, 11/9)$, $C = [0_2]$, $[-10\sqrt{3}/9, 0]$, $D = [0_2]$, $[-20/9]$; $V_{33}(4 \times 4)$: $A = K = 68/9$, $B = D = 34/9$; $V_{12}(16 \times 8)$: $A = -K^f$, $B = -H^f$, $C = -L^f$, D, $N = P$, $M = J^f$, $S = X$, T, $A = [0_2]$, $[0_2]$, $[0, 2\sqrt{5}/3]$, $[0_2]$; $B = [0_2]$, $[4\sqrt{5}/9, 0]$, $[0, -2\sqrt{5}/9]$, $[0_2]$; $C = [0_2]$, $[0_2]$, $[-2\sqrt{15}/9, 0]$, $[0, 2\sqrt{5}/3]$; $D = [0_2]$, $[0_2]$, $[2\sqrt{5}/9, 0]$, $[0, 2\sqrt{15}/9]$; $N = [0_2]$, $[0_2]$, $[0_2]$, $[-2\sqrt{5}/3, 0]$; $J = [0_2]$, $[0_2]$, $[4\sqrt{15}/9, 0]$, $[0_2]$; $S = [0, 2\sqrt{5}/3]$, $[0_2]$, $[0_2]$, $[0_2]$; $T = [-2\sqrt{15}/9, 0]$, $[0, -2\sqrt{5}/9]$, $[0_2]$, $[0_2]$; $V_{13}(16 \times 8)$: $A = K^f$, $B = H^f = D$, $C = L^f$, D, $C = N$, P, S, $X = L$, $M = J^f$, $T = H$, $A = [0]$, $[0]$, $[4\sqrt{30}/9]$ $[0]$, $B = [0]$, $[0]$, $[-2\sqrt{10}/9]$ $[0]$, $C = [0]$, $[0]$, $[0]$, $[2\sqrt{10}/3]$, $P = [0]$, $[0]$, $[0]$, $[4\sqrt{30}/9]$, $J = [0]$, $[0]$, $[-8\sqrt{10}/9]$, $[0]$, $S = [4\sqrt{30}/9]$, $[0]$, $[0]$, $[0]$; $V_{33}(4 \times 4)$: $A = -K^f$, $B = -H^f$, $D = -T^f$, $J = -M^f$, $A = [0]$, $[20\sqrt{6}/9]$, $B = [0]$, $[20\sqrt{2}/9]$, $D = [0]$, $[20\sqrt{2}/9]$, $J = [0]$, $[20\sqrt{2}/9]$; $V_{14}(16 \times 8) = -V_{12}$, $V_{24}(8 \times 8) = -V_{22}$, $V_{34}(4 \times 8) = -V_{32}$, $V_{44}(8 \times 8) = V_{22}$.

The zero rows and columns are absent. The invariant blocks are

1	$J_z = 3$
16	$J_z = -3$
2 5 17 29	$J_z = 2$
3 6 9 18 19 25 30 31	$J_z = 1$
4 7 10 13 20 21 26 27 32 33	$J_z = 0$
8 11 14 22 23 28 34 35	$J_z = -1$
12 15 24 36	$J_z = -2.$

The spectrum (multiplicity) is $-2\,(18)$, $0\,(15)$, $18\,(3)$.

For the $\tilde{\Omega}$ fields, we use the reduced sub-block versions of Eqs. (39) and (41) for the sum and difference potential matrices. For the sum and difference potentials, the upper triangular parts of the symmetric matrices are given in Eqs. (39) and (41), respectively.

For the sum potential, the spin ordering is that of $\tilde{\Lambda}_1$, $\tilde{\Lambda}_3$, $\tilde{\Lambda}_2$ for the first, second, and third blocks, respectively (such as $1 \ldots 16$, $25 \ldots 28$, and $17 \ldots 24$, respectively. Their invariant blocks and spectral properties are given in the text.

For the difference potential, the spin ordering for the first, second, and third blocks is that of $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$, $\tilde{\Lambda}_3$, respectively.

APPENDIX E: PERMANENT REPRESENTATION FOR THE EXCHANGE POTENTIAL

We now obtain a permanent representation for $D^{u(2)}/(1/12)$ of Eq. (31), namely, the formula for $W^{\sigma k \beta g}_{\rho h \alpha f}$ of Eq. (32). From Eq. (31), we evaluate the factors $\Phi_1 \equiv \langle \psi(e^1) B(e^1) \bar{\psi}(e^1) \bar{B}(e^1) \rangle_t^{(0)} = \langle \psi_{b\beta_1 r_1} \epsilon_{\vec{c}} \psi^3_{\vec{c}\,\vec{\alpha}\,\vec{f}} \bar{\psi}_{b\alpha_2 r_2} \epsilon_{\vec{d}} \bar{\psi}^3_{\vec{d}\,\vec{\beta}\,\vec{g}} \rangle_t^{(0)}$ and $\Phi_2 \equiv \langle \psi(0) B(0) \bar{\psi}(0) \bar{B}(0) \rangle_t^{(0)} = \langle \psi_{a\beta_2 r_2} \epsilon_{\vec{\ell}} \psi^3_{\vec{\ell}\,\vec{\rho}\,\vec{h}} \bar{\psi}_{a\alpha_1 r_1} \epsilon_{\vec{m}} \bar{\psi}^3_{\vec{m}\,\vec{\sigma}\,\vec{k}} \rangle_t^{(0)}$.

By direct calculation, we obtain

$$\begin{aligned}
\Phi_1 &= -6\delta_{\beta_1\beta_1^\ell}\delta_{r_1 g_1}\,\mathrm{perm}(\delta_{\vec{\alpha}(\alpha_2\beta_2^\ell\beta_3^\ell)}\delta_{\vec{f}(r_2 g_2 g_3)}) - 6\delta_{\beta_1\beta_2^\ell}\delta_{r_1 g_2}\,\mathrm{perm}(\delta_{\vec{\alpha}(\beta_1^\ell\alpha_2\beta_3^\ell)}\delta_{\vec{f}(g_1 r_2 g_3)}) \\
&\quad - 6\delta_{\beta_1\beta_3^\ell}\delta_{r_1 g_3}\,\mathrm{perm}(\delta_{\vec{\alpha}(\beta_1^\ell\beta_2^\ell\alpha_2)}\delta_{\vec{f}(g_1 g_2 r_2)}) \\
&\equiv F_1(\beta_1\alpha_2 r_1 r_2 \vec{\alpha}\,\vec{f}\,\vec{\beta}\,\vec{g}), \\
\Phi_2 &= -6\delta_{\beta_2\sigma_1^\ell}\delta_{r_2 k_1}\,\mathrm{perm}(\delta_{\vec{\rho}(\alpha_1\sigma_2^\ell\sigma_3^\ell)}\delta_{\vec{h}(r_1 k_2 k_3)}) - 6\delta_{\beta_2\sigma_2^\ell}\delta_{r_2 k_2}\,\mathrm{perm}(\delta_{\vec{\rho}(\sigma_1^\ell\alpha_1\beta_3^\ell)}\delta_{\vec{h}(k_1 r_1 k_3)}) \\
&\quad - 6\delta_{\beta_2\sigma_3^\ell}\delta_{r_2 k_3}\,\mathrm{perm}(\delta_{\vec{\rho}(\sigma_1^\ell\sigma_2^\ell\alpha_1)}\delta_{\vec{h}(k_1 k_2 r_1)}) \\
&\equiv F_0(\beta_2\alpha_1 r_2 r_1 \vec{\rho}\,\vec{h}\,\vec{\sigma}\,\vec{k}).
\end{aligned}$$

In the above, we have used Eq. (5).

For the $F_0()\,F_1()$ product, noting that $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, since only the identity term of $\Gamma^{\pm e^1}$ contributes, and these indices are lower, we obtain

$$\begin{aligned}
F_0(\beta\alpha r_2 r_1 \vec{\rho}\,\vec{h}\,\vec{\sigma}\,\vec{h}) F_1(\alpha\beta r_1 r_2 \vec{\alpha}\,\vec{f}\,\vec{\beta}\,\vec{g}) = 36[&\delta_{\beta\sigma_1^\ell}\delta_{r_2 k_1}\mathrm{perm}(\delta_{\vec{\rho}(\alpha\sigma_2^\ell\sigma_3^\ell)}\delta_{\vec{h}(r_1 k_2 k_3)}) + \delta_{\beta\sigma_2^\ell}\delta_{r_2 k_2}\,\mathrm{perm}(\delta_{\vec{\rho}(\sigma_1^\ell\alpha\sigma_3^\ell)}\delta_{\vec{h}(k_1 r_1 k_3)}) \\
&+ \delta_{\beta\sigma_3^\ell}\delta_{r_2 k_3}\,\mathrm{perm}(\delta_{\vec{\rho}(\sigma_1^\ell\sigma_2^\ell\alpha)}\delta_{\vec{h}(k_1 k_2 r_1)})][\delta_{\alpha\beta_1^\ell}\delta_{r_1 g_1}\,\mathrm{perm}(\delta_{\vec{\alpha}(\beta\beta_2^\ell\beta_3^\ell)}\delta_{\vec{f}(r_2 g_2 g_3)}) \\
&+ \delta_{\alpha\beta_2^\ell}\delta_{r_1 g_1}\,\mathrm{perm}(\delta_{\vec{\alpha}(\beta_1^\ell\beta\beta_3^\ell)}\delta_{\vec{f}(g_1 r_2 g_3)}) + \delta_{\alpha\beta_3^\ell}\delta_{r_1 g_3}\,\mathrm{perm}(\delta_{\vec{\alpha}(\beta_1^\ell\beta_2^\ell\beta)}\delta_{\vec{f}(g_1 g_2 r_2)})],
\end{aligned}$$

with sums over the lower indices α, β and r_1, r_2. Carrying out these sums gives $W^{\vec{\sigma}\,\vec{k}\,\vec{\beta}\,\vec{g}}_{\vec{\rho}\,\vec{h}\,\vec{\alpha}\,\vec{f}}$ of Eq. (32).

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