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Effective string theory and QCD scattering amplitudes

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QCD string is formed at distances larger than the confinement scale and can be described by the Polchinski–Strominger effective string theory with a nonpolynomial action, which has nevertheless a well-defined semiclassical expansion around a long-string ground state. We utilize modern ideas about the Wilson-loop/scattering-amplitude duality to calculate scattering amplitudes and show that the expansion parameter in the effective string theory is small in the Regge kinematical regime. For the amplitudes we obtain the Regge behavior with a linear trajectory of the intercept $(d-2)/24$ in d dimensions, which is computed semiclassically as a momentum-space Lüscher term, and discuss an application to meson scattering amplitudes in QCD.

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I. INTRODUCTION

Recently there has renewed interest in the long-standing problem of the relation between strings and QCD. On the one hand, some properties of Wilson loops have been understood via the AdS/CFT correspondence [1], where the Wilson loop in the $\mathcal{N}=4$ super Yang–Mills at large coupling constant is described by the supergravity approximation to an open superstring of type IIB on $AdS_5\times S^5$ background, whose worldsheet is a minimal surface extended to the 5th dimension. This approach has resulted in numerous applications of holographic duals to QCD. On the other hand, the lattice QCD simulations indicate (see the review [2], references therein and the subsequent paper [3]) that the Nambu–Goto string very well approximates the QCD string for a wide range of distances.

An old result [4] is that the Nambu–Goto string is not an exact solution to the loop equation of large-N QCD, but rather its asymptote—the area law—*is* a self-consistent solution for asymptotically large loops. Extra degrees of freedom, populating the string worldsheet, are required to reproduce a factorized structure on the right-hand side of the loop equation at intermediate distances and/or a proper behavior of Wilson loops for the case of self-intersections. These degrees become frozen for large loops = long strings (in the units of the QCD confinement scale), that makes it possible to perform an expansion in the inverse area of the minimal surface, spanned by the loop, which has the meaning of a semiclassical expansion. This leads us to an ideology of an effective QCD string, formed by fluxes of the Yang–Mills field, which is consistent [5] at large distances.

A beautiful example of how such an effective string theory works is a closed string winding along a compact direction of a large radius R. It is described by a nonpolynomial action [6]

$$S_{\rm eff}=2K\int d^2z\,\partial X\cdot\bar\partial X+\frac{d-26}{24\pi}\int d^2z\frac{\partial^2X\cdot\bar\partial^2X}{\partial X\cdot\bar\partial X}+\ldots,\tag{1}$$

where the conformal anomaly is expressed (modulo total derivatives and the constraints) via an induced metric

$$e^{\varphi_{\rm ind}}=2\partial X\cdot\bar\partial X\tag{2}$$

in the conformal gauge, which is not treated independently as distinct from the Polyakov formulation. This effective string theory has been analyzed using the conformal field theory technique order by order in $1/R$ [6,7], revealing the spectrum [8] of the Nambu–Goto string in d dimensions.

The goal of this Paper is to expand the effective string theory approach to calculations of QCD meson scattering amplitudes in the Regge kinematical regime, where a semiclassical expansion is applicable as will be momentarily explained. These scattering amplitudes are represented in the large-N limit (or in the quenched approximation) as sums over paths of the Wilson loops. Remarkably, large loops dominate the sum over paths in the Regge kinematical regime when Mandelstam's variable s is large and t is fixed, as it has been shown in Ref. [9], so an effective string theory ideology is then applicable.

In obtaining this result, it was crucial to use the manifestly reparametrization-invariant representation [10] of large Wilson loops in the form of the path integral over reparametrizations of the boundary contour $x^\mu(t)$:

$$W[x(\cdot)]=\int\mathcal{D}_{\rm diff}t(s)e^{-KS[x(t)]},\tag{3}$$

where $K=1/2\pi\alpha'$ is the string tension and

$$S[x(t)]=\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{ds_1ds_2}{(s_1-s_2)^2}[x(t(s_1))-x(t(s_2))]^2.\tag{4}$$

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We have used the notation $W[x(\cdot)]$ on the left-hand side of Eq. (3) to emphasize its reparametrization invariance.

The functional (4) is known in mathematics as the Douglas integral [11], whose minimum with respect to reparametrizing functions $t(s)$ coincides with the minimal area. The path integral in Eq. (3) is thus dominated for large loops by a saddle point, giving the area law. This remarkably holds for loops of an arbitrary shape, even not necessarily planar. The area-law behavior of the reparametrization path integral is of course associated with a classical string. As is shown in Ref. [12], quantum fluctuations around the saddle point reproduce in the quadratic approximation the Lüscher term in $d = 26$ dimensions, which is usually associated with quantum fluctuations around the minimal surface in the semiclassical approximation. This is perhaps not surprising because the ansatz (3) emerges as the Dirichlet disk amplitude for the Polyakov string in the critical dimension $d = 26$.

The fact that $S[x(t)]$ in Eq. (3) is quadratic in $x^\mu(t)$ makes it possible to perform its Fourier transformation by doing the Gaussian path integral over $x^\mu(t)$, which results in the scattering amplitude again of the type of the right-hand side of Eq. (3) with $x^\mu(t)$ substituted by the function $p^\mu(t)/K$, where $p^\mu(t)$ is a step function, whose discontinuities are momenta of colliding particles. Thus all nonlinearities are hidden in the reparametrization path integral which can be partially done, while the rest is represented as an integral over the Koba–Nielsen variables known from dual resonance models. This path integral over reparametrizations goes over *subordinated* functions (i.e. those having $dt(s)/ds \geq 0$) with a certain measure which respects reparametrization invariance and whose properties are considered in some detail in Refs. [9,13]. What is most important is that the resulting scattering amplitudes [9,14] possess projective invariance and are consistent off shell. Once again, this is intimately related to the presence of the reparametrization path integral in Eq. (3), which factorizes in the on-shell scattering amplitudes of the fundamental string (i.e. for tachyonic scalars, massless vectors etc.), reproducing the usual Koba–Nielsen amplitude, and correspondingly plays no role then. On the contrary, excitations of QCD string should reproduce the meson spectrum, i.e. the vector state is a massive ρ meson, which explains why scattering amplitudes are required off shell. Off-shell amplitudes of this kind were previously obtained [15] (see the review [16] and the subsequent papers [17]) for the Polyakov quantization of the critical string, using the Lovelace choice [18] of the N-Reggeon vertex instead of the usual vertex operator. However, their extension to $d = 4$ dimensions is still missing to my knowledge.

In the present paper, we derive scattering amplitudes for a noncritical effective open string theory with the action (1) in the semiclassical approximation justified by the Regge kinematical regime, where the expansion parameter $1/\ln(s/t)$ is small. The technique used is pretty much in the spirit of the Wilson-loop/scattering-amplitude duality recently elaborated [19,20] (for a review see Ref. [21]) for $\mathcal{N} = 4$ super Yang–Mills. The calculation is analogous to that of the Lüscher term for a rectangle, except it is performed in momentum space. As a result, we obtain the Regge behavior of scattering amplitudes with a linear trajectory

$$\alpha(t) = \frac{d-2}{24} + \alpha' t. \tag{5}$$

We then discuss an application of this result to large-N QCD, where meson scattering amplitudes are represented as sums over paths of the Wilson loop. We demonstrate that large loops dominate the sum over paths in the Regge kinematical regime of large s and fixed $-t$, so the effective string theory representation of the Wilson loop is expected to work. Alternatively, perturbative QCD is expected to work when both s and $-t$ are large. We also discuss how a linear Regge trajectory of the type in Eq. (5) appears for spinor quarks.

II. THE CLASSICAL LIMIT

A. Review of Douglas' minimization

Let us consider Eq. (3) as a representation of the disk amplitude for bosonic string with the Dirichlet boundary condition in $d = 26$ dimensions. As is already mentioned in the Introduction, this representation can be derived in the Polyakov formulation by integrating over $X^\mu(x, y)$ in the bulk with the boundary condition

$$X^\mu(s, 0) = x^\mu(t(s)). \tag{6}$$

The form of the boundary action (4) depends on the choice of coordinates parametrizing the world sheet, as the Green function of the Laplace operator does. Equation (4) is written for the upper half-plane (UHP): $z = x + iy \in$ UHP, bounded by the real axis. While the Polyakov action is invariant under conformal transformations, they change, in general, the shape of the boundary, so the Douglas integral (4) changes accordingly. The only conformal transformation that maps UHP onto itself is $SL(2;\mathbb{R})$, which results in the projective transformation at the boundary:

$$s \longrightarrow \frac{as+b}{cs+d}, \qquad ad - bc = 1. \tag{7}$$

The Douglas integral (4) is invariant under it.

It is instructive to compare the UHP parametrization with a more physical parametrization through worldsheet coordinates, which take values in a rectangle and are usually associated with propagation of an open string of the length R during the time T. These two coordinate choices are related by the Schwarz–Christoffel mapping, which will be extensively used below when calculating a semiclassical correction. For the purposes of the present paper, the former parametrization has some advantages over the latter. First, the Green function in Eq. (4) looks simpler for UHP than for a rectangle.[1] Second, after the decomposition

$$X^\mu = X^\mu_{\rm cl} + Y^\mu_{\rm q}, \tag{8}$$

where $X^\mu_{\rm cl}$ obeys the Laplace equation and the boundary condition, so that $Y^\mu_{\rm q} = 0$ at the boundary, the path integral over $Y^\mu_{\rm q}$ does *not* depend on the boundary contour x^μ for the UHP parametrization. This is the reason why the boundary path integral in Eq. (3) captures fluctuations of the critical string around the minimal surface, as was explicitly demonstrated in Ref. [12]. This is in contrast to the parametrization by a rectangle, when semiclassical stringy fluctuations reside in a determinant coming from the path integral over $Y^\mu_{\rm q}$ as is well-known. We shall return to this issue in Sect. III.

The minimum of the Douglas integral is reached for the function $t(s)$ obeying

$$\int dt_1 \frac{\dot{x}(t)\cdot\dot{x}(t_1)}{[s(t)-s(t_1)]} = 0, \tag{9}$$

where $s(t)$ denotes inverse to $t(s)$. The minimal surface can then be reconstructed in UHP from its boundary value $x^\mu(t(s))$ by the Poisson formula

$$\begin{aligned} X^\mu(x,y) &= \int_{-\infty}^{+\infty}\frac{ds}{\pi}\frac{x^\mu(t(s))y}{(x-s)^2+y^2} \\ &= \int_{-\infty}^{+\infty}\frac{dt}{\pi}\dot{x}^\mu(t)\arctan\frac{x-s(t)}{y}. \end{aligned} \tag{10}$$

This function is obviously harmonic in UHP and satisfies Eq. (6). The presence of the reparametrizing function $t(s)$ guarantees that (10) obeys the conformal gauge if $t(s) = t_*(s)$ with $t_*(s)$ being inverse to the minimizing function $s_*(t)$. This is demonstrated in Appendix A. While Douglas' theorem was originally proven for Euclidean space, the consideration of Appendix A shows that it applies for a spacelike surface in Minkowski space as well. The necessity of a reparametrization of the boundary for consistency with the conformal gauge in the Polyakov formulation of an open string was pointed out in Ref. [22].

B. Polygonal loop with $(x_{i+1} - x_i) = \Delta p_i/K$

Since the boundary action (4) is quadratic in $x^\mu(t)$, the functional Fourier transformation of Eq. (3) to momentum space equals [9]

$$A[p(\cdot)] \equiv \int \mathcal{D}x^\mu e^{i\int dt p\cdot\dot{x}} W[x(\cdot)] = W[p(\cdot)/K], \tag{11}$$

which looks exactly like the right-hand side of Eq. (3) with $x^\mu(t)$ substituted by the trajectory

$$x^\mu(t) = \frac{1}{K}p^\mu(t). \tag{12}$$

For piecewise constant $p^\mu(t)$ this disk amplitude is proportional to the scattering amplitude. We shall make use of this remarkable fact applying the technique, developed for a noncritical string with Dirichlet boundary conditions, to semiclassical calculations of the scattering amplitudes.

Substituting in Eq. (11) the smeared stepwise

$$p^\mu(t) = \frac{1}{\pi}\sum_i \Delta p_i^\mu \arctan\frac{(t-t_i)}{\zeta_i} \to \frac{1}{2}\sum_i \Delta p_i^\mu \,{\rm sign}(t-t_i) \tag{13}$$

for $(t_i - t_{i-1}) \gg \zeta_i, \zeta_{i-1}$, that results in polygonal $x^\mu(t)$, we have for the amplitude explicitly

$$A(\{\Delta p_i\}) = W[p(\cdot)/K] \tag{14}$$

with $p^\mu(t)$ given by Eq. (13). The discontinuities Δp_i^μ of $p^\mu(t)$ are the particle momenta.

Let us calculate the minimal area for such nonplanar contours. Since we are interested in the Regge limit of $s \gg -t \gg -\Delta p_i^2$, we can set $\Delta p_i^2 = 0$ to have lightlike edges as in Refs. [19,20]. The case of $\Delta p_i^2 \neq 0$ will be considered in Sect. IV.

The Douglas integral then reads

$$\begin{aligned} KS &= -\alpha' \int dt dt' \dot{p}(t)\cdot\dot{p}(t')\ln|s(t)-s(t')| \\ &= -\alpha' \sum_{i,j\neq i}\Delta p_i\cdot\Delta p_j \ln|s_i - s_j| \end{aligned} \tag{15}$$

with $s_i = s(t_i)$. Here the values t_i's, at which $p^\mu(t)$ has (smeared) discontinuities, are fixed by the initial parametrization, while the Douglas minimization is to be performed with respect to s_i. Nothing depends on $s(t)$ at the intermediate points $t \in (t_{i-1}, t_i)$, which is a zero mode as is explained in Ref. [9].

The Douglas minimization Eq. (9) is trivially satisfied for the given polygonal $x^\mu(t)$ at the intermediate points, when t is not close to t_i's, because then $\dot{x}^\mu(t) = 0$. For $t = t_i$ we rewrite Eq. (9) as

$$\sum_{j\neq i}\frac{\Delta p_i\cdot\Delta p_j}{s_i - s_j} = 0. \tag{16}$$

Only $M-3$ of these M equations are independent because of the invariance under the projective transformation of s_i's. Thus the Douglas minimization determines only $M-3$ values of s_i's, while three of them remain arbitrary. The minimal surface does not depend on these three values.

For $M=4$ we obtain from Eq. (16)

$$s_{2*} = s_1 + \frac{s s_{41} s_{31}}{s s_{41} + t s_{43}} = s_3 - \frac{t s_{43} s_{31}}{s s_{41} + t s_{43}} \tag{17}$$

with arbitrary s_1, s_3 and s_4. In the usual way we can set $s_1 = 0$, $s_3 = 1$, $s_4 = \infty$, after which the solution (17) simplifies to

[1]The Dirichlet Green functions for UHP and a rectangle are displayed below in Sect. III C (see Eqs. (56) and (57)).

$$s_{2*} = \frac{s}{s+t}. \tag{18}$$

This is nothing but the well-known saddle point of the Veneziano amplitude at large $-s$ and $-t$.

At the minimum we shall get the minimal area

$$KS_{\min} = \alpha' s \ln\frac{s}{s+t} + \alpha' t \ln\frac{t}{s+t} \overset{s\gg t}{\longrightarrow} -\alpha' t \ln\frac{s}{t} \tag{19}$$

whose exponential reproduces the classical Regge behavior of the scattering amplitude:

$$A(s,t) = e^{-KS_{\min}} \propto s^{\alpha' t}. \tag{20}$$

C. Reconstruction of the minimal surface

For polygonal $x^\mu(t)$ given by Eqs. (12) and (13), we can reconstruct the harmonic function in UHP by the Poisson formula (10) which satisfies the boundary condition (6).

From Eqs. (10) and (13), we have

$$X^\mu(x,y) = \frac{1}{\pi K}\sum_i \Delta p_i^\mu \arctan\frac{(x-s_i)}{y+\varepsilon_i}. \tag{21}$$

It is instructive to see how the boundary contour (13) is reproduced by this formula for $y=0$. For $t \approx t_i$ we have

$$\frac{s(t)-s(t_i)}{\varepsilon_i} \longrightarrow \frac{s'(t_i)(t-t_i)}{\varepsilon_i} = \frac{(t-t_i)}{\zeta_i}, \tag{22}$$

where $\varepsilon_i = s'(t_i)\zeta_i$ in accordance with the reparametrization covariance. As $\zeta_i \to 0$ we reproduce the step function (13) which results in the harmonic function (21) with $\varepsilon_i = 0$. It is used below in this Subsection because there are no divergences in the $\varepsilon_i \to 0$ limit at the classical level.

The domain of both $s<0$ and $t<0$ corresponds to scattering in the u channel:

$$\begin{aligned} \Delta p_1^\mu &= (E, p, 0, 0), \\ \Delta p_2^\mu &= (-E, -p\cos\theta, -p\sin\theta, 0), \\ \Delta p_3^\mu &= (E, -p, 0, 0), \\ \Delta p_4^\mu &= (-E, p\cos\theta, p\sin\theta, 0), \end{aligned} \tag{23}$$

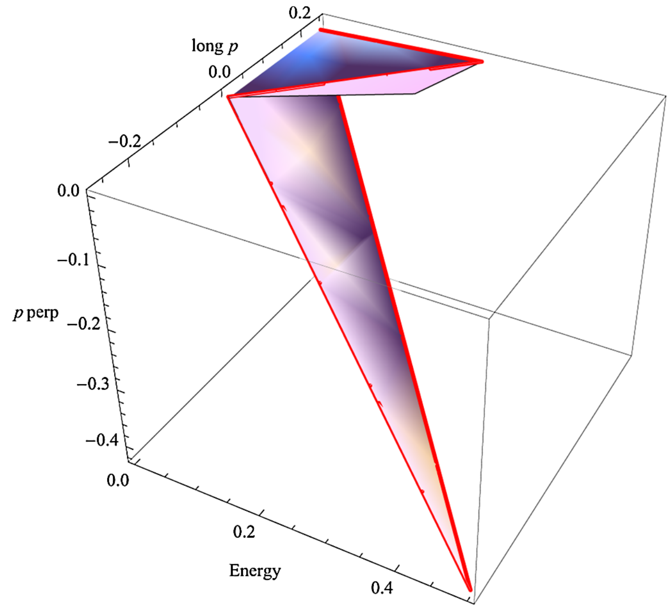
where

$$\cos\theta = \frac{t}{s+t}, \qquad (1-\cos\theta) = \frac{s}{s+t}. \tag{24}$$

From Eq. (17) we then have

$$\cos\theta = \frac{s_{32}s_{41}}{s_{42}s_{31}}, \qquad (1-\cos\theta) = \frac{s_{21}s_{43}}{s_{42}s_{31}}. \tag{25}$$

The minimal surface spanned by the contour (13) with Δp_i's given by Eq. (23) is depicted in Fig. 1 for $\theta = 1.0$ and $\theta = 0.2$. With decreasing the scattering angle θ, we move from the one in the left figure to the one in the right figure with decreasing the minimal area which tends to 0 as $\theta \to 0$. It is a spacelike surface embedded in Minkowski space. One-loop divergences, associated with its transverse fluctuations, will be regularized by setting $\varepsilon_i \neq 0$, as is described below.



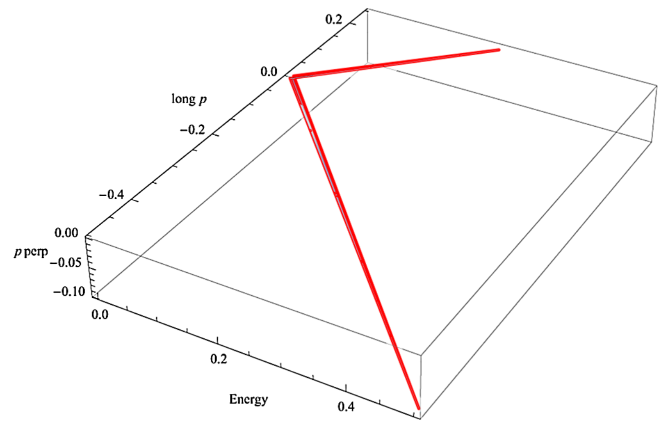


FIG. 1 (color online). Minimal surface spanned by the contour (13) for $\theta = 1.0$ (left) and $\theta = 0.2$ (right).

The induced metric $g_{ab} = \partial_a X \cdot \partial_b X$ of the minimal surface, spanned the polygon given by Eq. (13), reads

$$g_{12} = g_{21} = -\frac{1}{\pi^2 K^2}\sum_{i,j\neq i}\frac{\Delta p_i \cdot \Delta p_j (x-s_i)y}{[(x-s_i)^2+y^2][(x-s_j)^2+y^2]} \tag{26}$$

and

$$g_{11} = \frac{1}{\pi^2 K^2}\sum_{i,j\neq i}\frac{\Delta p_i \cdot \Delta p_j y^2}{[(x-s_i)^2+y^2][(x-s_j)^2+y^2]}, \tag{27}$$

$$g_{22} = \frac{1}{\pi^2 K^2}\sum_{i,j\neq i}\frac{\Delta p_i \cdot \Delta p_j (x-s_i)(x-s_j)}{[(x-s_i)^2+y^2][(x-s_j)^2+y^2]}. \tag{28}$$

Using the identities of Appendix A, it can be shown that g_{12}, given by Eq. (26), vanishes and g_{11}, given by Eq. (27), coincides with g_{22}, given by Eq. (28), if Eq. (16) is satisfied. Then the induced metric is conformal:

$$g_{ab} = e^{\varphi}\delta_{ab} \tag{29}$$

with

$$e^{\varphi(x,y)} = \frac{1}{\pi^2K^2}\sum_{i,j\neq i}\frac{\Delta p_i\cdot\Delta p_j y^2}{[(x-s_i)^2+y^2][(x-s_j)^2+y^2]}. \tag{30}$$

When $y\to 0$, this function vanishes except in the vicinities of s_i's. This implies that the boundary metric vanishes in the corners of the polygon and may nonvanish only along the edges.

When $M=4$ and the projective symmetry is not fixed, Eq. (17) holds and Eq. (18) changes to

$$\frac{s_{21}s_{43}}{s_{31}s_{42}} = \frac{s}{s+t}. \tag{31}$$

We then get for the induced metric

$$e^{\varphi(x,y)} = -\frac{sts_{42}^2s_{31}^2y^2}{\pi^2K^2(s+t)\prod_{i=1}^4[(x-s_i)^2+y^2]} \tag{32}$$

with s_2 given by Eq. (17). We see from Eq. (32) that the boundary metric vanishes except for $s=s_i$'s, where we have explicitly

$$e^{\varphi(s_1,0)/2} = \frac{1}{\pi K}\sqrt{\frac{-st}{s+t}}\frac{s_{42}}{s_{21}s_{41}}, \tag{33a}$$

$$e^{\varphi(s_2,0)/2} = \frac{1}{\pi K}\sqrt{\frac{-st}{s+t}}\frac{s_{31}}{s_{32}s_{21}}, \tag{33b}$$

$$e^{\varphi(s_3,0)/2} = \frac{1}{\pi K}\sqrt{\frac{-st}{s+t}}\frac{s_{42}}{s_{43}s_{32}}, \tag{33c}$$

$$e^{\varphi(s_4,0)/2} = \frac{1}{\pi K}\sqrt{\frac{-st}{s+t}}\frac{s_{31}}{s_{43}s_{41}}. \tag{33d}$$

Calculating the integral with φ given by Eq. (32) using the formula

$$\int dy\int_{-\infty}^{+\infty}dx\frac{y^2}{[(x-s_i)^2+y^2][(x-s_j)^2+y^2]} = \frac{\pi}{4}\ln[(s_i-s_j)^2+4y^2], \tag{34}$$

we obtain

$$K\int_0^{\infty}dy\int_{-\infty}^{+\infty}dxe^{\varphi(x,y)} = \alpha'\left(s\ln\frac{s_{43}s_{21}}{s_{42}s_{31}}+t\ln\frac{s_{32}s_{41}}{s_{42}s_{31}}\right) = \alpha'\left(s\ln\frac{s}{s+t}+t\ln\frac{t}{s+t}\right) \tag{35}$$

which reproduces Eq. (19).

Analogously, the length of the boundary contour equals

$$\int_{-\infty}^{+\infty}dxe^{\varphi(x,y=0)/2} = 0 \tag{36}$$

as it should for a polygon with lightlike edges.

III. SEMICLASSICAL LÜSCHER TERM FOR THE LIGHT-LIKE POLYGON

A. Semiclassical stringy fluctuations as the Lüscher term

The Regge behavior (20) with a linear trajectory $\alpha(t)=\alpha' t$ of zero intercept is associated with a classical string. Quantum fluctuations shift the intercept of the critical bosonic string to $\alpha(0)=1$. We shall perform in this Section the computation of the Regge trajectory for a noncritical string in $d<26$ in the semiclassical approximation.

For a long string, quantum fluctuations can be taken into account by a semiclassical expansion whose leading order is given by the minimal area and the semiclassical correction is known as the Lüscher term [23,24]. Its form is explicitly written for a plane contour via the conformal anomaly:

$$W(C)\overset{\text{plane}}{\underset{\propto}{}}{}^{C}\, e^{-KS_{\min}(C)+((d-2)/96\pi)\int d^2w(\partial_a\ln|(dz/dw)|)^2}, \tag{37}$$

where an analytic function $w(z)$ maps UHP onto a piece of the plane bounded by the contour C. For a $R\times T$ rectangle with $T\gg R$, Eq. (37) simplifies to

$$W(C)\overset{\text{rectangle}}{\underset{\propto}{}} e^{-KRT+((d-2)\pi/24)(\mathrm{T}/R)}, \tag{38}$$

which is more familiar. How the Lüscher term emerges for noncritical strings is demonstrated in Refs. [25–28]. We shall generalize this technique, applying it for the (non-plane) momentum-space polygonal loop (13).

B. Mapping onto rectangle

Let us map the upper half plane onto a rectangle for arbitrary s_1, s_2, s_3, s_4. By the Schwarz–Christoffel formula we get (see [29], Eq. (3.147.4))

$$\omega(z) = \sqrt{s_{42}s_{31}}\int_{s_2}^{z}\frac{dx}{\sqrt{(s_4-x)(s_3-x)(x-s_2)(x-s_1)}} = 2F\left(\sqrt{\frac{s_{31}(z-s_2)}{s_{32}(z-s_1)}},\sqrt{\frac{s_{32}s_{41}}{s_{42}s_{31}}}\right), \tag{39}$$

where F is the incomplete elliptic integral of the first kind and the normalization factor is introduced for the projective symmetry. The new variable ω takes values inside a rectangle, which has the meaning, as is already said, of the worldsheet parametrization.

Using the relations in [29], Eq. (3.147), we find

$$R = 2CK(\sqrt{1-r}), \qquad T = 2CK(\sqrt{r}), \tag{40}$$

where K is the complete elliptic integral of the first kind, C is a constant and

$$r = \frac{s_{43}s_{21}}{s_{42}s_{31}} \tag{41}$$

is the projective-invariant ratio. Therefore,

$$\frac{T}{R} = \frac{K(\sqrt{r})}{K(\sqrt{1-r})} \tag{42}$$

is projective invariant.

To reproduce the mapping of [12], that corresponds to the choice $s_1 = -1/\sqrt{\mu}$, $s_2 = -\sqrt{\mu}$, $s_3 = +\sqrt{\mu}$, $s_4 = +1/\sqrt{\mu}$, we note that

$$\sqrt{r} = \frac{1-\mu}{1+\mu}, \qquad \sqrt{1-r} = \frac{2\sqrt{\mu}}{1+\mu}. \tag{43}$$

Using the formulas in [29], Eqs. (8.126.1) and (8.126.3):

$$\begin{aligned} K\left(\frac{1-\mu}{1+\mu}\right) &= \frac{1+\mu}{2}K\left(\sqrt{1-\mu^2}\right), \\ K\left(\frac{2\sqrt{\mu}}{1+\mu}\right) &= (1+\mu)K(\mu), \end{aligned} \tag{44}$$

we then reproduce Eq. (20) of [12].

To calculate the Lüscher term, we decompose

$$X^\mu(\omega_1, \omega_2) = X^\mu_{\rm cl}(\omega_1, \omega_2) + Y^\mu_{\rm q}(\omega_1, \omega_2), \tag{45}$$

where $X^\mu_{\rm cl}$ is harmonic with the boundary value (13), so $Y^\mu_{\rm q}$ has the mode expansion

$$Y^\mu_{\rm q}(\omega_1, \omega_2) = \sum_{m,n} \chi^\mu_{mn} \sin\frac{\pi m \omega_1}{R} \sin\frac{\pi n \omega_2}{T}. \tag{46}$$

Now the Lüscher term results from the determinant coming from the path integral over $Y^\mu_{\rm q}$.

Using the asymptotes

$$K(\sqrt{r}) \stackrel{r\to 1}{\to} \frac{1}{2}\ln\frac{16}{1-r}, \qquad K(\sqrt{1-r}) \stackrel{r\to 1}{\to} \frac{\pi}{2}, \tag{47}$$

it is now clear that each set of modes results in the Lüscher term

$$\frac{\pi T}{24R} = \frac{1}{24}\ln\frac{16s}{t} \tag{48}$$

for $T \gg R$ and $r = r_* = s/(s+t)$. There are $(d-2)$ such sets, so their contribution to the intercept of the Regge trajectory is

$$\alpha(0) = \frac{d-2}{24}. \tag{49}$$

It is described in the next section how to get the same result within the framework of the effective string theory with the action (1).

C. The effective string theory calculation

As was already mentioned in Sect. II A, the way the Lüscher term emerges for the UHP parametrization differs from the one for the worldsheet parametrization, described in the previous Subsection. It comes now from the classical part $X^\mu_{\rm cl}$ in the decomposition (8), rather than from the quantum part $Y^\mu_{\rm q}$. How this happens for plane contours is described in the original paper [23], where the determinant of the Laplace operator in a domain given by the conformal map $w(z)$ was represented by the integral in the exponent in Eq. (37). For this reason the consideration of this Section is similar to that of Ref. [28] for the contribution of the Liouville field in the Polyakov formulation. This is because the Liouville field can be simply substituted to the given order of the semiclassical expansion by its value given by the induced metric (2).

The conformal symmetry that is maintained in noncritical dimension is

$$\delta X^\mu = \epsilon(\omega)\partial X^\mu - \frac{\beta a^2}{2}\partial^2\epsilon(\omega)\frac{\bar\partial X^\mu}{\partial X\cdot\bar\partial X}. \tag{50}$$

It transforms X^μ nonlinearly—the same as for the closed string—and the corresponding energy-momentum tensor is

$$T_{zz} = -\frac{1}{2a^2}\partial X\cdot\partial X + \frac{\beta}{2}\frac{\partial^3 X\cdot\bar\partial X}{\partial X\cdot\bar\partial X} \tag{51}$$

with $K = 1/4\pi a^2$, so that $2a^2 = \alpha'$. Expanding around the classical solution

$$X^1_{\rm cl} = \omega_1\sqrt{RT}, \qquad X^2_{\rm cl} = \omega_2\sqrt{RT}, \qquad X^0_{\rm cl} = X^3_{\rm cl} = 0 \tag{52}$$

or

$$X^\mu_{\rm cl} = (e^\mu\omega + \bar e^\mu\bar\omega)\sqrt{RT} \tag{53}$$

with

$$e^\mu = \left(0, \frac{1}{2}, -\frac{i}{2}, 0\right), \tag{54}$$

where ω takes values inside a $\sqrt{R/T}\times\sqrt{T/R}$ rectangle, we obtain

$$T_{zz} = -\frac{\sqrt{RT}}{a^2}e\cdot\partial Y_{\rm q} - \frac{1}{2a^2}\partial Y_{\rm q}\cdot\partial Y_{\rm q} + \frac{\beta}{\sqrt{RT}}\bar e\cdot\partial^3 Y_{\rm q}. \tag{55}$$

Using the Dirichlet Green function for UHP

$$G(z,\zeta) = -\frac{1}{2\pi K}\ln\left|\frac{z-\zeta}{z-\bar\zeta}\right|, \tag{56}$$

we get for the rectangle by the (inverse) conformal mapping (39):

$$G(\omega,\Omega) = -\frac{1}{2\pi K}\ln\left|\frac{\mathrm{sn}^2\frac{\omega}{2} - \mathrm{sn}^2\frac{\Omega}{2}}{\mathrm{sn}^2\frac{\omega}{2} - \overline{\mathrm{sn}^2\frac{\Omega}{2}}}\right| \tag{57}$$

with $\mathrm{sn}\alpha \equiv \mathrm{sn}(\alpha, \sqrt{1-r})$ being the Jacobi elliptic function.

Equations (53) and (55) look like those of Ref. [6] for a winding closed string with R replaced by $\sqrt{RT}$, so we repeat the computation of the central charge generated by the conformal transformation (50) to obtain analogously

$$\langle T_{zz}(\omega)T_{zz}(\Omega)\rangle_Y = \frac{d+12\beta}{2(\omega-\Omega)^4} + \mathcal{O}((\omega-\Omega)^{-2}), \quad (58)$$

where the averaging over the fluctuating field Y_{q}^{μ} is given by the Green function (57). This fixes

$$\beta = \frac{26-d}{12} \quad (59)$$

in our case of an open string as well.

Similar formulas can be obtained for the UHP parametrization, when $\omega(z)$ in Eq. (53) is given by mapping (39). Now it should be noted that the induced metric

$$e^{\varphi} \equiv 2\partial X_{\mathrm{cl}} \cdot \bar{\partial} X_{\mathrm{cl}} = 2RT \frac{s_{42}s_{31}}{\prod_{i=1}^{4}\sqrt{(x-s_i)^2+y^2}} \quad (60)$$

is not constant, as it is for the worldsheet parametrization. For a general choice of s_1, s_2, s_3, s_4 we have

$$\frac{\partial^2 X_{\mathrm{cl}} \cdot \bar{\partial}^2 X_{\mathrm{cl}}}{\partial X_{\mathrm{cl}} \cdot \bar{\partial} X_{\mathrm{cl}}} = \partial\varphi\bar{\partial}\varphi + \partial\bar{\partial}\varphi \quad (61)$$

and

$$\int_0^{\infty} dy \int_{-\infty}^{+\infty} dx (\partial_a \varphi)^2 = -\frac{\pi}{2}\sum_{i,j=1}^{4} \ln[(s_i - s_j)^2 + (\varepsilon_i + \varepsilon_j)^2]. \quad (62)$$

For $(s_i - s_{i-1}) \gg \varepsilon_i, \varepsilon_{i-1}$ this gives

$$\begin{aligned}\frac{(d-26)}{96\pi}\int_0^{\infty} dy \int_{-\infty}^{+\infty} dx(\partial_a\varphi)^2 &= -\frac{(d-26)}{96}\ln[16 s_{43}^2 s_{42}^2 s_{41}^2 s_{32}^2 s_{31}^2 s_{21}^2 \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4] \\ &= -\frac{(d-26)}{24}\ln[2 s_{43}s_{32}s_{21}s_{41}\varepsilon]\end{aligned} \quad (63)$$

provided

$$\varepsilon_i = \frac{(s_{i+1}-s_i)(s_i-s_{i-1})}{(s_{i+1}-s_{i-1})}\varepsilon \quad (64)$$

as is prescribed by the covariance, where ε is an invariant cutoff of the dimension of length. Equation (63) now reproduces the Lüscher term as $r \to 1$:

$$\frac{(d-26)}{96\pi}\int_0^{\infty} dy \int_{-\infty}^{+\infty} dx(\partial_a\varphi)^2 \to -\frac{(d-26)}{24}\ln[2(1-r)\varepsilon], \quad (65)$$

in view of Eqs. (42) and (47).

Actually, this calculation repeats the one of Ref. [28] for the Polyakov string because there is apparently no difference between the induced and intrinsic metrics through this order of the semiclassical expansion. Alternatively, in the worldsheet parametrization the Lüscher term comes from the determinant resulting from the path integration over Y_{q}^{μ}. As was originally pointed out in Ref. [23], this determinant equals precisely the left-hand side of Eq. (65). We have thus illustrated the statement already made in Sect. II A concerning the difference between the UHP and worldsheet parametrizations.

We are now in a position to compute a semiclassical correction to the Regge trajectory of the effective string theory in $d < 26$. Using Eq. (48) and substituting $\alpha(0) = 1$ for the intercept of the critical string, we obtain

$$\alpha(0) = 1 + \frac{d-26}{24} = \frac{d-2}{24}, \quad (66)$$

reproducing Eq. (49). It is worth emphasizing that the expansion of the effective string theory goes for the scattering amplitude in the parameter

$$\left(\ln\frac{1}{1-r}\right)^{-1} = \left(\ln\frac{s}{t}\right)^{-1}, \quad (67)$$

like it was R^{-1} in Ref. [6]. Therefore, the expansion is justified by the Regge kinematical regime and we assume that the semiclassical Regge trajectory (5) may turn out to be exact.

IV. GENERALIZATION TO $\Delta p_i^2 \neq 0$

If $\Delta p_i^2 \neq 0$, we have to keep the term with $j = i$ in the above equations. Then Eq. (16) is replaced by

$$-\sum_{j\neq i}\frac{2\Delta p_i \cdot \Delta p_j}{s_i - s_j} + \pi\sum_j \Delta p_j^2 \left\langle \frac{\partial G(s_j, s_j)}{\partial s_i}\right\rangle = 0, \quad (68)$$

where

$$\langle G(s_j, s_j)\rangle = \frac{\int \mathcal{D}_{\mathrm{diff}} s G(s_j, s_j)}{\int \mathcal{D}_{\mathrm{diff}} s} \quad (69)$$

with the reparametrization path integral going over functions obeying $s(t_i) = s_i$ which are zero modes of the Douglas minimization.

Substituting

$$\langle G(s_j, s_j)\rangle = \frac{1}{\pi}\ln\frac{(s_{j+1}-s_{j-1})}{(s_{j+1}-s_j)(s_j-s_{j-1})\varepsilon}, \quad (70)$$

that corresponds to the Lovelace choice as is discussed in Refs. [14,15], we get

$$
-\sum_{j\neq i}\frac{2\Delta p_i\cdot\Delta p_j}{s_i-s_j}+\Delta p_{i-1}^2\bigg[\frac{1}{(s_i-s_{i-2})}-\frac{1}{(s_i-s_{i-1})}\bigg]
-\Delta p_i^2\bigg[\frac{1}{(s_i-s_{i-1})}-\frac{1}{(s_{i+1}-s_i)}\bigg]
+\Delta p_{i+1}^2\bigg[\frac{1}{(s_{i+1}-s_i)}-\frac{1}{(s_{i+2}-s_i)}\bigg]=0. \tag{71}
$$

For $M=4$ this results in the same formula (17) with Δp_i^2 included in the definition of the Mandelstam variables.

For $\Delta p_i^2\neq 0$ the term

$$
\frac{1}{\pi^2K^2}\sum_i\frac{\Delta p_i^2y^2}{[(x-s_i)^2+y^2]^2} \tag{72}
$$

appears additionally in the induced metric. It is more singular at the boundary than (33), resulting in

$$
e^{\varphi(s,0)/2}=\sum_i\frac{\sqrt{\Delta p_i^2}}{\pi K}\frac{\varepsilon_i}{[(s-s_i)^2+\varepsilon_i^2]}
\to\sum_i\frac{\sqrt{\Delta p_i^2}}{K}\delta(s-s_i). \tag{73}
$$

This reproduces $\sqrt{\Delta p_i^2}/K$ for the lengths of the polygon edges.

It still has to be verified, however, whether or not the conformal gauge is maintained by this construction for $\Delta p_i^2\neq 0$.

V. APPLICATION TO QCD

As is already mentioned, the QCD string is stretched between quarks, when they are separated by large distances. The results of Refs. [9,14], which state that large loops dominate the sum-over-path representation of QCD scattering amplitudes in the Regge kinematical regime, assume therefore an applicability of the effective string theory ideology in this case.

To illustrate this issue, we start from the representation of M-particle scattering amplitudes in large-N QCD through the Wilson loops:

$$
A(\Delta p_1,\ldots,\Delta p_M)\propto\int_0^\infty d\mathcal{T}\,\mathcal{T}^{M-1}e^{-m\mathcal{T}}
\times\int_{-\infty}^{+\infty}\frac{dt_{M-1}}{1+t_{M-1}^2}\prod_{i=1}^{M-2}\int_{-\infty}^{t_{i+1}}\frac{dt_i}{1+t_i^2}
\times\int_{x^\mu(-\infty)=x^\mu(+\infty)=0}\mathcal{D}x^\mu(t)e^{i\int d\tau\dot{x}(t)\cdot p(t)}J[x(t)]W[x(t)], \tag{74}
$$

where $p^\mu(t)$ is the piecewise constant momentum-space loop (13). For spinor quarks and scalar operators, the weight for the path integration in Eq. (74) is

$$
J[x(t)]=\int\mathcal{D}k^\mu(t)\,\mathrm{sp}\boldsymbol{P}e^{i\int dt[\dot{x}(t)\cdot k(t)-\mathcal{T}\gamma\cdot k(t)/(1+t^2)]}, \tag{75}
$$

where sp and the path ordering refer to γ matrices. In Eq. (74) $W[x(t)]$ is the Wilson loop in pure Yang–Mills theory at large N (or quenched), m is the quark mass and $\mathcal{T}$ is the proper time. For finite N, correlators of several Wilson loops would have to be taken into account.

For the critical string, when Eqs. (3) and (4) are expected to hold for large contours, the path integral over $x^\mu(t)$ in Eq. (74) is Gaussian and can be done as is outlined in Refs. [9,14]. It is saturated in that case by the classical trajectory

$$
x_{\rm cl}^\mu(t)=i\alpha'\int_{-\infty}^{+\infty}dt'\dot{p}^\mu(t')\ln[s(t)-s(t')]^2
=i\alpha'\sum_j\Delta p_j^\mu\ln[s(t)-s_j]^2, \tag{76}
$$

which is T-dual in the sense of Ref. [19] to that given by Eq. (13), giving the same magnitude of the minimal area. It is purely imaginary as this often happens for a saddle point of integrals with an oscillating integrand. Doing the integral over $x^\mu(t)$, we finally obtain

$$
A(\Delta p_1,\ldots,\Delta p_M)\propto\int_0^\infty d\mathcal{T}\,\mathcal{T}^{M-1}e^{-m\mathcal{T}}
\times\int_{-\infty}^{+\infty}\frac{dt_{M-1}}{1+t_{M-1}^2}\prod_{i=1}^{M-2}\int_{-\infty}^{t_{i+1}}\frac{dt_i}{1+t_i^2}
\times\int\mathcal{D}k^\mu(t)\,\mathrm{sp}\boldsymbol{P}e^{-i\mathcal{T}\int dt\gamma\cdot k(t)/(1+t^2)}
\times W\Big[x_*(t)=\frac{1}{K}(p(t)+k(t))\Big]. \tag{77}
$$

For $d\leq 26$ we substitute the Wilson loop in the form of the disk amplitude for the effective string theory with the action (1):

$$
W[x(\cdot)]=\int\mathcal{D}_{\rm diff}t(s)\int_{X^\mu(x=s,0)=x^\mu(t(s))}\mathcal{D}X^\mu(x,y)e^{-KS_{\rm eff}}, \tag{78}
$$

which reproduces Eqs. (3) and (4) in $d=26$, when $S_{\rm eff}$ is quadratic in X^μ. The path integral over $x^\mu(t)$ in Eq. (74) can also be done for $d<26$ within the semiclassical expansion. Equation (76) is then modified in the semiclassical approximation as

$$
x^\mu(t)=x_{\rm cl}^\mu(t)+\alpha'\int_{-\infty}^{+\infty}dt'\ln[s(t)-s(t')]^2\frac{\delta S_{\rm eff}^{(2)}}{\delta x_\mu(t')}, \tag{79}
$$

where $S_{\rm eff}^{(2)}$ stands for the second term on the right-hand side of Eq. (1). Equation (77) remains unchanged with this accuracy. Details of the derivation are described in Appendix B.

As distinct from its stringy counterpart (14), the right-hand side of Eq. (77) has the additional path integration over $k^\mu(t)$, which emerges from Feynman's disentangling of the γ matrices. For small m and/or very large M, the integral over $\mathcal{T}$ in Eq. (74) is dominated by large $\mathcal{T} \sim (M-1)/m$. Then typical values of $k \sim 1/\mathcal{T}$ are essential for large $\mathcal{T}$ in the path integral over $k^\mu(t)$, and we can disregard $k(t)$ in the argument of W in Eq. (77), so the path integral over $k^\mu(t)$ factorizes. We finally obtain [9] from Eq. (74) the product of the string scattering amplitude $A[p(t)]$ times factors which do not depend on p. The substitution of the effective string theory representation (78) into Eq. (74) for $d < 26$ results in a more complicated path integral over $x^\mu(t)$ which is, however, Gaussian within the semiclassical expansion, reproducing again Eq. (77).

Thus, the scattering amplitude $A(\{\Delta p_i\})$ coincides for the ansatz (78) with $W[x_*(t)]$, where $x_*^\mu(t)$ is given by Eq. (13) and the reparametrization path integral goes over the functions $s(t)$, obeying $s(t_i) = t_i$. Therefore, Eq. (77) reproduces for piecewise constant $p^\mu(t)$ the Regge behavior of (off-shell) scattering amplitudes in the effective string theory as $m \to 0$ and/or $M \to \infty$. Since we are dealing with the disk amplitude, associated with planar diagrams, we identify this Regge trajectory with the quark-antiquark Regge trajectory[2] in large N QCD and thus conclude that it is linear in the semiclassical approximation, while the actual intercept can be larger than the value given by Eq. (5) owing to the breaking of the chiral symmetry, as is pointed out in Ref. [14]. The linear trajectory seems to disagree with the old results [26,30], where the path integral over the Liouville field was Gaussian with either Neumann or Dirichlet boundary conditions, so zero modes associated with reparametrizations were not taken into account, as is done now. To my understanding, this emphasizes the very important role played by the reparametrization path integral.

VI. CONCLUSION

We have shown in this paper that the Regge asymptote of scattering amplitudes can be obtained within the ideology of an effective string theory and is not affected by short distances. For this reason, these results are also applicable to QCD string which is generically not the Nambu–Goto one, but behaves like it at large distances. The expansion goes around a long-string configuration and has the meaning of a semiclassical expansion, whose parameter $1/\ln(s/t)$ is small in the Regge kinematical regime of $s \gg -t$.

A linear Regge trajectory (5) of a noncritical string had been vastly discussed in the literature.[3] The intercept $(d-2)/24$ is precisely the value which follows from the spectrum [8]. This result is most probably consistent because the anomaly emerging in the Virasoro quantization vanishes for long strings, as was pointed out in Ref. [5,8].

It is interesting to discuss the relation between our results on the Regge behavior of QCD scattering amplitudes in the framework of the effective string theory and similar known results on the Pomeron [32,33] and the Reggeon [34] (the one we consider) trajectories in the framework of the AdS/CFT correspondence in a confining background. While the minimal surfaces describing the classical part are constructed in both cases for a flat metric, they are apparently different because Refs. [32–34] use an impact-parameter representation of the scattering process and this paper deals with polygonal loops in momentum space. An advantage of our approach is the existence of a systematic expansion in the parameter (67). The way we have calculated the intercept in Sect. III B via semiclassical fluctuations of the minimal surface (given by the Lüscher term) is pretty much similar to the one in Refs. [33,34] except for the difference in the number of fluctuating transverse degrees of freedom, that equals 2 in our case for $d = 4$ from the consistency of the effective string theory in hand, which is also favored by the lattice simulations [2,3] as was already mentioned in the Introduction. This issue can be further clarified by extending our calculations to an annulus amplitude which is to be associated with the Pomeron exchange.

It is also worth mentioning that a semiclassical calculation of the intercept in the framework of the effective string theory, which is close spiritually to our calculation, was performed in Ref. [35] from the spectrum of a rotating string. We emphasize once again that the consideration of this paper refers to the scattering domain of $t < 0$ and deals with the scattering amplitudes.

As distinct from the Polyakov formulation, where the intrinsic metric is treated as an independent variable, in the effective string theory the worldsheet metric is induced. The path integration now goes only over the embedding-space coordinate X^μ and reparametrizations of the boundary contour. The former path integral turns out to be Gaussian within the semiclassical expansion, while the latter one has a well-defined measure and has been recently studied both analytically [9] and numerically [13]. Therefore, the issue of integrating over the Liouville field in the bulk, which was the subject of Ref. [36], does not emerge. It would be very interesting to calculate [37] the string susceptibility $\gamma_{\rm str}$ for large areas within the effective string theory approach and to compare with the existing results.

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[2] It is often called the Reggeon or the secondary Regge trajectory to be distinguished from the vacuum Regge trajectory = Pomeron.

[3] For a historical review see Ref. [31].

APPENDIX A: PROOF OF THE CONFORMAL GAUGE FOR DOUGLAS' MINIMIZATION

Let us show that $X^\mu(x, y)$ obeys the conformal gauge

$$\partial_x X \cdot \partial_y X = 0, \tag{A1a}$$

$$\partial_x X \cdot \partial_x X = \partial_y X \cdot \partial_y X, \tag{A1b}$$

if the Douglas minimization (9) is imposed on $s(t)$ for the given boundary contour $x^\mu(t)$.

We substitute

$$\partial_y \frac{y}{(x-s)^2 + y^2} = -\partial_x \frac{(x-s)}{(x-s)^2+y^2}, \tag{A2}$$

integrate by parts and use the following two identities

$$\frac{1}{(s_1-s_2)}\left\{\frac{1}{[(x-s_1)^2+y^2]} - \frac{1}{[(x-s_2)^2+y^2]}\right\} = \frac{(2x-s_1-s_2)}{[(x-s_1)^2+y^2][(x-s_2)^2+y^2]}, \tag{A3a}$$

$$\frac{1}{(s_1-s_2)}\left\{\frac{(x-s_2)}{[(x-s_1)^2+y^2]} - \frac{(x-s_1)}{[(x-s_2)^2+y^2]}\right\} = \frac{y^2+(x-s_1)^2+(x-s_2)^2+(x-s_1)(x-s_2)}{[(x-s_1)^2+y^2][(x-s_2)^2+y^2]} \tag{A3b}$$

for Eqs. (A1a) and (A1b), respectively. They are then satisfied if the minimization Eq. (9) is fulfilled.

For Eq. (A1a) we have

$$\partial_x X \cdot \partial_y X = -\int \frac{dt_1}{\pi}\frac{dt_2}{\pi} \frac{(x-s(t_1))y}{[(x-s(t_1))^2+y^2][(x-s(t_2))^2+y^2]} = -\int \frac{dt_1}{\pi}\frac{dt_2}{\pi} \frac{(2x-s(t_1)-s(t_2))y}{2[(x-s(t_1))^2+y^2][(x-s(t_2))^2+y^2]} = 0 \tag{A4}$$

in view of Eq. (A3a) and (9).

For Eq. (A1b) we have

$$\begin{aligned}\partial_x X \cdot \partial_x X - \partial_y X \cdot \partial_y X &= \int \frac{dt_1}{\pi}\frac{dt_2}{\pi} \frac{(x-s(t_1))(x-s(t_2)) - y^2}{[(x-s(t_1))^2+y^2][(x-s(t_2))^2+y^2]} \\ &= \int \frac{dt_1}{\pi}\frac{dt_2}{\pi} \frac{(x-s(t_1))(x-s(t_2)) + y^2 + (x-s(t_1))^2 + (x-s(t_2))^2}{[(x-s(t_1))^2+y^2][(x-s(t_2))^2+y^2]} = 0\end{aligned} \tag{A5}$$

in view of Eq. (A3b) and (9).

APPENDIX B: DERIVATION OF EQ. (77)

We collect in this Appendix some formulas which are used in the derivation of Eq. (77).

First of all, let us explain how to understand the variational derivative in Eq. (79). The point is that $S^{(2)}_{\rm eff}$ is a functional of the bulk variable $X^\mu(x, y)$, while $x^\mu(t(s))$ is its boundary value. At the classical level, Eq. (10) holds and we have

$$\frac{\delta X^\mu_{\rm cl}(x, y)}{\delta x^\nu(t)} = \delta^\mu_\nu \frac{\dot{s}(t)}{\pi} \frac{y}{(x-s(t))^2+y^2}, \tag{B1}$$

which reproduces the standard delta function as $y \to 0$.

To derive Eq. (77), it is convenient to use the short-hand notation

$$G * f(t) \equiv \int dt' G(s(t)-s(t')) f(t'). \tag{B2}$$

Then, for example, Eq. (76) takes the form

$$x^\mu_{\rm cl}(t) = -\frac{i}{K} G * \dot{p}^\mu(t) \tag{B3}$$

with

$$G(s) = -\frac{1}{\pi}\ln|s|. \tag{B4}$$

For the exponent in the path integral we have

$$ip_\mu * \dot{x}^\mu - S_{\rm eff}[X] \tag{B5}$$

whose Euler–Lagrange equation including a semiclassical correction reads

$$-i\dot{p}_\mu - KG^{-1} * x_\mu - \frac{\delta S^{(2)}_{\rm eff}}{\delta x^\mu} = 0, \tag{B6}$$

where the inverse to G is

$$G^{-1}(s(t)-s(t')) = \frac{d}{dt}\frac{d}{dt'} G(s(t)-s(t')). \tag{B7}$$

An iterative solution to Eq. (B6) is given by Eq. (79). Substituting this into the exponent (B5), we finally obtain the given order of the semiclassical expansion

$$(B5) = -\frac{1}{2K}\dot{p}_\mu * G * \dot{p}^\mu - S^{(2)}_{\rm eff}[X_{\rm cl}], \tag{B8}$$

where $X^\mu_{\rm cl}$ is reconstructed from (76) by Eq. (10).

The last step in proving Eq. (77) is to note that $X^{\mu}_{\rm cl}$, reconstructed from the boundary value (76), can be replaced by (21), reconstructed from the boundary value (13). The point is that $\ln\sqrt{(x-s_i)^2+y^2}$ and $\arctan[(x-s_i)/y]$ are real and imaginary parts of an analytic function $\ln(z-s_i)$ and obey the Cauchy–Riemann equations. Therefore, $S_{\rm eff}$ does not change under such a replacement. This is similar to the T duality transformation in Ref. [19].

We have thus proved Eq. (77).

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