Hamiltonian constraint in polymer parametrized field theory

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Recently, a generally covariant reformulation of two-dimensional flat spacetime free scalar field theory known as parametrized field theory was quantized using loop quantum gravity (LQG) type "polymer" representations. Physical states were constructed, without intermediate regularization structures, by averaging over the group of gauge transformations generated by the constraints, the constraint algebra being a Lie algebra. We consider classically equivalent combinations of these constraints corresponding to a diffeomorphism and a Hamiltonian constraint, which, as in gravity, define a Dirac algebra. Our treatment of the quantum constraints parallels that of LQG and obtains the following results, expected to be of use in the construction of the quantum dynamics of LQG: (i) the (triangulated) Hamiltonian constraint acts only on vertices, its construction involves some of the same ambiguities as in LQG and its action on diffeomorphism invariant states admits a continuum limit, (ii) if the regulating holonomies are in representations tailored to the edge labels of the state, all previously obtained physical states lie in the kernel of the Hamiltonian constraint, (iii) the commutator of two (density weight 1) Hamiltonian constraints as well as the operator correspondent of their classical Poisson bracket converge to zero in the continuum limit defined by diffeomorphism invariant states, and vanish on the Lewandowski-Marolf habitat, (iv) the rescaled density 2 Hamiltonian constraints and their commutator are ill-defined on the Lewandowski-Marolf habitat despite the well-definedness of the operator correspondent of their classical Poisson bracket there, (v) there is a new habitat which supports a nontrivial representation of the Poisson-Lie algebra of density 2 constraints.

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I. INTRODUCTION

One of the key open problems in canonical LQG is a satisfactory treatment of the Hamiltonian constraint operator. Problems stem from the tension between the local nature of the Hamiltonian constraint and the nonlocal nature of some of the basic operators used in its construction. As a result, intermediate regularization structures have to be introduced and the final operator definition depends on the (infinitely manifold) choice of the regulating structures. In our opinion, not much progress has been made on this issue in the canonical theory. Confronted with such a situation, we believe that the availability of a good toy model would go a long way in testing various proposals for the Hamiltonian constraint and suggest avenues to restrict the choices in its construction.

In recent work [1,2], we have developed just such a toy model which we call Polymer Parameterised Field Theory. In [2], we used polymer representations and group averaging techniques to construct the physical Hilbert space of the model and demonstrated that the quantization did encode the right classical limit. Here we use the arena provided by Ref. [2] to explore issues related to the definition of the Hamiltonian constraint operator.

Polymer Parameterised Field Theory is an LQG-type polymer quantization of classical Parameterised Field Theory (PFT) on the Minkowskian cylinder. PFT was introduced by Dirac [3] and its use as a toy model for quantum gravity was pioneered by Kuchař. PFT is just free field theory on flat spacetime, cast in a diffeomorphism invariant disguise. It offers an elegant description of free scalar field evolution on *arbitrary* (and in general curved) foliations of the background spacetime by treating the "embedding variables" which describe the foliation as dynamical variables to be varied in the action in addition to the scalar field. Specifically, let $X^A = (T, X)$ denote inertial coordinates on two-dimensional flat spacetime. In PFT, X^A are parametrized by a new set of arbitrary coordinates $x^\alpha = (t, x)$ such that for fixed t, the embedding variables $X^A(t, x)$ define a spacelike Cauchy slice of flat spacetime. General covariance of PFT ensues from the arbitrary choice of x^α and implies that in its canonical description, evolution from one slice of an arbitrary foliation to another is generated by a pair of constraints. Its field theoretic nature, general covariance and the fact that the dynamics of the true degrees of freedom is just that of a free scalar field make PFT a good toy model for gravity.

In [2], we took advantage of the simplicity of PFT when expressed in terms of light cone variables $X^\pm(x) = T(x) \pm X(x)$ and right and left moving matter variables to solve the quantum theory. Specifically, we used density weight 2

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constraints, H_+ and H_-. H_+ generates the dynamics of left moving matter fields and advances the foliation along '+' null direction and H_- generates the dynamics for right movers and advances the foliation along the '−' direction. However, one can equally well use as constraints, the generators of motions along the Cauchy slice and normal to the Cauchy slice (the flat spacetime metric defines this normal). We shall refer to these, for obvious reasons, as the diffeomorphism and Hamiltonian constraints of the model. These constraints are appropriate combinations of the density weight 2 ones and their Poisson brackets yield a Dirac algebra exactly as is the case for the spatial diffeomorphism and Hamiltonian constraints of 4-d gravity. Specifically, the Poisson bracket between two Hamiltonian constraints yields the diffeomorphism constraint smeared by a vector field which involves the induced spatial metric on the Cauchy slice. The spatial metric is constructed from the canonical embedding data and hence the constraint algebra is a Dirac algebra. In contrast, the density 2 constraints form a Lie algebra and this fact was used in Ref. [2] to construct physical states by group averaging [4]. The group averaging technique only uses the structure of the group of (unitary representations of) finite gauge transformation and, hence, does not require any auxiliary regularization structures, thus yielding an unambiguous construction of the physical Hilbert space of the theory.

Here we follow the strategy used in LQG to construct the quantum dynamics of the model and isolate the space of physical states. In doing so, we find a remarkably close structural similarity to corresponding constructions in LQG. We solve the diffeomorphism constraint by group averaging. Then we construct the Hamiltonian constraint operator at finite triangulation, show that it has a finite action on states in the kinematic Hilbert space and that its dual action admits a well defined continuum limit on diffeomorphism invariant states. Holonomies play a crucial role in the construction and there is an ambiguity in the choice of their representation just as in LQG [5]. The inverse square root of the determinant of the spatial metric also plays an essential role in the finiteness of the action of the Hamiltonian constraint and we show that, similar to LQG, it acts only at vertices.

Next, we enquire if the physical states constructed in [2] also solve the diffeomorphism and Hamiltonian constraints as defined above. It happens to be straightforward to see that these states solve the diffeomorphism constraint. However, contrary to the expectations in LQG, they are *not* normalizable in the Hilbert space inner product obtained through the group averaging procedure applied to the diffeomorphism constraint. While we do not claim the existence of a proof, it does seem unlikely from the structure of the Hamiltonian constraint and that of the physical states of Ref. [2] that an LQG type choice of regulating holonomies in a fixed weight representation would result in a Hamiltonian constraint operator which annihilates all the physical states of Ref. [2]. What we do show is that there does exist a regularization choice in which the holonomy labels are chosen to *depend on the edge labels of the state* in just the right way so that all physical states of [2] are annihilated by the Hamiltonian constraint.

Next, we turn our attention to the quantum constraint algebra. We evaluate the commutator between the smeared density weight 1 Hamiltonian constraints using the topology provided by the arena of diffeomorphism invariant states along the lines of Thiemann's seminal work [6]. We find that this commutator vanishes as in LQG. We then introduce a habitat similar to that of Lewandowski and Marolf [7,8] and construct the smeared density weight 1 Hamiltonian constraint as an operator on this LM habitat. We show that, as in LQG, the commutator between the density weight 1 Hamiltonian constraints as well as the operator corresponding to the classical right-hand side of the corresponding Poisson bracket, all vanish on this LM habitat. Our computations clearly indicate that the constraint algebra trivializes in this manner due to the density weight 1 character of the Hamiltonian constraint. Hence, we turn to an analysis of slightly "more singular" operators obtained by rescaling the density weight 1 operator by the determinant of the spatial metric to obtain a density weight 2 Hamiltonian constraint. We show that neither this density weight 2 Hamiltonian constraint nor the commutator of two such constraints is well defined on the LM habitat. However, the operator corresponding to the Poisson bracket of the corresponding classical quantities *is* well defined on the LM habitat and, in this sense, the algebra of these constraints is anomalous on the LM habitat.[1]

Finally, we introduce a new habitat geared to the physical state space constructed in [2]; these are "vertex smooth" generalizations of the physical states in [2]. We show that the smeared density weight 2 Hamiltonian (and spatial diffeomorphism) constraints are well defined on this new habitat and satisfy the correct constraint algebra.

All these results have important repercussions for LQG. Chief among them are (i) one should consider the possibility of allowing the representations of regulating holonomies to be state dependent (ii) the lack of weak continuity of operators on the kinematic Hilbert space is not necessarily a hindrance to defining their generators on an appropriate space of distributions through the mechanisms of triangulation and continuum limit of dual actions (iii) in order to analyze the quantum constraint algebra, it may be profitable to look beyond smeared density weight 1 operators towards ones which are more singular, i.e. of higher density weight. We shall discuss these points as well as

[1]We also show that there is a subspace of the LM habitat on which the commutator trivializes but the operator correspondent of its Poisson bracket is nontrivial, thus reinforcing our view that the algebra is anomalous on the LM habitat.

other possible lessons for LQG in the concluding section of this paper. There is still much to be learnt about the structure of the constraint algebras in this model. The close structural similarity with LQG ensures that the lessons learnt will provide strategies to probe the constraint algebra in LQG at a deeper level than the seminal works of [6,7].[2]

The layout of the paper is as follows. Since we have already reviewed the necessary material in [2], in the interests of brevity we shall not do so again. Instead, in Sec. II we provide a quick and not necessarily complete list of essential definitions so that the reader can follow the broad thrust of this paper. For a detailed understanding, familiarity with [2] is necessary and will be assumed. In Sec. III, we restrict attention to a certain physically relevant superselected subspace of states in the kinematic Hilbert space and construct solutions to the diffeomorphism constraint by group averaging. The group averaging procedure automatically defines an inner product on these solutions and the Cauchy completion of their finite span yields the diffeomorphism invariant Hilbert space $\mathcal{H}_{\rm diff}$. We show that none of the solutions of [2] lie in $\mathcal{H}_{\rm diff}$. In Sec. IV, we define the operators corresponding to the spatial volume and the inverse square root of the determinant of the metric. In Sec. V, we construct the action of the Hamiltonian constraint on diffeomorphism invariant states. As in LQG, a regulated operator on the kinematic Hilbert space involving a choice of "small edge" holonomies is constructed and the continuum limit of its action is obtained on the space of diffeomorphism invariant states. Section VI is devoted to the algebra of quantum constraints using the arena of diffeomorphism invariant states along the line of Thiemann's seminal work [6]. We probe the constraint algebra on the LM habitat in Sec. VII and on the new habitat in Sec. VIII. Section IX is devoted to a discussion of our results with a view to LQG.

Note on notation: A minor change with respect to [2] is that, here, we only use objects which respect the zero mode constraint. Accordingly, our notation is the same as that of [2] except that (i) we continue to use l_e to denote matter charge labels after the imposition of the zero mode constraint; in contrast, in [2], l_e was used for matter charges before imposition of the constraint and Δl_e was used after solving the zero mode constraint, and (ii) we will omit the subscript $\lambda^{\pm}$ of [2] which is relevant only for objects which do not respect the zero mode constraint.

Finally, we would like to bring to the attention of the reader a recent paper by Thiemann [10] which touches on issues similar to those discussed in this paper.

[2]Indeed, this work has already motivated a definition of the operator corresponding to (the finite triangulation approximant of) the curvature of the Ashtekar-Barbero connection in such a way as to render a satisfactory definition of the diffeomorphism constraint in LQG [9].

II. BRIEF REVIEW OF POLYMER PFT

A. Classical theory

Cauchy slices are oriented circles coordinatized by the angular coordinate $x \in [0, 2\pi]$, with the direction of angular increase agreeing with the orientation of the circle.[3]

Inertial time and space coordinates on the flat spacetime are T, X. Null coordinates are $X^{\pm} = T \pm X$. The length of the $T =$ constant circles in the flat spacetime is L. The scalar field is f.

Canonically conjugate embedding variables: $(X^{+}(x), \Pi_{+}(x))$, $(X^{-}(x), \Pi_{-}(x))$, $X^{\pm}(2\pi) = X^{\pm}(0) \pm 2\pi$

Matter variables: $Y^{\pm}(x) := \pi_f \pm f'$, $\{f(x), \pi_f(y)\} = \delta(x, y)$ $\{Y^{+}, Y^{-}\} = 0$, $\{Y^{\pm}(x), Y^{\pm}(y)\} = \pm(\partial_x \delta(x, y) - \partial_y \delta(y, x))$

Density weight 2 constraints: $H_{\pm}(x) = [\Pi_{\pm}(x) X^{\pm'}(x) \pm \frac{1}{4} Y^{\pm}(x)^2]$. The constraint algebra is isomorphic to the Lie Algebra of vector fields on the circle.

Diffeomorphism constraint: $C_{\rm diff}$ generates spatial diffeomorphisms.

$$C_{\rm diff}(x) = H_{+} + H_{-} = [\Pi_{+}(x) X^{+'}(x) + \Pi_{-}(x) X^{-'}(x) + \pi_f(x) f'(x)]. \tag{1}$$

Hamiltonian constraint: $C_{\rm ham}$ generates evolution normal to the Cauchy slice,

$$C_{\rm ham}(x) = \frac{1}{\sqrt{X^{+'}(x) X^{-'}(x)}} (H_{+} - H_{-}) = \frac{1}{\sqrt{X^{+'}(x) X^{-'}(x)}} \Big[\Pi_{+}(x) X^{+'}(x) - \Pi_{-}(x) X^{-'}(x) + \frac{1}{4}(\pi_f^2 + f'^2) \Big] \tag{2}$$

Constraint algebra: The Poisson algebra generated by $C_{\rm diff}$ and $C_{\rm ham}$ is the Dirac algebra:

$$\begin{aligned} \{C_{\rm diff}[\vec{N}], C_{\rm diff}[\vec{M}]\} &= C_{\rm diff}[\vec{N}, \vec{M}] \\ \{C_{\rm diff}[\vec{N}], C_{\rm ham}[M]\} &= C_{\rm ham}[L_{\vec{N}} M] \\ \{C_{\rm ham}[N], C_{\rm ham}[M]\} &= C_{\rm diff}[\vec{\beta}(N, M)] \end{aligned} \tag{3}$$

wherein $\vec{N}, \vec{M}$ are shift vectors, N, M are lapse functions and the structure function $\beta^a(N, M) := q^{ab}(N \nabla_b M - M \nabla_b N)$ in (3) is defined by the induced spatial metric q_{ab},

$$q_{ab} dx^a dx^b = -X^{+'} X^{-'} (dx)^2. \tag{4}$$

[3]In [1,2] we fixed an angular coordinate system once and for all. Here we allow any positively oriented angular coordinate system ranging between 0 and 2π such that the coordinate values 0 and 2π label the same point on the Cauchy slice, i.e. the Cauchy slice has a preferred point.

B. Quantum theory

A *charge network s* is a finite collection, $\gamma(s)$, of colored, nonoverlapping (except at vertices) edges, e, which span the range of the angular coordinate x, (i.e., $[0, 2\pi]$), the colors being referred to as charges, and the collection of edges being referred to as a graph. Charge network labels depend only on equivalence classes of graphs, similar to the situation for spin networks in LQG [11]. $s = s_1 + s_2$ is the charge network label associated to a fine enough graph underlying both s_1 and s_2. An edge e of this graph is colored by the sum of the charges of s_1 and s_2 which color e. Charge network states are in correspondence with charge networks and constitute an orthonormal basis similar to spin network states in LQG.

1. Embedding sector

Charge network: $s^\pm = \{\gamma(s^\pm), (k^\pm_{e_1^\pm}, \ldots, k^\pm_{e_n^\pm})\}$ where $k^\pm_{e_I^\pm}$ are embedding charges whose range is specified by $k^\pm_{e_I^\pm} \in \frac{2\pi L}{\hbar A}\mathbf{Z}\ \forall\, I$. Here A is a fixed, positive, integer-valued Barbero-Immirizi-like parameter. It is useful to define the "minimum length increment," a, as $a := \frac{2\pi L}{A}$.

Elementary variables: $X^\pm(x)$, $T_{s^\pm}[\Pi_\pm] := \exp[-i\sum_{e\in\gamma(s^+)} k^\pm_{e^\pm} \int_{e^\pm} \Pi_\pm]$.

Representation: $T_{s_1^\pm}$ denotes an embedding charge network state. $\hat{X}^\pm(x)$, $\hat{T}_{s^\pm}$ denote the operators corresponding to the classical quantities $X^\pm(x)$, $T_{s^\pm}[\Pi_\pm]$. Their action is given by

$$\hat{T}_{s^\pm} T_{s_1^\pm} = T_{s^\pm + s_1^\pm} \qquad \hat{X}^\pm(x) T_{s^\pm} := \lambda_{x,s^\pm} T_{s^\pm}, \quad (5)$$

where, for $\gamma(s^\pm)$ with $n^\pm$ edges,

$$\begin{aligned}\lambda_{x,s^\pm} &:= \hbar k^\pm_{e^\pm_{I^\pm}} \quad \text{if } x \in \text{Interior}(e^\pm_{I^\pm}) 1 \le I^\pm \le n^\pm \\ &:= \frac{\hbar}{2}\left(k^\pm_{e^\pm_{I^\pm}} + k^\pm_{e^\pm_{(I+1)^\pm}}\right) \quad \text{if } x \in e^\pm_{I^\pm} \cap e^\pm_{(I^\pm+1)} \\ &\quad 1 \le I^\pm \le (n^\pm - 1)\end{aligned} \quad (6)$$

$$\begin{aligned}&:= \frac{\hbar}{2}\left(k^\pm_{e^\pm_{n^\pm}} \mp \frac{1}{\hbar}2\pi L + k^\pm_{e_1^\pm}\right) \quad \text{if } x = 0 \\ &:= \frac{\hbar}{2}\left(k^\pm_{e_1^\pm} \pm \frac{1}{\hbar}2\pi L + k^\pm_{e^\pm_{n^\pm}}\right) \quad \text{if } x = 2\pi.\end{aligned} \quad (7)$$

2. Matter sector

Charge network: $s^\pm = \{\gamma(s^\pm), (l^\pm_{e_1^\pm}, \ldots, l^\pm_{e_n^\pm})\}$, $\sum_{I=1}^{n^\pm} l^\pm_{e_I^\pm} = 0$, $l^\pm_{e_I^\pm} \in \epsilon\mathbf{Z}\ \forall\, I$. Here ϵ is a fixed (real, positive) parameter with dimensions $(ML)^{-(1/2)}$. ϵ is also a Barbero-Immirizi-like parameter. The zero sum condition on the matter charges stems from technicalities related to the scalar field zero mode [2], an understanding of which is not essential for the discussion here.

Elementary variables: $W_{s^\pm_{\lambda^\pm}}[Y^\pm] = \exp[i\sum_{e^\pm\in\gamma(s^\pm)} l^\pm_{e^\pm} \int_{e^\pm} Y^\pm]$

Weyl algebra[4] of operators: $\hat{W}(s^\pm)\hat{W}(s'^\pm) = \exp[-i\frac{\hbar}{2}\alpha(s^\pm, s'^\pm)]\hat{W}(s^\pm + s'^\pm)$. Here the exponent in the phase-factor $\alpha(s^\pm, s'^\pm)$ is given by

$$\alpha(s^\pm, s'^\pm) := \sum_{e^\pm\in\gamma(s^\pm)} \sum_{e'^\pm\in\gamma(s'^\pm)} (l^\pm_{e^\pm})(l^\pm_{e'^\pm})\alpha(e^\pm, e'^\pm). \quad (8)$$

Here $\alpha(e^\pm, e'^\pm) = (\kappa_{e'^\pm}(f(e^\pm)) - \kappa_{e'^\pm}(b(e^\pm))) - (\kappa_{e^\pm}(f(e'^\pm)) - \kappa_{e^\pm}(b(e'^\pm)))$. Here $f(e)$, $b(e)$ are the final and initial points of the edge e respectively. κ_e is defined as

$$\begin{aligned}\kappa_e(x) &= 1 \text{ if } x \text{ is in the interior of } e \\ &= \frac{1}{2} \text{ if } x \text{ is a boundary point of } e.\end{aligned} \quad (9)$$

Representation: $\hat{W}(s^\pm)W(s'^\pm) = \exp(\frac{-i\hbar}{2}\alpha(s^\pm, s'^\pm)) W(s^\pm + s'^\pm)$.

3. Kinematic Hilbert space

The kinematic Hilbert space $\mathcal{H}_{\text{kin}}$ is the product of the plus and minus sectors, $\mathcal{H}^\pm_{\text{kin}}$, each of which is a product of the appropriate embedding and matter sectors. $\mathcal{H}^\pm_{\text{kin}}$ is spanned by an orthonormal basis of charge network states.

A charge network state in $\mathcal{H}^\pm_{\text{kin}}$ is denoted by $|\mathbf{s}^\pm\rangle := T_{s^\pm} \otimes W(s'^\pm)$.

The label $\mathbf{s}^\pm$ is specified by $\mathbf{s}^\pm_{\lambda^\pm} := \{\gamma(\mathbf{s}^\pm), (k^\pm_{e_1^\pm}, l^\pm_{e_1^\pm}), \ldots, (k^\pm_{e^\pm_{n^\pm}}, l^\pm_{e^\pm_{n^\pm}})\}$. Here we have used the equivalence of charge networks to set $\gamma(\mathbf{s}^\pm) := \gamma(s^\pm) = \gamma(s'^\pm)$ so that each edge of the charge network is labeled by an embedding charge and a matter charge.

4. Unitary Representation of gauge transformations

Finite gauge transformations generated by the density 2 constraints act, essentially, as 2 *independent* diffeomorphisms of the spatial manifold, one which acts only on the '+' fields and one which acts only on the '−' fields. Consequently, in analogy to spatial diffeomorphisms in LQG, their action on charge networks is to appropriately "drag" them around the circle. However, due to the quasiperiodic nature of $X^\pm$, it is more appropriate to think of these diffeomorphisms as being periodic diffeomorphisms of the real line. Consequently, the action of these gauge transformations in quantum theory also keeps track of 'factors of 2π' when embedding charge edges 'go past $x = 2\pi$'.

More precisely, the action of finite gauge transformations is specified by introducing the notion of an extension of a charge network s to the real line. Such an extension is

[4]The definition of the Weyl algebra follows in the standard way from the Poisson brackets between $Y^\pm(x)$, $Y^\pm(y)$ and an application of the Baker-Campbell-Hausdorff Lemma [12]

labeled by the graph $\gamma(s)_{\rm ext}$ which covers the real line and by charge labels on each edge of $\gamma(s)_{\rm ext}$. Let $T_N(x) \in R$ denote a rigid translation of the point $x \in [0, 2\pi]$ by $2N\pi$ so that $T_N(\gamma(s))$ spans $[2N\pi, 2(N+1)\pi]$. Then $\gamma(s)_{\rm ext} = \cup_{N\in \mathbf{Z}} T_N(\gamma(s))$. For the embedding charge network $s^{\pm}$, we define the *quasiperiodic extension* $\bar{s}^{\pm}_{\rm ext}$ by specifying the embedding charges on $T_N(\gamma(s))$ by $k^{\pm}_{T_N(e)} := k^{\pm}_e \pm 2N\pi \frac{L}{\hbar}$ for every edge $e \in \gamma(s)$. Similarly, for the matter charge network $s^{\pm}$ we define the *periodic extension* $s^{\pm}_{\rm ext}$ by setting $l^{\pm}_{T_N(e)} := l^{\pm}_e$.

The action of periodic diffeomorphisms, ϕ, of the real line on $\bar{s}^{\pm}_{\rm ext}$, $s^{\pm}_{\rm ext}$ is defined by mapping $\gamma(s)_{\rm ext}$ to $\phi(\gamma(s)_{\rm ext})$ and setting $k^{\pm}_{\phi(e)} = k^{\pm}_e, l^{\pm}_{\phi(e)} = l^{\pm}_e$ for every edge $e \in \gamma(s)_{\rm ext}$.

Then unitary representation of the gauge group is given by

$$\begin{aligned}
\hat{U}^{\pm}(\phi^{\pm})T_{s^{\pm}} &:= T_{\phi(\bar{s}^{\pm}_{\rm ext})|_{[0,2\pi]}}\\
\hat{U}^{\mp}(\phi^{\mp})T_{s^{\pm}} &:= T_{s^{\pm}}\\
\hat{U}^{\pm}(\phi^{\pm})W(s'^{\pm}) &:= W((\phi^{\pm})(s'^{\pm}_{\rm ext})|_{[0,2\pi]}).\\
\hat{U}^{\mp}(\phi^{\mp})W(s'^{\pm}) &:= W(s'^{\pm}).
\end{aligned} \tag{10}$$

Denoting $T_{s^{\pm}} \otimes W(s'^{\pm})$ by $|\mathbf{s}^{\pm}\rangle$ and $T_{\phi(\bar{s}^{\pm}_{\rm ext})|_{[0,2\pi]}} \otimes W((\phi^{\pm})(s'^{\pm}_{\rm ext})|_{[0,2\pi]})$ by $|\mathbf{s}^{\pm}_{\phi^{\pm}}\rangle$, the above equations can be written in a compact form as

$$|\mathbf{s}^{\pm}_{\phi^{\pm}}\rangle := \hat{U}^{\pm}(\phi^{\pm})|\mathbf{s}^{\pm}\rangle. \tag{11}$$

5. Physical Hilbert space

Physical states are obtained by group averaging the action of the finite gauge transformations discussed in the previous section. Henceforth, we restrict attention to a physically relevant superselected sector of the physical Hilbert space. This sector is obtained by group averaging a superselected subspace, $\mathcal{D}_{ss}$ of $\mathcal{H}_{\rm kin}$, $\mathcal{D}_{ss} = \mathcal{D}^{+}_{ss} \otimes \mathcal{D}^{-}_{ss}$.

$\mathcal{D}^{\pm}_{ss}$ is defined as follows. Fix a pair of graphs $\gamma^{\pm}$ with A edges. Place the embedding charges $\vec{k}^{\pm}$ such that $k^{\pm}_{e^{\pm}_{I^{\pm}}} - k^{\pm}_{e^{\pm}_{I^{\pm}-1}} = \frac{2\pi}{A\hbar}\ \forall\, I^{\pm}$. Consider the set of all charge network states $\{|\mathbf{s}^{\pm}\rangle = |\gamma^{\pm}, \vec{k}^{\pm}, (l^{\pm}_{e^{\pm}_1}, \ldots, l^{\pm}_{e^{\pm}_A})\rangle\}$, where $l^{\pm}_{e^{\pm}_I} \in \mathbf{Z}\epsilon$ are allowed to take all possible values subject to the zero sum condition $\sum_I l^{\pm}_{e^{\pm}_I} = 0$. Let $\mathcal{D}^{\pm}_{ss}$ be a finite span of charge network states of the type $\{|\mathbf{s}^{\pm}_{\lambda^{\pm}\phi^{\pm}}\rangle\ \forall\, \phi^{\pm}\}$.

The action of the group averaging map $\eta^{\pm}$ on a charge network state in $\mathcal{D}^{\pm}_{ss}$ yields the distribution,

$$\eta^{\pm}(|\mathbf{s}^{\pm}\rangle) = \sum_{\mathbf{s}^{\pm}\in[\mathbf{s}^{\pm}]} < \mathbf{s}'^{\pm}| = \sum_{\phi^{\pm}\in {\rm Diff}^{P}_{[\mathbf{s}^{\pm}]}\mathbf{R}} < \mathbf{s}^{\pm}_{\phi^{\pm}}|. \tag{12}$$

Here $[\mathbf{s}^{\pm}]$ is the equivalence class defined by $[\mathbf{s}^{\pm}] = \{\mathbf{s}'^{\pm}|\mathbf{s}'^{\pm} = \mathbf{s}^{\pm}_{\phi^{\pm}}\ \text{for some}\ \phi^{\pm}\}$, and ${\rm Diff}^{P}_{[\mathbf{s}^{\pm}]}\mathbf{R}$ is a set of gauge transformations such that for each $\mathbf{s}'^{\pm} \in [\mathbf{s}^{\pm}]$ there is precisely one gauge transformation in the set which maps $\mathbf{s}^{\pm}$ to $\mathbf{s}'^{\pm}$. The space of such gauge invariant distributions comes equipped with the inner product

$$< \eta^{\pm}(|\mathbf{s}^{\pm}_1\rangle), \quad \eta^{\pm}(|\mathbf{s}^{\pm}_2\rangle) >_{\rm phys} = \eta^{\pm}(|\mathbf{s}^{\pm}_1\rangle)[|\mathbf{s}^{\pm}_2\rangle], \tag{13}$$

which can be used to complete $\eta^{\pm}(\mathcal{D}^{\pm}_{ss})$ to the Hilbert space $\mathcal{H}^{ss\pm}_{\rm phy}$. We shall restrict attention to $\mathcal{H}^{ss}_{\rm phy} := \mathcal{H}^{ss+}_{\rm phy} \otimes \mathcal{H}^{ss-}_{\rm phy}$.

III. THE HILBERT SPACE OF DIFFEOMORPHISM INVARIANT DISTRIBUTIONS

In Sec. III A, we show that the solutions of [2] are invariant under the unitary action of spatial diffeomorphisms. In Sec. III B, we construct the Hilbert space of diffeomorphism invariant distributions, $\mathcal{H}_{\rm diff}$, by group averaging. In Sec. III C, we restrict attention to a physically relevant superselected subspace of $\mathcal{H}_{\rm diff}$ and show that (a basis of) this subspace is in correspondence with quantum matter states on discrete Cauchy slices of the flat spacetime. We then use this correspondence to show that no solution of [2] is normalizable in $\mathcal{H}_{\rm diff}$.

A. Diffeomorphism invariance of the solutions of [2]

The diffeomorphism constraint (see Eq. (1)) generates spatial diffeomorphisms of the circular Cauchy slice. As indicated in Sec. II B 4, due to the quasiperiodicity of $X^{\pm}$ it is useful to think of diffeomorphisms of the circle in terms of periodic diffeomorphisms of the real line.

Let ϕ be a periodic diffeomorphism of the real line so that $\phi(x + 2\pi m) = \phi(x) + 2\pi m$, $m \in \mathbf{Z}$. From Eq. (1), recall that $C_{\rm diff} = H_+ + H_-$. Also recall that $\{H_+, H_-\} = 0$. It follows from Sec. II B 4 that the unitary action of the finite spatial diffeomorphism labeled by ϕ on any charge network state $|\mathbf{s}^+, \mathbf{s}^-\rangle := |\mathbf{s}^+\rangle \otimes |\mathbf{s}^-\rangle$ is given by

$$|\mathbf{s}^{+}_{\phi}, \mathbf{s}^{-}_{\phi}\rangle := |\mathbf{s}^{+}_{\phi}\rangle \otimes |\mathbf{s}^{-}_{\phi}\rangle = \hat{U}^{+}(\phi)|\mathbf{s}^{+}\rangle \otimes \hat{U}^{-}(\phi)|\mathbf{s}^{-}\rangle. \tag{14}$$

Thus, spatial diffeomorphisms correspond to those gauge transformations for which $\phi^+ = \phi^- = \phi$. Since the solutions of [2] are invariant under the action of *all* gauge transformations, they are, in particular, invariant under the action of spatial diffeomorphisms.

B. The construction of $\mathcal{H}_{\rm diff}$

Spatial diffeomorphism invariant distributions are constructed by the action of the group averaging map, $\eta_{\rm diff}$ on the dense space of finite linear combinations of charge network states as follows. Let $[\mathbf{s}^+, \mathbf{s}^-]$ be the orbit of $\mathbf{s}^+$, $\mathbf{s}^-$ under all spatial diffeomorphisms so that $[\mathbf{s}^+, \mathbf{s}^-]$ is the set of all distinct charge network labels obtained by the action of spatial diffeomorphisms on $\mathbf{s}^+$, $\mathbf{s}^-$. Then

$$\eta_{\rm diff}(|\mathbf{s}^+\rangle \otimes |\mathbf{s}^-\rangle) = \eta_{[\mathbf{s}^+,\mathbf{s}^-]} \sum_{\mathbf{s}'^+,\mathbf{s}'^-\in[\mathbf{s}^+,\mathbf{s}^-]} \langle \mathbf{s}'^+| \otimes \langle \mathbf{s}'^-|. \tag{15}$$

Here, $\eta_{[\mathbf{s}^+,\mathbf{s}^-]}$ is a constant which depends only on the orbit of $\mathbf{s}^+$, $\mathbf{s}^-$. The arbitrariness in the choice of this constant can be reduced by requiring that $\eta_{\rm diff}$ commute with all diffeomorphism invariant observables. Specifically, consider the discrete time translation operator $\hat{T}_n$, $n \in \mathbf{Z}$, which translates t_e by an amount na and leaves x_e, $l_e^\pm$ untouched, i.e.

$$\hat{T}_n|\mathbf{s}^+,\mathbf{s}^-\rangle := |\mathbf{s}_n^+,\mathbf{s}_n^-\rangle,$$
$$\mathbf{s}_n^\pm = \Big(\gamma(\mathbf{s}^\pm), \Big(k^\pm_{e^\pm_{I^\pm}} + \frac{2\pi nL}{\hbar A}, l^\pm_{e^\pm_{I^\pm}}, I^\pm = 1,..,N^\pm\Big)\Big). \tag{16}$$

It is straightforward to check that $\hat{T}_n$ commutes with all finite gauge transformations and, hence, with diffeomorphisms. Requiring its commutativity with $\eta_{\rm diff}$ implies that

$$\eta_{[\mathbf{s}_n^+,\mathbf{s}_n^-]} = \eta_{[\mathbf{s}^+,\mathbf{s}^-]}. \tag{17}$$

While one could attempt to further restrict the choice of $\eta_{[\mathbf{s}^+,\mathbf{s}^-]}$, we shall not do so here in view of the fact that our subsequent considerations are independent of any such further restrictions.

C. Quantum Cauchy data from states in $\mathcal{H}_{\rm diff}$

We restrict attention to states in the superselected sector $\mathcal{D}_{ss} = \mathcal{D}_{ss}^+ \otimes \mathcal{D}_{ss}^-$ (see Sec. II B 5). It is straightforward to see that any charge network state $|\mathbf{s}^\pm\rangle \in \mathcal{D}_{ss}^\pm$ has charges which satisfy the conditions

$$\pm k^\pm_{e^\pm_{I^\pm}} - \pm k^\pm_{e^\pm_{I^\pm-1}} = \frac{2\pi L}{\hbar A}, \qquad I^\pm = 2,..,N^\pm, \tag{18}$$

$$\pm k^\pm_{e^\pm_{N^\pm}} - \pm k^\pm_{e^\pm_1} \leq \frac{2\pi L}{\hbar}, \tag{19}$$

$$\pm k^\pm_{e^\pm_{N^\pm}} - \pm k^\pm_{e^\pm_1} = \frac{2\pi L}{\hbar} \quad \text{iff } N^\pm = A+1 \quad \text{and} \quad l^\pm_{e^\pm_{N^\pm}} = l^\pm_{e^\pm_1}. \tag{20}$$

Here, $N^\pm$ are the number of edges of (the coarsest graphs underlying) $\mathbf{s}^\pm$ and $N^\pm = A$ if the strict inequality holds in Eq. (19).

Next, consider any charge network state $|\mathbf{s}^+,\mathbf{s}^-\rangle \in \mathcal{D}_{ss}$ and let $\gamma(\mathbf{s}^+,\mathbf{s}^-)$ be the coarsest graph underlying both $\mathbf{s}^+$ and $\mathbf{s}^-$. Denote the number of edges of $\gamma(\mathbf{s}^+,\mathbf{s}^-)$ by N. Each edge e of $\gamma(\mathbf{s}^+,\mathbf{s}^-)$ is labeled by the quadruple $(k_e^+, k_e^-, l_e^+, l_e^-)$ or, alternatively, by (t_e, x_e, l_e^+, l_e^-) where $t_e := \hbar\frac{k_e^+ + k_e^-}{2}$, $x_e := \hbar\frac{k_e^+ - k_e^-}{2}$. From Eqs. (18)–(20) it follows that

$$x_{e_I} < x_{e_J} \quad \text{iff } I < J \tag{21}$$

$$x_{e_N} - x_{e_1} \leq 2\pi L \tag{22}$$

$$x_{e_N} - x_{e_1} = 2\pi L \ \text{ iff } k^+_{e_N} - k^+_{e_1} = k^-_{e_1} - k^-_{e_N} = \frac{2\pi L}{\hbar} \tag{23}$$

$$\Rightarrow t_{e_1} = t_{e_N}, \quad l^\pm_{e_1} = l^\pm_{e_N} \ \text{ when } x_{e_N} - x_{e_1} = 2\pi L. \tag{24}$$

Each pair of embedding charges x_e, t_e defines a spacetime point with inertial coordinates $(X,T) = (x_e, t_e)$. If Eq. (22) holds with a strict inequality, then Eq. (21) implies that the set of pairs (x_{e_I}, t_{e_I}), $I = 1,..,N$ define N distinct points in the flat spacetime. If Eq. (23) holds, then Eqs. (21), (23), and (24) ensure that this set defines $N-1$ distinct spacetime points by virtue of the circular topology of space.

Denote the set of spacetime points associated to the charge network state $|\mathbf{s}^+,\mathbf{s}^-\rangle \in \mathcal{D}_{ss}$ in this way by $\mathcal{C}_{\mathbf{s}^+,\mathbf{s}^-}$. Next, associate the matter charge labels $(l^+_{e_I}, l^-_{e_I})$ to the corresponding spacetime point defined by (x_{e_I}, t_{e_I}) (if Eq. (23) holds, such an association is consistent by virtue of Eq. (24)). This association defines the set $\mathcal{M}_{\mathbf{s}^+,\mathbf{s}^-}$, each element of which is a spacetime point defined by $|\mathbf{s}^+,\mathbf{s}^-\rangle$ together with the matter charges associated to it. We shall refer to $\mathcal{C}_{\mathbf{s}^+,\mathbf{s}^-}$ as a discrete Cauchy slice (despite the existence of pairs of points which are light like separated [2]) and $\mathcal{M}_{\mathbf{s}^+,\mathbf{s}^-}$ as quantum matter on this discrete Cauchy slice.

It is then straightforward to see, from the action of $\hat{U}^\pm(\phi)$ and the properties of the extensions of charge network labels to the real line , that

(a) any spatial diffeomorphism preserves both $\mathcal{C}_{\mathbf{s}^+,\mathbf{s}^-}$ and $\mathcal{M}_{\mathbf{s}^+,\mathbf{s}^-}$,

(b) if $|\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}\rangle \in \mathcal{D}_{ss}$ is such that $\mathcal{C}_{\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}} = \mathcal{C}_{\mathbf{s}^+,\mathbf{s}^-}$, $\mathcal{M}_{\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}} = \mathcal{M}_{\mathbf{s}^+,\mathbf{s}^-}$, then $\mathbf{s}^{+\prime}, \mathbf{s}^{-\prime} \in [\mathbf{s}^+,\mathbf{s}^-]$.

It follows that $\eta_{\rm diff}(|\mathbf{s}^+,\mathbf{s}^-\rangle)$ is in unique correspondence with quantum matter data on a discrete Cauchy slice.

Next, consider the group average of $|\mathbf{s}^+,\mathbf{s}^-\rangle$ with respect to all finite gauge transformations generated by H_+, H_-. From Sec. II B 5, this yields the distribution $\eta^+(|\mathbf{s}^+\rangle) \otimes \eta^-(|\mathbf{s}^-\rangle)$

$$\eta^+(|\mathbf{s}^+\rangle) \otimes \eta^-(|\mathbf{s}^-\rangle) = \sum_{\mathbf{s}^{+\prime}\in[\mathbf{s}^+]} \sum_{\mathbf{s}^{-\prime}\in[\mathbf{s}^-]} \langle \mathbf{s}^{+\prime}| \otimes \langle \mathbf{s}^{-\prime}|, \tag{25}$$

where $[\mathbf{s}^\pm]$ is the orbit of $\mathbf{s}^\pm$ under the action of all finite gauge transformations. Clearly, the sum (25) contains the sum (15) since every diffeomorphism is a finite gauge transformation (see (14)). Moreover since $\eta^+(|\mathbf{s}^{+\prime}\rangle) \otimes \eta^-(|\mathbf{s}^{-\prime}\rangle) = \eta^+(|\mathbf{s}^+\rangle) \otimes \eta^-(|\mathbf{s}^-\rangle)$ if $\mathbf{s}^{\pm\prime} \in [\mathbf{s}^\pm]$, the sum also contains all the diffeomorphism images of any state $|\mathbf{s}^{+\prime}\rangle \otimes |\mathbf{s}^{-\prime}\rangle$ which is gauge-related to $|\mathbf{s}^+\rangle \otimes |\mathbf{s}^-\rangle$.

It is straightforward to see that generic gauge transformations do not preserve the discrete Cauchy slice $\mathcal{C}_{\mathbf{s}^+,\mathbf{s}^-}$ and that, in fact, a countable infinity of (states corresponding to) distinct discrete Cauchy slices are generated by the action of finite gauge transformations on

$|\mathbf{s}^+\rangle \otimes |\mathbf{s}^-\rangle$. In particular, it is easy to see that discrete Cauchy slices which are time translations by $2m\pi L$, $m \in \mathbf{Z}$ are in the sum (25).[5] This, together with the orthogonality of diffeomorphism invariant states corresponding to different discrete Cauchy slices and condition (17), implies that no solution of [2] is normalizable in $\mathcal{H}_{\rm diff}$.

IV. THE VOLUME AND INVERSE METRIC OPERATORS

The inverse (square root of the determinant of the) metric operator, $\widehat{\frac{1}{\sqrt{X^{+'}X^{-'}}}}$, is constructed using a Thiemann-like trick [6] to express $\frac{1}{\sqrt{X^{+'}X^{-'}}}$ in terms of Poisson brackets of the volume with holonomies and then replacing the Poisson brackets with quantum commutators. We construct the volume operator in Sec. IV A and the operator $\widehat{\frac{1}{\sqrt{X^{+'}X^{-'}}}}$ in Sec. IV B. Calculation details pertaining to Sec. IV B are in the Appendix.

A. The volume operator

We construct the volume operator corresponding to a small region centered around any point p_0 on the circular Cauchy slice. The volume of any other region can be built out of these. As mentioned in Sec. II, we use angular coordinate systems on S^1 whose range is $[0, 2\pi]$. Let us fix one such coordinate system and denote the shortest coordinate distance, in the angular coordinates, between two points in S^1 with coordinates y_1, $y_2 \in [0, 2\pi]$ by $d(y_1, y_2)$, i.e. $d(y_1, y_2)$ is the minimum of $(|y_1 - y_2|, |y_1 - y_2 + 2\pi|, |y_1 - y_2 - 2\pi|)$.

Let $\mathcal{U}_{y_0} \subset S^1$ be a closed interval of radius ϵ centered around y_0. Then its (one-dimensional) volume, $V_{\mathcal{U}_{y_0}}$, as measured by the spatial metric induced on the circular slice from the flat spacetime metric is given by

$$V_{\mathcal{U}_{y_0}} = \int_{\mathcal{U}_{y_0}} dy_1 \sqrt{|X^{+'}X^{-'}|}(y_1) = \int \kappa_\epsilon(y_1, y_0)\sqrt{|X^{+'}X^{-'}|}(y_1), \qquad (26)$$

where $\kappa_\epsilon(\cdot, y_0)$ is the characteristic function on S^1, defined as

$$\kappa_\epsilon(y_1, y_0) = 1 \quad \text{if } d(y_1, y_0) < \epsilon$$
$$\kappa_\epsilon(y_1, y_0) = \frac{1}{2} \quad \text{if } d(y_1, y_0) = \epsilon = 0 \text{ otherwise} \qquad (27)$$

Let T be a triangulation of $[0, 2\pi]$ whose 1-simplices we denote by $\triangle$, each simplex $\triangle$ being of length $|\triangle|$. We orient each simplex in the increasing x direction. Now consider the union of 1-simplices $\bar{\triangle}$ which are obtained by joining the midpoints of $\triangle \in T$. We denote the collection of $\bar{\triangle}$ as T^* and loosely refer to it as the triangulation dual to T. Notice that T^* does not completely cover $[0, 2\pi]$, and that $|\bar{\triangle}| = |\triangle|$. One can now approximate the right-hand side (rhs) of (26) by a Riemann sum over T

$$\int \kappa_\epsilon(y_1, y_0)\sqrt{|X^{+'}X^{-'}|}(y_1) \approx \sum_{\triangle\in T} |\triangle| \kappa_\epsilon(b(\triangle), y_0)\sqrt{|X^{+'}X^{-'}|}(b(\triangle)), \qquad (28)$$

where we have implicitly assumed that $|\triangle| \ll \epsilon$. The above sum can in turn be approximated by a sum over simplices in T^*, and the two remaining terms coming from the intervals belonging to $[0, 2\pi] - T^*$.

$$\int \kappa_\epsilon(y_1, y_0)\sqrt{|X^{+'}X^{-'}|}(y_1) \approx \sum_{\bar{\triangle}\in T^*} |\bar{\triangle}| \kappa_\epsilon(m(\bar{\triangle}), y_0)\sqrt{|X^{+'}X^{-'}|}(m(\bar{\triangle}) + \frac{|\bar{\triangle}|}{2}\kappa_\epsilon(0, y_0)\sqrt{|X^{+'}X^{-'}|}(0) + \frac{|\bar{\triangle}|}{2}\kappa_\epsilon(0, y_0)\sqrt{|^{+'}X^{-'}|}(2\pi) \qquad (29)$$

The last two terms are in fact equal to each other as the characteristic function is periodic and so is $\sqrt{|X^{+'}X^{-'}|}(x)$.

With a further approximation (which becomes exact in the limit that $|\triangle| \to 0$), we have that

$$\int \kappa_\epsilon(y_1, y_0)\sqrt{|X^{+'}X^{-'}|}(y_1) \approx \sum_{\bar{\triangle}\in T^*} \kappa_\epsilon(m(\bar{\triangle}), y_0)\sqrt{|X^+(f(\bar{\triangle})) - X^+(b(\bar{\triangle}))||X^-(f(\bar{\triangle})) - X^-(b(\bar{\triangle}))|} + \kappa_\epsilon(0, y_0)\sqrt{|X^+(b(\bar{\triangle}_1)) - (X^+(f(\bar{\triangle}_N)) - 2\pi)||X^-(b(\bar{\triangle}_1)) - (X^-(f(\bar{\triangle}_N)) + 2\pi)|}, \qquad (30)$$

where $\bar{\triangle}_1$ is the first (leftmost) simplex in T^* and $\bar{\triangle}_N$ the final (rightmost) simplex in T^*.

Next, we define the action of the operator corresponding to the right-hand side of (30) (and, eventually, its $|\triangle| \to 0$ limit) on the charge network basis. Let $T_{s^+} \otimes T_{s^-}$ be the

[5]It is straightforward to check that for $\phi^\pm$ chosen to be appropriate rigid translations on $\mathbf{R}$, we have that $\hat{U}(\phi^+)\hat{U}(\phi^-) = \hat{T}_n$ with $n = Am$.

embedding charge network state of interest and consider the graphs $\gamma(s^{\pm})$ underlying the state.[6] Let $v^{\pm}$ be a vertex of $\gamma(s^{\pm})$. Let $k^{\pm}_{e_{v^{\pm}}}$ denote the embedding charge in $s^{\pm}$ on an edge terminating at the vertex $v^{\pm}$ and $k^{\pm}_{e^{v^{\pm}}}$ denote the embedding charge in $s^{\pm}$ on an edge originating at vertex $v^{\pm}$. If $v^{\pm}=0$ (or 2π) we define $k^{\pm}_{e_{v^{\pm}}}$ (or $k^{\pm}_{e^{v^{\pm}}}$) by the quasiperiodic extension of the charge network (see Sec. II B 4), i.e. $k^{\pm}_{e_{v^{\pm}=0}} = k^{\pm}_{e_{v^{\pm}=2\pi}} \mp 2\pi$ and $k^{\pm}_{e^{v^{\pm}=2\pi}} = k^{\pm}_{e^{v^{\pm}=0}} \pm 2\pi$. We shall refer to vertices for which $k^{\pm}_{e^{v^{\pm}}} - k^{\pm}_{e_{v^{\pm}}} \neq 0$ as *nontrivial* embedding vertices of $\gamma(s^{\pm})$. Note that the set of such nontrivial vertices, $V_E(s^{\pm})$, only depends on the state and not on the particular representative of the equivalence class of charge networks which labels the state.

We shall (as in LQG) adapt the triangulation T to the graphs underlying the charge network state. Since we defined T in terms of our chosen angular coordinates on the slice, we shall also adapt our choice of coordinates to the state (see Footnote 3).

Note that a one-dimensional triangulation naturally defines a graph. We shall slightly abuse notation and denote the graph defined by T, also by T. We restrict our choice of coordinate system and the resulting choice of T (for small enough $|\triangle|$) to be such that all vertices of the graphs $\gamma(s^{\pm})$ are vertices of T. While our final result will be independent of the particular choice of $\gamma(s^{\pm})$, we find it convenient to choose fine enough graphs so that $\gamma(s^+) = \gamma(s^-) = T$. The action of the operator corresponding to the right-hand side of (30) is obtained by replacing classical embedding variables by the corresponding operators. From Eq. (7), it follows that only those $\bar{\triangle} \in T^*$ contribute to the operator action for which $m(\bar{\triangle}) \in V_E(\gamma^+) \cap V_E(\gamma^-)$. Thus, the sum over midpoints can be replaced by a sum over nontrivial vertices of the graph and we obtain an expression independent of $|\triangle|$ so that the $|\triangle| \to 0$ limit can be taken. It follows that

$$\hat{V}_{\mathcal{U}_{y_0}} T_{s^+} \otimes T_{s^-} = \int_{\mathcal{U}_{y_0}} dy_1 \widehat{\sqrt{|X^{+\prime} X^{-\prime}|}} T_{s^+} \otimes T_{s^-} = \hbar \sum_{v \in V(\gamma^+) \cap V(\gamma^-), v \neq 2\pi} \kappa_\epsilon(v, y_0) \times \sqrt{|k^+_{e^v} - k^+_{e_v}||k^-_{e^v} - k^-_{e_v}|} T_{s^+} \otimes T_{s^-}. \quad (31)$$

[6] We remind the reader charge networks "live" on $[0, 2\pi]$ and that $x = 0$, $x = 2\pi$ are always vertices of the graphs; the circular topology is built into the definition of the action of various operators of interest as detailed in Sec. 2 and in Ref. [2]. We also remind the reader that the vertex set depends on the specific member of the equivalence class of labeled graphs underlying the charge network state.

For future purposes, it is convenient to define the vertex volume operator, $\hat{V}_x x \in [0, 2\pi]$, as

$$\hat{V}_x T_{s^+} \otimes T_{s^-} := \hbar \sqrt{|k^+_{e^x} - k^+_{e_x}||k^-_{e^x} - k^-_{e_x}|} T_{s^+} \otimes T_{s^-}, \quad (32)$$

where it is understood that we always choose a member of the equivalence class of graphs underlying the state which has x as a vertex. Then the action of the volume operator can also be expressed as

$$\hat{V}_{\mathcal{U}_{y_0}} T_{s^+} \otimes T_{s^-} = \sum_{x \in T, x \neq 2\pi} \kappa_\epsilon(x, y_0) \hat{V}_x T_{s^+} \otimes T_{s^-}, \quad (33)$$

where we have chosen the graph naturally defined by T itself as the specific member of the equivalence class of graphs underlying the state and summed over all vertices of T. (While the sum is over all vertices of the triangulation, except $x = 2\pi$ to avoid double counting the point on the circle corresponding to $x = 0 \equiv x = 2\pi$, clearly only the nontrivial vertices contribute to the sum.)

B. The operator $\widehat{\frac{1}{\sqrt{X^{+\prime} X^{-\prime}}}}$

As above, let $\mathcal{U}_{y_0}$ be a closed interval of radius ϵ centered around y_0. Let $x_1 \in S^1$ be such that $x_1 \notin \mathcal{U}_{y_0}$. Then one can easily verify the following two identities.

$$\begin{aligned} -\frac{\mathrm{sgn}(X^{+\prime}(y_0))}{2} \sqrt{\left|\frac{X^{-\prime}}{X^{+\prime}}\right|}(y_0) &= \left\{ \int_{\mathcal{U}_{y_0}} dy_1 \sqrt{X^{+\prime} X^{-\prime}}(y_1), \int_{x_1}^{y_0} \Pi_+(y_2) dy_2 \right\} \\ &\quad \times \frac{\mathrm{sgn}(X^{+\prime} X^{-\prime})(y_0)}{4} \frac{1}{\sqrt{|X^{+\prime} X^{-\prime}|}}(y_0) \\ &= \left\{ \left\{ \int_{\mathcal{U}_{y_0}} dy_1 \sqrt{X^{+\prime} X^{-\prime}}(y_1), \int_{\mathcal{U}_{y_0}} dy \int_{x_1}^{y} \Pi_+(y_2) dy_2 \right\}, \right. \\ &\quad \left. \times \int_{x_1}^{y_0} \Pi_-(y_3) dy_3 \right\}, \end{aligned} \quad (34)$$

where sgn stands for the signum function. The second equation can be written in a manner that treats the $(+)$ and $(-)$ sectors more symmetrically.

$$\begin{aligned} &\frac{\mathrm{sgn}(X^{+\prime} X^{-\prime})(y_0)}{4} \frac{1}{\sqrt{|X^{+\prime} X^{-\prime}|}}(y_0) \\ &= \frac{1}{2} \Bigg(\left\{ \left\{ \int_{\mathcal{U}_{y_0}} dy_1 \sqrt{X^{+\prime} X^{-\prime}}(y_1), \int_{\mathcal{U}_{y_0}} dy \int_{x_1}^{y} \Pi_+(y_2) dy_2 \right\}, \times \int_{x_1}^{y_0} \Pi_-(y_3) dy_3 \right\} \\ &\quad + \left\{ \left\{ \int_{\mathcal{U}_{y_0}} dy_1 \sqrt{X^{+\prime} X^{-\prime}}(y_1), \int_{\mathcal{U}_{y_0}} dy \int_{x_1}^{y} \Pi_-(y_2) dy_2 \right\}, \times \int_{x_1}^{y_0} \Pi_+(y_3) dy_3 \right\} \Bigg) \end{aligned} \quad (35)$$

We shall turn the classical expression above into a quantum operator. Before doing so, we need to define the operator, $\widehat{\mathrm{sgn}(X^{+\prime}X^{-\prime})}(y_0)$, corresponding to the signum function $\mathrm{sgn}(X^{+\prime}X^{-\prime}(y_0))$. A natural definition is

$$\begin{aligned}&\widehat{\mathrm{sgn}(X^{+\prime}X^{-\prime})}(y_0)T_{s^+}\otimes T_{s^-}\\&\quad=\mathrm{sgn}(k^+_{e^{y_0}}-k^+_{e_{y_0}})\mathrm{sgn}(k^-_{e^{y_0}}-k^-_{e_{y_0}})T_{s^+}\otimes T_{s^-}\\&\quad\text{if } y_0\in V_E(s^+)\cap V_E(s^-).\end{aligned}\tag{36}$$

We do not specify the action of $\widehat{\mathrm{sgn}(X^{+\prime}X^{-\prime})}(y_0)$ on the state if y_0 is not a nontrivial vertex of both $\gamma(s^+)$ and $\gamma(s^-)$. As we shall see, such a specification is not required.

There are a host of quantization ambiguities involved in defining $\frac{1}{4}\frac{1}{\sqrt{|X^{+\prime}X^{-\prime}|}}(y_0)$, However, in analogy with the inverse volume operator in LQG, we want the operator to be such that

(i) $\widehat{\frac{1}{\sqrt{X^{+\prime}X^{-\prime}(y_0)}}}T_{s^+}\otimes T_{s^-}=0$, if y_0 is a vertex of triangulation but does not belong to $V_E(\gamma^+)\cup V_E(\gamma^-)$.

(ii) $\widehat{\frac{1}{\sqrt{X^{+\prime}X^{-\prime}(y_0)}}}T_{s^+}\otimes T_{s^-}\neq 0$ if $y_0\in V_E(\gamma^+)\cup V_E(\gamma^-)$.

(iii) For embedding data $(\vec{k}^+,\vec{k}^-)$ which suitably approximate the classical continuum data (X^+,X^-), the spectrum of $\widehat{\frac{1}{\sqrt{X^{+\prime}X^{-\prime}(y_0)}}}$ should be well approximated by the classical expression $\frac{1}{\sqrt{|X^{+\prime}X^{-\prime}|(y_0)}}$.[7]

We shall implicitly tune our quantization choices to meet these three requirements.

It is easy to check that Eq. (35) can be rewritten as

$$\begin{aligned}\frac{1}{\sqrt{|X^{+\prime}X^{-\prime}|}}(y_0)=&\left(\frac{\hbar}{a}\right)^2\mathrm{sgn}(X^{+\prime}X^{-\prime})(y_0)\Bigg[h_{e_{x_1,y_0}}[\Pi_-]^{-1}\Bigg\{\int_{\mathcal{U}_{y_0}}dy h_{e_{x_1,y}}[\Pi_+]\\&\times\Bigg\{\int_{\mathcal{U}_{y_0}}dy_1\sqrt{X^{+\prime}X^{-\prime}}(y_1),h_{e_{x_1,y}}[\Pi_+]^{-1}\Bigg\},h_{e_{x_1,y_0}}[\Pi_-]\Bigg\}-h_{e_{x_1,y_0}}[\Pi_-]^{-1}\Bigg\{\int_{\mathcal{U}_{y_0}}dy h_{e_{x_1,y}}[\Pi_+]^{-1}\\&\times\Bigg\{\int_{\mathcal{U}_{y_0}}dy_1\sqrt{X^{+\prime}X^{-\prime}}(y_1),h_{e_{x_1,y}}[\Pi_+]\Bigg\},h_{e_{x_1,y_0}}[\Pi_-]\Bigg\}\Bigg]-\mathrm{sgn}(X^{+\prime}X^{-\prime})(y_0)\Bigg[h_{e_{x_1,y_0}}[\Pi_+]\\&\times\Bigg\{\int_{\mathcal{U}_{y_0}}dy h_{e_{x_1,y}}[\Pi_-]\Bigg\{\int_{\mathcal{U}_{y_0}}dy_1\sqrt{X^{+\prime}X^{-\prime}}(y_1),h_{e_{x_1,y}}[\Pi_-]^{-1}\Bigg\},h_{e_{x_1,y_0}}[\Pi_+]^{-1}\Bigg\}-h_{e_{x_1,y_0}}[\Pi_+]\\&\times\Bigg\{\int_{\mathcal{U}_{y_0}}dy h_{e_{x_1,y}}[\Pi_-]^{-1}\Bigg\{\int_{\mathcal{U}_{y_0}}dy_1\sqrt{X^{+\prime}X^{-\prime}}(y_1),h_{e_{x_1,y}}[\Pi_-]\Bigg\},h_{e_{x_1,y_0}}[\Pi_+]^{-1}\Bigg\}\Bigg].\end{aligned}\tag{37}$$

Here, $h_{e_{x,y}}$ refers to the embedding momentum holonomy with unit charge $k_{e_{x,y}}=\frac{a}{\hbar}$ (see Sec. II B 1 for the definition of the parameter a) along the edge starting at the point x and ending at the point y, the edge being oriented along the orientation of the circle. As we shall see, the specific choice of the classical expression above ensures that its quantum correspondent acts trivially on those vertices of triangulation, which *are not* nontrivial vertices of the acted-upon state.

We now approximate $\int_{\mathcal{U}_{y_0}}dy h_{e_{x_1,y}}[\Pi_\pm]$ by a Riemann sum over simplices of the triangulation T (recall that T is a triangulation of the interval $[0,2\pi]$, and replace the classical objects by the corresponding quantum ones. This yields a finite triangulation approximant to the operator $\widehat{\frac{1}{\sqrt{|X^{+\prime}X^{-\prime}|}}}(y_0)$. We shall refer to this approximant as $\widehat{\frac{1}{\sqrt{|X^{+\prime}X^{-\prime}|}}}(y_0)|_T$. We have that[8]

$$\begin{aligned}\widehat{\frac{1}{\sqrt{|X^{+\prime}X^{-\prime}|}}}(y_0)|_T=&-a^{-2}(\hat{h}^{(-)-1}_{e_{x_1,y_0}}\widehat{\mathrm{sgn}(X^{+\prime}X^{-\prime})}(y_0)\sum_{\triangle}|\triangle|\kappa_\epsilon(b(\triangle),y_0)([\hat{h}^{(+)}_{e_{x_1,b(\triangle)}}[\hat{V}_{\mathcal{U}_{y_0}},\hat{h}^{(+)-1}_{e_{x_1,b(\triangle)}}],\hat{h}^{(-)}_{e_{x_1,y_0}}]\\&-[\hat{h}^{(+)-1}_{e_{x_1,b(\triangle)}}[\hat{V}_{\mathcal{U}_{y_0}},\hat{h}^{(+)}_{e_{x_1,b(\triangle)}}],\hat{h}^{(-)}_{e_{x_1,y_0}}]))-(\hat{h}^{(+)}_{e_{x_1,y_0}}\widehat{\mathrm{sgn}(X^{+\prime}X^{-\prime})}(y_0)\sum_{\triangle}|\triangle|\kappa_\epsilon(b(\triangle),y_0)\\&\times([\hat{h}^{(-)}_{e_{x_1,b(\triangle)}}[\hat{V}_{\mathcal{U}_{y_0}},\hat{h}^{(-)-1}_{e_{x_1,b(\triangle)}}],\hat{h}^{(+)-1}_{e_{x_1,y_0}}]-[\hat{h}^{(-)-1}_{e_{x_1,b(\triangle)}}[\hat{V}_{\mathcal{U}_{y_0}},\hat{h}^{(-)}_{e_{x_1,b(\triangle)}}],\hat{h}^{(+)-1}_{e_{x_1,y_0}}])).\end{aligned}\tag{38}$$

[7] We shall be more precise about the sense in which classical data are well approximated at the end of this section

[8] Because of the negative density weight of the inverse metric, the approximant will have an overall factor of $|\triangle|$ but, as in LQG, this factor will cancel with other factors which appear in the definition of the Hamiltonian constraint and, as we shall see in the next section, yield a well-defined $|\triangle|\to 0$ action of the constraint on diffeomorphism invariant states.

As in the previous section, we choose the triangulation T to be adapted to the charge network state it acts on by requiring that every vertex of the state is a vertex of T. We further restrict T by requiring that x_1, y_0 be vertices of T. In order to obtain a concise operator action, we shall also tailor the choice of $\epsilon \gg |\triangle|$ to the location of the point y_0, as well as the vertex structure of the graphs underlying the charge network state as follows.

As before, let the state be $T_{s^+} \otimes T_{s^-}$ and let $V_E(s^+) \cup V_E(s^-)$ be the set of nontrivial vertices. Given y_0, we choose ϵ to be small enough that $\mathcal{U}_{y_0}$ contains no nontrivial vertex of the graph other than one at y_0, if such a vertex exists. It is then straightforward to check, using Eq. (33), that

$$\widehat{\frac{1}{\sqrt{|X^{+'}X^{-'}|}}}(y_0)|_T T_{s^+} \otimes T_{s^-} = -a^{-2}(\hat{h}^{(-)-1}_{e_{x_1,y_0}} \mathrm{sgn}(\widehat{X^{+'}X^{-'}})(y_0) \sum_{\triangle,\triangle_1} |\triangle| \kappa_\epsilon(b(\triangle), y_0)\kappa_\epsilon(b(\triangle_1), y_0)$$
$$\times ([\hat{h}^{(+)}_{e_{x_1,y_0}}[\hat{V}_{b(\triangle_1)}, \hat{h}^{(+)-1}_{e_{x_1,b(\triangle)}}], \hat{h}^{(-)}_{e_{x_1,y_0}}] - [\hat{h}^{(+)-1}_{e_{x_1,y_0}}[\hat{V}_{b(\triangle_1)}, \hat{h}^{(+)}_{e_{x_1,b(\triangle)}}], \hat{h}^{(-)}_{e_{x_1,y_0}}]))$$
$$- (\hat{h}^{(+)}_{e_{x_1,y_0}} \mathrm{sgn}(\widehat{X^{+'}X^{-'}})(y_0) \sum_{\triangle,\triangle_1} |\triangle| \kappa_\epsilon(b(\triangle), y_0)\kappa_\epsilon(b(\triangle_1), y_0)([\hat{h}^{(-)}_{e_{x_1,y_0}}[\hat{V}_{b(\triangle_1)}, \hat{h}^{(-)-1}_{e_{x_1,b(\triangle)}}], \hat{h}^{(+)-1}_{e_{x_1,y_0}}]$$
$$- [\hat{h}^{(-)-1}_{e_{x_1,y_0}}[\hat{V}_{b(\triangle_1)}, \hat{h}^{(-)}_{e_{x_1,b(\triangle)}}], \hat{h}^{(+)-1}_{e_{x_1,y_0}}]))T_{s^+} \otimes T_{s^-}. \quad (39)$$

It is easy to show that the double commutators vanish if the points $b(\triangle)$, $b(\triangle_1)$, y_0 do not coincide on the circle. This simplifies the above expression to

$$\widehat{\frac{1}{\sqrt{|X^{+'}X^{-'}|}}}(y_0)|_T T_{s^+} \otimes T_{s^-} = -a^{-2}(\hat{h}^{(-)-1}_{e_{x_1,y_0}} \mathrm{sgn}(\widehat{X^{+'}X^{-'}})(y_0)|\triangle|([\hat{h}^{(+)}_{e_{x_1,y_0}}[\hat{V}_{y_0}, \hat{h}^{(+)-1}_{e_{x_1,y_0}}], \hat{h}^{(-)}_{e_{x_1,y_0}}] - [\hat{h}^{(+)-1}_{e_{x_1,y_0}}[\hat{V}_{y_0}, \hat{h}^{(+)}_{e_{x_1,y_0}}], \hat{h}^{(-)}_{e_{x_1,y_0}}]))$$
$$- (\hat{h}^{(+)}_{e_{x_1,y_0}} \mathrm{sgn}(\widehat{X^{+'}X^{-'}})(y_0)|\bar{\triangle}|([\hat{h}^{(-)}_{e_{x_1,y_0}}[\hat{V}_{y_0}, \hat{h}^{(-)-1}_{e_{x_1,y_0}}], \hat{h}^{(+)-1}_{e_{x_1,y_0}}]$$
$$- [\hat{h}^{(-)-1}_{e_{x_1,y_0}}[\hat{V}_{y_0}, \hat{h}^{(-)}_{e_{x_1,y_0}}], \hat{h}^{(+)-1}_{e_{x_1,y_0}}]))T_{s^+} \otimes T_{s^-}. \quad (40)$$

Note that the operator $\hat{h}^{(\pm)-1}_{e_{x_1,y_0}}[\hat{V}_{y_0}, \hat{h}^{(\pm)}_{e_{x_1,y_0}}]$ is only sensitive to that part of e_{x_1,y_0} which overlaps with the 1-simplex, which ends at y_0 (for $y_0 = 0 \equiv 2\pi$, this would be the "last" simplex ending at 2π). It follows that the operator action (40) is independent of the choice of x_1 (provided, of course, that $x_1 \notin \mathcal{U}_{y_0}$), just as is the case for the classical expression.

As detailed in the Appendix, a straightforward calculation shows that charge network states are eigenstates of the inverse metric operator. Specifically, we have that

$$\widehat{\frac{1}{\sqrt{X^{+'}X^{-'}(y_0)}}}|_T T_{s^+} \otimes T_{s^-} = |\Delta|\hbar a^{-2}\lambda(s^+, s^-, y_0)T_{s^+} \otimes T_{s^-}, \quad (41)$$

where $\lambda(s^+, s^-, y_0)$ is as follows (see Equation (A5) in the Appendix). $\lambda(s^+, s^-, y_0)$ vanishes if y_0 is not a nontrivial vertex, i.e. if $y_0 \notin V_E(s^+) \cup V_E(s^-)$. If $y_0 = v \in V_E(s^+) \cup V_E(s^-)$, then we have that

$$\lambda(s^+, s^-, v) = -\mathrm{sgn}(k^+_{e^v} - k^+_{e_v})\mathrm{sgn}\left(k^-_{e^v} - \left(k^-_{e_v} + \frac{a}{\hbar}\right)\right)\left[\left[\sqrt{\left|k^+_{e^v} - \left(k^+_{e_v} - \frac{a}{\hbar}\right)\right|\left|k^-_{e^v} - \left(k^-_{e_v} + \frac{a}{\hbar}\right)\right|}\right.\right.$$
$$\left. - \sqrt{\left|k^+_{e^v} - \left(k^+_{e_v} - \frac{a}{\hbar}\right)\right|\left|k^-_{e^v} - k^-_{e_v}\right|}\right] - \left[\sqrt{\left|k^+_{e^v} - \left(k^+_{e_v} + \frac{a}{\hbar}\right)\right|\left|k^-_{e^v} - \left(k^-_{e_v} + \frac{a}{\hbar}\right)\right|}\right.$$
$$\left.\left. - \sqrt{\left|k^+_{e^v} - \left(k^+_{e_v} + \frac{a}{\hbar}\right)\right||k^-_{e^v} - k^-_{e_v}|}\right]\right] + \mathrm{sgn}\left(k^+_{e^v} - \left(k^+_{e_v} - \frac{a}{\hbar}\right)\right)\mathrm{sgn}(k^-_{e^v} - k^-_{e_v})$$
$$\times \left[\left[\sqrt{\left|k^+_{e^v} - \left(k^+_{e_v} - \frac{a}{\hbar}\right)\right|\left|k^-_{e^v} - \left(k^-_{e_v} - \frac{a}{\hbar}\right)\right|} - \sqrt{\left|k^+_{e^v} - (k^+_{e_v})\right|\left|k^-_{e^v} - \left(k^-_{e_v} - \frac{a}{\hbar}\right)\right|}\right]\right.$$
$$\left. - \left[\sqrt{\left|k^+_{e^v} - \left(k^+_{e_v} - \frac{a}{\hbar}\right)\right|\left|k^-_{e^v} - \left(k^-_{e_v} + \frac{a}{\hbar}\right)\right|} - \sqrt{|k^+_{e^v} - k^+_{e_v}|\left|k^-_{e^v} - \left(k^-_{e_v} + \frac{a}{\hbar}\right)\right|}\right]\right]. \quad (42)$$

The discrete analog of the classical restrictions $\pm X^{\pm\prime} > 0$ are the "positivity" conditions $\pm(k^{\pm}_{e^v} - k^{\pm}_{e_v}) \geq 0$. Indeed, the sector of the polymer Hilbert space we shall be interested in satisfies these conditions and for such states it is straightforward to see that

(a) $\lambda(s^+, s^-, v) \neq 0$ iff $v \in V_E(\gamma^+) \cup V_E(\gamma^-)$. This property is intimately tied to the operator ordering we have chosen, specifically, the positioning of the signum operator in the expression.[9]

(b) $\lambda(s^+, s^-, v = 0) = \lambda(s^+, s^-, v = 2\pi)$.

(c) Whenever the argument of the signum function vanishes, so does the factor multiplying it, thus obviating the necessity of defining the signum function for vanishing arguments.

Finally, it is straightforward to verify that for states which satisfy the condition $\pm\hbar(k^{\pm}_{e^v} - k^{\pm}_{e_v}) \gg a$ (this condition partially captures the continuum condition of existence of the second (spatial) derivative of the embedding variables), the expression (42) has the correct continuum limit. The verification consists in showing that it defines a discrete approximant to the continuum expression $-4\frac{d\sqrt{X^{+\prime}}}{dX^{+\prime}}\frac{d\sqrt{-X^{-\prime}}}{dX^{-\prime}}$, which is just another way to write $\frac{1}{\sqrt{-X^{+\prime}X^{-\prime}}}$

V. THE HAMILTONIAN CONSTRAINT OPERATOR

This section is devoted to the construction of the Hamiltonian constraint as an operator on the space of diffeomorphism invariant states. We follow the strategy used in LQG. Our aim is to first define a discrete approximant to the Hamiltonian constraint on a triangulation of the spatial manifold, promote the expression to an operator on the kinematic Hilbert space and then show that its dual action on diffeomorphism invariant distributions, admits a well-defined continuum limit.

From Eq. (2), the smeared Hamiltonian constraint with lapse $N(x)$ is

$$C_{\mathrm{ham}}[N] = \int N(x)\Big[\Pi_+(x)X^{+\prime}(x) - \Pi_-(x)X^{-\prime}(x) + \frac{1}{4}(\pi_f^2 + f^{\prime 2})\Big]\frac{1}{\sqrt{X^{+\prime}(x)X^{-\prime}(x)}}.$$

On a triangulation T, a discrete approximant to the above expression is given by

$$\begin{aligned} C_{\mathrm{ham},T}[N] = \sum_{\triangle\in T}|\triangle|N(b(\triangle))\Big[\Pi_+(b(\triangle)) \\ \times\left(\frac{X^+(m(\triangle)) - X^+(m(\triangle-1) + L\delta_{b(\triangle),0})}{|\triangle|}\right) \\ - \Pi_-(b(\triangle))\left(\frac{X^-(m(\triangle)) - X^-(m(\triangle-1) - L\delta_{b(\triangle),0})}{|\triangle|}\right) \\ + \frac{1}{4}(Y^+)^2(b(\triangle)) + \frac{1}{4}(Y^-)^2(b(\triangle))\Big]\frac{1}{\sqrt{X^{+\prime}X^{-\prime}}}(b(\triangle)) \end{aligned} \quad (43)$$

where we have used the notation of Sec. IV so that $b(\triangle)$ is the beginning vertex of simplex $\triangle$, $|\triangle|$ is its length and $m(\triangle)$ its midpoint. The symbol $\triangle - 1$ denotes the simplex to the left of $\triangle$, and it is understood that if $\triangle_1$ is the leftmost simplex with $b(\triangle_1) = 0$ then $m(\triangle_1 - 1) = m(\triangle_N)$, with $\triangle_N$ being the rightmost simplex such that $f(\triangle_N) = 2\pi$ where $f(\triangle)$ is the ending vertex of $\triangle$. The Kronecker delta terms $L\delta_{b(\triangle),0}$ take into account the quasiperiodic nature of the embedding variables and, similar to the term in the second line of Eq. (30), come into play only for the first cell of the triangulation. Since only the holonomies of $\Pi_\pm$, $Y^\pm$ are well-defined operators on $\mathcal{H}_{\mathrm{kin}}$, the local fields $\Pi_\pm$, $(Y^\pm)^2$ need to be approximated on T by appropriate combinations of holonomies.

Recall that the solutions of [2] are obtained via averaging over the unitary action of finite gauge transformations. Since the unitary representation of gauge transformations is not weakly continuous on the kinematic Hilbert space, we cannot directly define their putative generators as operators there. Since the classical constraints are the generators of such transformations it seems impossible to define the Hamiltonian constraint in such away that it kills the solutions of [2]. We get around this potential obstruction by the pursuing the following *key* idea.

Note that the solutions of Ref. [2] are invariant under the action of any finite gauge transformation and hence would be annihilated by the difference of a *finite* gauge transformation and the identity operator. Note also that the LQG strategy is to first define an operator on the kinematic Hilbert space at finite triangulation and then take the continuum limit. This suggests that we seek finite triangulation holonomy approximants to the various local fields of interest in such a way that $C_{\mathrm{ham},T}[N]$ is proportional to a combination of finite gauge transformations minus the identity, with the finite gauge transformations being parametrized by $|\triangle|$ so that at $|\triangle| = 0$ the gauge transformations are just identity.

We display exactly such approximants to $\Pi_\pm$ in Sec. VA. The holonomies turn out to be state dependent,[10]

[9]In LQG, the inverse triad operator annihilates the vertices which are annihilated by the volume operator. In contrast, we are defining $\widehat{\frac{1}{\sqrt{|X^{+\prime}X^{-\prime}|}}}(y_0)$ such that it has nonvanishing action on all nontrivial (embedding) vertices of the underlying graph. Without this property, all zero volume states would be in the kernel of the constraint, similar to LQG. Such a kernel is much larger than the solution space constructed by group averaging in [2].

[10]As we will see shortly, this dependence involves the eigenvalues of the embedding operators $\hat{X}^\pm$. Since $\hat{X}^\pm$ are the analogs of the LQG densitized triad operators, this feature is reminiscent of the $\bar{\mu}$ scheme employed in the improved dynamics in LQC[13].

the approximants yield $\Pi_{\pm}(b(\triangle))$ to leading order in $|\triangle|$ and the resulting '$\Pi_{\pm}X^{\pm\prime}$' terms are proportional to a the difference of a finite gauge transformation and identity. It is then easy to guess the form of the matter terms which contributes the correct finite gauge transformation on the matter sector. However the tricky part is to realize these terms as approximants to the $(Y^{\pm})^2$ terms. As far as we can see, the terms we need do not correspond to (the replacements by quantum operators of) classical functions which yield $(Y^{\pm})^2(b(\triangle))$ up to higher order terms in $|\triangle|$. Instead, they may be derived as operators whose action approximates that of the Hamiltonian vector fields of $(Y^{\pm})^2$. We present these derivations in Sec. V B. In our opinion, this provides a useful lesson for LQG: We may have to be similarly open-minded there in our search for finite triangulation approximants to local fields such as the curvature of the Ashtekar-Barbero connection if we want a satisfactory definition of the quantum dynamics of LQG.

We use the approximants defined in Secs. V A and IV Bto define finite triangulation approximants to the Hamiltonian constraint in Sec. V C and show the resulting operator action has a continuum limit on diffeomorphism invariant states.

A. Embedding momenta approximants

We shall, as usual, focus on the left-moving variables. Let v be a vertex of T such that $v = b(\triangle)$. The traditional choice in LQG corresponds to the embedding momentum approximant $\Pi_{+}^{\triangle,k}(v)$ defined through the holonomy, $h_{\triangle}^{(+),k}$ with charge k over the edge $\triangle$ as

$$\hat{\Pi}_{+}^{\triangle,k}(v) = \frac{i}{|\triangle|k}(\hat{h}_{\triangle}^{+,k} - 1), \tag{44}$$

where $\hat{h}_{\triangle}^{+k} = e^{-ik\widehat{\int_{\triangle}\Pi_{+}}}$. Clearly, Eq. (44) can be promoted to an operator on $\mathcal{H}_{\text{kin}}$.

Another possible choice is to allow the charge label of the holonomy to depend on that of the embedding charge network state it acts on. Accordingly, we fix the state-dependent triangulation T of Sec. IV (recall that every nontrivial embedding vertex of the state is a vertex of T). We further restrict T to be fine enough that for any pair of successive vertices of T only one, at most, is nontrivial and define $\hat{\Pi}_{+}^{\triangle}(v)$ as

$$\hat{\Pi}_{+}^{\triangle}(v)T_{s^+} = \frac{i}{|\triangle|(k_{e_v}^+ - k_{e^v}^+)}(\hat{h}_{\triangle}^{(k_{e_v}^+ - k_{e^v}^+)} - 1)T_{s^+} \quad \text{if } v \in V_E(\gamma^+) \tag{45}$$

$$\hat{\Pi}_{+}^{\triangle}(v)T_{s^+} = \frac{i}{|\triangle|}(\hat{h}_{\triangle}^{(+)} - 1)T_{s^+} \quad \text{if } v \notin V_E(\gamma^+).$$

Here, as in Sec. IV A, e_v (e^v) refer to the edges which terminate (originate) at v, $V(\gamma^+)$ refers to the set of nontrivial vertices where $k_{e_v}^+ - k_{e^v}^+ \neq 0$ where $k_{e_v}^+$ $(k_{e^v}^+)$ for the first (last) edge are defined through the quasiperiodic extension of the state (see Sec. II B 4) and we have set $\hat{h}_{\triangle}^{k_{e_v}^+ - k_{e^v}^+} := e^{-i\widehat{(k_{e_v}^+ - k_{e^v}^+)\int_{\triangle}\Pi_{+}}}$, $\hat{h}_{\triangle}^{+} := e^{-i\widehat{\int_{\triangle}\Pi_{+}}}$.

Next, we show that the above choice (45) directly leads to an operator action of the "+" embedding part of the $C_{\text{ham},T}$, which is a finite diffeomorphism on the + part of the embedding state. As we shall see, this will finally lead to a satisfactory definition of the Hamiltonian constraint in Sec. V C. With the above definition, the approximant to the $\Pi_{+}X^{+\prime}$ term is

$$\begin{aligned}\hat{\Pi}_{+}\hat{X}^{+\prime}(b(\triangle))|_{T}T_{s^+} &:= \frac{1}{|\triangle|}\hat{\Pi}_{+}^{\triangle}(b(\triangle))(\hat{X}^{+}(m(\triangle)) - \hat{X}^{+}(m(\triangle - 1)))T_{s^+} \\ &= \frac{-i\hbar}{|\triangle|^2}(\hat{h}_{\triangle}^{k_{e_v}^+ - k_{e^v}^+} - 1)T_{s^+} \quad \text{if } b(\triangle) \in V_E(\gamma^+) \\ &= 0 \quad \text{if } b(\triangle) \notin V_E(\gamma^+).\end{aligned} \tag{46}$$

Whence for $x \in V_E(\gamma^+)$,

$$\begin{aligned}\hat{\Pi}_{+}\hat{X}^{+\prime}(b(\triangle))|_{T}T_{s^+} &= \frac{-i\hbar}{|\triangle|^2}(\hat{h}_{\triangle}^{k_{e_v}^+ - k_{e^v}^+} - 1)T_{s^+} \\ &= \frac{-i\hbar}{|\triangle|^2}(T_{s^+_{\phi_{\triangle}}} - T_{s^+}) \\ &= \frac{-i\hbar}{|\triangle|^2}(\hat{U}^{+,E}(\phi_{\triangle}) - 1)T_{s^+}.\end{aligned} \tag{47}$$

Here,$\phi_{\triangle}$ is a diffeomorphism of the circle (more precisely, $\phi_{\triangle}$ is a periodic diffeomorphism of the real line) which is identity in the neighborhood of all the vertices of T except $b(\triangle)$, $f(\triangle)$.[11] Further, $\phi_{\triangle}$ maps $b(\triangle)$ to $f(\triangle)$ and its action on the charge network label s^+ is denoted by $s^+_{\phi_{\triangle}}$ as in Sec. II B 4. $\hat{U}^{+,E}(\phi_{\triangle})$ is the restriction of the unitary action of the finite gauge transformation $\hat{U}^{+}(\phi^+ = \phi_{\triangle})$ (see Sec. II B 4) to the left-moving embedding Hilbert space. Finally, note that we could as well have chosen $\hat{\Pi}_{+}^{\triangle}(v^+)T_{s^+} = \frac{-i}{|\triangle|(k_{e_v}^+ - k_{e^v}^+)}((\hat{h}_{\triangle}^{(k_{e_v}^+ - k_{e^v}^+)})^{\dagger} - 1)T_{s^+}$ and we would have obtained the inverse diffeomorphism with a negative sign. We shall use this flexibility according to our convenience.

The analysis of the right-moving mode proceeds in a similar way.

B. Matter field approximants

Let T be the triangulation of Sec. IV A with the restriction that every nontrivial matter vertex of the charge network state on which $\hat{C}_{\text{ham},T}$ acts is a vertex of T and no two successive vertices of T are nontrivial. A nontrivial left-moving (or right-moving) matter vertex v is one for which $l_{e_v}^+ - l_{e^v}^+ \neq 0$ (or $l_{e_v}^- - l_{e^v}^- \neq 0$). Here, similar to Sec. IV A,

[11] In addition, $\phi_{\triangle}$ also differs from identity at $x = 2\pi$ (or $x = 0$) if $b(\triangle) = 0$ (or $f(\triangle) = 2\pi$); this is just a consequence of the circular topology of space.

e_v (e^v) refer to the edges which terminate (originate) at v and $l^{\pm}_{e_v}$ ($l^{\pm}_{e^v}$) for the first (last) edge are defined through the periodic extension of the matter charge network (see Sec. II B 4). We shall refer to the set of nontrivial matter vertices as $V_M(\gamma^{\pm})$.

The traditional LQG type approximant, similar to how curvature terms are quantized in lattice gauge theories, is of the form

$$(Y^{\pm})^2(x = b(\triangle))|_T = \frac{e^{im\int_{\triangle} Y^+} + e^{-im\int_{\triangle} Y^+} - 2}{m^2|\triangle|^2}. \quad (48)$$

We are unable to see how such a choice could lead to an operator action which is that of a finite diffeomorphism. Hence, it seems unlikely that with this choice, the physical states of [2] are in the kernel of the Hamiltonian constraint.

Instead, as we shall see, the following choice yields a satisfactory definition of the Hamiltonian constraint:

$$\widehat{(Y^+)}^2(b(\triangle))_T W_{s^+} = \frac{-4i\hbar}{|\triangle|^2}\left[e^{-i(\hbar/2)(l^+_{e^+_I} - l^+_{e^+_{I+1}})^2}\, \hat{h}_{\triangle}^{\,l^+_{e^+_I} - l^+_{e^+_{I+1}}} - 1\right]W_{s^+}, \quad (49)$$

which can also be rewritten as

$$\widehat{(Y^+)}^2(b(\triangle))_T = \frac{-4i\hbar}{|\triangle|^2}(\hat{U}^{+,M}(\phi_{\triangle}) - 1), \quad (50)$$

where $\phi_{\triangle}$ has been defined in the previous section and $\hat{U}^{+,M}(\phi_{\triangle})$ is the restriction of the finite gauge transformation operator $\hat{U}^{+}(\phi^+ = \phi_{\triangle})$ to the matter Hilbert space. This choice does not arise straightforwardly as the evaluation of some finite $\triangle$ approximant to $(Y^+(x))^2$, as in the case of the embedding momentum. Rather, as we now argue, its justification lies in an analysis of the Hamiltonian vector field generated by the corresponding classical quantity. In what follows, we use the notation $A \approx B$ to indicate that A and B agree to leading order in $|\triangle|$.

Let $\bar{\triangle}$ be the interval obtained by joining the points $m(\triangle - 1)$ and $m(\triangle)$ on the circle. Let x be the coordinate system which defines T so that the 1-simplices of T are of equal length $|\triangle|$. Let s^+ be a matter charge network label such that every vertex of s^+ is a vertex of T and no pair of successive vertices of T are nontrivial vertices of s^+. (Since we are ultimately interested in the $|\triangle| \to 0$ limit, this is not an unreasonable restriction.)

Then it is straightforward to check the following:

$$Y^{+2}(b(\triangle)) \approx \frac{1}{|\triangle|}\int_{\bar{\triangle}}(Y^+)^2 dx, \quad (51)$$

$$\begin{aligned}\left\{\int_{\bar{\triangle}}(Y^+)^2 dx, W(s^+)(Y^+)\right\} &\approx \frac{-4}{|\triangle|}(W(s^+_{\phi_{\triangle}})(Y^+) - W(s^+)(Y^+)) \\ &\approx \frac{-4}{|\triangle|}(W(s^+_{\phi_{\triangle}}) - W(s^+))\left(1 - \frac{|\triangle|^2}{4i\hbar}Y^{+2}(b(\triangle))\right),\end{aligned} \quad (52)$$

where the validity of the last line lies in the fact that $\hbar$ remains nonzero in the continuum limit defined by $|\triangle| \to 0$.

If we define $(Y^+\widehat{(b(\triangle))})^2_T$ by

$$(Y^+\widehat{(b(\triangle))})^2_T := \frac{-4i\hbar}{|\triangle|^2}(\hat{U}_{\phi_{\triangle}} - 1), \quad (53)$$

it is easy to check that

$$[(Y^+\widehat{(b(\triangle))})^2_T, \hat{W}(s^+)] = \frac{-4i\hbar}{|\triangle|}(\hat{W}(s^+_{\phi_{\triangle}}) - \hat{W}(s^+)) \times\left(1 - \frac{|\triangle|^2}{4i\hbar}(Y^+\widehat{(b(\triangle))})^2_T\right). \quad (54)$$

Thus, the definition (49) yields a representation of the approximant to the Poisson bracket $\{(Y^+(b(\triangle))^2_T, W(s^+)\}$, and this is the justification for the choice (49).

The following observations offer further evidence that as far as the continuum limit of the ensuing Hamiltonian constraint operator is concerned, our choice (49) is a reasonable one. First, consider the one parameter family of gauge transformations labeled by the one parameter family of diffeomorphisms $\phi^+(\lambda)$ and generated by the function $\int_{S^1} N_+(x)(Y^+(x))^2$ for some smooth smearing function N_+. Then, we have that $\lim_{\lambda\to 0}\frac{U_{\phi^+(\lambda)} - 1}{\lambda}|\circ\rangle = 0$, where $|\circ\rangle$ is the state with vanishing matter charges. This is supportive of the putative operator identity $(\hat{Y}^+)^2(x)|\circ\rangle = 0$ which implies that it is reasonable to impose the condition

$$(Y^+\widehat{(b(\triangle))})^2_T|\circ\rangle = 0 \quad \forall\ \triangle \in T. \quad (55)$$

Next, restrict attention to matter charge nets whose underlying graph can be chosen to be (coarser than or equal to) T. Let the Weyl algebra of matter holonomies labeled by such charge nets be $\mathcal{W}^M_T$. Let the Hilbert space of states labeled by such charge nets be $\mathcal{H}^M_T$. Clearly, $\mathcal{H}^M_T$ supports a cyclic representation of $\mathcal{W}^M_T$ with the cyclic state $|\circ\rangle$. By virtue of the cyclicity of $|\circ\rangle$, it is easy to see that $(Y^+\widehat{(b(\triangle))})^2_T$ is uniquely specified as an operator on $\mathcal{H}^M_T$ by Eqs. (54) and (55).

C. The Hamiltonian constraint operator at finite triangulation

Recall that our aim is to write the Hamiltonian constraint operator in terms of the difference of a finite gauge transformation and identity. From Eq. (2), the Hamiltonian constraint is proportional to the difference, $H_+ - H_-$, of the generators of the "+" and "−" gauge transformations.

From considerations similar to those of Sec. III A, it follows that the transformations generated by the combination $H_+ - H_-$ correspond to gauge transformations for which $\phi^+ = (\phi^-)^{-1}$. Given that the left-moving operators of Eqs. (47) and (50) are associated with diffeomorphisms which displace vertices to their right (i.e., anticlockwise on the circle), this suggests that we construct the right moving operators in terms of diffeomorphisms which displace vertices to their left (i.e., clockwise on the circle).

Accordingly, we define

$$\hat{\Pi}_-^{\triangle}(v)T_{s^-} = \frac{-i}{|\triangle|(k^-_{e_v} - k^-_{e^v})}((\hat{h}_{\triangle}^{(k^-_{e_v}-k^-_{e^v})})^\dagger - 1)T_{s^-} \quad \text{if } v \in V_E(\gamma^-)$$
$$\hat{\Pi}_-^{\triangle}(v)T_{s^-} = \frac{i}{|\triangle|}(\hat{h}_{\triangle}^{(-)} - 1)T_{s^-} \quad \text{if } v \notin V_E(\gamma^-), \tag{56}$$

and

$$\widehat{(Y^-)^2}(b(\triangle))_T = \frac{-4i\hbar}{|\triangle|^2}(\hat{U}^{-,M}(\phi^{-1}_{\triangle-1}) - 1). \tag{57}$$

It is straightforward to derive Eqs. (56) and (57) along the lines of Sec. V A and V B and to see that for $x \in V_E(\gamma^-)$,

$$\hat{\Pi}_-\hat{X}^{-'}(b(\triangle))|_T T_{s^-} = \frac{i\hbar}{|\triangle|^2}(T_{s^-_{\phi^{-1}_{\triangle-1}}} - T_{s^-}) = \frac{i\hbar}{|\triangle|^2}(\hat{U}^{-,E}(\phi^{-1}_{\triangle-1}) - 1)T_{s^-}. \tag{58}$$

Here, $(\triangle - 1) \in T$ refers to the edge immediately preceding $\triangle$ with $\triangle - 1 = \triangle_N$ if $b(\triangle) = 0$. The diffeomorphism $\phi_\triangle$ for any $\triangle \in T$ has already been defined in Sec. V A and $\phi_\triangle^{-1}$ denotes the inverse of $\phi_\triangle$. Thus, $\phi^{-1}_{\triangle-1}$ maps $b(\triangle)$ to $b(\triangle - 1)$ and is identity on all vertices other $b(\triangle)$, $b(\triangle - 1)$ (modulo the identifications $x = 0 \sim x = 2\pi$, see Footnote [11]). Finally, $\hat{U}^{-,M}(\phi^{-1}_{\triangle-1})$, $\hat{U}^{-,E}(\phi^{-1}_{\triangle-1})$ refer to the restriction of the unitary action of the finite gauge transformation $\hat{U}^-(\phi^- = \phi^{-1}_{\triangle-1})$ to the matter and embedding sectors.

Next, recall that the finite gauge transformation labeled by ϕ^+ moves the right-moving embedding fields and the right-moving matter fields *together*. The same is true for the corresponding objects in the "$-$" sector. If the classical quantities in (43) are replaced by their corresponding operators through Eqs. (45), (50), (56), and (57), it is immediate to see that the action of the resulting constraint operator on a charge network state is the *sum* of gauge transformations each acting only on the matter part of the state or only on the embedding part of the state. This is not what we desire and is remedied as follows.

Equation (43) is a discrete approximant to the continuum expression and we may modify it by terms which vanish in the continuum limit, $|\triangle| \to 0$. It is straightforward to see that the following expression is one such modification:

$$C_{\mathrm{ham},T}[N] = \sum_{\triangle\in T} \frac{-i\hbar N(b(\triangle))}{|\triangle|\sqrt{X^{+'}X^{-'}}} \times \left(\left[1 + \frac{|\triangle|^2}{-i\hbar}\Pi_+(b(\triangle))X^{+'}(b(\triangle))\right]\right.$$
$$\times\left[1 + \frac{|\triangle|^2}{i\hbar}\Pi_-(b(\triangle))(b(\triangle))\right] \times\left[1 + \frac{|\triangle|^2}{-4i\hbar}(Y^+)^2(b(\triangle))\right] \times\left.\left[1 + \frac{|\triangle|^2}{-4i\hbar}(Y^-)^2(b(\triangle))\right]\right). \tag{59}$$

Replacing the classical quantities in the above equation by their quantum operators through Eqs. (47), (50), (57), and (58) and ordering the constraint operator so that the inverse metric is rightmost, the action of the quantum Hamiltonian constraint at finite triangulation on the state $|\mathbf{s}^+, \mathbf{s}^-\rangle$ is

$$\hat{C}_{\mathrm{ham},T}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle = \sum_{\triangle\in T, b(\triangle)\in V_E(s^+)\cup V_E(s^-)} N(b(\triangle)) \times[\hat{U}^{+,E}(\phi_\triangle)\otimes\hat{U}^{-,E}(\phi^{-1}_{\triangle-1}) \otimes\hat{U}^{+,M}(\phi_\triangle)\otimes\hat{U}^{-,M}(\phi^{-1}_{\triangle-1}) - 1] \times\frac{-i\hbar}{|\triangle|\widehat{\sqrt{X^{+'}X^{-'}}}}|\mathbf{s}^+,\mathbf{s}^-\rangle. \tag{60}$$

Using Eq. (42) and the fact that the unitary operators in the above equation are just restricted actions of unitary operators associated with finite gauge transformations, we obtain

$$\hat{C}_{\mathrm{ham},T}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle = \sum_{\triangle\in T, b(\triangle)\in V_E(s^+)\cup V_E(s^-)} N(b(\triangle)) \times\frac{-i\hbar}{a^2}\lambda(s^+,s^-,b(\triangle))[\hat{U}^+(\phi_\triangle) \otimes\hat{U}^-(\phi^{-1}_{\triangle-1}) - 1]|\mathbf{s}^+,\mathbf{s}^-\rangle. \tag{61}$$

Clearly, the above action kills any state invariant under all finite gauge transformations generated by H_+, H_- and thus provides a satisfactory definition of the Hamiltonian constraint at finite triangulation. We now show that the above action admits a continuum limit on the space of diffeomorphism invariant distributions.

D. The continuum limit of the action of $\hat{C}_{\mathrm{ham},T}$ on $\mathcal{H}_{\mathrm{kin}}$

Let us summarize the properties of the triangulation T:

(a) T depends on the (coarsest) graph $\gamma := \gamma(\mathbf{s}^+, \mathbf{s}^-)$ underlying the state $|\mathbf{s}^+, \mathbf{s}^+\rangle$ on which $\hat{C}_{\mathrm{ham},T}$ acts.
(b) Every vertex of the graph is a vertex of T.
(c) No two successive vertices of T are nontrivial (matter or embedding) vertices of $\gamma(\mathbf{s}^+, \mathbf{s}^-)$.
(d) There is a coordinate system in which every edge $\triangle$ of T has the same length $|\triangle|$.

We shall often emphasize (i) and (iv) above by setting $\delta := |\triangle|$ and denoting T by $T(\gamma, \delta)$. Consider a

1-parameter family of triangulations $T(\gamma, \delta)$, parameterized by $\delta > 0$ for fixed γ. The continuum limit of any quantity defined on $T(\gamma, \delta)$ is its limiting behavior as $|\triangle| \rightarrow 0$. In what follows, it is convenient to change our notation for $\phi_{\triangle}, \phi^{-1}_{\triangle-1}$. Accordingly, for $v := b(\triangle)$ we set $\phi_{\triangle} =: \phi_{v,\delta}$ and $\phi^{-1}_{\triangle-1} =: \phi_{v,-\delta}$. The notation signifies that $\phi_{v,\delta}$ moves the point v to the point $v+\delta$ on the circle and $\phi_{v,-\delta}$ moves the point v to the point $v-\delta$ on the circle.

Let $\Psi \in \mathcal{H}_{\rm diff}$ be a diffeomorphism invariant distribution. From Eq. (61), in our new notation, we have that

$$\Psi(\hat{C}_{{\rm ham},T(\gamma,\delta)}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle) = \sum_{v\in V_E(s^+)\cup V_E(s^-)} N(v)\frac{-i\hbar}{a^2}\lambda(s^+,s^-,v)\Psi([\hat{U}^+(\phi_{v,\delta})\otimes\hat{U}^-(\phi_{v,-\delta})-1]|\mathbf{s}^+,\mathbf{s}^-\rangle)$$
$$= \sum_{v\in V_E(s^+)\cup V_E(s^-)} N(v)\frac{-i\hbar}{a^2}\lambda(s^+,s^-,v)[\Psi(|\mathbf{s}^+{}_{\phi_{v,\delta}},\mathbf{s}^-{}_{\phi_{v,-\delta}}\rangle)-\Psi(|\mathbf{s}^+,\mathbf{s}^-\rangle)]. \quad (62)$$

It is easy to see, for any two triangulations $T(\gamma, \delta_1)$, $T(\gamma, \delta_2)$ and any vertex $v \in V_E(s^+) \cup V_E(s^-)$, that there exists a diffeomorphism $\phi(v, \delta_1, \delta_2)$ such that

$$|\mathbf{s}^+{}_{\phi_{v,\delta_1}},\mathbf{s}^-{}_{\phi_{v,-\delta_1}}\rangle = \hat{U}(\phi(v,\delta_1,\delta_2))|\mathbf{s}^+{}_{\phi_{v,\delta_2}},\mathbf{s}^-{}_{\phi_{v,-\delta_2}}\rangle, \quad (63)$$

where $\hat{U}(\phi(v, \delta_1, \delta_2))$ is the unitary operator corresponding to the spatial diffeomorphism $\phi(v, \delta_1, \delta_2)$ so that, from Sec. III A, $\hat{U}(\phi(v,\delta_1,\delta_2)) := \hat{U}^+(\phi(v,\delta_1,\delta_2)) \otimes \hat{U}^-(\phi(v,\delta_1,\delta_2))$. It is then immediate from Eq. (62) that

$$\Psi(\hat{C}_{{\rm ham},T(\gamma,\delta_1)}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle) = \Psi(\hat{C}_{{\rm ham},T(\gamma,\delta_2)}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle), \quad (64)$$

so that

$$\lim_{\delta\rightarrow 0}\Psi(\hat{C}_{{\rm ham},T(\gamma,\delta)}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle) = \Psi(\hat{C}_{{\rm ham},T(\gamma,\delta_0)}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle). \quad (65)$$

Here, the choice of $\delta_0 > 0$ is arbitrary (subject, of course, to the restriction that $T(\gamma, \delta_0)$ satisfies properties (i)–(iv) above). Equation (65) shows that the dual action of the Hamiltonian constraint operator (61) possesses a well-defined continuum limit. Indeed, this conclusion is unchanged if Ψ is any diffeomorphism invariant distribution i.e. Ψ has a finite action on the (dense) space of finite linear combinations of charge network states and Ψ is invariant under the (dual) action of the unitary operators of Sec. III A which implement spatial diffeomorphisms on $\mathcal{H}_{\rm kin}$. In particular, it follows that the continuum limit of the Hamiltonian constraint operator annihilates the physical states of [2] (see Sec. II B 5) which are obtained by group averaging over the action of $H_{\pm}$.

Finally, we note that as explained beautifully by Thiemann in [14], Eq. (65) shows that the one parameter family of triangulated operators $\hat{C}_{{\rm ham}T(\gamma,\delta)}[N]$ converges to a (nonunique) densely defined operator $\hat{C}_{\rm ham}[N]$ on the *kinematic* Hilbert space in the so-called Uniform Rovelli-Smolin (URS) topology. Specifically, in the notation used above, we may choose the limit of the one parameter family $\hat{C}_{{\rm ham}T(\gamma,\delta)}[N]$ to be the operator $\hat{C}_{\rm ham}[N]$ where

$$\hat{C}_{\rm ham}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle := \hat{C}_{{\rm ham}T(\gamma,\delta_0)}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle. \quad (66)$$

VI. THE CONSTRAINT ALGEBRA AND THE ARENA OF DIFFEOMORPHISM INVARIANT DISTRIBUTIONS

Given a distribution Ψ (more precisely, an element of the algebraic dual to the superselected sector $\mathcal{D}_{ss}$ of Sec. II B 5) and a charge network $|\mathbf{s}^+,\mathbf{s}^-\rangle$, we would like to check if

$$\Psi([\hat{C}_{\rm ham}[N_2],\hat{C}_{\rm ham}[N_1]]|\mathbf{s}^+,\mathbf{s}^-\rangle) = \Psi(\widehat{C_{\rm diff}[\vec{\beta}(N,M)]}|\mathbf{s}^+,\mathbf{s}^-\rangle). \quad (67)$$

where the embedding dependent structure function $\vec{\beta}(N,M)$ has been defined in (3). Section VI A is devoted to the definition and evaluation of the left-hand side of the above equation and Sec. VI B to the right-hand side when Ψ is diffeomorphism invariant. We find that both sides of the equation vanish as is the case in LQG [6]. We structure our computations so that they are of use for evaluations in which Ψ is not diffeomorphism invariant, but lies in a suitable "habitat." We shall explore the constraint algebra on such habitats in Secs. VII and VIII.

A. The commutator of two Hamiltonian constraints

In this section, we compute the continuum limit of the commutator between two Hamiltonian constraints on the space of diffeomorphism invariant distributions. Since the Hamiltonian constraint does not map the space of such distributions to itself, we proceed along the lines of Thiemann's seminal work [6]. Specifically, we define the left-hand side of Eq. (67) through

$$\Psi([\hat{C}_{\rm ham}[N_2],\hat{C}_{\rm ham}[N_1]]|\mathbf{s}^+,\mathbf{s}^-\rangle) := \lim_{\delta'\rightarrow 0}\lim_{\delta\rightarrow 0}\Psi(\hat{C}_{{\rm ham},T'(\delta')}[N_2]\hat{C}_{{\rm ham},T(\delta)}[N_1] - \hat{C}_{{\rm ham},T'(\delta')}[N_1]\hat{C}_{{\rm ham},T(\delta)}[N_2]|\mathbf{s}^+,\mathbf{s}^-\rangle). \quad (68)$$

Here, $T(\delta) := T(\gamma, \delta)$ is a triangulation adapted to $|\mathbf{s}^+, \mathbf{s}^-\rangle$. $T'(\delta')$ is a refinement of $T(\delta)$ and has 1-cells of size $\delta' \ll \delta$ in the same coordinate system in which $T(\delta)$ has 1-cells of size δ. Further, $T'(\delta')$ is adapted to the charge networks which appear on the right-hand side of Eq. (61) (with the appropriate replacement of N by N_1 or N_2). It is easy to see that for small enough $\delta' \ll \delta$, such triangulations always exist.

Recall that T, T' are subject to the conditions (i)–(iv) of Sec. V D. In addition we shall, for simplicity, require that δ be small enough that (iii) is strengthened to the condition that nontrivial vertices of $|\mathbf{s}^+, \mathbf{s}^-\rangle$ are seperated by a large number of 1- cells of T.

Using the notation of Sec. V D in conjunction with Eq. (61), a straightforward computation yields

$$\hat{C}_{\mathrm{ham},T'(\delta')}[N_2]\hat{C}_{\mathrm{ham},T(\delta)}[N_1]|\mathbf{s}^+, \mathbf{s}^-\rangle = \left(\frac{-i\hbar^2}{a^2}\right)^2 \sum_{v\in V_E(s^+)\cup V_E(s^-)} N_1(v)\lambda(s^+, s^-, v)$$
$$\times\Bigg[\sum_{v'\in V_E(s^+_{\phi_{v,\delta}})\cup V_E(s^-_{\phi_{v,-\delta}})} N_2(v')\lambda(s^+_{\phi_{v,\delta}}, s^-_{\phi_{v,-\delta}}, v')(|(\mathbf{s}^+{}_{\phi_{v,\delta}})_{\phi_{v',\delta'}}, (\mathbf{s}^-{}_{\phi_{v,-\delta}})_{\phi_{v',\delta'}}\rangle - |\mathbf{s}^+{}_{\phi_{v,\delta}}, \mathbf{s}^+{}_{\phi_{v,-\delta}}\rangle)$$
$$\times \sum_{v'\in V_E(s^+)\cup V_E(s^-)} N_2(v')\lambda(s^+, s^-, v')(|\mathbf{s}^+{}_{\phi_{v',\delta'}}, \mathbf{s}^-{}_{\phi_{v',-\delta'}}\rangle - |\mathbf{s}^+, \mathbf{s}^-\rangle)\Bigg]. \tag{69}$$

Next, we restrict attention to N_i, $i = 1, 2$ of compact support. Specifically, let N_i be supported in a neighborhood $U_i(v_i)$ of the vertex $v_i \in V_E(s^+) \cup V_E(s^-)$ such that $U_i(v_i) \cap (V_E(s^+) \cup V_E(s^-)) = v_i$, $i = 1, 2$. The linear dependence of $C_{\mathrm{ham}}(N_i)$ on the lapse N_i, together with the fact that an arbitrary lapse function can be obtained by linear combinations of ones which have the above compact support property, imply that the restriction to lapses of compact support entail no loss of generality. In Sec. VI A 1, we consider the case $v_1 \neq v_2$ and in Sec. VI A 2, the case $v_1 = v_2$.

1. *The case* $v_1 \neq v_2$

Note that:

$$[\phi_{v_i,\pm\delta}, \phi_{v_j,\pm\delta'}] = 0, \qquad i \neq j \tag{70}$$

$$U_i(v_i) \cap (V(s^+_{\phi_{v_j,\delta}}) \cup V(s^-_{\phi_{v_j,-\delta}})) = v_i, \qquad i \neq j \tag{71}$$

$$\lambda(s^+_{\phi_{v_i,\delta}}, s^-_{\phi_{v_i,-\delta}}, v_j) = \lambda(s^+, s^-, v_j) \qquad i \neq j \tag{72}$$

Using this in conjunction with Eq. (69), it is straightforward to see that

$$\hat{C}_{\mathrm{ham},T'(\delta')}[N_2]\hat{C}_{\mathrm{ham},T(\delta)}[N_1]|\mathbf{s}^+, \mathbf{s}^-\rangle$$
$$= \left(\frac{-i\hbar^2}{a^2}\right)^2 N_1(v_1)\lambda(s^+, s^-, v_1)N_2(v_2)\lambda(s^+, s^-, v_2)$$
$$\times [(|(\mathbf{s}^+{}_{\phi_{v_1,\delta}})_{\phi_{v_2',\delta'}}, (\mathbf{s}^-{}_{\phi_{v_1,-\delta}})_{\phi_{v_2',-\delta'}}\rangle - |\mathbf{s}^+{}_{\phi_{v_1,\delta}}, \mathbf{s}^-{}_{\phi_{v_1,-\delta}}\rangle)$$
$$- (|\mathbf{s}^+{}_{\phi_{v_2,\delta'}}, \mathbf{s}^-{}_{\phi_{v_2,-\delta'}}\rangle - |\mathbf{s}^+, \mathbf{s}^-\rangle)]. \tag{73}$$

The second term in the commutator is obtained by interchanging $N_1(v_1)$, v_1 with $N_2(v_2)$, v_2 in the above equation so that the commutator evaluates to

$$\hat{C}_{\mathrm{ham},T'(\delta')}[N_2]\hat{C}_{\mathrm{ham},T(\delta)}[N_1] - \hat{C}_{\mathrm{ham},T'(\delta')}[N_1]\hat{C}_{\mathrm{ham},T(\delta)}[N_2]|\mathbf{s}^+, \mathbf{s}^-\rangle$$
$$= \left(\frac{-i\hbar^2}{a^2}\right)^2 N_1(v_1)\lambda(s^+, s^-, v_1)N_2(v_2)\lambda(s^+, s^-, v_2)[(|(\mathbf{s}^+{}_{\phi_{v_1,\delta}})_{\phi_{v_2',\delta'}}, (\mathbf{s}^-{}_{\phi_{v_1,-\delta}})_{\phi_{v_2',-\delta'}}\rangle$$
$$- |(\mathbf{s}^+{}_{\phi_{v_2,\delta}})_{\phi_{v_1',\delta'}}, (\mathbf{s}^-{}_{\phi_{v_2,-\delta}})_{\phi_{v_1',-\delta'}}\rangle) - (|\mathbf{s}^+{}_{\phi_{v_1,\delta}}, \mathbf{s}^-{}_{\phi_{v_1,-\delta}}\rangle - |\mathbf{s}^+{}_{\phi_{v_2,\delta}}, \mathbf{s}^-{}_{\phi_{v_2,-\delta}}\rangle)$$
$$- (|\mathbf{s}^+{}_{\phi_{v_2,\delta'}}, \mathbf{s}^-{}_{\phi_{v_2,-\delta'}}\rangle - |\mathbf{s}^+{}_{\phi_{v_1,\delta'}}, \mathbf{s}^-{}_{\phi_{v_1,-\delta'}}\rangle)] \tag{74}$$

From Eq. (68), the continuum limit of the commutator on the distribution Ψ is:

$$\Psi([\hat{C}_{\mathrm{ham}}[N_2], \hat{C}_{\mathrm{ham}}[N_1]]|\mathbf{s}^+, \mathbf{s}^-\rangle) := \left(\frac{-i\hbar^2}{a^2}\right)^2 N_1(v_1)\lambda(s^+, s^-, v_1)N_2(v_2)\lambda(s^+, s^-, v_2))$$
$$\times \lim_{\delta'\to 0}\lim_{\delta\to 0}(\Psi_1(\delta, \delta') + \Psi_2(\delta, \delta') + \Psi_3(\delta, \delta')), \tag{75}$$

where, using (70),

$$\Psi_1(v_1, v_2, \delta, \delta') := \Psi(|\mathbf{s}^+{}_{\phi_{v_1,\delta}})_{\phi_{v_2',\delta'}}, (\mathbf{s}^-{}_{\phi_{v_1,-\delta}})_{\phi_{v_2',-\delta'}}\rangle - |(\mathbf{s}^+{}_{\phi_{v_1,\delta'}})_{\phi_{v_2,\delta}}, (\mathbf{s}^-{}_{\phi_{v_1,-\delta'}})_{\phi_{v_2,-\delta}}\rangle) \tag{76}$$

$$\Psi_2(v_1, v_2, \delta, \delta') := -\Psi(|\mathbf{s}^+{}_{\phi_{v_1,\delta}}, \mathbf{s}^-{}_{\phi_{v_1,-\delta}}\rangle - |\mathbf{s}^+{}_{\phi_{v_1,\delta'}}, \mathbf{s}^-{}_{\phi_{v_1,-\delta'}}\rangle) \quad (77)$$

$$\Psi_3(v_1, v_2, \delta, \delta') := -\Psi(|\mathbf{s}^+{}_{\phi_{v_2,\delta'}}, \mathbf{s}^-{}_{\phi_{v_2,-\delta'}}\rangle - |\mathbf{s}^+{}_{\phi_{v_2,\delta}}, \mathbf{s}^-{}_{\phi_{v_2,-\delta}}\rangle). \quad (78)$$

It is easy to see that irrespective of the nature of the nontrivial vertices v_1, v_2 (i.e., whether $v_i \in V(s^+) \cap V(s^-)$ or not) each of the two charge network states in Eqs. (76)–(78) are diffeomorphic. Thus, if Ψ is a diffeomorphism invariant distribution, we have that $\Psi_I(v_1, v_2, \delta, \delta') = 0$, $I = 1, 2, 3$ for all δ, δ' under consideration which, in turn, implies that the commutator (75) vanishes.

2. *The case* $v_1 = v_2 = v$

We note that the set $U_i(v) \cap V(\mathbf{s}^+_{\phi_{v,\delta}})$ is either empty or consists of the single point $v + \delta$ (recall that $U_i(v)$ is the support of the lapse function N_i). Similarly, the set $U_i(v) \cap V(\mathbf{s}^-_{\phi_{v,-\delta}})$ is either empty or consists of the single point $v - \delta$. This implies that

$$\lambda(s^+_{\phi_{v,\delta}}, s^-_{\phi_{v,-\delta}}, v + \delta) = \lambda(s^+_{\phi_{v,\delta}}, s^-, v + \delta) \quad (79)$$

$$\lambda(s^+_{\phi_{v,\delta}}, s^-_{\phi_{v,-\delta}}, v - \delta) = \lambda(s^+, s^-_{\phi_{v,-\delta}}, v - \delta). \quad (80)$$

Using the remarks above, together with Eq. (69), a straightforward computation leads to the result

$$\Psi([\hat{C}_{\rm ham}[N_2], \hat{C}_{\rm ham}[N_1]]|\mathbf{s}^+, \mathbf{s}^-\rangle) = \left(\frac{-i\hbar^2}{a^2}\right)^2 \lim_{\delta'\to 0}\lim_{\delta\to 0}(\Psi_1(N_1, N_2, v, \delta, \delta') + \Psi_2(N_1, N_2, v, \delta, \delta')), \quad (81)$$

where

$$\begin{aligned}\Psi_1(N_1, N_2, v, \delta, \delta') &:= \lambda(s^+_{\phi_{v,\delta}}, s^-, v + \delta)\lambda(s^+, s^-, v)(N_1(v)N_2(v + \delta) \\ &\quad - N_1(v + \delta)N_2(v))\Psi(|(\mathbf{s}^+_{\phi_{v,\delta}})_{\phi_{v+\delta,\delta'}}, \mathbf{s}^-_{\phi_{v,-\delta}}\rangle \\ &\quad - |\mathbf{s}^+_{\phi_{v,\delta}}, \mathbf{s}^-_{\phi_{v,-\delta}}\rangle),\end{aligned} \quad (82)$$

$$\begin{aligned}\Psi_2(N_1, N_2, v, \delta, \delta') &:= \lambda(s^+, s^-_{\phi_{v,-\delta}}, v - \delta)\lambda(s^+, s^-, v)(N_1(v)N_2(v - \delta) \\ &\quad - N_1(v - \delta)N_2(v))\Psi(|(\mathbf{s}^+_{\phi_{v,\delta}}), (\mathbf{s}^-_{\phi_{v,-\delta}})_{\phi_{v-\delta,-\delta'}}\rangle \\ &\quad - |\mathbf{s}^+_{\phi_{v,\delta}}, \mathbf{s}^-_{\phi_{v,-\delta}}\rangle).\end{aligned} \quad (83)$$

It is easy to see that the two charge networks in each of the above equations are diffeomorphic. Hence, if Ψ is a diffeomorphism invariant distribution, $\Psi_I(N_1, N_2, v, \delta, \delta') = 0$, $I = 1, 2$ for all δ, δ' under consideration. This, in turn, implies that the commutator (75) vanishes.

B. The diffeomorphism constraint $C_{\rm diff}[\widehat{\beta(N, M)}]$

In this section, we analyze the right-hand side of (67). Recall that

$$C_{\rm diff}[\beta(N_1, N_2)] = \int_\Sigma\left(\Pi_+ X^{+'} + \Pi_- X^{-'} + \frac{1}{4}((Y^+)^2 - (Y^-)^2)\right) \times q^{xx}(N_1\partial_x N_2 - N_2\partial_x N_1). \quad (84)$$

From Eq. (4), it follows that $q^{xx}(x) = -(X^{+\prime}(x)X^{-\prime}(x))^{-1}$. The operator corresponding to $C_{\rm diff}[\beta(N_1, N_2)]$ can be obtained by using the same ideas that we employed for the Hamiltonian constraint. Thus, we first define action of the operator at finite triangulation T on $|\mathbf{s}^+, \mathbf{s}^-\rangle$ where, as before $T = T(\gamma, \delta)$ satisfies conditions (i)–(iv) of Sec. V D. The operator correspondent of q^{xx} is obtained by squaring Eq. (41). The operators for the other fields in Eq. (84) can be constructed along the lines of Sec. V A and V Band chosen in such a way that both the "+" and the "−" parts of the constraint are replaced by unitary operators labeled by the *same* gauge transformation so that the constraint operator at finite triangulation kills diffeomorphism invariant states. Specifically, it is easy to show that

$$C_{\rm diff}[\widehat{\vec{\beta}(N_1, N_2)}]|_T(|\mathbf{s}^+\rangle\otimes|\mathbf{s}^-\rangle) = (-i\hbar)\sum_{v\in V_E(s^+)\cup V_E(s^-)}\left(\frac{\hbar}{a^2}\right)^2(\lambda(s^+, s^-, v))^2\delta(N_1(v)N_2'(v) - N_2(v)N_1'(v))[\hat{U}^+(\phi_{v,\delta})\otimes\hat{U}^-(\phi_{v,\delta}) - 1]|\mathbf{s}^+, \mathbf{s}^+\rangle. \quad (85)$$

The action of this operator on the distribution Ψ is then

$$\lim_{\delta\to 0}\Psi(C_{\rm diff}[\widehat{\vec{\beta}(N_1, N_2)}]|_T(|\mathbf{s}^+\rangle\otimes|\mathbf{s}^-\rangle) = (-i\hbar)\sum_{v\in V_E(s^+)\cup V_E(s^-)}\left(\frac{\hbar}{a^2}\right)^2(\lambda(s^+, s^-, v))^2 \times\lim_{\delta\to 0}\delta(N_1(v)N_2'(v) - N_2(v)N_1'(v))(\Psi(N_1, N_2, v, \delta), \quad (86)$$

where

$$\Psi(N_1, N_2, v, \delta) = \Psi(|\mathbf{s}^+_{\phi_{v,\delta}}, \mathbf{s}^-_{\phi_{v,\delta}}\rangle) - \Psi(|\mathbf{s}^+, \mathbf{s}^-\rangle). \quad (87)$$

Clearly, $\Psi(N_1, N_2, v, \delta)$ vanishes if Ψ is a diffeomorphism invariant distribution so that action of operator $C_{\rm diff}[\widehat{\vec{\beta}(N_1, N_2)}]|_T$ at any finite triangulation of the type under consideration vanishes. Thus, for diffeomorphism invariant distributions the continuum limit of this operator, although trivial, exists in the same sense as for the Hamiltonian constraint (see Eq. (65)). Note also that, by virtue of the factor of δ, the right-hand side of (86) vanishes, for a large class of nondiffeomorphism invariant distributions Ψ. This is in exact analogy to what happens in LQG [8].

VII. THE ALGEBRA OF QUANTUM CONSTRAINTS ON THE LM HABITAT

In Sec. VII A, we define Lewandowski-Marolf habitat [7] for PFT. In the LQG context, the LM habitat is a specific enlargement of the space of spatial diffeomorphism group averages of charge networks (see Sec. III B) such that the (continuum limit of the triangulated) Hamiltonian constraint operator maps the habitat into itself. In Sec. VII B we show that the same is true here. We also show that the commutator of a pair of smeared Hamiltonian constraints, $[\hat{C}_{\rm ham}(N_1), \hat{C}_{\rm ham}(N_2)]$, as well as the operator corresponding to their classical Poisson bracket,$C_{\rm diff}[\vec{\beta}(N_1, N_2)]$, annihilate all states in the habitat. This is the exact analog of the result [8] for LQG. As we shall see, these operators kill states in the habitat for a very trivial reason stemming from the density weight 1 character of the Hamiltonian constraint: at finite triangulation, these operators do not have enough factors of δ in the denominator to obtain nontrivial action on the habitat. (Note that this is already apparent for $\hat{C}_{\rm diff}[\beta(N_1, N_2)]$ from the discussion at the end of Sec. VI B.) This motivates the exploration, in Sec. VII C, of slightly more singular constraint operators, namely, those corresponding to the smeared density weight 2 Hamiltonian constraint, $H_+ - H_-$, their commutator and the operator corresponding to their Poisson bracket. We show that while the last is a well-defined operator on the habitat, neither the smeared density 2 constraint operators, nor their commutator is well defined on the habitat. Our calculations indicate that a key role is played by states of nonzero volume in this discrepancy. In Sec. VII D, we shrink both the habitat as well as the space of charge networks by removing such states from their construction, and show that the constraint algebra is represented in an anomaly-free manner on this smaller set of states.

A. The LM habitat

Let $V_E(\mathbf{s}^+, \mathbf{s}^-)$ be the set of nontrivial embedding vertices of the state $|\mathbf{s}^+, \mathbf{s}^-\rangle$ so that

$$V_E(\mathbf{s}^+, \mathbf{s}^-) = V_E(s^+) \cup V_E(s^-), \tag{88}$$

where s^+, s^- are the embedding charge network labels of the state (see Sec. IV A for a definition of $V_E(s^\pm)$). Note that $v = 0$ is a nontrivial vertex if $v = 2\pi$ is a nontrivial vertex. This is simply a consequence of the circular topology of space. Note also that the elements of $V_E(s^\pm)$ are points in the interval $[0, 2\pi]$ and hence can be mapped to points on the circle via the identification $x = 0 \sim x = 2\pi$. Let the set of images in S^1 of the elements of $V_E(s^\pm), V_E(\mathbf{s}^+, \mathbf{s}^-)$ be denoted by $V_E^{S^1}(s^\pm)$, $V_E^{S^1}(\mathbf{s}^+, \mathbf{s}^-)$ so that

$$V_E^{S^1}(\mathbf{s}^+, \mathbf{s}^-) = V_E^{S^1}(s^+) \cup V_E^{S^1}(s^-). \tag{89}$$

It is easy to check that if $V_E(s^\pm)$ define $m^\pm$ points on the circle so that $V_E^{S^1}(s^\pm) = \{p_i^\pm, p_2^\pm, \ldots, p_{m^\pm}^\pm \in S^1\}$ then $V_E^{S^1}(s^\pm_{\phi^\pm})$ also defines $m^\pm$ points on the circle and is given by

$$V_E^{S^1}(s^\pm_{\phi^\pm}) = \{\phi^\pm p_i, \phi^\pm p_2, \ldots, \phi^\pm p_{m^\pm} \in S^1\}. \tag{90}$$

Here $s^\pm_{\phi^\pm}$ denotes the embedding charge network label of the gauge-related state $|s^\pm_{\phi^\pm}\rangle$ for a gauge transformation labeled by $\phi^\pm$ and $\phi^\pm(p)$ denotes the image of $p \in S^1$ under $\phi^\pm$.[12] It is also easy to see that, since the charge nets are in the superselected sector $\mathcal{D}_{ss}$, the $\pm$ embedding charges can be arranged in increasing/decreasing order, thus inducing an ordering of vertices. This implies a unique identification of vertices in $V_E^{S^1}(s^\pm)$ with those in $V_E^{S^1}(s^\pm_{\phi^\pm})$.

Let $V_E^{S^1}(\mathbf{s}^+, \mathbf{s}^-)$ consist of the points q_i, $i = 1, .., n$, i.e.

$$V_E^{S^1}(\mathbf{s}^+, \mathbf{s}^-) = \{q_1, q_2, .., q_n \in S^1\}, \tag{91}$$

and let f be a smooth (real valued) function of n points on the circle. Then the LM habitat, $\mathcal{V}_{\rm LM}$, is defined as the linear span of the distributions $\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}$, where

$$\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]} := \sum_{\mathbf{s}'^+,\mathbf{s}'^- \in [\mathbf{s}^+,\mathbf{s}^-]} f(V_E^{S^1}(\mathbf{s}'^+, \mathbf{s}'^-))\langle \mathbf{s}^+, \mathbf{s}^-|. \tag{92}$$

Note that by virtue of the discussion centering on Eq. (90), the cardinality of $V_E^{S^1}(\mathbf{s}'^+, \mathbf{s}'^-)$ is independent of $\mathbf{s}'^+, \mathbf{s}'^-$ if $\mathbf{s}'^+, \mathbf{s}'^+ \in [\mathbf{s}^+, \mathbf{s}^-]$ (we remind the reader that $[\mathbf{s}^+, \mathbf{s}^-]$ is the orbit of $\mathbf{s}^+$, $\mathbf{s}^-$ under diffeomorphisms). Further, that discussion also indicates that we can uniquely define the orbit of points in $V_E^{S^1}(\mathbf{s}'^+, \mathbf{s}'^-)$ under the action of some 1-parameter set of diffeomorphisms. This fact will be implicitly used in our considerations below.

B. Density one constraints

1. Continuum limit of the Hamiltonian constraint on $\mathcal{V}_{\rm LM}$

We show that Eq. (62) has a well-defined continuum limit if $\Psi \in \mathcal{V}_{\rm LM}$. As described in Sec. VI A, without loss of generality, we restrict attention to lapses N of compact support around the nontrivial vertex v of the state $|\mathbf{s}^+, \mathbf{s}^-\rangle$. From Eq. (62), we have that

$$\begin{aligned}&\Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]}(\hat{C}_{{\rm ham},T(\gamma,\delta)}[N]|\mathbf{s}^+, \mathbf{s}^-\rangle)\\ &= \sum_{v\in V_E(s^+)\cup V_E(s^-)} N(v)\frac{-i\hbar}{a^2}\lambda(s^+, s^-, v)\\ &\quad\times \sum_{\mathbf{s}''^+,\mathbf{s}''^- \in[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]} f(V_E^{S^1}(\mathbf{s}''^+, \mathbf{s}''^-))\\ &\quad\times[\delta_{\mathbf{s}^{+\prime\prime},\mathbf{s}^+_{\phi_{v,\delta}}}\delta_{\mathbf{s}^{-\prime\prime},\mathbf{s}^-_{\phi_{v,-\delta}}} - \delta_{\mathbf{s}^{+\prime\prime},\mathbf{s}^+}\delta_{\mathbf{s}^{-\prime\prime},\mathbf{s}^-}].\end{aligned} \tag{93}$$

[12]Recall that $\phi^\pm$ is a periodic diffeomorphism of the real line and hence can be naturally identified with a diffeomorphism of the circle.

If $v \in V(s^+) \cap V(s^-)$ and $[\mathbf{s}^{+\prime}, \mathbf{s}^{-\prime}] = [\mathbf{s}^+, \mathbf{s}^-]$, we have that

$$\Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]}(\hat{C}_{\mathrm{ham},T(\gamma,\delta)}[N]|\mathbf{s}^+, \mathbf{s}^-\rangle) = -N(v)\frac{-i\hbar}{a^2}\lambda(s^+, s^-, v)f(V_E^{S^1}(\mathbf{s}^+, \mathbf{s}^-)) = \lim_{\delta\to 0}\Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]}(\hat{C}_{\mathrm{ham},T(\gamma,\delta)}[N]|\mathbf{s}^+, \mathbf{s}^-\rangle). \quad (94)$$

If $v \in V(s^+) \cap V(s^-)$ and $[\mathbf{s}^{+\prime}, \mathbf{s}^{-\prime}] = [\mathbf{s}^+{}_{\phi_{v,\delta}}, \mathbf{s}^-{}_{\phi_{v,-\delta}}]$, we have that

$$\lim_{\delta\to 0}\Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]}(\hat{C}_{\mathrm{ham},T(\gamma,\delta)}[N]|\mathbf{s}^+, \mathbf{s}^-\rangle). = \lim_{\delta\to 0}f(\vec{v}', v+\delta, v-\delta) = f(\vec{v}', v, v), \quad (95)$$

where $\vec{v}'$ denotes all the nontrivial vertices of $|\mathbf{s}^+, \mathbf{s}^-\rangle$ outside the support of N.

If $v \notin V(s^+) \cap V(s^-)$, then $[\mathbf{s}^+{}_{\phi_{v,\delta}}, \mathbf{s}^-{}_{\phi_{v,-\delta}}] = [\mathbf{s}^+, \mathbf{s}^-]$. It follows that

$$\lim_{\delta\to 0}\Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]}(\hat{C}_{\mathrm{ham},T(\gamma,\delta)}[N]|\mathbf{s}^+, \mathbf{s}^-\rangle) = -N(v)\frac{-i\hbar}{a^2}\lambda(s^+, s^-, v)\lim_{\delta\to 0}(f(\vec{v}', v\pm\delta) - f(\vec{v}', v)) = 0, \quad (96)$$

where the $\pm$ signs refer to the cases $v \in V(s^{\pm})$.

In all the above, the continuum limit of the action of Hamiltonian constraint is well defined. It is also straightforward to see that, due to the diffeomorphism covariance of the operator $\hat{C}_{\mathrm{ham},T}[N]$, the continuum limit of the Hamiltonian constraint operator maps $\mathcal{V}_{\mathrm{LM}}$ into itself.

Note that, as emphasized by Thiemann [14], the continuum limit of $\hat{C}_{\mathrm{ham},T(\gamma,\delta)}[N]$ on the LM habitat as defined above, and the continuum limit of $\hat{C}_{\mathrm{ham},T(\gamma,\delta)}[N]$ in the URS topology as defined in Sec. VI, are distinct from each other in that the latter is implemented via uniform convergence in $\mathcal{H}_{\mathrm{kin}}$, whereas the latter is implemented via pointwise convergence in $\mathcal{V}_{\mathrm{LM}}$.

That the convergence of the one parameter family of operators $\hat{C}_{\mathrm{ham},T(\gamma,\delta)}[N]$ on $\mathcal{H}_{\mathrm{kin}}$ defined in the URS topology is uniform, follows directly from Eq. (65) by virtue of the fact that, with respect to the URS topology, the sequence is a constant one. We now show through an example that the convergence of the one parameter family of operators on the LM habitat is pointwise, i.e. that given $\mu > 0$, $\Psi \in \mathcal{V}_{\mathrm{LM}}$, $|\mathbf{s}^+, \mathbf{s}^-\rangle \in \mathcal{D}_{ss}$ $\exists\delta(\mu, \Psi, (\mathbf{s}^+, \mathbf{s}^-))$ such that

$$|(\hat{C}_{\mathrm{ham}}[N]\Psi)|\mathbf{s}^+, \mathbf{s}^-\rangle - \Psi(\hat{C}_{\mathrm{ham},T(\gamma,\delta)}[N]|\mathbf{s}^+, \mathbf{s}^-\rangle)| < \mu \quad (97)$$

$\forall\ \delta < \delta(\mu, \Psi, (\mathbf{s}^+, \mathbf{s}^-))$.

Consider the charge network state $|\mathbf{s}^+, \mathbf{s}^-\rangle$ with a vertex v such that $v \in V_E(s^+)$ but $v \notin V_E(s^-)$ so that $[\mathbf{s}^+{}_{\phi_{v,\delta}}, \mathbf{s}^-{}_{\phi_{v,-\delta}}] = [\mathbf{s}^+{}_{\phi_{v,\delta}}, \mathbf{s}^-] = [\mathbf{s}^+, \mathbf{s}^-]$. It follows that

$$\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}(\hat{C}_{\mathrm{ham},T(\gamma,\delta)}[N]|\mathbf{s}^+, \mathbf{s}^-\rangle) = -N(v)\frac{-i\hbar}{a^2}\lambda(s^+, s^-, v)(f(\vec{v}', v+\delta) - f(\vec{v}', v)). \quad (98)$$

$$\Rightarrow \Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}(\hat{C}_{\mathrm{ham}}[N]|\mathbf{s}^+, \mathbf{s}^-\rangle) = 0. \quad (99)$$

It follows that

$$|\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}(\hat{C}_{\mathrm{ham}}[N]|\mathbf{s}^+, \mathbf{s}^-\rangle) - \Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}(\hat{C}_{\mathrm{ham},T(\gamma,\delta)}[N]|\mathbf{s}^+, \mathbf{s}^-\rangle)| = N(v)\frac{\hbar}{a^2}\lambda(s^+, s^-, v)|f(\vec{v}', v+\delta) - f(\vec{v}', v))|. \quad (100)$$

Equation (100) implies that Eq. (97) (with $\Psi := \Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}$) is satisfied for δ which depends on Ψ (through the function f) and on $(\mathbf{s}^+, \mathbf{s}^-)$ (through the position of argument v), thus indicating pointwise convergence.

Despite the notion of convergence on $\mathcal{V}_{\mathrm{LM}}$ being (seemingly) much weaker than that with respect to the URS topology, our considerations below illustrate the usefulness of habitats such as $\mathcal{V}_{\mathrm{LM}}$ in exploring the off shell closure of the quantum constraint algebra.

2. The constraint algebra

Consider, first, the commutator of two Hamiltonian constraints (68) with $\Psi = \Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]} \in \mathcal{V}_{\mathrm{LM}}$. Let the cardinality of $V(s^+) \cup V(s^-)$ be n'. Recall that the function $f(V_E^{S^1}(\mathbf{s}'^+, \mathbf{s}'^-))$ is a smooth function from $(S^1)^{n'}$ (i.e., n' copies of the circle) to the complex numbers. Next, note that whenever nontrivial, the terms in Eqs. (76)–(78), (82), and (83) consist of the difference of the evaluation of the function $f(V_E^{S^1}(\mathbf{s}'^+, \mathbf{s}'^-))$ at nearby points in $(S^1)^{n'}$ which coincide in the continuum limit. Hence, by virtue of the smoothness of $f(V_E^{S^1}(\mathbf{s}'^+, \mathbf{s}'^-))$ all these terms vanish in the continuum limit. Clearly, the commutator trivializes due to the absence of factors of δ, δ' in the denominator. Had such factors been present, there could be the possibility that the terms which vanished now yield derivatives of $f(V_E^{S^1}(\mathbf{s}'^+, \mathbf{s}'^-))$. Such factors could arise if we considered higher density constraints. This motivates the analysis of the density two constraints in Sec. VII C.

What about the left-hand side of Eq. (67) with $\Psi = \Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]} \in \mathcal{V}_{\mathrm{LM}}$? It is easy to see, from Sec. VI B and from arguments identical to those above, that the continuum limit of the diffeomorphism constraint $C_{\mathrm{diff}}[\widehat{\beta(N_1, N_2)}]|_T$ vanishes on $\mathcal{V}_{\mathrm{LM}}$. Indeed it vanishes "doubly": first, due to the extra factor of δ in Eq. (86) and second, by virtue of the fact that, similar to the case of the Hamiltonian constraint commutator discussed above, Eq. (87) consists of the evaluation of the habitat state on the difference of a pair of charge networks related by a small diffeomorphism which approaches the identity in the continuum limit.

Thus, both sides of Eq. (67) vanish on the LM habitat in exact analogy, and, in fact, for exactly the same reasons as in LQG: namely, the absence of suitable factors of δ in the denominator. The considerations of Secs. VIII and IX will make this remark precise.

For later use, we conclude this section with an explicit evaluation of the commutator on the LM habitat for two specific cases outlined below.

Case 1: See Sec. VI A 1. Let $v_1 \neq v_2$. Set $\Psi = \Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]} \in \mathcal{V}_{\rm LM}$. Let $v_1,\ v_2 \in V(s^+)\cap V(s^-)$ and let $[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}] = [(\mathbf{s}^+_{\phi_{v_1,\delta}})_{\phi_{v_2,\delta'}}, (\mathbf{s}^-_{\phi_{v_1,-\delta}})_{\phi_{v_2,-\delta'}}]$ for sufficiently small $\delta,\ \delta'$. Note that for sufficiently small $\delta,\ \delta'$, $[(\mathbf{s}^+_{\phi_{v_1,\delta}})_{\phi_{v_2,\delta'}}, (\mathbf{s}^-_{\phi_{v_1,-\delta}})_{\phi_{v_2,-\delta'}}]$ is independent of $\delta,\ \delta'$. Also note that the vertices v_i, $i = 1, 2$ of $|\mathbf{s}^+,\mathbf{s}^-\rangle$ each split into 2 vertices around v_i, a '+' vertex and a '−' vertex, to yield $|(\mathbf{s}^+_{\phi_{v_1,\delta}})_{\phi_{v_2,\delta'}}, (\mathbf{s}^-_{\phi_{v_1,-\delta}})_{\phi_{v_2,-\delta'}}\rangle$. This immediately implies that $\Psi_2(v_1, v_2, \delta, \delta') = \Psi_3(v_1, v_2, \delta, \delta') = 0$. Further, we have that

$$\Psi_3(v_1, v_2, \delta, \delta') = f(\vec{v}', v_1 + \delta, v_1 - \delta, v_2 + \delta', v_2 - \delta') - f(\vec{v}', v_1 + \delta', v_1 - \delta', v_2 + \delta, v_2 - \delta), \tag{101}$$

which vanishes in the continuum limit.

Case 2: See Sec. VI A 2. Let $v_1 = v_2 = v \in V(s^+)\cap V(s^-)$ and set $\Psi = \Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]} \in \mathcal{V}_{\rm LM}$ so that we are interested in the case where the diffeomorphism class which labels the habitat state is the same as that of the charge network state on which the commutator acts. Since the terms in Eqs. (82) and (83) involve the charge nets in which the joint +, − vertex at v splits into a "+" one and a "−" one, we have that $\Psi_1(N_1, N_2, v, \delta, \delta') = \Psi_2(N_1, N_2, v, \delta, \delta') = 0$.

C. Density 2 constraints

Rescaling the density weight 1 Hamiltonian constraint $C_{\rm ham}$ (given in Eq. (2)) by the square root of the determinant of the spatial metric yields the density weight 2 Hamiltonian constraint $H := H_+ - H_-$, which on smearing with the density weight -1 lapse, N yields

$$H(N) := \int dx\Big[\Pi_+(x)X^{+\prime}(x) - \Pi_-(x)X^{-\prime}(x) + \frac{1}{4}(\pi_f^2 + f'^2)\Big]. \tag{102}$$

In 1 spatial dimension, a scalar of density weight -1 transforms in the same way as vector field. Thus, N in the above equation can equally well be thought of as a vector field. We shall use this equivalence to denote N by $\vec{N}$ whenever it is convenient. Replacing $C_{\rm ham}$ by H in the Dirac algebra (3) yields the Lie algebra

$$\begin{aligned}\{C_{\rm diff}(\vec{N}_1), C_{\rm diff}(\vec{N}_2)\} &= C_{\rm diff}([\vec{N}_1, \vec{N}_2])\\ \{C_{\rm diff}(\vec{N}_1), H(N_2)\} &= H[L_{\vec{N}_1}N_2]\\ \{H(N_1), H(N_2)\} &= C_{\rm diff}([\vec{N}_1, \vec{N}_2]),\end{aligned} \tag{103}$$

where in the last equation we have used the equivalence $N_i \equiv \vec{N}_i$ between density weight -1 scalars and vectors.

In Sec. VII C 1, we show that $\hat{C}_{\rm diff}(\vec{N})$ is a well-defined operator on $\mathcal{V}_{\rm LM}$ and that the Poisson bracket (103) is represented in an anomaly-free manner on $\mathcal{V}_{\rm LM}$. In Sec. VII C 2, we construct the smeared density two Hamiltonian constraint operator at finite triangulation, $\hat{H}_T(N)$, and show that neither $\hat{H}_T(N)$ nor the commutator between a pair of such operators admits a continuum limit on all of $\mathcal{V}_{\rm LM}$. We also show, through an example, that there exist states in $\mathcal{V}_{\rm LM}$ on which the action of the commutator admits a continuum limit but is anomalous. The example shows that the anomaly can be traced to the existence of charge network states with nonvanishing volume and motivates the considerations of Sec. VII D.

1. The diffeomorphism constraint and its commutator

The analysis of $\hat{C}_{\rm diff}(\vec{N})$ parallels that of Sec. VI B. It is straightforward to see that, due to the absence of the metric dependent factor, there are now no factors of λ and an overall factor of δ^{-1} instead of δ (see Eq. (85)). In detail, we have that

$$\hat{C}_{{\rm diff},T(\delta)}(\vec{N})|\mathbf{s}^+,\mathbf{s}^-\rangle = (-i\hbar)\sum_{v\in V_E(s^+)\cup V_E(s^-)} N^x(v) \times \frac{|(\mathbf{s}^+_{\phi_{v,\delta}}, \mathbf{s}^-_{\phi_{v,\delta}}\rangle - |\mathbf{s}^+,\mathbf{s}^-\rangle}{\delta}, \tag{104}$$

where N^x is the component of the shift vector in the coordinate system $\{x\}$ for which the length of each edge of T is δ.[13]

$$\Rightarrow \Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]}(\hat{C}_{{\rm diff},T(\delta)}(\vec{N})|\mathbf{s}^+,\mathbf{s}^-\rangle) = 0 \quad \text{if } [\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}] \neq [\mathbf{s}^+,\mathbf{s}^-], \tag{105}$$

and

$$\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}(\hat{C}_{{\rm diff},T(\delta)}(\vec{N})|\mathbf{s}^+,\mathbf{s}^-\rangle) = -i\hbar\sum_{v\in V_E(s^+)\cup V_E(s^-)} N^x(v)\frac{f(\vec{v}', v+\delta) - f(\vec{v}', v)}{\delta} \tag{106}$$

$$\lim_{\delta\to 0}\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}(\hat{C}_{{\rm diff},T(\delta)}(\vec{N})|\mathbf{s}^+,\mathbf{s}^-\rangle) = -i\hbar\sum_{v\in V_E(s^+)\cup V_E(s^-)} N^x(v)\partial_x f(\vec{v}', v), \tag{107}$$

[13] In the interest of clarity, we denote $T(\gamma, \delta)$ by $T(\delta)$ from now on. We hope to have conveyed to the reader by now that the triangulation is graph dependent and hence hope that omitting the label γ will not create any confusion.

so that $\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]} \in \mathcal{V}_{\mathrm{LM}}$ is mapped to $\Psi_{g_{\vec{N}},[\mathbf{s}^+,\mathbf{s}^-]} \in \mathcal{V}_{\mathrm{LM}}$ with

$$g_{\vec{N}}(V_E^{S^1}(\mathbf{s}^+,\mathbf{s}^-)) := -i\hbar \sum_{v\in V_E(s^+)\cup V_E(s^-)} N^x(v)(\partial_x f(\vec{v}',x))|_{x=v}. \tag{108}$$

Here (and in an obvious fashion, below) the argument $\vec{v}'$ of $f(\vec{v}',x)$ indicates the set of nontrivial vertices other than the vertex x under consideration. This immediately implies that

$$\Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]}(\hat{C}_{\mathrm{diff}}(\vec{N}_1)\hat{C}_{\mathrm{diff}}(\vec{N}_2)|\mathbf{s}^+,\mathbf{s}^-\rangle) = 0 \quad \text{if } [\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}] \neq [\mathbf{s}^+,\mathbf{s}^-], \tag{109}$$

$$\begin{aligned}\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}(\hat{C}_{\mathrm{diff}}(\vec{N}_1)\hat{C}_{\mathrm{diff}}(\vec{N}_2)|\mathbf{s}^+,\mathbf{s}^-\rangle) &= \Psi_{g_{\vec{N}_1},[\mathbf{s}^+,\mathbf{s}^-]}(\hat{C}_{\mathrm{diff}}(\vec{N}_2)|\mathbf{s}^+,\mathbf{s}^-\rangle) = (-i\hbar)\sum_{v\in V_E(s^+)\cup V_E(s^-)} N_2^x\partial_x g_{\vec{N}_1}(\vec{v}',x)|_{x=v}\\ &= (-i\hbar)^2 \sum_{v,\bar{v}\in V_E(s^+)\cup V_E(s^-),\bar{v}\neq v} N_2^x(v)N_1^{\bar{x}}(\bar{v})\partial_x\partial\bar{x} f(\vec{v}',x,\bar{x})|_{x=v,\bar{x}=\bar{v}}\\ &\quad + (-i\hbar)^2\sum_{v\in V_E(s^+)\cup V_E(s^-)}(N_2^x(v)\partial_x(N_1^x(x)\partial_x f(\vec{v}',x)))|_{x=v}. \end{aligned}\tag{110}$$

From Eqs. (109) and (110), it is easy to see that

$$\begin{aligned}&\Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]}([\hat{C}_{\mathrm{diff}}(\vec{N}_1),\hat{C}_{\mathrm{diff}}(\vec{N}_2)]|\mathbf{s}^+,\mathbf{s}^-\rangle)\\ &\quad = (-i\hbar)\Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]}(\hat{C}_{\mathrm{diff}}([\vec{N}_2,\vec{N}_1])|\mathbf{s}^+,\mathbf{s}^-\rangle), \end{aligned}\tag{111}$$

which is an antirepresentation of the Poisson bracket algebra (103).

2. The Hamiltonian constraint and its commutator

The smeared density weight 2 Hamiltonian constraint $\hat{H}_T$ at finite triangulation is constructed along the lines of Sec. VI. As for the diffeomorphism constraint, the rescaling of the density 1 constraint and the consequent absence of factors of the square root of determinant of the spatial metric imply that there are no longer any factors of λ (see Eq. (61)), and that there is now an overall factor of δ^{-1}. More in detail, it is straightforward to see that for $\Psi \in \mathcal{V}_{\mathrm{LM}}$, we have that

$$\begin{aligned}&\Psi(\hat{H}_{T(\delta)}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle)\\ &\quad = (-i\hbar)\sum_{v\in V_E(s^+)\cup V_E(s^-)} N(v)\Psi\bigg(\frac{|\mathbf{s}^+_{\phi_{v,\delta}},\mathbf{s}^-_{\phi_{v,-\delta}}\rangle - |\mathbf{s}^+,\mathbf{s}^-\rangle}{\delta}\bigg). \end{aligned}\tag{112}$$

Let v, $\mathbf{s}^+$, $\mathbf{s}^-$ be such that $v \in V(s^+)\cap V(s^-)$. Let N be compactly supported around v with support of the type discussed in Sec. VI A and let $\Psi = \Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}$. Then Eq. (112) implies that

$$\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}(\hat{H}_{T(\delta)}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle) = i\hbar N(v)\frac{f(V_E^{S^1}(\mathbf{s}'^+,\mathbf{s}'^-))}{\delta}, \tag{113}$$

which does not admit a $\delta\to 0$ continuum limit. Thus $\hat{H}[N]$ is not well defined on (all of) $\mathcal{V}_{\mathrm{LM}}$.

Can we make sense of the commutator of a pair of density 2 Hamiltonian constraints on (all of) $\mathcal{V}_{\mathrm{LM}}$? The example below shows that the answer is in the negative. Let N_1, N_2, v_1, v_2 be as in Sec. VI A 1 and let $\Psi \in \mathcal{V}_{\mathrm{LM}}$. It is straightforward to see, from Eqs. (112) and (74), that

$$\begin{aligned}&\Psi(\hat{H}_{T'(\delta')}[N_2]\hat{H}_{T(\delta)}[N_1] - \hat{H}_{T'(\delta')}[N_1]\hat{H}_{T(\delta)}[N_2]|\mathbf{s}^+,\mathbf{s}^-\rangle)\\ &\quad = (-i\hbar)^2 N_1(v_1)N_2(v_2)\frac{\sum_{i=1}^3\Psi_i(v_1,v_2,\delta,\delta')}{\delta\delta'}, \end{aligned}\tag{114}$$

where $\Psi_i(v_1,v_2,\delta,\delta')$, $i = 1, 2, 3$ are given by Eqs. (76)–(78). Clearly, the existence of the continuum limit is tied to that of the limit $\lim_{\delta\to 0}\lim_{\delta'\to 0}\frac{\sum_{i=1}^3\Psi_i(v_1,v_2,\delta,\delta')}{\delta\delta'}$. Now, consider Case 1 of Sec. VII B 2. Clearly,

$$\lim_{\delta'\to 0}\lim_{\delta\to 0}\frac{\sum_{i=1}^3\Psi_i(v_1,v_2,\delta,\delta')}{\delta\delta'} = \lim_{\delta'\to 0}\lim_{\delta\to 0}\frac{\Psi_1(v_1,v_2,\delta,\delta')}{\delta\delta'}. \tag{115}$$

Since the limit $\lim_{\delta'\to 0}\frac{f(\vec{v}',v_1+\delta,v_1-\delta,v_2+\delta',v_2-\delta') - f(\vec{v}',v_1+\delta',v_1-\delta',v_2+\delta,v_2-\delta)}{\delta'}$ does not exist for generic f, the commutator does not admit a continuum limit on (all of) $\mathcal{V}_{\mathrm{LM}}$.

Nevertheless, as the following calculation suggests, such a limit may exist for a subset of states in $\mathcal{V}_{\mathrm{LM}}$. Consider the setting of Sec. VI A 2, where $v_1 = v_2 = v$. It is straightforward to see that

$$\Psi(\hat{H}_{T'(\delta')}[N_2]\hat{H}_{T(\delta)}[N_1] - \hat{H}_{T'(\delta')}[N_1]\hat{H}_{T(\delta)}[N_2]|\mathbf{s}^+, \mathbf{s}^-\rangle) = (-i\hbar)^2 \frac{\Psi_1(N_1, N_2, v, \delta, \delta') + \Psi_2(N_1, N_2, v, \delta, \delta')}{\delta\delta'}, \tag{116}$$

where $\Psi_i(N_1, N_2, v, \delta, \delta')$, $i = 1, 2$ are defined in Eqs. (82) and (83). Now let us consider Case 2 of Sec. VII B 2. From the discussion there, we have that the right-hand side of the above equation vanishes. However, from Eq. (107), we see that for generic f that the particular evaluation of the commutator (116), while possessing a continuum limit, is anomalous. More generally, if we restrict attention to habitat states $\Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]}$ for which $|\mathbf{s}^{+\prime}, \mathbf{s}^{-\prime}\rangle$ is such that $V(s^{+\prime}) = V(s^{-\prime})$, the commutator always vanishes, and, for generic f is anomalous.

The three sets of calculations above all involve states for which $V(s^+) \cap V(s^-)$ is nonempty. This suggests that perhaps the problems with ill-definedness and the presence of anomalies could disappear by removing such states from our considerations. This is the subject of the next section.

D. The zero volume sector

From Sec. IV A, it follows that given a charge network state $|\mathbf{s}^+, \mathbf{s}^-\rangle$, the operator corresponding to the volume of some spatial region $\mathcal{R} \subset S^1$ acts nontrivially only on those vertices which are in the set $V_E(s^+) \cap V_E(s^-) \cap \mathcal{R}$. Hence, we shall refer to a charge network $|\mathbf{s}^+, \mathbf{s}^-\rangle$ as a zero volume charge network if $V_E(s^+) \cap V_E(s^-)$ is empty. We define the zero volume sector, $\mathcal{D}^0_{ss}$, of $\mathcal{D}_{ss}$ to be the finite span of all "zero volume" charge networks in $\mathcal{D}_{ss}$.

It is easy to see that if $V_E(s^+) \cap V_E(s^-)$ is empty, then $V_E(s^{+\prime}) \cap V_E(s^{-\prime})$ is also empty for any $\mathbf{s}^{+\prime}, \mathbf{s}^{-\prime} \in [\mathbf{s}^+, \mathbf{s}^-]$ so that the zero volume property extends to spatial diffeomorphism classes of charge networks. We define the zero volume sector $\mathcal{V}^0_{\rm LM} \subset \mathcal{V}_{\rm LM}$ as the finite span of those states $\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]} \in \mathcal{V}_{\rm LM}$ for which $V_E(s^+) \cap V_E(s^-)$ is empty.

In the rest of this section, we shall restrict attention to charge nets in $\mathcal{D}^0_{ss}$ and distributions in $\mathcal{V}^0_{\rm LM}$. Thus, we shall think of $\mathcal{V}^0_{\rm LM}$ as a subset of $(\mathcal{D}^0_{ss})^*$, where $(\mathcal{D}^0_{ss})^*$ is the algebraic dual to $\mathcal{D}^0_{ss}$.

Next, note that, for $|\mathbf{s}^+, \mathbf{s}^-\rangle \in \mathcal{D}^0_{ss}$, $v \in V_E(s^+) \cap V_E(s^-)$ and sufficiently small δ, the charge network states $|\mathbf{s}^+_{\phi_{v,\delta}}, \mathbf{s}^-_{\phi_{v,-\delta}}\rangle$, $|\mathbf{s}^+_{\phi_{v,\delta}}, \mathbf{s}^-_{\phi_{v,\delta}}\rangle$, and $|\mathbf{s}^+, \mathbf{s}^-\rangle$, are related to each other by the action of spatial diffeomorphisms so that

$$[\mathbf{s}^+_{\phi_{v,\delta}}, \mathbf{s}^-_{\phi_{v,-\delta}}] = [\mathbf{s}^+_{\phi_{v,\delta}}, \mathbf{s}^-_{\phi_{v,\delta}}] = [\mathbf{s}^+, \mathbf{s}^-]. \tag{117}$$

Since $|(\mathbf{s}^+_{\phi_{v,\delta}}, \mathbf{s}^-_{\phi_{v,-\delta}}\rangle$ and $|(\mathbf{s}^+_{\phi_{v,\delta}}, \mathbf{s}^-_{\phi_{v,\delta}}\rangle$ are generated from $|\mathbf{s}^+, \mathbf{s}^-\rangle$ by the action of the Hamiltonian and diffeomorphism constraints, and since the zero volume property holds for diffeomorphism classes of charge networks, it follows that the constraints at finite triangulation map $\mathcal{D}^0_{ss}$ to itself so that it is consistent to restrict attention to $\mathcal{D}^0_{ss}$.

We have already shown, in Sec. VII C 1, that the diffeomorphism constraint has a well-defined continuum limit on $\mathcal{V}_{\rm LM}$ and hence also on $\mathcal{V}^0_{\rm LM}$. We now show that $\hat{H}_T[N]$ also has a well-defined continuum limit on $\mathcal{V}^0_{\rm LM}$. We shall denote a vertex of $|\mathbf{s}^+, \mathbf{s}^-\rangle$ which is in $V_E(s^+)$ by v^+ and one which is in $V_E(s^+)$ by v^-. Then from Eqs. (112) and (117), it follows that, for $\Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]} \in \mathcal{V}^0_{\rm LM}$,

$$\Psi_{f,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]}(\hat{H}_{T(\delta)}(N)|\mathbf{s}^+, \mathbf{s}^-\rangle) = 0 \quad \text{if } [\mathbf{s}^{+\prime}, \mathbf{s}^{-\prime}] \neq [\mathbf{s}^+, \mathbf{s}^-], \tag{118}$$

and that, in obvious notation,

$$\lim\delta \to 0\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}(\hat{H}_{T(\delta)}(N)|\mathbf{s}^+, \mathbf{s}^-\rangle) = (-i\hbar)\Bigg(\sum_{v^+ \in V_E(s^+)} N(v^+)\frac{\partial f}{\partial v^+} - \sum_{v^- \in V_E(s^-)} N(v^-)\frac{\partial f}{\partial v^-}\Bigg) := \Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}(\hat{H}(N)|\mathbf{s}^+, \mathbf{s}^-\rangle). \tag{119}$$

It is straightforward to see, similar to the case of the diffeomorphism constraint (107), that the state $\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}$ is mapped into $\Psi_{g_N,[\mathbf{s}^+,\mathbf{s}^-]} \in \mathcal{V}^0_{\rm LM}$ where

$$g_N(V_E^{S^1}(\mathbf{s}^+, \mathbf{s}^-)) := -i\hbar\Bigg(\sum_{v^+ \in V_E(s^+)} N(v^+)\frac{\partial f}{\partial v^+} - \sum_{v^- \in V_E(s^-)} N(v^-)\frac{\partial f}{\partial v^-}\Bigg). \tag{120}$$

It is then also straightforward to see that a calculation, almost identical to that for the commutator of the diffeomorphism constraint (111) then yields the following result:

$$\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}([\hat{H}(N_2), \hat{H}(N_1)]|\mathbf{s}^+, \mathbf{s}^-\rangle) = (-i\hbar)\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}(\hat{C}_{diff}[\vec{N}_2, \vec{N}_1]|\mathbf{s}^+, \mathbf{s}^-\rangle), \tag{121}$$

where as in Eq. (103), we have used the equivalence between density weight -1 scalars and vectors to denote N_1, N_2 by $\vec{N}_1, \vec{N}_2$. It is easy to see that Eqs. (118) and (121) imply that the Poisson-Lie algebra (103) of density 2 constraints is represented in an anomaly-free manner on $\mathcal{V}^0_{\rm LM} \subset (\mathcal{D}^0_{ss})^*$. We now determine the kernel of $\hat{H}[N]$ inside $\mathcal{V}^0_{\rm LM}$. From (119) and (120), it follows rather straightforwardly that diffeomorphism invariant distribution which lie inside $\mathcal{V}^0_{\rm LM}$ are certainly in the kernel of $\hat{H}[N]$. We now show that these are the only states in the kernel.

Lemma $\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}$ is in the kernel of $\hat{H}[N]$ if f is a constant function.

Proof: In light of (118), we want to show that

$$\Psi_{f,[\mathbf{s}^+,\mathbf{s}^-]}(\hat{H}[N]|\mathbf{s}^{+\prime}, \mathbf{s}^{-\prime}\rangle) = 0 \tag{122}$$

$\forall\, N$ and $\forall |\mathbf{s}^{+\prime}, \mathbf{s}^{-\prime}\rangle$ for which $[\mathbf{s}^{+\prime}, \mathbf{s}^{-\prime}] = [\mathbf{s}^+, \mathbf{s}^-]$. (120) essentially implies that this will be true if

$$g_N(V_E^{S^1}(\mathbf{s}_\phi^+, \mathbf{s}_\phi^-)) := -i\hbar\bigg(\sum_{v^+\in V_E(s^+)} N(\phi(v^+))\frac{\partial f}{\partial\phi(v^+)} - \sum_{v^-\in V_E(s^-)} N(\phi(v^-))\frac{\partial f}{\partial\phi(v^-)}\bigg)\bigg) = 0 \tag{123}$$

$\forall N$ and $\forall\phi$.

Clearly this will be true if f are constant functions.

Whence the kernel of density 2 Hamiltonian constraint inside $\mathcal{V}^0_{\rm LM}$ is analogous to the kernel of Hamiltonian constraint in LQG when the domain of the operator is restricted to planar spin-networks.

VIII. THE ALGEBRA OF QUANTUM CONSTRAINTS ON THE NEW HABITAT

The computations of Sec. VI and VII B indicate that due to their density 1 character, the quantum constraints $\hat{C}_{\rm ham}[N]$ are "too nonsingular" to give rise to a nontrivial commutator algebra either on $\mathcal{H}_{\rm kin}$ (when working in the URS), or on the LM Habitat (where one looks at a net of regulated dual operators).

This motivates the consideration of the density 2 constraints in Secs. VII C and VII D. Section VII C throws up an apparent paradox. On the one hand, it is easy to see that the "correct" physical states (see Sec. II B 5) lie in the kernel of the constraints at any finite triangulation, thus indicating that the constraints have been correctly constructed. On the other, the continuum limit of the smeared density 2 constraint is ill-defined on states in the LM habitat[14] and that of its commutator, anomalous. Section VII D shows that, if one throws states of nonzero volume out of the description, there does exist a smaller habitat on which the density 2 constraints are well defined and their commutator anomaly-free. However, even this is not completely satisfactory for two reasons: (i) our aim is to preserve contact with the physical states constructed in [2] (and reviewed in Sec. II B 5), and the states of nonzero volume are retained in their construction, (ii) our aim is to uncover lessons for LQG, and in LQG a key role is played by states of nonvanishing volume in semiclassical considerations at the kinematic level [15].

For these reasons, in this section we construct a new habitat where the continuum limit of density 2 (Hamiltonian and diffeomorphism) constraint operators is well defined, their Poisson-Lie algebra is faithfully represented and their kernel in this new habitat is precisely the set of physical states of Sec. II B 5.

In Sec. VIII A, we define the new habitat. In Sec. VIII B, we show that the diffeomorphism constraint at finite triangulation has a well-defined continuum limit on the habitat and that its commutator is anomaly-free. In Sec. VIII C, we prove identical results for the density 2 Hamiltonian constraint. In Sec. VIII D, we show that the (joint) kernel of the density 2 Hamiltonian constraint (and the diffeomorphism constraint) is precisely the set of physical states of Sec. II B 5.

A. The new habitat

Given a pair of charge networks$(\mathbf{s}^+, \mathbf{s}^-)$, let

$$[\mathbf{s}^+, \mathbf{s}^-]_{+-} = \{(\mathbf{s}^{+\prime}, \mathbf{s}^{-\prime}) | (\mathbf{s}^{+\prime}, \mathbf{s}^{-\prime}) = (\mathbf{s}^+_{\phi^+}, \mathbf{s}^-_{\phi^-}) \quad \text{for some } \phi^\pm\}, \tag{124}$$

so that $[\mathbf{s}^+, \mathbf{s}^-]_{+-}$ is the set of all charge networks related by the finite gauge transformations generated by H_+ and H_-.[15]

We define the new habitat, $\mathcal{V}_{+-}$, as the finite linear span of distributions (over $\mathcal{D}_{ss}$) of the type,

$$\Psi_{f^+,f^-,[\mathbf{s}^+,\mathbf{s}^-]_{+-}} = \sum_{(\mathbf{s}^{+\prime},\mathbf{s}^{-\prime})\in[\mathbf{s}^+,\mathbf{s}^-]_{+-}} f^+(V_E^{S^1}(\mathbf{s}^{+\prime})) \times f^-(V_E^{S^1}(\mathbf{s}^{-\prime}))\langle \mathbf{s}^{+\prime}, \mathbf{s}^{-\prime}|. \tag{125}$$

Note that as claimed in Sec. VII A the cardinality of the sets $V_E^{S^1}(s^+)$, $V_E^{S^1}(s^+_{\phi^+})$ is identical for any ϕ^+, a similar result being true for the "$-$" sector and that each $v^\pm \in V_E^{S^1}(s^\pm)$ has the unique image $(\phi^\pm(v^\pm)) \in V_E^{S^1}(s^\pm_{\phi^\pm})$. This implies that we can uniquely define the orbit of points in $V_E^{S^1}(s^\pm)$ under the action of some 1 parameter family of gauge transformations $\phi^\pm$. We shall implicitly use this fact in our considerations below.

B. The diffeomorphism constraint and its commutator

Since the computations here are very similar to those encountered in Sec. VII C 1, we shall be brief in our presentation. Using the fact that finite diffeomorphisms are gauge transformations (see Eq. (14)) in conjunction with Eq. (104), it follows that

$$\Psi_{f^+,f^-,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]_{+-}}(\hat{C}_{\rm diff}[\vec{N}]|\mathbf{s}^+, \mathbf{s}^-\rangle) = 0 \quad \text{if } (\mathbf{s}^+, \mathbf{s}^-) \notin [\mathbf{s}^{+\prime}, \mathbf{s}^{-\prime}]_{+-}. \tag{126}$$

Next, it is straightforward to see that from Eq. (104) we have, in obvious notation,

$$\lim_{\delta\to 0}\Psi_{f^+,f^-,[\mathbf{s}^+,\mathbf{s}^-]_{+-}}(\hat{C}_{{\rm diff},T(\delta)}(\vec{N})|\mathbf{s}^+, \mathbf{s}^-\rangle) = -i\hbar\sum_{v\in V_E(s^+)\cup V_E(s^-)} N^x(v)\partial_x(f^+f^-)|_{x=v}. \tag{127}$$

[14]This naturally implies that in the URS, the continuum limit will certainly not be well defined on $\mathcal{H}_{\rm kin}$ either).

[15]The astute reader will recognize that a marginally simpler treatment of the material in this section would ensue if we worked with $H_\pm$ and the sets $[\mathbf{s}^\pm]$ of Sec. II B 5, and derived the results for the density 2 constraints $H = H_+ - H_-$, $C_{\rm diff} = H_+ + H_-$ as immediate consequences. The reason for our presentation of H, $C_{\rm diff}$ (and hence, $[\mathbf{s}^+, \mathbf{s}^-]_{+-}$) as primary structures is to preserve, as far as possible, structural similarity with LQG.)

It is easy to see that the above equation implies that $\Psi_{f^+,f^-,[\mathbf{s}^+,\mathbf{s}^-]_\pm} \in \mathcal{V}_{+-}$ is mapped to the linear combination $(\Psi_{f^+,g^-_{\vec{N}},[\mathbf{s}^+,\mathbf{s}^-]_{+-}} + \Psi_{g^+_{\vec{N}},f^-,[\mathbf{s}^+,\mathbf{s}^-]_{+-}}) \in \mathcal{V}_{+-}$ where

$$g^{\pm}_{\vec{N}} = (-i\hbar) \sum_{v\in V_E(s^\pm)} N^x(v)\partial_x(f^\pm)|_{x=v} \tag{128}$$

It is then straightforward to compute action of the commutator on the habitat state $\Psi_{f^+,f^-,[\mathbf{s}^+,\mathbf{s}^-]_{+-}}$ along the lines of Sec. VII C 1 and verify that

$$\begin{aligned}\Psi_{f^+,f^-,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]_{+-}}&([\hat{C}_{\rm diff}(\vec{N}_1), \hat{C}_{\rm diff}(\vec{N}_2)]|\mathbf{s}^+,\mathbf{s}^-\rangle)\\ &= (-i\hbar)\Psi_{f^+,f^-,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]_{+-}}(\hat{C}_{\rm diff}([\vec{N}_2,\vec{N}_1])|\mathbf{s}^+,\mathbf{s}^-\rangle),\end{aligned} \tag{129}$$

which is an antirepresentation of the corresponding Poisson brackets.

C. The density 2 Hamiltonian constraint and its commutator

The computations parallel that of the previous section.

It is easy to see that Eq. (112) holds for any distribution Ψ, and, in particular, for $\Psi \in \mathcal{V}_{+-}$. Using the fact that the charge nets on the right-hand side of Eq. (112) are related by the action of finite gauge transformations, it follows that

$$\begin{aligned}\Psi_{f^+,f^-,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]_{+-}}(\hat{H}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle) = 0\\ \text{if } (\mathbf{s}^+,\mathbf{s}^-) \notin [\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]_{+-}.\end{aligned} \tag{130}$$

Next, it is straightforward to see that from Eq. (112) we have, in obvious notation,

$$\begin{aligned}\lim_{\delta\to 0}&\Psi_{f^+,f^-,[\mathbf{s}^+,\mathbf{s}^-]_{+-}}(\hat{H}_{T(\delta)}(N)|\mathbf{s}^+,\mathbf{s}^-\rangle)\\ &= -i\hbar \sum_{v\in V_E(s^+)\cup V_E(s^-)} N(v)\bigg(f^-\frac{\partial f^+}{\partial v} - f^+\frac{\partial f^-}{\partial v}\bigg).\end{aligned} \tag{131}$$

It is easy to see that the above equation implies that $\Psi_{f^+,f^-,[\mathbf{s}^+,\mathbf{s}^-]_\pm} \in \mathcal{V}_{+-}$ is mapped to the linear combination $(\Psi_{f^+,h^-_N,[\mathbf{s}^+,\mathbf{s}^-]_{+-}} + \Psi_{h^+_N,f^-,[\mathbf{s}^+,\mathbf{s}^-]_{+-}}) \in \mathcal{V}_{+-}$ where

$$h^{\pm}_N = (-i\hbar) \sum_{v\in V_E(s^\pm)} \pm N(v)\frac{\partial f^\pm}{\partial v}. \tag{132}$$

It is then straightforward to compute action of the commutator on the habitat state $\Psi_{f^+,f^-,[\mathbf{s}^+,\mathbf{s}^-]_{+-}}$ along the lines of Sec. VII C 1 (or VII D) and verify that

$$\begin{aligned}\Psi_{f^+,f^-,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]_{+-}}&([\hat{H}(\vec{N}_1), \hat{H}(\vec{N}_2)]|\mathbf{s}^+,\mathbf{s}^-\rangle)\\ &= (-i\hbar)\Psi_{f^+,f^-,[\mathbf{s}^{+\prime},\mathbf{s}^{-\prime}]_{+-}}(\hat{C}_{\rm diff}([\vec{N}_2,\vec{N}_1])|\mathbf{s}^+,\mathbf{s}^-\rangle),\end{aligned} \tag{133}$$

which is an antirepresentation of the corresponding Poisson brackets. (Recall that the density weight -1 lapses can equally well be thought of as vector fields).

It is also easy, using Eqs. (127), (131), (128), and (132), to see that the Poisson bracket between the diffeomorphism constraint and the density weight 2 Hamiltonian constraint is represented faithfully on the new habitat.

D. The kernel of the density 2 constraints on the new habitat

Lemma: Given any state $\Psi_{f^+,f^-,[s^+,s^-]_{+-}}$, it will be in the kernel of $\hat{H}[N]\forall N$ if f^+, f^- are constant functions.

Proof: "If side" is trivial in light of (131), we now prove the "only-if" side.

Let $\Psi_{f^+,f^-,[s^{+\prime},s^{-\prime}]_{+-}}$ be in the kernel of $\hat{H}[N]\,\forall\, N$. Notice that given any $[\mathbf{s}^{+'},\mathbf{s}^{-'}]_{+-}$, there exists infinitely many $(\mathbf{s}^+,\mathbf{s}^-)$ such that

(i) $V_E(\mathbf{s}^+)\cap V_E(\mathbf{s}^-) = \Phi$, where Φ denotes an empty set and

(ii) $[\mathbf{s}^{+'},\mathbf{s}^{-'}]_{+-} = [\mathbf{s}^+,\mathbf{s}^-]_{+-}$.

We choose one such $(\mathbf{s}^+,\mathbf{s}^-)$. As

$$\begin{aligned}\Psi_{f^+,f^-,[\mathbf{s}^+,\mathbf{s}^-]_{+-}}&(\hat{H}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle)\\ &= (-i\hbar)\sum_{v\in V_E(\mathbf{s}^+)\cup V_E(\mathbf{s}^-)} N(v)\bigg(f^-\frac{\partial f^+}{\partial v} - f^+\frac{\partial f^-}{\partial v}\bigg)\\ &= (-i\hbar)\bigg(\sum_{v\in V_E(\mathbf{s}^+)} N(v)f^-(V_E(\mathbf{s}^-))\frac{\partial f^+}{\partial v}\\ &\quad - \sum_{v'\in V_E(\mathbf{s}^-)} N(v')f^+(V_E(\mathbf{s}^+))\frac{\partial f^-}{\partial v'}\bigg) = 0.\end{aligned} \tag{134}$$

As the above equation is true for all N, it implies that $f^\pm$ are constant in the neighborhood of each vertex $v \in V_E(\mathbf{s}^+)\cup V_E(\mathbf{s}^-)$.

Now consider the set $\{(\mathbf{s}^+_\phi,\mathbf{s}^-_\phi)\}$ for all periodic diffeomorphisms ϕ which do not keep $(\mathbf{s}^+,\mathbf{s}^-)$ invariant.

As $V_E(\mathbf{s}^+_\phi)\cap V_E(\mathbf{s}^-_\phi) = \{\Phi\}$ and as $(\mathbf{s}^+_\phi,\mathbf{s}^-_\phi)\in[\mathbf{s}^+,\mathbf{s}^-]_{+-}$ for all ϕ,

$$\Psi_{f^+,f^-,[\mathbf{s}^+,\mathbf{s}^-]_{+-}}(\hat{H}[N]|\mathbf{s}^+_\phi,\mathbf{s}^-_\phi\rangle) = 0 \tag{135}$$

for all N implies that $\frac{\partial f^\pm(V_E^{S^1}(\mathbf{s}^\pm_\phi))}{\partial\phi\cdot v^\pm} = 0\;\forall\,(\phi\cdot v^\pm)\in V_E(\mathbf{s}^\pm_\phi)$. These conditions for all diffeomorphisms ϕ show that $f^\pm$ are constant everywhere. This completes the proof.

IX. DISCUSSION

A key open issue in canonical LQG relates to the definition of the Hamiltonian constraint operator. This operator is constructed as the continuum limit of its finite triangulation approximant. The latter is the quantum correspondent of a classical approximant which is uniquely defined only up to terms which vanish in the classical continuum limit of infinitely fine triangulation. In contrast to the classical continuum limit, the continuum limit of the quantum operator is not independent of the choice of finite triangulation approximant, thus implying an unacceptable (infinitely manifold) choice in the definition of the quantum dynamics of LQG. On the other hand, a necessary condition for the very consistency of the quantum theory is an anomaly-free representation of the constraint algebra. Therefore, one possible way to restrict the choice of quantum dynamics is to demand that the ensuing algebra of quantum constraints is free from anomalies. Unfortunately, irrespective of the specific choice of quantum dynamics made in the current state of art in LQG, the quantum constraint algebra trivializes, i.e. the commutator of a pair of Hamiltonian constraints as well as the operator corresponding to their classical Poisson bracket vanish. The question then is: Is there any way out, i.e. is it still possible to use the anomaly-free requirement on the constraint algebra to single out a (hopefully almost) unique choice for the quantum dynamics? Below, we argue that the PFT results derived in this work suggest a strategy to answer this question.

Let us first summarize the state of art in LQG in more technical terms. In what follows we shall refer to the commutator between a pair of Hamiltonian constraints as the left-hand side (lhs) and the operator correspondent of their Poisson bracket (which is proportional to the diffeomorphism constraint) as the rhs.

The quantum dynamics of LQG was first defined rigorously in the seminal work of Thiemann [6] in terms of the smeared density weight one Hamiltonian constraint. The continuum limit of its action is defined either with respect to the URS topology [6] or directly on an appropriate habitat of distributions [7]. In both cases, the continuum limit of the Hamiltonian constraint operator is well defined and its commutator (i.e., the lhs) trivializes as does the rhs [6]. In the URS topology, the rhs vanishes doubly, first due to a factor of "δ" and second by virtue of the fact that the rhs is proportional to the diffeomorphism constraint. The rhs vanishes doubly also on the LM habitat, first due to an overall factor of δ and second due to the absence of an additional factor of δ^{-1} which could have converted the difference in the evaluation of vertex smooth functions at points separated by δ into a derivative in the continuum limit (here, δ is a parameter which measures the fineness of the triangulation, $\delta \rightarrow 0$ being the continuum limit). As we have seen, *exactly* the same situation prevails in PFT.

In PFT additional factors of δ^{-1} can be introduced by replacing the density 1 constraints by density 2 constraints. As we have seen in Secs. VII D and VIII, this leads to a nontrivial representation of the ensuing constraint algebra on appropriate spaces of distributions. The lesson we draw from this is that the choice of density 1 constraints in PFT and LQG hides the underlying nontriviality of the constraint algebra. The key issue is then: Can we handle the algebra of appropriately chosen higher density weight Hamiltonian constraints in LQG? We first discuss the rhs and then the lhs.

The rhs: In PFT, the rhs corresponding to the Poisson bracket between a pair of density 2 Hamiltonian constraints is just the diffeomorphism constraint smeared with a c- number, metric independent shift constructed out of the lapses. In the terminology used in LQG, the rhs can no longer be defined as a finite operator on the kinematic Hilbert space. Nevertheless, as we have seen, the operator is perfectly well defined on the LM habitat in terms of (Lie) derivatives of vertex smooth functions. Moreover, the commutator between a pair of diffeomorphism constraints is anomaly-free on this habitat. We take this as indicative of being on the right track with reference to the definition of the various finite triangulation approximants to the local fields which comprise the constraint. In LQG, most of the ambiguities in the Hamiltonian constraint arise from those involved in the choice of finite triangulation approximant to the curvature, F^i_{ab}, of the Ashtekar-Barbero connection. The question then arises as to whether we can define the curvature operator at finite triangulation in such a way that the diffeomorphism constraint (smeared with a c- number shift) has a well defined continuum limit on the LM habitat in such a way that the Poisson-Lie algebra of diffeomorphism constraints is represented in an anomaly-free manner on the LM habitat. This is the subject of work now in progress which suggests that the answer may indeed be in the affirmative. That we have even contemplated such a possibility is already evidence of the usefulness of PFT.

The lhs: In PFT, the density 2 Hamiltonian constraint does not admit a continuum limit (both with respect to the URS topology as well as on the LM habitat). Nevertheless, a sign that the strategy of using density 2 constraints may be a profitable one is provided by the existence of the zero volume habitat of Sec. VII D, which is closely related to the LM habitat. The final solution requires the new habitat of Sec. VIII, which is geared to the physical state space of Sec. II B 5. In LQG, one can check that, modulo some subtleties, if the Hamiltonian constraint is rescaled by a factor of the determinant of the metric to the power $\frac{1}{6}$, the rhs obtains 2 factors of δ^{-1} which then could perhaps yield a nontrivial action of the rhs on the LM habitat. However, these higher density (smeared) Hamiltonian constraints are not themselves well defined on the LM habitat by virtue of the extra factor of δ^{-1} at finite triangulation. Further, as indicated in the beautiful analysis of Ref. [8], wherein the authors simply rescale the action of the density weight 1 Hamiltonian constraint and attempt to evaluate the action of rescaled commutator on the habitat, there is no way to obtain the diffeomorphism generated by the rhs unless the Hamiltonian constraint moves the vertices of the state on which it acts. The current proposals for the Hamiltonian constraint do not involve movement of vertices and this can again be traced to the inadequacy of the choice of approximant to F^i_{ab} at finite triangulation.

In PFT, despite the fact that the density 2 Hamiltonian constraint does move vertices, the lhs is not well defined on $\mathcal{V}_{\rm LM}$. However it is well defined on $\mathcal{V}^0_{\rm LM}$ or $\mathcal{V}_{+-}$. The lesson we draw from this is to

(i) search for a better finite triangulation approximant to F^i_{ab} which can move vertices around; our work in progress [9] should feed into this.

(ii) search for some analog of $\mathcal{V}^0_{\rm LM}$ or $\mathcal{V}_{+-}$. Of course $\mathcal{V}_{+-}$ could be constructed precisely because we already know the correct physical state space through Ref. [2]. In LQG, the physical state space is presumably only known once the Hamiltonian constraint is defined so the situation is far more involved. However, the aim, at least is clear, namely, that we need a satisfactory habitat and a definition of the Hamiltonian constraint such that the higher density constraints are well defined on this habitat and the constraint algebra, anomaly-free. A more modest question, relevant to see if a candidate definition of the constraint operator may be viable, is to look for the analog of $\mathcal{V}^0_{\rm LM}$. The direct analog does not help at all because, in contrast to our PFT construction (see Footnote [9]), the present construction of the "inverse volume" operator in LQG is such that all the zero volume states are also annihilated by the inverse volume operator. A natural question is: Is it possible to find an alternate construction of the inverse in LQG so that it does not annihilate zero volume states? This also brings us to a natural question in PFT: Can the restriction to zero volume states still yield a physically sensible theory? This actually may be the case because (a) zero volume states are gauge-related to nonzero volume ones, and (b) as can easily be checked, the Dirac observables of [2] preserve the space of zero volume charge nets.

In conclusion, while the situation in LQG is far more complicated, we are convinced that the structures which permit the construction of a nontrivial representation of the constraint algebra of PFT (higher density constraints, alternate habitats, operator definitions through an analysis of Hamiltonian vector fields, holonomies in representations attuned to the edge labels of the state on which they act, the coexistence of discontinuous unitary operators on the kinematic Hilbert space with the well definedness of the action of their generators on a suitable space of distributions) will open up new directions with regard to the problem of a consistent definition of the quantum dynamics of LQG.

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APPENDIX A: SPECTRUM OF THE INVERSE METRIC OPERATOR

Our starting point is Eq. (40). We adopt the following notation in this section. Let y_0 be located at the vertex v of the graph (which is naturally associated with the triangulation) T. We remind the reader that we have set $\gamma(s^\pm) = T$. The charges $k^\pm_{e_v}$, $k^\pm_{e^v}$ are defined as in Sec. IV A (see the discussion after Eq. (30)). Let the edge (i.e., the 1 simplex of the triangulation) which ends at v be $\triangle$. If $v = 0$, $\triangle$ denotes the edge which ends at 2π. We shall denote signum operator at the vertex v by $\widehat{\rm sgn}(v)$ i.e. $\widehat{\rm sgn}(v) := {\rm sgn}\widehat{X^{+\prime} X^{-\prime}}(v)$. It is also convenient to change the notation of embedding charge network states. We shall denote the charge network state $T_{s^\pm}$ by $|s^\pm\rangle$ and, whenever required, expand out the charge network label $s^\pm$ in terms of its defining data, i.e. its underlying graph and charge labels (see Sec. II B 1 and [2]). Finally, in this section we choose units in which $\hbar = a = 1$.

As noted in Sec. IV B, the operator $\hat{h}^{(\pm)-1}_{e_{x_1,y_0}}[\hat{V}_{y_0}, \hat{h}^{(\pm)}_{e_{x_1,y_0}}]$ is only sensitive to that part of e_{x_1,y_0} which overlaps with $\triangle$. Setting $y_0 = v$ and expanding the double commutators in the right-hand side of Eq. (40), it is straightforward to see that,

$$\widehat{\frac{1}{\sqrt{|X^{+\prime}X^{-\prime}|}}}(v)|_T(|s^+\rangle \otimes |s^-\rangle) = -|\triangle|\hat{h}^{(-)-1}_\triangle \widehat{\rm sgn}(v)[\hat{h}^{(+)}_\triangle \hat{V}_v \hat{h}^{(+)-1}_\triangle \hat{h}^{(-)}_\triangle - \hat{h}^{(-)}_\triangle \hat{h}^{(+)}_\triangle \hat{V}_v \hat{h}^{(+)-1}_\triangle - \hat{h}^{(+)-1}_\triangle \hat{V}_v \hat{h}^{(+)}_\triangle \hat{h}^{(-)}_\triangle$$
$$+ \hat{h}^{(-)}_\triangle \hat{h}^{(+)-1}_\triangle \hat{V}_v \hat{h}^{(+)}_\triangle]|s^+\rangle \otimes |s^-\rangle + |\bar{\triangle}|\hat{h}^{(+)}_{\bar{\triangle}} \widehat{\rm sgn}(v)[\hat{h}^{(-)}_\triangle \hat{V}_v \hat{h}^{(-)-1}_\triangle \hat{h}^{(+)-1}_\triangle$$
$$- \hat{h}^{(+)-1}_\triangle \hat{h}^{(-)}_\triangle \hat{V}_v \hat{h}^{(-)-1}_\triangle - \hat{h}^{(-)-1}_\triangle \hat{V}_v \hat{h}^{(-)}_\triangle \hat{h}^{(+)-1}_\triangle + \hat{h}^{(+)-1}_\triangle \hat{h}^{(-)-1}_\triangle \hat{V}_v \hat{h}^{(-)}_\triangle]|s^+\rangle \otimes |s^-\rangle \quad \text{(A1)}$$

Since the edge $\triangle$ ends at v, the action of the embedding holonomies in the expression above is to change $k^\pm_{e_v}$ by unity. It is then straightforward to obtain

$$\widehat{\frac{1}{\sqrt{|X^{+'}X^{-'}|}}}(v)|_T(|s^+\rangle\otimes|s^-\rangle) = -|\triangle|\hat{h}_{\triangle}^{(-)-1}\widehat{\mathrm{sgn}}(v)\Big[\Big[\sqrt{|k^+_{e^v}-(k^+_{e_v}-1)||k^-_{e^v}-(k^-_{e_v}+1)|}$$
$$-\sqrt{|k^+_{e^v}-(k^+_{e_v}-1)||k^-_{e^v}-k^-_{e_v}|}\Big]-\Big[\sqrt{|k^+_{e^v}-(k^+_{e_v}+1)||k^-_{e^v}-(k^-_{e_v}+1)|}-\sqrt{|k^+_{e^v}-(k^+_{e_v}+1)||k^-_{e^v}-k^-_{e_v}|}\Big]\Big]|s^+\rangle\otimes|\gamma(s^-)$$
$$= T,(\ldots k^-_{e_v}, k^-_{e_v}+1, k^-_{e^v},\ldots)\rangle + |\triangle|\hat{h}_{\triangle}^{(+)}\widehat{\mathrm{sgn}}(v)\Big[\Big[\sqrt{|k^+_{e^v}-(k^+_{e_v}-1)||k^-_{e^v}-(k^-_{e_v}-1)|}$$
$$-\sqrt{|k^+_{e^v}-(k^+_{e_v})||k^-_{e^v}-(k^-_{e_v}-1)|}\Big]-\Big[\sqrt{|k^+_{e^v}-(k^+_{e_v}-1)||k^-_{e^v}-(k^-_{e_v}+1)|}-\sqrt{|k^+_{e^v}-k^+_{e_v}||k^-_{e^v}-(k^-_{e_v}+1)|}\Big]\Big]|\gamma(s^+)$$
$$= T,(\ldots k^+_{e_v}, k^+_{e_v}-1, k^+_{e^v},\ldots)\rangle\otimes|s^-\rangle. \quad \text{(A2)}$$

Using

$$\mathrm{sgn}(\widehat{X^{+'}X^{-'}})(v)|s^+\rangle\otimes|s^-\rangle = \mathrm{sgn}(k^+_{e^v}-k^+_{e_v})\mathrm{sgn}(k^-_{e^v}-k^-_{e_v})|s^+\rangle\otimes|s^-\rangle, \quad \text{(A3)}$$

we get

$$\widehat{\frac{1}{\sqrt{|X^{+'}X^{-'}|}}}(v)|_T(|s^+\rangle\otimes|s^-\rangle) = -\mathrm{sgn}(k^+_{e^v}-k^+_{e_v})\mathrm{sgn}(k^-_{e^v}-(k^-_{e_v}+1))\Big[\Big[\sqrt{|k^+_{e^v}-(k^+_{e_v}-1)||k^-_{e^v}-(k^-_{e_v}+1)|}$$
$$-\sqrt{|k^+_{e^v}-(k^+_{e_v}-1)||k^-_{e^v}-k^-_{e_v}|}\Big]-\Big[\sqrt{|k^+_{e^v}-(k^+_{e_v}+1)||k^-_{e^v}-(k^-_{e_v}+1)|}$$
$$-\sqrt{|k^+_{e^v}-(k^+_{e_v}+1)||k^-_{e^v}-k^-_{e_v}|}\Big]\Big]|s^+\rangle\otimes|s^-\rangle + \mathrm{sgn}(k^+_{e^v}-(k^+_{e_v}-1))\mathrm{sgn}(k^-_{e^v}-k^-_{e_v})\Big[\Big[\sqrt{|k^+_{e^v}-(k^+_{e_v}-1)||k^-_{e^v}-(k^-_{e_v}-1)|}$$
$$-\sqrt{|k^+_{e^v}-(k^+_{e_v})||k^-_{e^v}-(k^-_{e_v}-1)|}\Big]-\Big[\sqrt{|k^+_{e^v}-(k^+_{e_v}-1)||k^-_{e^v}-(k^-_{e_v}+1)|}-\sqrt{|k^+_{e^v}-k^+_{e_v}||k^-_{e^v}-(k^-_{e_v}+1)|}\Big]\Big]|s^+\rangle\otimes|s^-\rangle.$$
$$\text{(A4)}$$

Whence,

$$\lambda(s^+,s^-,v) = -\mathrm{sgn}(k^+_{e^v}-k^+_{e_v})\mathrm{sgn}(k^-_{e^v}-(k^-_{e_v}+1))\Big[\Big[\sqrt{|k^+_{e^v}-(k^+_{e_v}-1)||k^-_{e^v}-(k^-_{e_v}+1)|}$$
$$-\sqrt{|k^+_{e^v}-(k^+_{e_v}-1)||k^-_{e^v}-k^-_{e_v}|}\Big]-\Big[\sqrt{|k^+_{e^v}-(k^+_{e_v}+1)||k^-_{e^v}-(k^-_{e_v}+1)|}-\sqrt{|k^+_{e^v}-(k^+_{e_v}+1)||k^-_{e^v}-k^-_{e_v}|}\Big]\Big]$$
$$+\,\mathrm{sgn}(k^+_{e^v}-(k^+_{e_v}-1))\mathrm{sgn}(k^-_{e^v}-k^-_{e_v})\Big[\Big[\sqrt{|k^+_{e^v}-(k^+_{e_v}-1)||k^-_{e^v}-(k^-_{e_v}-1)|}-\sqrt{|k^+_{e^v}-(k^+_{e_v})||k^-_{e^v}-(k^-_{e_v}-1)|}\Big]$$
$$-\Big[\sqrt{|k^+_{e^v}-(k^+_{e_v}-1)||k^-_{e^v}-(k^-_{e_v}+1)|}-\sqrt{|k^+_{e^v}-k^+_{e_v}||k^-_{e^v}-(k^-_{e_v}+1)|}\Big]\Big]. \quad \text{(A5)}$$

[1] A. Laddha and M. Varadarajan, Phys. Rev. D **78**, 044008 (2008).
[2] A. Laddha and M. Varadarajan, Classical Quantum Gravity **27**, 175010 (2010).
[3] P. Dirac, *Lectures on Quantum Mechanics* (Library of Congress, New York, 1964).
[4] D. Giulini and D. Marolf, Classical Quantum Gravity **16**, 2479 (1999).
[5] A. Perez, Phys. Rev. D **73**, 044007 (2006).
[6] T. Thiemann, Classical Quantum Gravity **15**, 839 (1998).
[7] J. Lewandowski and D. Marolf, Phys. Rev. D **7**, 299 (1998).
[8] R. Gambini, J. Lewandowski, D. Marolf, and J. Pullin, Int. J. Mod. Phys. D **7**, 97 (1998).
[9] A. Laddha and M. Varadarajan, Diffeomorphism Constraint in LQG (to be published).
[10] T. Thiemann, arXiv:1010.2426.
[11] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, and T. Thiemann, J. Math. Phys. (N.Y.) **36**, 6456 (1995).
[12] See for example E. Merzbacher, *Quantum Mechanics* (John Wiley and Sons, New York, 1970).
[13] A. Ashtekar, T. Pawlowski, and P. Singh, Phys. Rev. D **74**, 084003 (2006).
[14] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge Monographs on Mathematical Physics, Cambridge, 2008).
[15] A. Ashtekar, C. Rovelli, and L. Smolin, Phys. Rev. Lett. **69**, 237 (1992).