Gauge invariant treatment of γ_5 in the scheme of ’t Hooft and Veltman

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We propose moving all the γ_5 matrices to the rightmost position before continuing the dimension and show that this simple prescription will enable the dimension regularization scheme proposed by ’t Hooft and Veltman to be consistent with gauge invariance.

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I. INTRODUCTION

The dimensional regularization scheme of ’t Hooft and Veltman [1] has proven to be successful not only for its theoretical implication in renormalizing gauge field theories but also for its practical application in simplifying the calculations of Feynman amplitudes. Only one difficulty of the dimensional regularization scheme remains. As pointed out by ’t Hooft and Veltman in Sec. 6 of [1], the breakdown of Ward identities [2] in the presence of

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1}$$

is caused by the inability of maintaining the validity of the basics identity

$$\not{k}\gamma_5 = (\not{p} + \not{k})\gamma_5 + \gamma_5\not{p} \tag{2}$$

beyond dimension $n = 4$. In this paper, we shall present a simple method to deal with γ_5 which reserves the validity of the basic Ward identity (2) so that the dimensional regularization scheme can still be useful in yielding gauge invariant regularized amplitudes for gauge theories involving γ_5.

In a four-dimensional space, any matrix product

$$\hat{M} = \gamma_{\omega_1}\gamma_{\omega_2}\cdots\gamma_{\omega_n} \quad \text{with} \quad \omega_i \in \{0, 1, 2, 3, 5\}$$

may be reduced, by anticommuting γ_5 to the rightmost position, to either the form of $\pm\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_m}$ with $\mu_i \in \{0, 1, 2, 3\}$ if $\hat{M}$ contains even γ_5 factors, or the form $\pm\gamma_{\nu_1}\gamma_{\nu_2}\cdots\gamma_{\nu_p}\gamma_5$ with $\nu_i \in \{0, 1, 2, 3\}$ if the γ_5 count is odd. The matrix product $\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_m}$ is unambiguously defined under dimensional regularization when the components μ_i run out of the range $\{0, 1, 2, 3\}$ of the first four dimensions. The product $\gamma_{\nu_1}\gamma_{\nu_2}\cdots\gamma_{\nu_p}\gamma_5$ with one γ_5 on the right is continued by defining it to be the product of the analytically continued $\gamma_{\nu_1}\gamma_{\nu_2}\cdots\gamma_{\nu_p}$ and the γ_5 in (1). Before analytic continuation is made, a γ_5-odd matrix product may always be reduced to a matrix product with only one γ_5 factor. To analytically continue such a matrix product, we adopt the default continuation by anticommuting the γ_5 matrix to the rightmost position before making the continuation.

Let us introduce the notation $\underline{p}^\mu$ for the component of the p^μ vector in the first 4 dimensions and the notation p^μ_Δ for the component in the remaining $n - 4$ dimensions; i.e., $p^\mu = \underline{p}^\mu + p^\mu_\Delta$ with $p^\mu_\Delta = 0$ if $\mu \in \{0, 1, 2, 3\}$ and $\underline{p}^\mu = 0$ if $\mu \notin \{0, 1, 2, 3\}$. The Dirac matrix γ^μ is also decomposed as $\gamma^\mu = \underline{\gamma}^\mu + \gamma^\mu_\Delta$ with $\gamma^\mu_\Delta = 0$ when $\mu \notin \{0, 1, 2, 3\}$ and $\underline{\gamma}^\mu = 0$ when $\mu \notin \{0, 1, 2, 3\}$. We then have

$$\gamma_5\gamma^\mu + \gamma^\mu\gamma_5 = 2\gamma^\mu_\Delta\gamma_5, \tag{3}$$

which means that γ_5 does not anticommute with γ^μ when $\mu \notin \{0, 1, 2, 3\}$.

For the QED theory, the identity

$$\frac{1}{\not{\ell} - m}\not{k}\frac{1}{\not{\ell} - \not{k} - m} = \frac{1}{\not{\ell} - \not{k} - m} - \frac{1}{\not{\ell} - m} \tag{4}$$

is the foundation that a Ward identity is built upon. For a gauge theory involving γ_5, there is a basic identity similar to (4) for verifying Ward identities:

$$\begin{aligned}&\frac{1}{\not{\ell} + \not{k} - m}(\not{k} - 2m)\gamma_5\frac{1}{\not{\ell} - m}\\&\quad = \gamma_5\frac{1}{\not{\ell} - m} + \frac{1}{\not{\ell} + \not{k} - m}\gamma_5.\end{aligned} \tag{5}$$

The above identity valid at $n = 4$ is derived by decomposing the vertex factor $(\not{k} - 2m)\gamma_5$ into $(\not{\ell} + \not{k} - m)\gamma_5$ and $\gamma_5(\not{\ell} - m)$ that annihilate, respectively, the propagators of the outgoing fermion with momentum $\ell + k$ and the incoming fermion with momentum ℓ. Positioning γ_5 at the rightmost site, the above identity becomes

$$\begin{aligned}&\frac{1}{\not{\ell} + \not{k} - m}(\not{k} - 2m)\frac{1}{-\not{\ell} - m}\gamma_5\\&\quad = \left(\frac{1}{-\not{\ell} - m} + \frac{1}{\not{\ell} + \not{k} - m}\right)\gamma_5.\end{aligned} \tag{6}$$

If we disregard the rightmost γ_5 on both sides of the above identity, we obtain another identity that is valid at $n = 4$. This new identity, which is void of γ_5, may be analytically continued to hold when $n \neq 4$. We then multiply γ_5 on the right to every analytically continued term of this γ_5-free identity to yield the analytic continuation of the identity (5).

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As a side remark, we note that when we go to the dimension of $n \neq 4$, (5) in the form presented above is not valid. This is because γ_5 defined in (1) does not always anticommute with γ^μ if $n \neq 4$. Adopting the rightmost γ_5 ordering avoids this difficulty, as the validity of the identity in the form of rightmost γ_5 ordering no longer depends on γ_5 anticommuting with the γ matrices.

For an amplitude corresponding to a diagram involving no fermion loops, we shall move all γ_5 matrices to the rightmost position before we continue analytically the dimension n. Subsequent application of dimensional regularization gives us regulated amplitudes satisfying the Ward identities.

If a diagram has one or more fermion loops, the amplitude corresponding to this diagram can be regulated in more than one way. This is because there are different ways to assign the starting position on a fermion loop. Once we have chosen a starting point, we define the matrix product inside the trace by rightmost γ_5 ordering before making the analytic continuation. In general, continuations from different starting points give different values for the trace when $n \neq 4$. An identity relating the traces of matrix products without γ_5 at $n = 4$ can always be analytically continued to hold when $n \neq 4$. Therefore, the portion of an amplitude in which the count of γ_5 on every loop is even has no γ_5 difficulty [3]. But to calculate amplitudes with an odd count of γ_5, we need an additional prescription. This is because, as we have mentioned, the rightmost position on a fermion loop is not defined *a priori*.

Although we have multiple continuations for the trace of a matrix product, they differ from one another by terms containing at least a factor of γ_Δ^μ. In the tree order and in the limit $n \to 4$, they are all restored to the same result because γ_Δ^μ will disappear when $n \to 4$. For higher loop orders, γ_Δ^μ contribution may not be ignored. This is because the factor $\gamma_\Delta^\mu \gamma_\Delta^\nu g_{\mu\nu} = (n - 4)$ multiplied by a divergent integral, which generates a simple pole factor $\frac{1}{(n-4)}$ or a higher-order pole term, is finite or even infinite in the limit $n \to 4$. Thus divergent diagrams with fermion loops are the only type of diagrams that may be ambiguous with respect to the γ_5 positioning. Not incidentally, they are also the diagrams that may be plagued by the anomaly problem [4].

In the dimensional regularization scheme of Breitenlohner and Maison (BM) [5], the γ_5 matrix is also defined as (1). By continuing the Lagrangian to dimension $n \neq 4$ and carefully handling the evanescent terms that are proportional to $(n - 4)$, Breitenlohner and Maison were able to show that dimensional regularization and minimal-subtraction renormalization can be implemented consistently for theories involving γ_5. But there is a major deficiency in this BM scheme: it is not a gauge invariant scheme. Consequently, amplitudes obtained therewith do not satisfy Ward identities and finite counterterms are required to restore the validities of these identities [6–11]. This in fact renders the application of dimensional regularization for chiral gauge theories rather complicated in practical calculation.

There is a naive dimensional regularization (NDR) scheme which assumes that γ_5 satisfies $\gamma_5\gamma^\mu + \gamma^\mu\gamma_5 = 0$ for all μ even when $n \neq 4$. Since no such γ_5 exists, this scheme is not without fault. In particular, it is not capable of producing the triangular anomaly term. While regulated amplitudes satisfying Ward identities have often been obtained in the past with the use of the NDR scheme [12–14], it is because all the γ_5 matrices have been tacitly moved outside of divergent subdiagrams in these calculations. This is to say that the rightmost γ_5 scheme has been employed in actuality.

In contrast to the NDR scheme, the triangular diagrams that are responsible for the anomaly can be handled by the rightmost γ_5 scheme. We shall find that, while there are many choices for the rightmost position on a loop, some choices are in violation of gauge invariance. When no choice obeying all symmetry requirements is available, the Ward identity involving such amplitudes may be broken to give rise to an anomaly.

Körner, Kreimer, and Schilcher (KKS) [15] have shown that if we relinquish the cyclicity of the trace for a matrix product, an anticommuting γ_5 as the one adopted in the NDR scheme can be defined under dimensional regularization. In this KKS formalism, the noncyclic trace becomes the spoiler of Ward identities because the trace of a matrix product with an odd number of γ_5 depends on where the ''reading point'' is designated. For a fermion loop, the reading point in the KKS scheme plays a similar role as the rightmost γ_5 position in our scheme. In fact, they both yield the same dimensionally regularized amplitudes for the fermion loop if the reading point and the rightmost γ_5 are identically positioned. But the prescription for choosing the reading point given in [15] to resolve the noncyclicity difficulty is flawed because the effect of the pole terms arising from loop integrals of divergent subdiagrams or overall diagram on the $O(n - 4)$ difference due to different reading points has been mistakenly ignored. As a consequence, the KKS method presented in [15] does not lead to regularized amplitudes that obey multiloop Ward identities. This problem can be solved, as we shall verify in this paper, by using only the reading points or γ_5 positions that are located outside all self-energy-insertion or vertex-correction subdiagrams and are not at any vertices connecting to external field lines.

By taking advantage of the rightmost γ_5 scheme, which allows us to employ identities such as (2) and (5) given above, we will be able to choose the rightmost positions for the γ_5 to be consistent with the gauge invariance so that Ward identities are preserved. In particular, we shall demonstrate how to apply the rightmost γ_5 scheme to account for the 1-loop anomaly and the preservation of the 2-loop triangular Ward identity in the chiral Abelian-Higgs theory defined in the following section. This Abelian theory has no infrared problem and has a nonfree ghost field due to the particular gauge fixing term that we choose to use. The γ_5 treatment that we shall present for the Abelian theory is also applicable to non-Abelian theories which must be accompanied by ghost interactions.

II. ABELIAN-HIGGS GAUGE THEORY WITH CHIRAL FERMION

The Lagrangian for the Abelian-Higgs gauge theory [16] with chiral fermion is

$$\begin{aligned} L = &-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) - \tfrac{1}{2}\lambda g^2(\phi^\dagger\phi - \tfrac{1}{2}v^2)^2 \\ &+ \bar{\psi}_L(i\not{D})\psi_L + \bar{\psi}_R(i\not{\partial})\psi_R \\ &- \sqrt{2}f(\bar{\psi}_L\phi\psi_R + \bar{\psi}_R\phi^\dagger\psi_L), \end{aligned} \tag{7}$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad D_\mu\phi \equiv (\partial_\mu + igA_\mu)\phi,$$
$$\psi_L = L\psi, \qquad \psi_R = R\psi,$$

with the chiral projection operators L and R defined as

$$L = \tfrac{1}{2}(1 - \gamma_5), \qquad R = \tfrac{1}{2}(1 + \gamma_5).$$

We define two Hermitian fields H and ϕ_2 for the real and imaginary parts of the complex scalar field by

$$\phi = \frac{H + i\phi_2 + v}{\sqrt{2}}. \tag{8}$$

We also introduce two mass parameters M and m defined by

$$M = gv, \qquad m = fv. \tag{9}$$

Both M and m will be regarded as zero order quantities in perturbation. To quantize this theory, we add to the Lagrangian L gauge fixing terms as well as the associated ghost terms. The sum will be called the effective Lagrangian $L_{\rm eff}$, and is invariant under the following BRST [17,18] variations:

$$\begin{aligned} &\delta A_\mu = \partial_\mu c, && \delta\phi_2 = -Mc - gcH, \\ &\delta H = gc\phi_2, && \delta\psi_L = -igc\psi_L, \\ &\delta\psi_R = 0, && \delta\bar{c} = -\frac{i}{\alpha}(\partial^\mu A_\mu - \alpha\Lambda\phi_2), \\ &\delta c = 0, && \end{aligned} \tag{10}$$

where c is the ghost field and $\bar{c}$ is the antighost field. The gauge fixing term is

$$L_{\rm gf} = -\frac{1}{2\alpha}(\partial_\mu A^\mu - \alpha\Lambda\phi_2)^2, \tag{11}$$

and the ghost term is

$$\begin{aligned} L_{\rm ghost} &= i\bar{c}\delta(\partial_\mu A^\mu - \alpha\Lambda\phi_2) \\ &= i\bar{c}(\partial_\mu\partial^\mu + \alpha\Lambda M)c + ig\alpha\Lambda\bar{c}cH. \end{aligned} \tag{12}$$

The BRST invariant effective Lagrangian is

$$L_{\rm eff} = L + L_{\rm gf} + L_{\rm ghost}. \tag{13}$$

A. Graphical identities

The prescription of the rightmost γ_5 ordering under dimensional regularization offers a scheme to construct amplitudes when $n \neq 4$. We now introduce some graphical notations for verifying diagrammatically if the regularized amplitudes so obtained satisfy Ward identities.

According to the Feynman rules, one assigns the factor $-igR\gamma^\mu L$ to the vertex $\bar{\psi} - A^\mu - \psi$ and the factor $-f(L - R)$ to the vertex $\bar{\psi} - \phi_2 - \psi$, as these factors correspond to the terms $-g\bar{\psi}_L\not{A}\psi_L$ and $-if(\bar{\psi}_L\phi_2\psi_R - \bar{\psi}_R\phi_2\psi_L)$ in the interaction Lagrangian of (7). Let us define the following two graphical notations for these two vertices:

$$\underset{\mu}{\times} = -igR\gamma^\mu L, \qquad \bullet = -f(L - R). \tag{14}$$

We also introduce the notation

$$\underset{k}{\otimes} = -gR\not{k}L + mg(L - R), \tag{15}$$

which represents the sum of $-ik_\mu$ times the $\bar{\psi} - A^\mu - \psi$ vertex factor, with k the momentum of the vector particle flowing into the vertex and $-M$ times the $\bar{\psi} - \phi_2 - \psi$ vertex factor. Note that Mf may be equated to mg according to (9). The identity

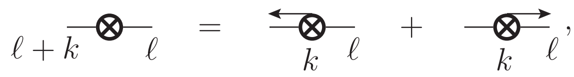
$$R\not{k}L - m(L - R) = (\not{\ell} + \not{k} - m)L - R(\not{\ell} - m), \tag{16}$$

valid in a four-dimensional space, will be our building block for verifying various Ward identities involving fermion lines. Indeed, if we set $L = R = 1$, the identity above becomes the familiar identity used in verifying Ward identities in QED.

Sandwiching Eq. (16) between two fermion propagators, we get

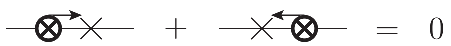
$$\begin{aligned} &\frac{1}{(\not{\ell} + \not{k} - m)}(R\not{k}L - m(L - R))\frac{1}{(\not{\ell} - m)} \\ &= L\frac{1}{(\not{\ell} - m)} - \frac{1}{(\not{\ell} + \not{k} - m)}R. \end{aligned} \tag{17}$$

Note the similarity of this identity with its familiar counterpart (4) in QED. As noted before, when we go to the dimension of $n \neq 4$, (17) in the form presented above is not valid. This is because γ_5 does not always anticommute with γ^μ if $n \neq 4$. Adopting the rightmost γ_5 ordering avoids this difficulty. The above equation multiplied by the coupling constant g may be expressed graphically as



(18)

where the double line emitting from the composite vertex ⊗ indicates that the fermion propagator is annihilated. In addition, the double line together with the composite vertex is to be replaced by $-igL$ if the arrow points to the left, and to be replaced by igR if the arrow points to the right. Thus the following two diagrams cancel each other if the corresponding external momenta are the same:



(19)

In our convention, the direction of any horizontal fermion line is assumed to be pointing to the left side unless indicated otherwise.

B. Cut point

We note that a fermion loop opens up and becomes a fermion line if we make a cut at some point on the loop. We shall always choose as the cut point either the beginning point or the end point of an internal fermion line on the loop. An internal fermion line begins from a vertex and ends at another vertex. When the cut point is chosen to be the end point of an internal fermion line, the vertex factor is then assigned to appear as the beginning factor and stands at the right end of the matrix product for the entire open fermion line. And when the cut point is chosen to be the beginning point of an internal fermion line that emits from a vertex, the matrix factor corresponding to that vertex will be assigned to be the terminating factor and stands at the left end of the matrix product for the entire open fermion line. With the cut point on a fermion loop chosen and with the fermion loop turned into a fermion line, we may apply the rule of rightmost ordering for γ_5.

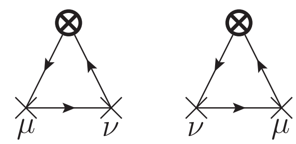
The diagrams in this paper are often cut open at the end of an internal fermion line flowing into a vertex, and as a matter of convenience, we will speak of such a vertex as the cut point. If the vertex is $\bar{\psi} - H - \psi$ with the vertex factor $-if$ or $\bar{\psi} - \phi_2 - \psi$ with the vertex factor $-f\gamma_5$, choosing the cut point to be either the end point of the fermion line flowing into the vertex or the beginning point of the fermion line leaving the vertex gives us identical rightmost γ_5 positioning and therefore the same dimensionally regularized amplitude. Furthermore, if the vertex is $\bar{\psi} - A^\mu - \psi$ and the polarization μ falls within the first 4 dimensions such that its vertex factor anticommutes with γ_5, we also get the same regularized amplitude whether or not the cut point is in the immediate front or rear of the vertex.

III. ONE-LOOP TRIANGULAR DIAGRAMS

Let $\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho; k_1, k_2, k_3)$ denote the one particle irreducible (1PI) AAA amplitude with one fermion loop and three external fields A^μ, A^ν, A^ρ, with k_1, k_2, $k_3 = -k_1 - k_2$ the momenta of A^μ, A^ν, A^ρ, respectively. We may omit the momentum variables k_1, k_2, k_3 if there is no confusion. The superscript (1) signifies that the amplitude is of one loop, while the subscript F signifies the presence of a fermion loop. The directions of the external momenta are inward. Similarly, $\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2; k_1, k_2, k_3)$ denotes the 1PI $AA\phi_2$ amplitude with one fermion loop. The Ward identity that relates $\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)$ to $\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)$ is

$$-ik_3^\rho \Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho; k_1, k_2, k_3) - M\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2; k_1, k_2, k_3) = 0. \qquad (20)$$

Formally, the amplitude for the left side of the above identity (20) is represented by the sum of the following two Feynman diagrams:



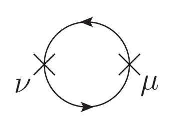
(21)

If we replace the circled cross ⊗ in the two diagrams above by the uncircled cross $\underset{\rho}{\times}$ defined in (14), then these two diagrams become the 1-loop diagrams for the AAA amplitude in (20). Similarly, if we replace the circled cross ⊗ by the black dot • defined in (14), then the two diagrams become the 1-loop diagrams for the $AA\phi_2$ amplitude in (20). Thus the two diagrams in (21) represent the left side of (20). Furthermore, according to Appendix A, only Levi-Civita tensor terms survive in the regularized amplitudes for both AAA and $AA\phi_2$. A 3-point 1PI function is linearly divergent, and the 2nd order term of its Taylor series expanded with respect to its external momenta, henceforth called the T_2 term, is convergent. Thus $T_2[\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)]$ is convergent and may be easily evaluated with any cut point. The result

$$\lim_{n\to 4} T_2[\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)] = \frac{ig^3}{12\pi^2 M}\epsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma} \qquad (22)$$

is unambiguously defined.

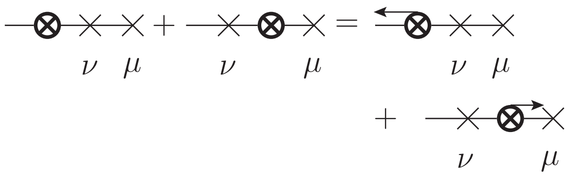
If the composite vertex ⊗ is detached from each of the two diagrams in (21), both diagrams then become identical to



(23)

Since the component diagrams in (21) may be generated by all possible insertions of the composite vertex ⊗ into the internal lines of (23), the diagram in (23) will be called the generator for the Ward identity (20).

By making a cut at the $\bar{\psi} - A^\mu - \psi$ vertex and then by repeated use of (18) and (19), the sum of the two diagrams in (21) becomes



(24)

in which the horizontal line is supposed to be an open fermion line flowing to the left. We emphasize that the identity (24) remains satisfied when $n \neq 4$ if we adopt the rightmost γ_5 dimensional regularization for every term in the identity. Calling the momentum for the fermion line entering the cut point as ℓ, we find that these two amplitudes are, respectively,

$$(-ig^3 L)\frac{1}{\ell + k_1 + k_2 - m}\gamma^\nu L \frac{1}{\ell + k_2 - m}\gamma^\mu L = L\hat{M}(\ell)L$$

and

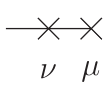
$$\frac{1}{\ell - m}\gamma^\nu L \frac{1}{\ell + k_2 + k_3 - m}(ig^3 R)\gamma^\mu L = -\hat{M}(\ell + k_3)L,$$

where $\hat{M}(\ell)$ stands for

$$-ig^3 \frac{1}{\ell + k_1 + k_2 - m}\gamma^\nu L \frac{1}{\ell + k_2 - m}\gamma^\mu.$$

Since the fermion lines form a closed loop, the trace of the expressions above will be taken. With γ_5 rightmost positioned, $\mathrm{Tr}(L\hat{M}(\ell)L)$ may be reduced to $\mathrm{Tr}(\hat{M}(\ell)L)$ and the amplitudes corresponding to the last two diagrams in (24) are related by a shift of the momentum variable. Since it is legitimate to shift the loop momentum by a finite amount after regularization, the regularized amplitude of (24) vanishes after integration.

Note that the first two cut diagrams in (24) may be generated by attaching the composite vertex ⊗ in all possible manners consistent with Feynman rules to the cut diagram



(25)

obtained by cutting the generator diagram (23) at the $\bar{\psi} - A^\mu - \psi$ vertex. It is convenient to view the identity in which the regularized amplitude of (24) vanishes as being generated by the cut generator in (25). To summarize, if we choose the $\bar{\psi} - A^\mu - \psi$ vertex as the cut point for the generator (23), construct the component diagrams by attaching ⊗, anticommute γ_5 to the rightmost position, and then dimensionally regularize the coefficients in front of γ_5, the regularized amplitudes so obtained satisfy the Ward identity (20).

Similarly, we may open up the fermion loop by choosing the $\bar{\psi} - A^\nu - \psi$ vertex as the cut point, follow through the same arguments, and arrive at another set of amplitudes for $\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)$ and $\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)$. Such amplitudes may also be obtained from the interchange of $(\mu, k_1) \Leftrightarrow (\nu, k_2)$ on the previously defined $\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)$ and $\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)$. Since $\epsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma}$ is invariant under such exchange, the amplitude $T_2[\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)]$ remains the same. Therefore, it may appear in order that the result of (22) is consistent with the Ward identity (20), $T_1[\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)]$ (1st order term in the Taylor series expansion) should be defined such that

$$-ik_3^\rho \lim_{n\to 4} T_1[\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)] = \frac{ig^3}{12\pi^2}\epsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma}. \tag{26}$$

However, we will show, immediately following, that the above condition (26) for AAA amplitude is inconsistent with the Bose permutation symmetry. By definition, $T_1[\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)]$ is a product of a Levi-Civita tensor and a linear combination of the independent external momenta k_1 and k_2. From relativistic covariance, we must have

$$T_1[\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho; k_1, k_2, k_3)] = \epsilon^{\mu\nu\rho\sigma}(C_1 k_{1\sigma} + C_2 k_{2\sigma}), \tag{27}$$

where C_1 and C_2 are dimensionless constants. The Bose symmetry under the exchange of $(A^\mu, k_1) \Leftrightarrow (A^\nu, k_2)$ gives

$$T_1[\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho; k_1, k_2, k_3)] = T_1[\Gamma_F^{(1)}(A^\nu, A^\mu, A^\rho; k_2, k_1, k_3)],$$

which is equivalent to

$$\epsilon^{\mu\nu\rho\sigma}(C_1 k_{1\sigma} + C_2 k_{2\sigma}) = \epsilon^{\nu\mu\rho\sigma}(C_1 k_{2\sigma} + C_2 k_{1\sigma}) = \epsilon^{\mu\nu\rho\sigma}(-C_2 k_{1\sigma} - C_1 k_{2\sigma}). \tag{28}$$

Similarly, the Bose symmetry for the exchange of $(A^\nu, k_2) \Leftrightarrow (A^\rho, k_3)$ yields

$$T_1[\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho; k_1, k_2, k_3)] = T_1[\Gamma_F^{(1)}(A^\mu, A^\rho, A^\nu; k_1, k_3, k_2)],$$

and

$$\epsilon^{\mu\nu\rho\sigma}(C_1 k_{1\sigma} + C_2 k_{2\sigma}) = \epsilon^{\mu\rho\nu\sigma}(C_1 k_{1\sigma} + C_2 k_{3\sigma}) = \epsilon^{\mu\nu\rho\sigma}((C_2 - C_1)k_{1\sigma} + C_2 k_{2\sigma}). \tag{29}$$

To meet (28) and (29), we must have $C_1 = C_2 = 0$. Consequently,

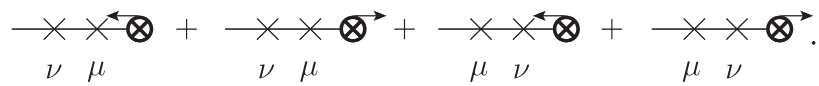
$$T_1[\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)] = 0. \tag{30}$$

This result contradicts the result (26) derived on the basis of the validity of the Ward identity (20), showing that this Ward identity for the triangular diagrams is not consistent with the Bose permutation symmetry.

Note that the diagrams used in the graphical identity (24) with the $\bar{\psi} - A^\mu - \psi$ vertex as the cut point are not Bose symmetric. Nor are those with the $\bar{\psi} - A^\nu - \psi$ vertex as the cut point. Nor are the sum of these two sets of diagrams. This is because we have left out the third vertex as a cut point. In fact, if we choose the vertex $\bar{\psi} - A^\rho - \psi$ to be the cut point for $\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)$ and the vertex $\bar{\psi} - \phi_2 - \psi$ to be the cut point for $\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)$, the left side of (20) is now diagrammatically expressed as


(31)

which, by making use of (18), is expanded into


(32)

Let ℓ be the momentum of the fermion line entering the cut point. The symbol ⊗→ on the right of either the second diagram or the fourth diagram in Eq. (32) is to be replaced by $gR(\ell\!\!/ - m)$ at the right end before moving γ_5 to the rightmost position. If the factor of $(\ell\!\!/ - m)$ at the right end of the second (fourth) diagram annihilates the fermion propagator $\frac{i}{\ell\!\!/ - m}$ at the left end, the amplitude so obtained will cancel the amplitude of the third (first) diagram. But in our scheme of dimensional regularization, we continue to $n \neq 4$ after positioning γ_5 at the rightmost site. This rightmost γ_5 may stand between the $\frac{1}{\ell\!\!/ - m}$ at the left end and the $(\ell\!\!/ - m)$ at the right end to prevent their annihilation in the trace. In fact, when $n \neq 4$, the symbol ⊗→ should be replaced by the expression

$$\begin{aligned}(gR(\ell\!\!/ - m))|_{\text{rightmost}\gamma_5} &= g(\ell\!\!/ L - mR)\\ &= gR(\ell\!\!/ - m) - g\ell\!\!/_\Delta \gamma_5. \end{aligned} \quad (33)$$

The last term $-g\ell\!\!/_\Delta\gamma_5$ in Eq. (33) is the leftover after the cancellation. To evaluate the total amplitude of (32) by dimensional regularization, we only need to take into account the contribution from the leftover terms. Furthermore, due to the presence of $\ell\!\!/_\Delta$, only the divergent orders in the Taylor series expansion with respect to the external momenta may contribute to the $n \to 4$ limit. In particular, the leftover amplitude from the second diagram of (32) is

$$ig^3 \int \frac{d^n \ell}{(2\pi)^n} \times \mathrm{Tr}\left(\frac{1}{\ell\!\!/ - m}\gamma^\mu \frac{\ell\!\!/ - k\!\!/_1}{(\ell - k_1)^2 - m^2}\gamma^\nu \frac{\ell\!\!/ + k\!\!/_3}{(\ell + k_3)^2 - m^2}\ell\!\!/_\Delta \gamma_5\right).$$

The evaluation of the above amplitude in the limit $n \to 4$ is greatly simplified by keeping only the T_2 order term to yield the result

$$-\frac{ig^3}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma}. \quad (34)$$

In Eq. (32), the fourth diagram may be obtained from the second diagram by the exchange $(\mu, k_1) \Leftrightarrow (\nu, k_2)$. Therefore, the leftover amplitude due to the former diagram is the same as that due to the latter diagram and the total amplitude of (31) in the limit $n \to 4$ is equal to twice the amount of (34),

$$-\frac{ig^3}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma}, \quad (35)$$

which is also equal to $-M$ times thrice the amplitude of $T_2[\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)]$ in (22). We have thus shown that the Ward identity (20) is not satisfied if the fermion loops are cut open as in (31).

The values of $T_2[-ik_3^\rho \Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)]$ and $-MT_2[\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)]$ evaluated with respect to the three possible cut points, after factoring out $-ig^3\epsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma}$, are summarized as follows:

cut point	$-ikAAA$	$-MAA\phi_2$	sum
$\times_\mu$	$-\frac{1}{12\pi^2}$	$\frac{1}{12\pi^2}$	0,
$\times_\nu$	$-\frac{1}{12\pi^2}$	$\frac{1}{12\pi^2}$	0,
⊗	$\frac{1}{6\pi^2}$	$\frac{1}{12\pi^2}$	$\frac{1}{4\pi^2}$.

(36)

We have learned that $T_1[\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)]$ vanishes if total permutation symmetry is built into its component diagrams. The amplitude $\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)$ obtained by averaging over the three amplitudes corresponding to the three different cut points chosen at the vertices of $\bar{\psi} - A^\mu - \psi$, $\bar{\psi} - A^\nu - \psi$, and $\bar{\psi} - A^\rho - \psi$ satisfies the permutation symmetry. The amplitude $T_1[\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)]$ so obtained therefore vanishes as can be verified by summing over the three coefficients of the second column in Eq. (36). The amplitude $T_2[\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)]$ is convergent and its value, which is independent of the cut point chosen, remains equal to (22). If we take the average of the T_2 order term for the left-hand side of the Ward identity (20) over the three cuts, this average value does not vanish but is equal to $-MT_2[\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)]$. Since the permutation symmetry of Bose statistics must be obeyed, we have to pay the price of losing the validity of a Ward identity and conclude that there exists an anomaly.

Anomaly compensating fermion field

We have just observed that the ambiguity in choosing the cut point for the 1-loop *AAA* amplitude owes its origin to the singular behavior of its integrand. As a result, the Ward identity for this amplitude is not obeyed and there is an anomaly. Let us add to the theory another fermion field ψ' with a coupling constant $-g$ for ψ'_L equal to the negative of the coupling constant g for ψ_L. The covariant derivative for ψ'_L is

$$D_\mu \psi'_L = (\partial_\mu - igA_\mu)\psi'_L$$

in contrast to the covariant derivative for ψ_L:

$$D_\mu \psi_L = (\partial_\mu + igA_\mu)\psi_L.$$

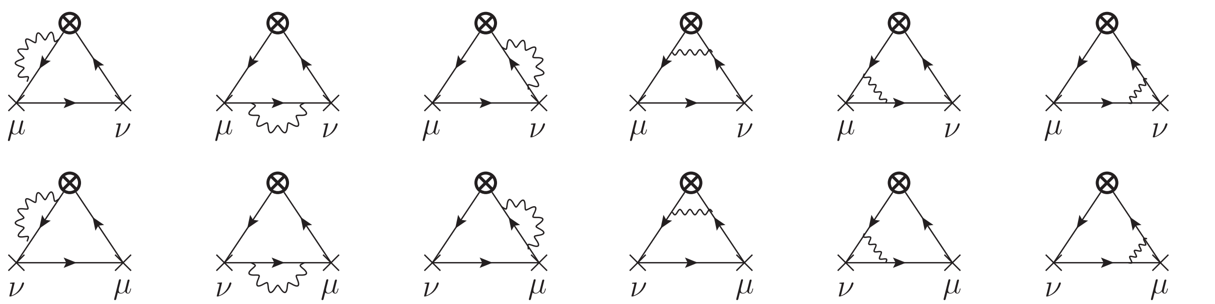
The Lagrangian for such a theory is given by

$$L'_{\rm eff} = L_{\rm eff} + \bar{\psi}'_L(i\not{D})\psi'_L + \bar{\psi}'_R(i\not{\partial})\psi'_R - \sqrt{2}f'(\bar{\psi}'_L\phi^\dagger\psi'_R + \bar{\psi}'_R\phi\psi'_L), \qquad (37)$$

where $L_{\rm eff}$ is defined in (13). Note the coupling f' does not need to be the same as the f in (7) and the masses for the two fermion fields may not be equal. The amplitude for a 1-loop *AAA* diagram with the fermion loop due to the ψ' field is proportional to $(-g)^3$ and cancels the logarithmically divergent term of the amplitude due to the ψ field provided that we have synchronized the cut-point positions for both the ψ and ψ' fermion loops. The 1-loop *AAA* amplitude is convergent and cut point independent. Therefore the theory defined by (37) is free of the 1-loop anomaly.

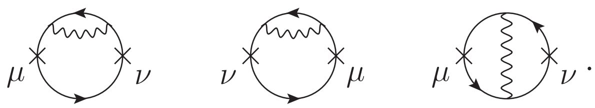
IV. TWO-LOOP TRIANGULAR DIAGRAMS

A straightforward calculation of the 2-loop anomaly is lengthy [19–22] without incorporating a gauge invariant regularization. By utilizing the basic diagrammatic identities (18) and (19), we will be able to choose all rightmost positions for the γ_5 to be in consistency with gauge invariance and prove, without laborious calculation, the vanishing of the 2-loop anomaly. To simplify the presentation in this section, we will consider only the subset of 2-loop triangular diagrams with one fermion loop and one internal vector meson line. Other types of 2-loop triangular diagrams can be handled similarly without additional difficulty and will be addressed in Appendix B. For this restricted type of diagram, the triangular Ward identity is the identity that equates the sum of amplitudes for the following 12 diagrams to zero:



(38)

In the above equation, the fermion loop is the arrowed loop and the wavy lines are vector meson lines. The above 12 diagrams are also the ones generated by attaching the composite ⊗ vertex in all possible manners consistent with Feynman rules to the following three generator diagrams:



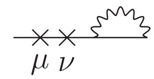
(39)

In order not to give an asymmetric treatment to any of the external fields, we will refrain from using cut points at the vertices connecting to external fields. A cut point is deemed illegitimate if it is positioned at a vertex connecting to an external field line. For the diagrams in (38), cut points at $\underset{\mu}{\times}$, $\underset{\nu}{\times}$, or ⊗ vertices are illegitimate.

Because we do not position γ_5 inside a self-energy or vertex-correction subdiagram on an open fermion line in our prescription, it is also appropriate to avoid cutting the fermion loops at such positions. A position for γ_5 will be called proper if it is not located within a divergent 1PI subdiagram such as a self-energy insertion or a vertex correction. For a fermion loop, a cut and the corresponding cut point will be called proper if the cut is not made within a divergent self-energy insertion or vertex-correction subdiagram.

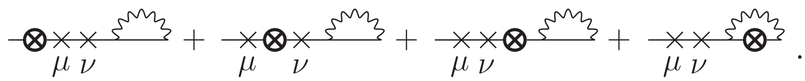
It will be shown in Appendix A that we only need to use cut points at the end points of fermion lines to evaluate the Levi-Civita tensor terms. Therefore, for each of the 12 diagrams in (38), we may choose the cut point to be the one and the only one that is proper, legitimate, and located at the end of an internal fermion line. For convenience, the sum of regularized amplitudes for the 12 cut diagrams so obtained will be denoted by S_{12}. Since none of the external fields is given a preferential treatment, the AAA amplitude obtained by replacing the composite vertex $\otimes$ with the vertex $\underset{\rho}{\times}$ in each of the 12 component diagrams in S_{12} is symmetric with respect to the permutation of the three external vector fields A^μ, A^ν, and A^ρ. This permutation symmetry ensures that the T_1 order term of the AAA amplitude vanishes and therefore S_{12} is superficially convergent.

There are many cancellations among the amplitudes for the 12 component diagrams in S_{12}. For example, making the cut at the end point of the fermion line connecting to the 1-loop fermion self-energy insertion of the second diagram in (39) yields the cut generator



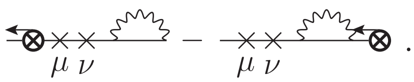
(40)

which gives, after attaching $\otimes$ in all possible manners that are consistent with Feynman rules, the following four component diagrams:



(41)

Using (18) and (19) repeatedly, the above expression can be reduced to



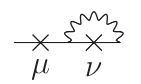
(42)

If we identify $f(\ell_1, \ell_2)$ as the Feynman integrand for the last diagram, where ℓ_2 is the momentum of the vector meson line and ℓ_1 is the momentum of the leftmost fermion line, the Feynman integrand corresponding to (42) is the difference of two terms related by a shift of the loop momentum ℓ_1. Specifically, this sum is

$$f(\ell_1 - k_3, \ell_2) - f(\ell_1, \ell_2), \tag{43}$$

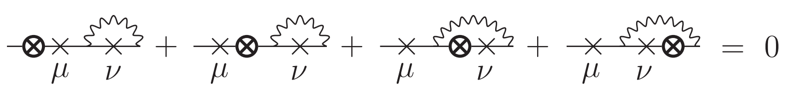
which vanishes upon carrying out the integration $\int d^n\ell_1 d^n\ell_2$ under our scheme of rightmost γ_5 dimensional regularization. The sum of amplitudes for the 4 diagrams in (41) therefore vanishes. Likewise, the sum of the 4 diagrams obtained from (41) by making the exchange $(\mu, k_1) \Leftrightarrow (\nu, k_2)$ also vanishes. By deleting those 4 + 4 diagrams from the 12 diagrams of S_{12}, we are left with 4 diagrams, each of which has a 1-loop vertex-correction subdiagram. Let us define S_4 to be the sum of amplitudes for these 4 remaining diagrams. S_4 is equal to S_{12} and is superficially convergent as well.

Unlike the first two diagrams in (39), no proper cut point is available for the third diagram. Making the cut at the end point of the fermion line connecting to the 1-loop radiative correction for the vertex $\bar{\psi} - A^\nu - \psi$ on the third diagram in (39), we get the cut generator



(44)

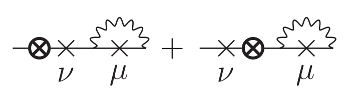
that, after attaching $\otimes$, yields the identity



(45)

For the first two diagrams in the above, the cut points are proper. But for each of the last two diagrams, if we reconnect the beginning point and the end point of the open fermion line to restore the original fermion loop, we see that there is a subdiagram of radiative correction for the vertex $\bar{\psi} - A^\mu - \psi$. The cut point, being the end point of the fermion line in this vertex-correction subdiagram, is improper. From here on in this section, we will identity S_2 as the sum of the last two diagrams in (45). The subdiagram of radiative correction for the vertex $\bar{\psi} - A^\mu - \psi$ in each diagram of S_2 will be denoted by H.

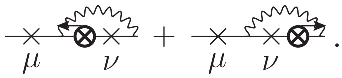
For both diagrams in S_2, if the improper cut point inside H is moved out of H to the end point of the fermion line connecting to H, the relocated cut becomes proper and S_2 becomes



(46)

Since all the fermion lines and vertex factors sandwiched between the original cut points in S_2 and the relocated ones in (46) lie within the subdiagram H, the difference between S_2 and (46) may be expressed as a combination of terms with γ_Δ factors stemming from the matrix product in H. These γ_Δ factors may not be ignored if they are multiplied by pole terms arising from the logarithmically divergent loop integrations due to the subdiagram H or the overall diagram.

Making repetitive use of (18) and (19), S_2 can be transformed into



(47)

There is a similar transformation for (46). The divergent loop integration of the subdiagram H only occurs in the T_0 term of H, denoted by $T_0[H]$, which is the amplitude of H with all the external momenta relative to H set to zero. If we substitute $T_0[H]$ for H in either diagram of (47), the resulting amplitude must vanish because it depends on only one external momentum k_2 and two external polarizations μ and ν, which are insufficient to form a Levi-Civita tensor term. Thus the divergent loop integral of H does not contribute to S_2. Similarly, the divergence of H does not contribute to (46).

Let us define $\bar{S}_2$ to be the first two diagrams in (45). The two diagrams of (46) may be obtained from $\bar{S}_2$ by making the interchange $(\mu, k_1) \Leftrightarrow (\nu, k_2)$. Since S_4 consists of the two diagrams of $\bar{S}_2$ and the two diagrams of (46), it is symmetric under $(\mu, k_1) \Leftrightarrow (\nu, k_2)$. In the Taylor series expansion with respect to the external momenta for S_2, $\bar{S}_2$, or S_4, only the second order T_2 term may have superficially divergent Levi-Civita tensor terms. In addition, any such T_2 order term is equal to some constant times $\epsilon^{\mu\nu\rho\sigma} k_{1\rho} k_{2\sigma}$ which is invariant under $(\mu, k_1) \Leftrightarrow (\nu, k_2)$. Thus $T_2[S_4]$ is equal to twice $T_2[\bar{S}_2]$. Since we have shown that S_4 is superficially convergent, $\bar{S}_2$ is also superficially convergent. Furthermore, the identity (45), which is equivalent to $\bar{S}_2 + S_2 = 0$, ensures that S_2 is superficially convergent as well. There is no divergent pole term to prevent the difference between S_2 and (46) from vanishing in the limit $n \to 4$. As a consequence, both S_4 and S_{12} vanish in the limit $n \to 4$, and we have succeeded in regularizing and preserving the 2-loop triangular Ward identity under dimensional regularization.

In the 1-loop case, one of the three external vertices must be used as the cut point and we have shown it is impossible to construct a set of diagrams to satisfy both the triangular Ward identity and Bose permutation symmetry. For the 2-loop diagrams we have discussed here, there is the additional freedom of choosing cut points at vertices connecting to internal vector meson lines. As a result, we are able to construct diagrams that satisfy both the triangular Ward identity and Bose permutation symmetry.

The $AA\phi_2$ function is superficially convergent and its renormalized amplitude can be calculated with any convenient choice of proper cut point. Since the 2-loop triangular Ward identity can be regularized and renormalized by minimal subtractions without violating Bose permutation symmetry, the T_2 term of the renormalized $AA\phi_2$ amplitude can be expressed as a linear combination of the T_1 term of the renormalized AAA amplitude. Knowing that $T_1[AAA]$ vanishes on the sole account of permutation symmetry, $T_2[AA\phi_2]$ must vanish as well. This condition has been verified by direct calculation [22] without using dimensional regularization.

V. CONCLUSION

In this paper, we have found a simple and natural way to treat γ_5 in dimensional regularization: moving all γ_5 matrices to the rightmost position before analytically continuing the dimension. For amplitude corresponding diagrams without fermion loops, the amplitudes obtained with our prescription automatically satisfy the Ward identities without further ado.

The rightmost position on a fermion loop is not defined. For this reason, we introduce the concept of a cut point. We have found that the choice of a cut point often conflicts with gauge invariance. From this vantage point, this lack of a rightmost position is what breaks the Ward identities, leading to triangular anomalies.

Applying our prescriptions to 1-loop triangular amplitudes, we reproduce correctly the value of the triangular anomaly, verifying that our prescription is applicable to diagrams with anomalies. For a 1-loop fermion self-energy diagram or a 1-loop vertex-correction diagram, positioning γ_5 within the divergent 1PI diagram gives an amplitude differing from the amplitude obtained with rightmost γ_5 by a finite amount, even after subtraction of pole terms. Thus for a 2-loop diagram with a fermion loop and with a 1-loop self-energy insertion or a 1-loop radiative vertex insertion, we do not assign a point inside a divergent 1-loop subdiagram as a cut point. Furthermore, in order not to give a preferential role to any of the external lines, we do not choose the point of the vertex connecting to an external field line as a cut point. We have shown that this prescription of utilizing proper and legitimate cut points enables us to regulate amplitudes in a gauge invariant manner.

APPENDIX A: CHARGE CONJUGATION TRANSFORMATION

If we disregard terms involving fermion fields, the effective Lagrangian (13) is invariant under the following charge conjugation transformation:

$$H \to H, \qquad \phi_2 \to -\phi_2, A^\mu \to -A^\mu, \bar{c} \to -\bar{c}, c \to -c. \tag{A1}$$

Fields that have odd (even) charge parity shorthanded as C parity under this transformation are classified as C-odd (C-even) fields. For nonfermion fields, H is C even and non-H fields are C odd. We define the C parity of a Feynman diagram to be the product of the C parities of its nonfermion external lines. A vertex without fermion lines attached is always C even, and so is a fermionless Feynman diagram. It is therefore impossible to construct a C-odd diagram without including fermion lines or loops. For a theory that does not involve γ_5, such as QED, the charge conjugation transformation is a symmetry of its Lagrangian. A consequence of this symmetry is the Furry theorem which states that any amplitude for an odd number of external vector fields such as the AAA amplitude vanishes in QED.

In four-dimensional space, the charge conjugation transformation $\psi \to C\bar{\psi}^T$ for fermion fields is effected by the matrix

$$C = i\gamma^2\gamma^0 \tag{A2}$$

that satisfies

$$C\gamma^\mu C^{-1} = -(\gamma^\mu)^T. \tag{A3}$$

The above identity is based on the property that γ^0 and γ^2 are symmetric matrices while γ^1 and γ^3 are antisymmetric in four-dimensional space. It is not guaranteed that this property specific to $n = 4$ may be dimensionally continued such that (A3) holds when $n \neq 4$.

We shall not assume the validity of (A3) when $n \neq 4$ and define instead the charge conjugation for a matrix product of N γ matrices $\hat{M} = \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_N}$ as $\hat{M}^C = (-\gamma^{\mu_N})\cdots(-\gamma^{\mu_2})(-\gamma^{\mu_1})$ which is the product of the negative of these N γ matrices in reversed order. When $n = 4$, we may make use of (A3) to verify straightforwardly that the trace of $\hat{M}$ is the same as that of $\hat{M}^C$ or

$$\mathrm{Tr}\,(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_N}) = \mathrm{Tr}((-\gamma^{\mu_N})\cdots(-\gamma^{\mu_2})(-\gamma^{\mu_1})). \tag{A4}$$

Since both sides in (A4) consist of terms that are products of $g^{\mu_i\mu_j}$ metric tensors, the polarizations $\mu_1, \mu_2, \ldots, \mu_N$ may be dimensionally continued beyond the first 4 dimensions so that (A4) is also valid when $n \neq 4$. The validity of

$$\mathrm{Tr}\,(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_N}\gamma^0\gamma^1\gamma^2\gamma^3) = \mathrm{Tr}(\gamma^3\gamma^2\gamma^1\gamma^0(-\gamma^{\mu_N})\cdots(-\gamma^{\mu_2})(-\gamma^{\mu_1}))$$

also yields

$$\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_N}\gamma_5) = \mathrm{Tr}(\gamma_5(-\gamma^{\mu_N})\cdots(-\gamma^{\mu_2})(-\gamma^{\mu_1})), \tag{A5}$$

where $\mu_1, \mu_2, \ldots, \mu_N$ are allowed to be polarizations in arbitrary n dimensional space.

We will use the notation $\hat{M}_{DR}$ with the subindex DR to indicate that $\hat{M}_{DR}$ is the matrix product obtained from $\hat{M}$ by anticommuting all the γ_5 matrices with γ matrices to the right and then continuing to $n \neq 4$. Conditions (A4) and (A5) in the above may be summarized as

$$\mathrm{Tr}\,(\hat{M}_{DR}) = \mathrm{Tr}((\hat{M}_{DR})^C). \tag{A6}$$

The accumulated sign change resulting from moving a γ_5 in $\hat{M}$ to the rightmost position is equal to that from moving the corresponding γ_5 in $\hat{M}^C$ to the leftmost position. We thus have

$$(\hat{M}_{DR})^C = (\hat{M}^C)_{DL}, \tag{A7}$$

where the subscript DL means that the analytical continuation to $n \neq 4$ starts from the expression obtained after anticommuting all the γ_5 factors to the leftmost position. If the count of γ^μ matrices with $\mu \in \{0, 1, 2, 3\}$ in a matrix product is odd, the trace of the matrix product is zero and so is its continuation from any form. Hence, the γ_5 factor at the leftmost position of $(\hat{M}^C)_{DL}$ in a trace may be moved to the rightmost position to yield

$$\mathrm{Tr}\,((\hat{M}^C)_{DL}) = \mathrm{Tr}((\hat{M}^C)_{DR}). \tag{A8}$$

Combining (A6)–(A8), in the above, we get

$$\mathrm{Tr}\,(\hat{M}_{DR}) = \mathrm{Tr}((\hat{M}^C)_{DR}). \tag{A9}$$

Let G be a Feynman diagram with a fermion loop that has been cut open at the point P. The conjugate diagram G^C is defined to be the diagram obtained by reversing the direction of the fermion loop in G. The point P remains to be the cut point of G^C. If the cut point P on G is the end point of a certain fermion line on the loop, it becomes the beginning point of the reversed fermion line in G^C, and vice versa.

The identity (A9) may be utilized to show that dimensionally regularized amplitudes for G and G^C are related. To be more specific, let F be a fermion loop attached by fields in the sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$ with inward momenta $k_1, k_2, \ldots, k_n$ and the cut point is chosen to be the end point of the internal fermion line flowing into the vertex of φ_1. The Feynman integrand $I(F)$ for F may be written as

$$I(F) = \mathrm{Tr}\left(\frac{i}{\not\ell - m}\varpi(\varphi_n)\frac{i}{\not\ell + \not k_1 + \not k_2 \cdots + \not k_{n-1} - m}\varpi(\varphi_{n-1})\cdots\frac{i}{\not\ell + \not k_1 + \not k_2 - m}\varpi(\varphi_2)\frac{i}{\not\ell + \not k_1 - m}\varpi(\varphi_1)\right)_{DR}, \tag{A10}$$

where ℓ is the loop momentum variable and the vertex factors are

$$\varpi(A^\mu) = -igR\gamma^\mu L, \qquad \varpi(\phi_2) = f\gamma_5, \quad \text{and} \quad \varpi(H) = -if.$$

According to the identity (A9), performing the charge conjugation operation on the matrix product inside the trace of (A10) leaves the value of $I(F)$ unchanged. Thus,

$$I(F) = \mathrm{Tr}\left(\tilde{\varpi}(\varphi_1)\frac{i}{-(\not\ell + \not k_1) - m}\tilde{\varpi}(\varphi_2)\frac{i}{-(\not\ell + \not k_1 + \not k_2) - m}\cdots\tilde{\varpi}(\varphi_{n-1})\frac{i}{-(\not\ell + \not k_1 + \not k_2 \cdots + \not k_{n-1}) - m}\tilde{\varpi}(\varphi_n)\frac{i}{-\not\ell - m}\right)_{DR},$$

where

$$\tilde{\varpi}(A^\mu) = igL\gamma^\mu R, \qquad \tilde{\varpi}(\phi_2) = f\gamma_5, \quad \text{and} \quad \tilde{\varpi}(H) = -if.$$

We are allowed to make the transformation $\ell \to -\ell$ in carrying out the $\int d^n\ell$ loop integration and arrive at

$$\int d^n\ell I(F) = \int d^n\ell \,\mathrm{Tr}\left(\tilde{\varpi}(\varphi_1)\frac{i}{\ell - k_1 - m}\tilde{\varpi}(\varphi_2)\frac{i}{\ell - k_1 - k_2 - m}\cdots\tilde{\varpi}(\varphi_{n-1})\frac{i}{\ell - k_1 - k_2\cdots - k_{n-1} - m}\tilde{\varpi}(\varphi_n)\frac{i}{\ell - m}\right)_{DR}. \tag{A11}$$

On the other hand, the conjugate diagram F^C is the fermion loop with the external fields attached on the loop in the order of $\varphi_n, \varphi_{n-1}, \ldots, \varphi_1$, and with the cut point being the beginning point of the fermion line that leaves the vertex of φ_1. The Feynman integrand for F^C may be written as

$$I(F^C) = \mathrm{Tr}\left(\varpi(\varphi_1)\frac{i}{\ell - k_1 - m}\varpi(\varphi_2)\frac{i}{\ell - k_1 - k_2 - m}\cdots\varpi(\varphi_{n-1})\frac{i}{\ell - k_1 - k_2\cdots - k_{n-1} - m}\varpi(\varphi_n)\frac{i}{\ell - m}\right)_{DR}. \tag{A12}$$

Let us observe that

$$\tilde{\varpi}(A^\mu) = -\varpi(A^\mu)|_{\gamma_5\to-\gamma_5},$$
$$\tilde{\varpi}(\phi_2) = -\varpi(\phi_2)|_{\gamma_5\to-\gamma_5},$$
$$\tilde{\varpi}(H) = \varpi(H)|_{\gamma_5\to-\gamma_5}.$$

These relationships demonstrate that if we insert an additional negative sign in front of every γ_5, the vertex factors $\tilde{\varpi}(A^\mu)$, $\tilde{\varpi}(\phi_2)$, and $\tilde{\varpi}(H)$ become $-\varpi(A^\mu)$, $-\varpi(\phi_2)$, and $\varpi(H)$, respectively. Note also that the integrand in (A11) becomes the integrand $I(F^C)$ in (A12) if all the vertex factors $\tilde{\varpi}(\varphi)$ in (A11) are replaced by $\varpi(\varphi)$. Thus we have

$$\int d^n\ell I(F^C) = (-1)^{N_C(F)}\int d^n\ell I(F)|_{\gamma_5\to-\gamma_5}, \tag{A13}$$

where $N_C(F)$ is the number of C-odd fields in $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ and $(-1)^{N_C(F)}$ is the C parity of the diagram F or F^C. Decomposing the identity (A13) into the γ_5-even part and the γ_5-odd part, we get

$$\gamma_5\text{-even part of } \int d^n\ell I(F^C) = \gamma_5\text{-even part of } (-1)^{N_C(F)} \times \int d^n\ell I(F) \tag{A14}$$

and

$$\gamma_5\text{-odd part of } \int d^n\ell I(F^C) = \gamma_5\text{-odd part of } (-1)^{N_C(F)+1} \times \int d^n\ell I(F). \tag{A15}$$

If the fermion loop F is a subdiagram of a larger diagram G that contains no other fermion lines than those in F, then the C parity of G is equal to the C parity of F. Since the Feynman integrand for the complement of F in G is the same as that for the complement of F^C in G^C, (A13)–(A15) are also valid if we replace F with G. If G is C even, the γ_5-even part of the dimensionally regularized amplitude of G is equal to the γ_5-even part of G^C but the γ_5-odd part of G is the negative of the γ_5-odd part of G^C. If G is C odd, the γ_5-odd part of G is equal to the γ_5-odd part of G^C and the γ_5-even part of G is the negative of the γ_5-even part of G^C.

We will require that if a diagram G is included as a component diagram, the conjugate diagram G^C must also be included (with, of course, suitable adjustment of weighting factors). Since the γ_5-odd parts are canceled between G and G^C when G is C even, no Levi-Civita tensor term is possible for C-even functions.

For C-odd functions, the γ_5-even parts are canceled between G and G^C. If we discard the γ_5-even part, either G or G^C suffices for the evaluation of the C-odd function. We will use the one whose cut point is located at the end point of an internal fermion line. In other words, the Levi-Civita tensor terms for C-odd functions may be evaluated by diagrams whose cut points are restricted to the subset of end points of internal fermion lines on the fermion loops.

APPENDIX B: GREEN FUNCTIONS AND WARD IDENTITIES

In the main context, we only consider Feynman diagrams in which the nonfermion internal lines are the vector meson lines. To handle other types of diagrams, we will make use of Green functions.

The Green function G is the vacuum expectation value of a time-ordered product. Specifically,

$$G(O_1(x_1), O_2(x_2), \ldots, O_n(x_n)) = T\langle O_1(x_1)O_2(x_2)\cdots O_n(x_n)\rangle, \tag{B1}$$

where the operator $O_i(x_i)$ is either a field operator or a product of field operators at the same space-time point x_i. The connected Green function, denoted by G_c, is

$$G_c(\cdots) = \text{all connected diagrams of } G(\cdots).$$

We need a notation to indicate that some external lines of a Green function are amputated. To denote a truncated external line, we underline the corresponding field variable in the Green function. That is,

$$G(\dots,\varphi_i,\dots) = D(\varphi_i,\varphi_j)G(\dots,\underline{\varphi_j},\dots),$$
$$G_c(\dots,\varphi_i,\dots) = D(\varphi_i,\varphi_j)G_c(\dots,\underline{\varphi_j},\dots), \tag{B2}$$

where the propagator $D(\varphi_i, \varphi_j)$ is also the two-point Green function,

$$D(\varphi_i,\varphi_j) = G(\varphi_i,\varphi_j).$$

Note that in (B2) the space-time dependence of the field variable φ_i is lumped into the index i, and the Einstein summation convention for the repeated index j is extended to include the summation over all possible field types and integration of space-time points. The fully truncated Green function Γ is the connected Green function with all field variables underlined.

$$\Gamma(\varphi_1,\varphi_2,\dots,\varphi_n) = G_c(\underline{\varphi_1,\varphi_2,\dots,\varphi_n}).$$

In particular, $\Gamma(\varphi_i,\varphi_j)$ is the inverse propagator,

$$\Gamma(\varphi_i,\varphi_j) = G(\underline{\varphi_i,\varphi_j}) = D^{-1}(\varphi_j,\varphi_i).$$

For a composite operator $\hat{O}$, which is a product of field operators at the same space-time point, we define

$$\Gamma(\varphi_1,\varphi_2,\dots,\varphi_n,\hat{O}) = G_c(\underline{\varphi_1,\varphi_2,\dots,\varphi_n},\hat{O}).$$

Note that to avoid misinterpretations, $\hat{O}$ is forbidden to be a single field operator in the above identification. The tree order part of a Green function $\mathcal{F}$, which may be any of the above G, G_c, or Γ function, will be denoted by the notation $\mathcal{F}^{(0)}$ with the superscript (0). The Fourier transform of a Green function $\mathcal{F}$ is labeled by an additional group of momentum variables and is related to its counterpart in the coordinate space by

$$\begin{aligned}\mathcal{F}(\varphi_1(x_1),\varphi_2(x_2),\dots,\varphi_n(x_n)) &= \int\frac{dk_1}{(2\pi)^4}\frac{dk_2}{(2\pi)^4}\cdots\frac{dk_{n-1}}{(2\pi)^4}e^{-i(k_1x_1+k_2x_2+\cdots k_nx_n)}\\ &\quad\times\mathcal{F}(\varphi_1,\varphi_2,\dots,\varphi_n;k_1,k_2,\dots,k_n),\end{aligned}$$

where $k_1+k_2+\cdots+k_n=0$. We will omit the momentum variables $k_1,k_2,\dots,k_n$ for the Fourier transform if there is little chance of confusion.

1. Basic graphical identities

The BRST invariance leads to a number of Ward identities which form an important part of the foundation on which renormalizability is based. These identities can be formally derived in the following way. The vacuum state $|0\rangle$ in the theory satisfies

$$Q|0\rangle = 0, \tag{B3}$$

where Q is the BRST charge. The commutator (anticommutator) of iQ with a nonghost (ghost) field is equal to the BRST variation of the field. Because of (B3), we have

$$T\langle 0|iQ\varphi_1(x_1)\varphi_2(x_2)\cdots|0\rangle = 0,$$

where φ_i is a field operator. By moving iQ to the right until it operates on $|0\rangle$ and vanishes, we get

$$\begin{aligned}T\langle 0|\delta(\varphi_1(x_1)\varphi_2(x_2)\cdots)|0\rangle &= T\langle 0|\delta\varphi_1(x_1)\varphi_2(x_2)\cdots|0\rangle\\ &\quad\pm T\langle 0|\varphi_1(x_1)\delta\varphi_2(x_2)\cdots|0\rangle\\ &\quad\pm\cdots = 0.\end{aligned} \tag{B4}$$

The relative sign between terms is determined by the positions of the ghost fields. The above BRST identity is formal and its renormalized version may not be satisfied when anomaly exists. But the tree order terms are finite and always satisfy the BRST identity provided the Lagrangian is BRST invariant. To facilitate the discussions for higher loop order terms, we will introduce graphical notations for some basic tree order identities.

The BRST variation for any field variable $\varphi(x)$ in general may be decomposed as

$$\delta\varphi(x) = \delta_1\varphi(x) + \delta_2\varphi(x), \tag{B5}$$

in which $\delta_1\varphi(x)$ is a linear superposition of field variables and $\delta_2\varphi(x)$ is a product of the ghost field $c(x)$ and another field variable at the same space-time point x. For the Abelian-Higgs theory with the Lagrangian (13), nonvanishing $\delta_1\varphi$ are $\delta_1A^\mu = \partial^\mu c$ and $\delta_1\phi_2 = -Mc$, and nonvanishing $\delta_2\varphi$ are $\delta_2H = gc\phi_2$, $\delta_2\phi_2 = -gcH$, and $\delta_2\psi_L = -igc\psi_L$.

By (B4), we have

$$\begin{aligned}T\langle 0|(\delta\bar{c}(z))\varphi_i|0\rangle_{(0)} &= T\langle 0|\bar{c}(z)\delta\varphi_i|0\rangle_{(0)}\\ &= \frac{\partial\delta_1\varphi_i}{\partial c(z')}D^{(0)}(\bar{c}(z),c(z')),\end{aligned} \tag{B6}$$

where the subscript and superscript (0) refer to tree order terms and $\frac{\partial\delta_1\varphi_i}{\partial c}$ is a constant or constant operator. Next, let us assume that φ_j and φ_k are nonghost fields. Then (B4) yields

$$\begin{aligned}T\langle 0|(\delta\bar{c}(z))\varphi_j\varphi_k|0\rangle_{(0)} &= T\langle 0|\bar{c}(z)(\delta\varphi_j)\varphi_k|0\rangle_{(0)}\\ &\quad+ T\langle 0|\bar{c}(z)\varphi_j(\delta\varphi_k)|0\rangle_{(0)}.\end{aligned} \tag{B7}$$

According to the definition (B2) for the Green function with underlined arguments, the left side of (B7) may be expressed as

$$\begin{aligned} T\langle 0|(\delta\bar{c}(z))\varphi_j\varphi_k|0\rangle_{(0)} &= D^{(0)}(\delta\bar{c}(z),\varphi_i)G^{(0)}(\underline{\varphi_i},\varphi_j,\varphi_k) \\ &= D^{(0)}(\bar{c}(z),c(z'))\frac{\partial\delta_1\varphi_i}{\partial c(z')}G^{(0)}(\underline{\varphi_i},\varphi_j,\varphi_k). \end{aligned}$$

In the tree order, the antighost $\bar{c}(z)$ field in $T\langle 0|\bar{c}(z)\times(\delta\varphi_j)\varphi_k|0\rangle_{(0)}$, which is the first term on the right side of (B7), must be paired under Wick contraction with the ghost c field in $\delta\varphi_j$ or with the c field from the interaction Lagrangian, and we have

$$\begin{aligned} T\langle 0|\bar{c}(z)(\delta\varphi_j)\varphi_k|0\rangle_{(0)} &= D^{(0)}(\bar{c}(z),c(z')) \\ &\times\left[D^{(0)}\left(\frac{\partial\delta_2\varphi_j}{\partial c(z')},\varphi_k\right)+G^{(0)}(\underline{c(z')},\delta_1\varphi_j,\varphi_k)\right]. \end{aligned} \tag{B8}$$

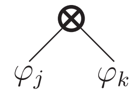
Note that we have discarded $D^{(0)}(\frac{\partial\delta_1\varphi_j}{\partial c},\varphi_k)$ owing to the vanishing vacuum expectation $\langle\varphi_k\rangle=0$. For the 2nd term on the right side of (B7), there is an expression similar to (B8). The identity (B7), after factoring out the common ghost propagator $D^{(0)}(\bar{c}(z),c(z'))$ and then replacing z' by z, becomes

$$\begin{aligned} \frac{\partial\delta_1\varphi_i}{\partial c(z)}G^{(0)}(\underline{\varphi_i},\varphi_j,\varphi_k) &= D^{(0)}\left(\frac{\partial\delta_2\varphi_j}{\partial c(z)},\varphi_k\right) \\ &+ D^{(0)}\left(\varphi_j(x),\frac{\partial\delta_2\varphi_k}{\partial c(z)}\right) \\ &+ G^{(0)}(\underline{c(z)},\delta_1\varphi_j,\varphi_k) \\ &+ G^{(0)}(\underline{c(z)},\varphi_j,\delta_1\varphi_k). \end{aligned} \tag{B9}$$

The definition (15) for the composite vertex $\otimes$ on a fermion line may be extended to include other types of vertices. The extended composite vertex is defined as

$$\begin{aligned} \otimes &= \frac{\partial\delta_1\varphi_i}{\partial c}\Gamma^{(0)}(\varphi_i,\varphi,\varphi') \\ &= -ik_\mu\Gamma^{(0)}(A^\mu,\varphi,\varphi')-M\Gamma^{(0)}(\phi_2,\varphi,\varphi'), \end{aligned} \tag{B10}$$

where the tree order amplitude $\Gamma^{(0)}(\varphi_i,\varphi,\varphi')$ stands for the vertex factor of $\varphi_i-\varphi-\varphi'$ and k is the incoming momentum of the vector field A_μ or scalar field ϕ_2. Note that this definition is the same as the restricted one of (15) when φ and φ' are the fermion fields ψ and $\bar{\psi}$. The amplitude $\frac{\partial\delta_1\varphi_i}{\partial c}G^{(0)}(\underline{\varphi_i},\varphi_j,\varphi_k)$ can then be diagrammatically expressed as a composite vertex $\otimes$ connected with two propagator lines to fields φ_j and φ_k:

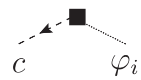


$$= \frac{\partial\delta_1\varphi_i}{\partial c}G^{(0)}\left(\underline{\varphi_i},\varphi_j,\varphi_k\right). \tag{B11}$$

Let us use a solid black box to graphically represent the $c-\bar{c}-\varphi$ vertex. Then the Green function

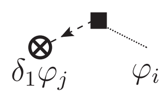
$$\begin{aligned} G^{(0)}(\underline{c},c,\varphi_k;k_1,k_2,k_3) &= D^{(0)}(c,\bar{c};k_2) \\ &\times\Gamma^{(0)}(c,\bar{c},\varphi_i)D^{(0)}(\varphi_i,\varphi_k;k_3), \end{aligned} \tag{B12}$$

can be diagrammatically expressed as



$$= G^{(0)}\left(\underline{c},c,\varphi_i\right), \tag{B13}$$

where the dotted arrowed line corresponds to the ghost propagator $D^{(0)}(c,\bar{c})$. Let us also define

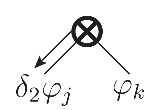


$$\begin{aligned} &= G^{(0)}\left(\underline{c},\delta_1\varphi_j,\varphi_i\right) \\ &= \frac{\partial\delta_1\varphi_j}{\partial c}G^{(0)}\left(\underline{c},c,\varphi_i\right). \end{aligned} \tag{B14}$$

Since $\delta_2\varphi_j$ is a product of one ghost c field and another nonghost field, taking the partial derivative with respect to c as in $\frac{\partial\delta_2\varphi_j}{\partial c}$ is equivalent to factoring out the c field to retain the nonghost factor. $D^{(0)}(\frac{\partial\delta_2\varphi_j}{\partial c},\varphi_k)$ is thus proportional to the free propagator that propagates the field φ_k to the nonghost field in $\delta_2\varphi_j$. In particular, if $\varphi_j=\phi_2$ and $\varphi_k=H$, then $\delta_2\phi_2=-gcH$ and

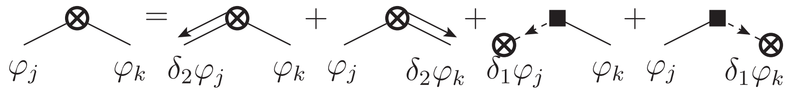
$$D^{(0)}\left(\frac{\partial\delta_2\phi_2}{\partial c},H\right)=-gD^{(0)}(H,H).$$

We now graphically represent $D^{(0)}(\frac{\partial\delta_2\varphi_j}{\partial c},\varphi_k)$ by



$$, \tag{B15}$$

where the single line stands for the free propagator from φ_k to the nonghost field in $\delta_2\varphi_j$ and the arrowed double line emitting from the composite vertex $\otimes$ is interpreted as that the original propagator connecting to field φ_j as in (B11) is annihilated and the composite vertex with the arrowed double line is to be replaced by the constant coefficient of the nonghost field in $\frac{\partial\delta_2\varphi_j}{\partial c}$. With the graphical elements defined in (B10)–(B15), the identity (B9) can be diagrammatically expressed as

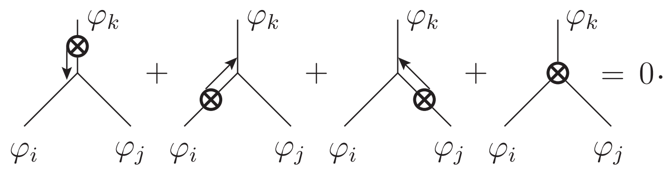


(B16)

Likewise, by expanding

$$T\langle 0|\delta(\bar{c}(z)\varphi_i\varphi_j\varphi_k)|0\rangle_{(0)} = 0$$

and utilizing (B16), we get the identity

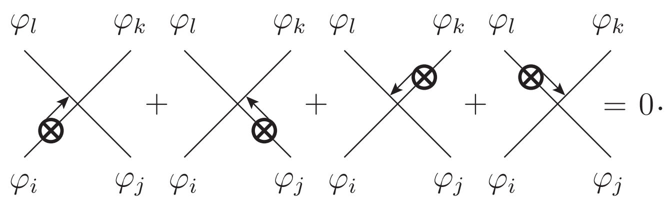


(B17)

The above two graphic identities (B16) and (B17) together with the condition

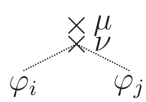
$$T\langle 0|\delta(\bar{c}(z)\varphi_i\varphi_j\varphi_k\varphi_l)|0\rangle_{(0)} = 0$$

can be combined to yield the identity



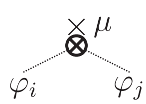
(B18)

We will need graphical notations to express two amputated external fields in a four-point function. In



, (B19)

the amputated A^μ and A^ν fields are represented by two crosses that are stacked together. Similarly,



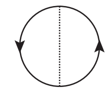
(B20)

represents a four-point function with an amputated external A^μ and a composite vertex ⊗.

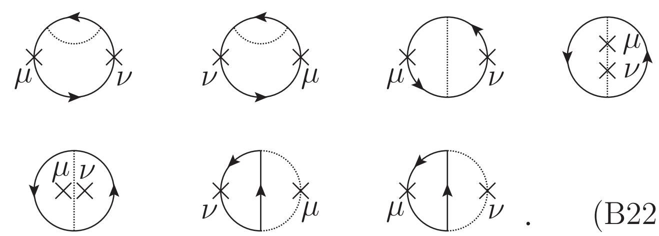
We are now equipped with the graphical notations and identities needed to construct component diagrams for Ward identities without the restriction on the type of internal field lines.

2. Two-loop triangular Ward identity

If all the vertices for external fields are detached, a 3-point 2-loop 1PI diagram in the presence of one fermion-loop subdiagram becomes a 2-loop supergenerator diagram

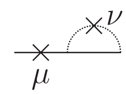

, (B21)

which is composed of a fermion loop and a nonfermion internal line. Seven topologically different generator diagrams will result from all possible attachments of the vertices for A^μ and A^ν consistent with Feynman rules to this supergenerator:



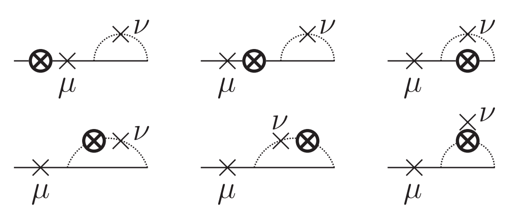
. (B22)

For each diagram in the above, either of the two legitimate cut points at the two vertices connecting to the nonfermion internal line is available to yield a cut generator for a regularized Ward identity. For example, if the cutting is made at the end point of the internal fermion line connecting to the lowest vertex of the last diagram in (B22), we obtain the generator



. (B23)

The vertex for A^μ is attached to the fermion line and the vertex for A^ν is attached to the arc above the fermion line. We may attach ⊗ to the above cut generator (B23) in all possible manners to obtain the following collection of component diagrams:

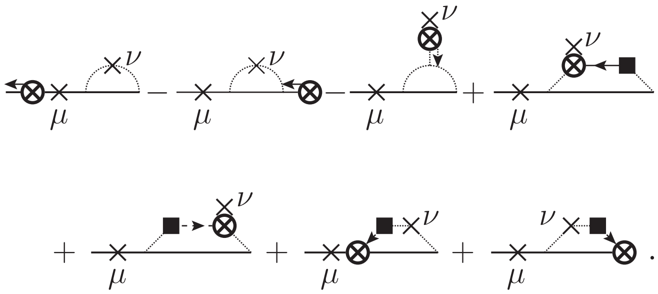


. (B24)

A component diagram is constructed when we insert ⊗ in consistency with Feynman rules into one of the internal lines or vertices in the generator. The momentum entering the open fermion line from the right side is assumed to be equal to the momentum leaving the fermion line at the left end. Since the original closed fermion loop is restored by fusing the open fermion line, the amplitude of the cut diagram is calculated by taking the trace and carrying out the fermion-loop momentum integration.

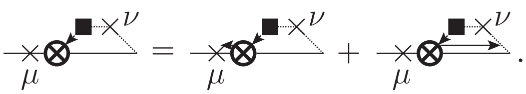
There are many cancellations for the sum of component diagrams constructed from a cut generator. Making use of

(B16) and (B17), the sum of the six diagrams in (B24) becomes



(B25)

The integrals for the first two diagrams in the above cancel each other after loop momentum shifting which is allowed under dimensional regularization. The amplitude for the 3rd diagram also vanishes because the fermion loop that may produce Levi-Civita tensor terms is essentially embedded in a two-point function that lacks sufficient indices to form a Levi-Civita tensor. The propagators for the two internal lines that are attached to the fermion line in the 4th or 5th diagram must both be $D(H, H)$. The amplitude for this type of diagram is absent of a Levi-Civita tensor term because a triangular fermion loop with one vector A and two scalar H lines attached does not have enough indices available to make up a Levi-Civita tensor. By (18), the 6th diagram in (B25) can be decomposed as



(B26)

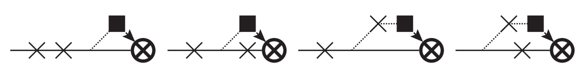
The fermion loop for either diagram on the right side of the above identity has only two vertices effectively and will not have enough indices to give rise to any Levi-Civita tensor term.

The last diagram in (B25) may be problematic because its cut point is located next to the composite vertex $\otimes$ as in (31) of which the nonvanishing amplitude invalidates the basic identity (18) to result in the 1-loop anomaly. Let us recall that a cut point not residing in a divergent subdiagram of self-energy insertion or vertex correction is said to be proper. Since a proper cut point for a generator diagram remains to be a proper one for any of the component diagrams constructed by attaching $\otimes$ to the generator, we will choose to cut each generator in (B22) at a legitimate and proper point if it is available. For the seven generators in (B22), only the third diagram does not have such a cut point when the nonfermion internal line corresponds to a wavy line representing an internal vector meson line. But this is the situation that we have already encountered in Sec. IV in constructing proper component diagrams from the third generator diagram of (39).

To violate a Ward identity in our γ_5 scheme, a cut point must be positioned next to the $\otimes$ vertex, such as the one for the last diagram in (B25). With γ_5 positioned immediately to the right of the composite vertex $\otimes$, the $\otimes$ vertex together with the double line pointing to the right in the identity (18) is no longer equal to igR but should be interpreted as

$$(igR(\ell - m))|_{\text{rightmost}\gamma_5} \frac{1}{(\ell - m)} = igR - ig\gamma_5 \ell_\Delta \frac{1}{(\ell - m)}. \quad \text{(B27)}$$

The extra term $-ig\gamma_5\ \ell_\Delta \frac{1}{(\ell - m)}$ may contribute to the violation of the Ward identity and give rise to an anomaly. If we restrict ourselves to legitimate and proper cut points, only the following four kinds of diagrams, one of which is the last diagram in (B25), may be responsible for the violation of the 2-loop triangular Ward identity.



(B28)

For the theory of (37) in which we have added another fermion field to cancel the 1-loop anomaly, the Λ factor of mass dimension 1 from the solid black box, which represents the vertex factor of $c - \bar{c} - H$, reduces the power counting such that the extra term with the ℓ_Δ factor cannot survive the $n \to 4$ limit in any diagram of (B28). The theory defined by (37) is therefore also free of the 2-loop anomaly.

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