PHYSICAL REVIEW D **83,** 025014 (2011)

Translational-invariant noncommutative gauge theory

F. Ardalan[1,2,*] and N. Sadooghi[1,†]

[1]*Department of Physics, Sharif University of Technology, P.O. Box 11155-9161, Tehran-Iran*

[2]*Institute for Studies in Theoretical Physics and Mathematics (IPM), School of Physics, P.O. Box 19395-5531, Tehran-Iran*

(Received 20 September 2010; published 20 January 2011)

A generalized translational-invariant noncommutative field theory is analyzed in detail, and a complete description of translational-invariant noncommutative structures is worked out. The relevant gauge theory is described, and the planar and nonplanar axial anomalies are obtained.

DOI: 10.1103/PhysRevD.83.025014 PACS numbers: 11.10.Nx, 11.10.Lm, 11.15.Bt, 11.15.Kc

I. INTRODUCTION

Noncommutative geometry [1] has a long history. The advent of the flurry of activity in this field related to physics was the discovery of noncommutativity in string theory [2]. Subsequently, noncommutative field theories, which appear in a decoupling limit of string theories, have been the focus of extensive research. Noncommutative geometry is most elegantly described in the context of noncommutative algebra [1], and, in particular, noncommutative algebra of functions, which is an ingenious generalization of the commutative C^* algebra of ordinary function as the Gelfand-Naimark dual of ordinary commutative geometry. Thus, general noncommutative star algebras of functions are primary objects in noncommutative geometry and, in particular, in the field theories based on such geometries. The first example of the noncommutative geometry arising from string theory was constructed on the basis of the canonical commutation relation

$$[x_\mu, x_\nu]_\star = i\theta_{\mu\nu}, \tag{1.1}$$

where, in the simplest case, $\theta_{\mu\nu}$ is a constant real antisymmetric matrix, and the Moyal product of functions derived from (1.1) is [3]

$$(f \star g)(x) = f(x) \exp\left(\frac{i}{2}\theta_{\mu\nu}\overleftarrow{\partial}^{\mu}\overrightarrow{\partial}^{\nu}\right)g(x). \tag{1.2}$$

This noncommutative product of functions has the additional properties of star algebra

$$(f \star g)^* = g^* \star f^*, \quad \text{and} \quad f^{**} = f. \tag{1.3}$$

In addition the function space is usually a unital algebra,

$$f \star 1 = 1 \star f = f. \tag{1.4}$$

The three properties (1.1)–(1.3) are essential properties of a noncommutative star algebra which, when added with a norm, define a Banach algebra, a cornerstone of noncommutative geometry.

At this stage, let us notice that the Moyal star product (1.2) is not the unique choice compatible with (1.1). A second alternative to quantize the classical Poisson structure is the Wick-Voros product [4]. In [5], a noncommutative $\lambda\varphi^4$ theory is formulated using these two products, and the differences between them are studied. It turns out that whereas the Lagrangian densities of these two apparently different theories, and consequently their tree level vertices and propagators as well as their one-loop Green's functions are different, they have the same S-matrix element. They are therefore "physically" equivalent. Moreover, it is shown that since both products are the realization of the same canonical commutation relation (1.1), the one-loop Feynman integrals arising from these two formulations have the same ultraviolet (UV) behavior. This is why the UV/IR mixing [6], appearing in the more elaborated Moyal formulation cannot be cured in the Wick-Voros formulation. To have a satisfactory interpretation of these remarkable results, the authors in [5] use a symmetry argument. They relate the equivalence between Moyal and Wick-Voros formulations at the level of S-matrix elements to the invariance of the physical observables, in general, and the S-matrix elements, in particular, under the Poincaré transformation. To prove this equivalence, they use instead of (1.1), which is not invariant under the "ordinary" Poincaré transformation, a twisted theory [7], which is formulated so that it is invariant under a certain twisted Poincaré symmetry [8]. The latter is based on a deformed Poincaré Lie algebra, that builds a noncommutative, noncocommutative Hopf algebra. Using a consistent twisting procedure, where the field operators (oscillators) of the theory are also deformed and new commutation relations between creation and annihilation operators are defined, they finally recalculate the twisted S-matrix element, which is shown to have the same expression as in the Moyal and Wick-Voros case. Different twisted formulations of quantum field theories are discussed in [9]. The deformation of field oscillators in a more general framework of braided algebras is discussed in [10].

In this paper, we will use a third, more general product than Moyal and/or Wick-Voros products. It is based on the

*ardalan@ipm.ir
†sadooghi@physics.sharif.ir

crucial requirement, that ensures the existence of energy-momentum conservation in the usual sense, and that is the property of translation invariance

$$\mathcal{T}_a(f) \star \mathcal{T}_a(g) = \mathcal{T}_a(f \star g), \tag{1.5}$$

where

$$\mathcal{T}_a(f)(x) \equiv f(x + a). \tag{1.6}$$

The general translational-invariant associative star product was originally introduced in [11,12] in terms of a certain function $\alpha(p, q)$, which is constrained mainly by the associativity requirement of the product. Constructing a simple scalar field theory using this new product, it was further shown that the nonplanar Feynman integrals of the theory are mainly modified by a combination of $\alpha(p, q)$ that reproduces, in particular, the same antisymmetric phase factor that appears in the Feynman integrals of noncommutative gauge theories constructed by the ordinary Moyal product. This phase factor, given by the commutator of coordinates (1.1) is responsible for the famous UV and IR connection of noncommutative field theory [6]. In the present work, we study these theories further. Apart from generalizing the results arising from noncommutative translational-invariant bosonic formulation to $U(1)$ gauge theory, the goal is to present the general structure of the characteristic function $\alpha(p, q)$ in order to understand the relation between the novel translational-invariant noncommutative product with the other two, Moyal and Wick-Voros, products. This is something which is not completely discussed in [11]. In light of this comparison, and following the line of arguments in [5] (see also our descriptions above), the fact that even the translational-invariant formulation is not able to cure the UV/IR mixing of noncommutative field theories will be clarified.

The paper is organized as follows: In Sec. II, after determining the general structure of the new translational-invariant noncommutative product in terms of the real and the imaginary part of $\alpha(p, q)$, we perform a complete specification of the product structure for two noncommutative directions, and determine in this way the general solution of the cocycle relation which is, as before mentioned, the main restriction on $\alpha(p, q)$. The main result of this section is Eq. (2.53), which states that the noncommutative structure function $\alpha(p, q)$ is the sum of a quadratic term $\omega(p, q)$ which enters in the loop diagrams and an arbitrary complex function $\eta(p)$, with real part even and imaginary part odd parity, that does not appear in the loop integrations. In particular, the real even part of $\eta(p)$ seems to be the generalization of the phase factor appearing in the three-level propagator of noncommutative field theory formulated with the Wick-Voros product. In Secs. III and IV, we construct the noncommutative gauge theory and its one-loop Feynman diagrams. Our goal is to study the effect of the elements of the characteristic function $\alpha(p, q)$ on the divergence properties of the Feynman integrals. We will determine the loop integrations of one-loop corrections to the fermion and photon propagators and vertex function and present general arguments to show that loop integrations in any order of perturbation theory involve only $\omega(p, q)$ and not $\eta(p)$. We conclude that only $\omega(p, q)$ is responsible for the well-known noncommutative UV/IR mixing [6], and $\eta(p)$ does not play any role in the divergence properties of Feynman integrals. Section V consists of a study of axial anomalies of these gauge theories. Quantum anomalies of the ordinary Moyal noncommutative gauge theories are studied intensively in [13–15], where it is shown that they consist of a planar (covariant) as well as a nonplanar (invariant) anomaly. As in the ordinary Moyal noncommutative gauge theory, we will show in Sec. V that whereas the planar anomaly is a noncommutative generalization of the well-known Adler-Bell-Jackiw axial anomaly, the nonplanar axial anomaly consists of a generalized star product [16] now modified with a phase factor consisting of a symmetric function in the momenta. Section VI is devoted to concluding remarks.

II. TRANSLATIONAL-INVARIANT PRODUCT

A. General structure

In a translationally invariant noncommutative product [11], the kernel of the product as defined by

$$f(x) \star g(x) \equiv \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \times e^{-irx} \tilde{f}(p) \tilde{g}(q) K(r, p, q), \tag{2.1}$$

has the following form:

$$K(r, p, q) = e^{\alpha(r,p)} \delta^d(p + q - r). \tag{2.2}$$

Translational invariance is defined by (1.5) and (1.6). The noncommutative star product is therefore characterized by the complex function $\alpha(p, q)$. The main restriction on $\alpha(p, q)$ follows from associativity of the star product

$$f \star (g \star h) = (f \star g) \star h, \tag{2.3}$$

which restricts the kernel function $K(r, p, q)$ by

$$\int d^d \ell K(p, \ell, q) K(\ell, r, s) = \int d^d \ell K(p, r, \ell) K(\ell, s, q), \tag{2.4}$$

and imposes the associativity condition

$$\alpha(p, q) + \alpha(q, r) = \alpha(p, r) + \alpha(p - r, q - r), \tag{2.5}$$

on the characterizing function $\alpha(p, q)$. There is another significant restriction on $\alpha(p, q)$ coming from the requirement of the existence of conjugation on the function space (1.3), which imposes the condition

$$\alpha(p, q)^* = \alpha(-p, q - p), \tag{2.6}$$

on the function $\alpha(p,q)$. Here, we will be requiring the star algebra to have the constant function 1 as its identity [see (1.4)], resulting in

$$\alpha(p,p) = \alpha(p,0) = 0. \tag{2.7}$$

The primary example of a noncommutative star product is the Moyal product defined by (1.2), which is equivalent to

$$\alpha(p,q) = -\frac{i}{2}\theta_{\mu\nu}p^\mu q^\nu, \tag{2.8}$$

with $\theta_{\mu\nu}$ a constant antisymmetric matrix defined in (1.1). In the rest of this section, we will find the most general form of the complex function $\alpha(p,q)$, with the additional assumption that $\alpha(p,q)$ can be expanded in a series in the components of p and q

$$\alpha(p,q) = \sum_{n=0}^{\infty}\sum_{\substack{\{i_1,\cdots,i_n\}\\ \{j_1,\cdots,j_n\}}} a_{i_1,\cdots,i_n;j_1,\cdots j_n} p_1^{i_1}\cdots p_n^{i_n} q_1^{j_1}\cdots q_n^{j_n}. \tag{2.9}$$

In the expression (2.9), there are no constant terms. This is because of the unitality condition (2.7). Moreover, each term in the series contains at least one power of p_i and one of q_i.

With these restrictions, the main constraint to be satisfied will be the condition of associativity (2.5), which we will proceed to analyze. The first step in the analysis of associativity condition is to separate the function $\alpha(p,q)$ in its real and imaginary part

$$\alpha(p,q) = \alpha_1(p,q) + i\alpha_2(p,q), \tag{2.10}$$

where $\alpha_1(p,q)$ and $\alpha_2(p,q)$ are now real functions.

1. The real part of $\alpha(p,q)$

The condition of conjugation (2.6) implies

$$\alpha_1(p,q) = \tfrac{1}{2}[\alpha(p,q) + \alpha(-p,q-p)]. \tag{2.11}$$

Using the associativity condition (2.5) by substituting $p \to 0$, $q \to q$, and $r \to p$, we get

$$\alpha(0,q) + \alpha(q,p) = \alpha(0,p) + \alpha(-p,q-p). \tag{2.12}$$

Using the associativity condition (2.5) again, but this time with the substitutions $p \to p$, $q \to q$, and $r \to p$, we have

$$\alpha(p,q) + \alpha(q,p) = \alpha(0,q-p), \tag{2.13}$$

where we have used the condition for existing of identity (2.7), $\alpha(p,p) = 0$. The net result is

$$\alpha_1(p,q) = \eta_1(q) - \eta_1(p) + \eta_1(p-q), \tag{2.14}$$

where $\eta_1(p) \equiv \frac{1}{2}\alpha(0,p)$. We note that $\eta_1(p)$ is real, by complex conjugation (2.6), and an even function of p,

$$\eta_1(-p) = \eta_1(p), \tag{2.15}$$

from the associativity condition (2.5), again, this time with the substitution $p \to r$ and $q \to 0$. It can be readily verified that $\alpha_1(p,q)$ as given by (2.14) satisfies the associativity condition identically for arbitrary even function $\eta_1(p)$. Moreover,

$$\eta_1(0) = 0, \tag{2.16}$$

by the existence of the unit of the algebra (2.17). Therefore, the real part of $\alpha(p,q)$ is given by (2.14) in terms of an arbitrary real even function η_1 satisfying (2.16). Note that the function $\eta_1(p)$ plays the role of a weighting function for the integral of the trace relation

$$\int d^dx f(x)\star g(x) = \int d^dx g(x)\star f(x) = \int \frac{d^dp}{(2\pi)^d} e^{2\eta_1(p)} f(p)g(p). \tag{2.17}$$

For that reason, it effectively determines the function space on which the star algebra is built.

2. The imaginary part of $\alpha(p,q)$

The determination of the imaginary part of $\alpha(p,q)$ is more involved. It was observed in [11] that only a certain part of $\alpha_2(p,q)$ in (2.10) defined by

$$-2i\omega(p,q) = \alpha(p+q,p) - \alpha(p+q,q), \tag{2.18}$$

appears in the loop integrals of the scalar $\lambda\varphi^4$ theory.[1] We will show in the subsequent section that this persists for the gauge theory also. However, we will find that $\alpha_2(p,q)$, the imaginary part of $\alpha(p,q)$, has an additional contribution that we call $\xi(p,q)$,

$$\alpha_2(p,q) = \omega(p,q) + \xi(p,q). \tag{2.19}$$

We will proceed to determine the form of both $\omega(p,q)$ and $\xi(p,q)$. In [11], the form of $\omega(p,q)$ was correctly identified; however, the arguments required are more rigorous and we will provide it.

To begin with, it is straightforward to see that $\omega(p,q)$ is real, as can be seen from the fact that the real part of $\alpha(p+q,p)$ from (2.14) is symmetrized in the exchange of p and q. It is also clear that $\omega(p,q)$ is antisymmetric in $p \leftrightarrow q$. Thus,

$$\omega(p,q)\ \text{is real,} \tag{2.20}$$

$$\omega(p,q) = -\omega(q,p), \quad \omega(p,q)\ \text{is antisymmetric in}\ p\ \text{and}\ q. \tag{2.21}$$

We can also show that $\omega(p,q)$ is an even function of p and q. First, we have

$$-2i\omega(-p,-q) = \alpha(-p-q,-p) - \alpha(-p-q,-q).$$

But from associativity (2.5), with the substitution $p \to 0$, $q \to p$, and $r \to p+q$, we get

$$\alpha(-p-q,-q) = \alpha(p,p+q) + \alpha(0,p) - \alpha(0,p+q). \tag{2.22}$$

[1] Note that in the definition of $\omega(p,q)$, there is an additional $-2i$ in Ref. [11].

Then, using (2.13), we get

$$\omega(-p,-q)=\omega(p,q), \quad \omega(p,q)\,\text{is odd in}\,p\,\text{and}\,q. \tag{2.23}$$

More significantly $\omega(p,q)$ satisfies the same associativity relation (2.5) as $\alpha(p,q)$:

$$\omega(p,q)+\omega(q,r)=\omega(p,r)+\omega(p-r,q-r). \tag{2.24}$$

To prove this, first we note that using the associativity relation as,

$$\alpha(p+q,p)+\alpha(p,r)=\alpha(p+q,r)+\alpha(p+q-r,p-r),$$

and substituting $r\rightarrow q$, we get

$$-2i\omega(p,q)=\alpha(p,p-q)-\alpha(p,q). \tag{2.25}$$

Then, proving associativity for $\omega(p,q)$ reduces to proving that $\alpha(p,p-q)+\alpha(q,q-r)-\alpha(p,p-r)-\alpha(p-r,p-q)$ vanishes. Using now $-\alpha(p,p-r)=\alpha(p-r,p)-\alpha(0,r)$, we get $\alpha(p-r,p)+\alpha(p,p-q)-\alpha(p-r,p-q)+\alpha(q,q-r)-\alpha(0,r)$, which upon using associativity again becomes $\alpha(q-r,q)+\alpha(q,q-r)-\alpha(0,r)$, which vanishes by (2.13). Thus we have proved (2.24).

Now clearly as $\alpha(p,q)$ satisfies associativity and also $\alpha_1(p,q)$, we conclude that so does $\xi(p,q)$,

$$\xi(p,q)+\xi(q,r)=\xi(p,r)+\xi(p-r,q-r). \tag{2.26}$$

Antisymmetry of $\xi(p,q)$,

$$\xi(p,q)=-\xi(q,p), \tag{2.27}$$

follows from (2.13) and antisymmetry of $\omega(p,q)$, together with the relation

$$\alpha_1(p,q)+\alpha_1(q,p)=\alpha(0,p-q).$$

The parity of $\xi(p,q)$,

$$\xi(-p,-q)=-\xi(p,q), \tag{2.28}$$

is obtained from the definition of $\omega(p,q)$ and $\xi(p,q)$, which gives, using both (2.18) and (2.25),

$$\xi(p+q,p)=\xi(p+q,q). \tag{2.29}$$

B. Determination of $\omega(p,q)$ and $\xi(p,q)$ for two noncommutative dimensions

From now on we assume that the noncommutativity occurs for only two spatial coordinates (x_1,x_2), and, assuming a series expansion for $\omega(\vec{p},\vec{q})$ and $\xi(\vec{p},\vec{q})$, find their most general form. Of the three basics conditions on $\omega(\vec{p},\vec{q})$ and $\xi(\vec{p},\vec{q})$ the most restrictive is the associativity condition (2.24) and (2.26), which follows from associativity condition on $\alpha(\vec{p},\vec{q})$, (2.5), and the complex conjugation condition (2.6).[2] We will in fact see shortly that condition (2.6) is indeed incorporated in (2.24) and (2.26). There remains for us to impose the condition of the unit of star algebra, (2.7). Imposing this condition implies that in each term in the series (2.9)

$$\sum_{\substack{\{i_1,i_2,j_1,j_2\}\\ i_1+i_2+j_1+j_2=N}} a_{i_1,i_2;j_1,j_2}\,p_1^{i_1}p_2^{i_2}q_1^{j_1}q_2^{j_2}, \tag{2.30}$$

we have, $i_1+i_2>0$ as well as $j_1+j_2>0$. Note that we have picked a particular term in the series (2.9) of total degree of N, as the associativity condition being a linear operation within the terms of a fixed total degree N.

The task on hand is therefore to extract the restriction of associativity conditions (2.24) and (2.26) on the coefficients $a_{\vec{i},\vec{j}}$, where we denote $\vec{i}=(i_1,i_2)$ and $\vec{j}=(j_1,j_2)$. We will show that for total degree N even, appropriate for $\omega(p,q)$, the only solution of the (2.24) is

$$\omega(\vec{p},\vec{q})=\vec{p}\wedge\vec{q}\equiv\frac{\theta}{2}(p_1q_2-p_2q_1), \tag{2.31}$$

with $N=2$. Here, θ is a multiplicative constant. But, we see that there are many solutions for N odd, appropriate for $\xi(\vec{p},\vec{q})$, and we will determine them. We will use the polynomial (2.30) for both $\omega(\vec{p},\vec{q})$ and $\xi(\vec{p},\vec{q})$, corresponding to even and odd N, respectively. We will then insert the sum (2.30) into the corresponding associativity condition (2.24) and (2.26), not in the original form

$$\zeta(\vec{p},\vec{q})+\zeta(\vec{q},\vec{r})=\zeta(\vec{p},\vec{r})+\zeta(\vec{p}-\vec{r},\vec{q}-\vec{r}),$$

where $\zeta(\vec{p},\vec{q})$ stands generically for $\omega(\vec{p},\vec{q})$ and $\xi(\vec{p},\vec{q})$, but with a change of variable $\vec{q}-\vec{r}\rightarrow\vec{q}$,

$$\zeta(\vec{p},\vec{q}+\vec{r})+\zeta(\vec{q}+\vec{r},\vec{r})=\zeta(\vec{p},\vec{r})+\zeta(\vec{p}-\vec{r},\vec{q}), \tag{2.32}$$

and substitute

$$\zeta(\vec{p},\vec{q})=\sum_{\vec{i},\vec{j}}a_{\vec{i},\vec{j}}\mathbf{p}^{i}\mathbf{q}^{j}, \tag{2.33}$$

where we are using two-dimensional vector notation $\mathbf{p}^i\equiv p_1^{i_1}p_2^{i_2}$, $\mathbf{q}^j\equiv q_1^{j_1}q_2^{j_2}$, and the sum is over $\{i_1,i_2,j_1,j_2\}$ with $i_1+i_2+j_1+j_2=N$. Equation (2.32) then becomes:

$$\sum_{\vec{i},\vec{j},\vec{k}}a_{\vec{i},\vec{j}}\mathbf{p}^{i}\mathbf{q}^{j-k}\mathbf{r}^{k}\begin{pmatrix} j\\ k\end{pmatrix}+\sum_{\vec{i},\vec{j},\vec{k}}a_{\vec{i},\vec{j}}\mathbf{q}^{i-k}\mathbf{r}^{j+k}\begin{pmatrix} i\\ k\end{pmatrix} -\sum_{\vec{i},\vec{j},\vec{k}}a_{\vec{i},\vec{j}}\mathbf{p}^{i-k}\mathbf{q}^{j}\mathbf{r}^{k}(-1)^{k}\begin{pmatrix} i\\ k\end{pmatrix}=\sum_{\vec{i},\vec{j},\vec{k}}a_{\vec{i},\vec{j}}\mathbf{p}^{i}\mathbf{r}^{j}, \tag{2.34}$$

where

[2]In two dimensions, $\vec{p}$ denotes (p_1,p_2).

$$\binom{i}{k} \equiv \binom{i_1}{k_1}\binom{i_2}{k_2}$$

etc., and $(-l)^k \equiv (-1)^{k_1+k_2}$. It is not hard to derive the recurrence relation for $a_{\vec{i},\vec{j}}$ from this equation. However, the limits on the indices require careful attention. We will not go into this tedious discussion and simply write down the solution

$$\binom{i+k}{k} a_{\vec{i}+\vec{k},\vec{j}-\vec{k}} = (-1)^k \binom{j}{k} a_{\vec{i},\vec{j}}, \quad \text{with } \vec{k} \leq \vec{j}. \tag{2.35}$$

We note that $a_{\vec{i},\vec{j}}$ are antisymmetric

$$a_{\vec{j},\vec{i}} = -a_{\vec{i},\vec{j}}, \tag{2.36}$$

by the antisymmetry of $\omega(\vec{p},\vec{q})$ and $\xi(\vec{p},\vec{q})$, (2.21) and (2.27), respectively. Imposing this condition on the recurrence relation (2.35) by letting $\vec{i}+\vec{k} \to \vec{j}, \vec{j}-\vec{k} \to \vec{i}$, we get

$$a_{\vec{j},\vec{i}} = (-1)^{j-i} a_{\vec{i},\vec{j}}, \tag{2.37}$$

which implies

$$(j_1 + j_2) - (i_1 + i_2) \quad \text{is odd.} \tag{2.38}$$

But this means that the function has odd parity which is only satisfied by $\xi(\vec{p},\vec{q})$. However, there is an except,

$$\vec{i} = (0,1), \qquad \vec{j} = (1,0), \quad \text{and} \quad \vec{k} = (0,0), \tag{2.39}$$

where the generic function $\zeta(\vec{p},\vec{q})$ is even and the recurrence relation (2.35) yields only an identity

$$a_{0,1,1,0} = a_{0,1,1,0}. \tag{2.40}$$

This is in fact the single possible solution of $\omega(\vec{p},\vec{q}) = \vec{p}\wedge\vec{q}$ from (2.30) observed in [11].[3] We will now proceed to solve the recurrence relation (2.35) and find the most general form for $\xi(\vec{p},\vec{q})$: Eq. (2.35) is

$$a_{\vec{i}+\vec{k},\vec{j}-\vec{k}} = (-1)^k \frac{i!j!}{(i+k)!(j+k)!} a_{\vec{i},\vec{j}}, \qquad \vec{k} \leq \vec{j}, \tag{2.41}$$

which written in the components is

$$a_{i_1+k_1,i_2+k_2;j_1-k_1,j_2-k_2} = (-1)^{k_1+k_2} \times \frac{i_1!i_2!j_1!j_2!}{(i_1+k_1)!(i_2+k_2)!(j_1-k_1)!(j_2-k_2)!} a_{i_1,i_2;j_1,j_2}, \tag{2.42}$$

with $0 \leq k_1 \leq j_1$, $0 \leq k_2 \leq j_2$, and $k_1 + k_2 < j_1 + j_2$. We can start from $a_{0,1;j_1,2n-j_1}$, where $N = 2n+1$ is the degree of the term in the expansion of $\xi(\vec{p},\vec{q})$, and apply (2.42) to find all the terms required by associativity, which results in the expansion

$$\frac{1}{j_2+1}\{(q_1-p_1)^{j_1}[q_2^{j_2+1} - (q_2-p_2)^{j_2+1}] - p_1^{j_1}p_2^{j_2+1}\}, \tag{2.43}$$

with $j_1 + j_2 = 2n$. We have taken care of the intricacies of the limits in the summations over k_1 and k_2. There is a further subtlety related to the generation of various terms and their antisymmetric partner terms in the expansion (2.43) in the application of (2.42). The point is that for each k_1 and k_2, by applying (2.42) there is another term (k_1', k_2') generated likewise which is the antisymmetric partner of (k_1, k_2), provided

$$k_1 + k_1' = j_1, \qquad k_2 + k_2' = j_2 - 1. \tag{2.44}$$

However, in the special cases $j_2 = 0$ or $k_2 = j_2$ when $j_2 \neq 0$, there are no antisymmetric partners present in (2.43), via (2.42); thus they should be added to (2.43) to complete the polynomial of order $N = 2n+1$ satisfying both associativity (2.24) and antisymmetry (2.27) and of course unitality (2.7). The final result is

$$\xi_{j_1j_2}(\vec{p},\vec{q}) = (q_1-p_1)^{j_1}(q_2-p_2)^{j_2+1} + p_1^{j_1}p_2^{j_2+1} - q_1^{j_1}q_2^{j_2+1}, \tag{2.45}$$

where $j_1 + j_2 = 2n$. Noting that there is no distinction between direction 1 and 2 in the two-dimensional space we are considering, we could start from $a_{1,0;j_1,j_2}$ and arrive at

$$\xi_{j_1j_2}(\vec{p},\vec{q}) = (q_1-p_1)^{j_1+1}(q_2-p_2)^{j_2} + p_1^{j_1+1}p_2^{j_2} - q_1^{j_1+1}q_2^{j_2}, \tag{2.46}$$

which leads to the general solution for $\xi_N(\vec{p},\vec{q})$ including all polynomials of order N,

$$\xi_N(\vec{p},\vec{q}) = \sum_{n_1,n_2} C_{n_1n_2}\xi_{n_1n_2}(\vec{p},\vec{q}), \tag{2.47}$$

with $n_1 + n_2 = N$, and

$$\xi_{n_1n_2}(\vec{p},\vec{q}) = (q_1-p_1)^{n_1}(q_2-p_2)^{n_2} + p_1^{n_1}p_2^{n_2} - q_1^{n_1}q_2^{n_2}. \tag{2.48}$$

We have verified that the expression (2.48) agrees with the computer generated polynomials ξ_3, ξ_5 of order $N = 3, 5$. For $N = 3$, $\xi_{j_1j_2}$ are given by

[3] In [11], the form (2.31) was obtained from the equation $\omega(p,q) = \omega(p-q,p)$, which is readily derived from the associativity condition. However, this equation can be shown to have a multitude of solutions of the form $\omega(p,q) = \sum_n c_n(p\wedge q)^n$. (We are grateful to M. Alishahiha and H Arfaei for pointing this to us.) One has to use the associativity condition (2.5) in its entirety to prove that only $n = 1$ is permissible.

$$\xi_{03}(\vec{p},\vec{q}) = 3p_2^2q_2 - 3p_2q_2^2, \qquad \xi_{12}(\vec{p},\vec{q}) = p_2^2q_1 + 2p_1p_2q_2 - 2p_2q_1q_2 - p_1q_2^2,$$
$$\xi_{21}(\vec{p},\vec{q}) = 2p_1p_2q_1 - p_2q_1^2 + p_1^2q_2 - 2p_1q_1q_2, \qquad \xi_{30}(\vec{p},\vec{q}) = 3p_1^2q_1 - 3p_1q_1^2, \tag{2.49}$$

whereas for $N = 5$, they read

$$\xi_{05}(\vec{p},\vec{q}) = 5p_2^4q_2 - 10p_2^3q_2^2 + 10p_2^2q_2^3 - 5p_2q_2^4,$$
$$\xi_{14}(\vec{p},\vec{q}) = p_2^4q_1 + 4p_1p_2^3q_2 - 4p_2^3q_1q_2 - 6p_1p_2^2q_2^2 + 6p_2^2q_1q_2^2 + 4p_1p_2q_2^3 - 4p_2q_1q_2^3 - p_1q_2^4,$$
$$\xi_{23}(p,q) = 2p_1p_2^3q_1 - p_2^3q_1^2 + 3p_1^2p_2^2q_2 - 6p_1p_2^2q_1q_2 + 3p_2^2q_1^2q_2 - 3p_1^2p_2q_2^2 + 6p_1p_2q_1q_2^2 - 3p_2q_1^2q_2^2 + p_1^2q_2^3 - 2p_1q_1q_2^3,$$
$$\xi_{32}(\vec{p},\vec{q}) = 3p_1^2p_2^2q_1 - 3p_1p_2^2q_1^2 + p_2^2q_1^3 + 2p_1^3p_2q_2 - 6p_1^2p_2q_1q_2 + 6p_1p_2q_1^2q_2 - 2p_2q_1^3q_2 - p_1^3q_2^2 + 3p_1^2q_1q_2^2 - 3p_1q_1^2q_2^2,$$
$$\xi_{41}(\vec{p},\vec{q}) = 4p_1^3p_2q_1 - 6p_1^2p_2q_1^2 + 4p_1p_2q_1^3 - p_2q_1^4 + p_1^4q_2 - 4p_1^3q_1q_2 + 6p_1^2q_1^2q_2 - 4p_1q_1^3q_2,$$
$$\xi_{50}(\vec{p},\vec{q}) = 5p_1^4q_1 - 10p_1^3q_1^2 + 10p_1^2q_1^3 - 5p_1q_1^4. \tag{2.50}$$

Noting that any odd function of a two-dimensional vector $\vec{p}$ can be expanded as

$$\eta_2(\vec{p}) \equiv \sum_{n=1}^{\infty}\sum_{\ell=0}^{2n+1} C_{\ell,2n+1-\ell}\,p_1^{\ell}p_2^{2n+1-\ell}, \tag{2.51}$$

from (2.48), we observe that, in general, $\xi(\vec{p},\vec{q})$ is given by

$$\xi(\vec{p},\vec{q}) = \eta_2(\vec{q}) - \eta_2(\vec{p}) + \eta_2(\vec{p}-\vec{q}), \tag{2.52}$$

with $\eta_2(\vec{p})$ an arbitrary odd function in the form (2.51). Thus, we have found the most general form for $\alpha(\vec{p},\vec{q})$, describing the translational-invariant star product of a function of two variables, as

$$\alpha(\vec{p},\vec{q}) = \sigma(\vec{p},\vec{q}) + i\omega(\vec{p},\vec{q}), \tag{2.53}$$

with

$$\sigma(\vec{p},\vec{q}) = \eta(\vec{q}) - \eta(\vec{p}) + \eta(\vec{p}-\vec{q}), \quad \text{where}$$
$$\eta(\vec{p}) \equiv \eta_1(\vec{p}) + i\eta_2(\vec{p}). \tag{2.54}$$

In (2.53), $\omega(\vec{p},\vec{q}) = \vec{p}\wedge\vec{q}$, and in (2.54), $\eta_1(\vec{p})$ is an arbitrary even function of $\vec{p}$ and $\eta_2(\vec{p})$ an arbitrary odd function $\vec{p}$, satisfying $\eta_1(\vec{0}) = \eta_2(\vec{0}) = 0$, and $\eta(-\vec{p}) = \eta^*(\vec{p})$.

At the end it is of interest to obtain the form of the star commutation of coordinates x_1 and x_2, derived for the above star algebra product (see (4.25) in [11])

$$[x_i, x_j]_\star = i\theta_{ij},$$

as there is no quadratic term in $\xi(\vec{p},\vec{q})$, and in the real part of $\alpha(\vec{p},\vec{q})$ only $\eta_1(\vec{p}-\vec{q})$ can contribute but not as $\eta_1(p)$ is even. In the next section, we will introduce the noncommutative gauge theory using the general translational-invariant noncommutative star product (2.1) and (2.2). The goal is to study the effect of the elements of the noncommutative structure function $\alpha(\vec{p},\vec{q})$, i.e., $\eta(\vec{p})$ and $\omega(\vec{p},\vec{q})$ on the divergence properties of Feynman integrals of this theory.

III. TRANSLATIONAL-INVARIANT NONCOMMUTATIVE $U(1)$ GAUGE THEORY

Let us start with the Lagrangian density of translational-invariant noncommutative $U(1)$ gauge theory, which is given by the ordinary Lagrangian of QED with the commutative products replaced by the translational-invariant noncommutative star product (2.1) and (2.2). The full Lagrangian density consists of a gauge/ghost and a fermionic part, $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_f$. The gauge/ghost Lagrangian is given by

$$\mathcal{L}_g = -\frac{1}{4}F_{\mu\nu}\star F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)\star(\partial_\nu A^\nu) + \frac{1}{2}(i\bar{c}\star\partial^\mu D_\mu c - i\partial^\mu D_\mu c\star\bar{c}), \tag{3.1}$$

where the non-Abelian field strength tensor is defined by

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)]_\star. \tag{3.2}$$

The fermionic part of $\mathcal{L}$ reads

$$\mathcal{L}_f = i\bar{\psi}\star\gamma^\mu\partial_\mu\psi - g\bar{\psi}\star\gamma^\mu A_\mu\star\psi - m\bar{\psi}\star\psi. \tag{3.3}$$

It arises from the commutative Dirac Lagrangian $\mathcal{L}_D = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)$ and the minimal coupling

$$\partial_\mu\psi(x)\to D_\mu\psi(x)\equiv\partial_\mu\psi(x)+igA_\mu(x)\star\psi(x). \tag{3.4}$$

Note that similar to the case of Moyal noncommutativity, the minimal coupling (3.4) is not unique. There are two other possibilities for introducing the gauge fields in the Lagrangian,

$$\partial_\mu\psi(x)\to D_\mu\psi(x)\equiv\partial_\mu\psi(x)-ig\psi(x)\star A_\mu(x), \tag{3.5}$$

and

$$\partial_\mu\psi(x)\to D_\mu\psi(x)\equiv\partial_\mu\psi(x)+ig[A_\mu(x),\psi(x)]_\star. \tag{3.6}$$

Whereas in (3.4) the fermions are in the fundamental representation and the resulting noncommutative action is invariant under the transformation

$$\psi(x)\rightarrow e^{ig\alpha(x)}\star\psi(x),\ \text{ and }\ A_\mu(x)\rightarrow A_\mu(x)-D_\mu\alpha(x), \tag{3.7}$$

the fermions in (3.5) and (3.6) are in the antifundamental and adjoint representations, respectively, and the resulting noncommutative actions are invariant under

$$\psi(x)\rightarrow\psi(x)\star e^{ig\alpha(x)},\ \text{ and }\ A_\mu(x)\rightarrow A_\mu(x)+D_\mu\alpha(x), \tag{3.8}$$

and

$$\begin{aligned}&\psi(x)\rightarrow e^{ig\alpha(x)}\star\psi(x)\star e^{-ig\alpha(x)}, \quad \text{and}\\ &A_\mu(x)\rightarrow A_\mu(x)-D_\mu\alpha,\end{aligned} \tag{3.9}$$

respectively. Here $D_\mu\alpha(x)\equiv\partial_\mu\alpha-ig[\alpha(x),A_\mu(x)]_\star$. In this paper, we will work with fermions in the fundamental representation with $\mathcal{L}_f$ from (3.3). The Lagrangian density of translational-invariant $U(1)$ gauge theory is of course invariant under the global $U(1)$ transformation

$$\delta_\alpha\psi(x)=i\alpha\psi(x), \quad \text{and} \quad \delta_\alpha\bar{\psi}(x)=-i\alpha\bar{\psi}(x). \tag{3.10}$$

Following the standard procedure, the Noether currents corresponding to the global $U(1)$ transformation can be determined, and it can be shown that the noncommutative gauge theory described by (3.1) possesses two different Noether currents[4]

$$J_\mu(x)=\psi(x)\star\bar{\psi}(x)\gamma_\mu,\ \text{ and }\ j_\mu(x)=\bar{\psi}(x)\gamma_\mu\star\psi(x). \tag{3.11}$$

Depending on their transformation properties under local $U(1)$ gauge transformation (3.7), they will be designated, in the rest of this article, as covariant and invariant currents, respectively. Using the equations of motion for $\bar{\psi}(x)$ and $\psi(x)$

$$\begin{aligned}&\partial_\mu\bar{\psi}\gamma^\mu=ig\bar{\psi}\gamma^\mu\star A_\mu+im\bar{\psi}, \quad \text{and}\\ &\gamma^\mu\partial_\mu\psi=-igA_\mu\star\gamma^\mu\psi-im\psi,\end{aligned} \tag{3.12}$$

the classical continuity equations of the invariant and covariant currents read

$$D_\mu J^\mu(x)=0, \quad \text{and} \quad \partial_\mu j^\mu(x)=0, \tag{3.13}$$

where the covariant derivative $D_\mu=\partial_\mu+ig[A_\mu,\cdot]_\star$. Using further the trace property of the star product

$$\int d^dxf(x)\star g(x)=\int d^dxg(x)\star f(x); \tag{3.14}$$

it is easy to check that both currents from (3.11) lead to the same conserved charge

$$Q\equiv\int d^{d-1}xj^0(x)=\int d^{d-1}xJ^0(x), \quad \text{with} \quad \partial_0Q=0. \tag{3.15}$$

Similarly, there are two different axial vector currents

$$J_{\mu,5}(x)=\psi(x)\star\bar{\psi}(x)\gamma_\mu\gamma_5, \tag{3.16}$$

$$j_{\mu,5}(x)=\bar{\psi}(x)\gamma_\mu\gamma_5\star\psi(x), \tag{3.17}$$

arising from the invariance of the Lagrangian density (3.1) under global $U_A(1)$ axial transformation $\delta_\alpha\psi=i\alpha\gamma_5\psi$. In the chiral limit, $m\rightarrow0$, similar classical conservation laws as in (3.13) hold also for axial vector currents (3.16) and (3.17). We will compute the quantum corrections (anomalies)
to the vacuum expectation values $\partial^\mu J_{\mu,5}$ and $D^\mu J_{\mu,5}$ in Sec. V.

IV. PERTURBATIVE DYNAMICS OF TRANSLATIONAL-INVARIANT $U(1)$ GAUGE THEORY

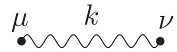
In this section, we will first present the Feynman rules of translational-invariant $U(1)$ gauge theory and determine eventually the Feynman integrals of one-loop quantum corrections corresponding to fermion and photon propagators and the three-point vertex function. The goal is to clarify the role played by $\alpha(p,q)$, that characterizes the translational-invariant star product (2.1) and (2.2). In particular, we will show that $\eta(p)$ from (2.53) does not appear in the internal loop integrals, and, similar to the case of scalar $\lambda\varphi^4$ theory discussed in [11], the divergence properties of the Feynman integrals are only affected by the antisymmetric function $\omega(p,q)$ which is given in two-dimensional noncommutative space by (2.31). To start, we present the Feynman rules corresponding to the translational-invariant $U(1)$ gauge theory described by (3.1).

Fermion Propagator:

$$S_{\alpha\beta}(p)=\left(\frac{i}{\gamma\cdot p-m}\right)_{\alpha\beta}e^{-2\eta_1(p)}, \tag{4.1}$$

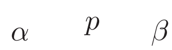
where $\gamma\cdot p\equiv\gamma_\mu p^\mu$.

Photon propagator (in Feynman gauge $\xi=1$):



$$D_{\mu\nu}(k)=-\frac{ig_{\mu\nu}}{k^2}e^{-2\eta_1(k)}. \tag{4.2}$$

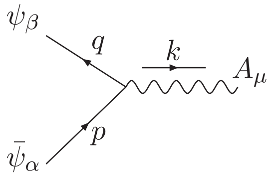
Ghost propagator:



$$G(p)=\frac{i}{p^2}e^{-2\eta_1(p)}. \tag{4.3}$$

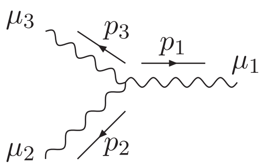
[4]See [14] for the arguments leading to the invariant and covariant currents in Moyal noncommutative $U(1)$ gauge theory.

$\bar{\psi}_\alpha A_\mu \psi_\beta$ *Vertex:*



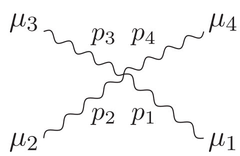
$$
\begin{aligned}
V_{\mu,\alpha\beta}(p,q;k) &= ig(2\pi)^4\delta^4(p-k-q)(\gamma_\mu)_{\alpha\beta}e^{\alpha(0,-q)}e^{\alpha(-q,-p)},\\
&= ig(2\pi)^4\delta^4(p-k-q)(\gamma_\mu)_{\alpha\beta}e^{[\eta(-p)+\eta(q)+\eta(k)]}\ e^{-i\omega(p,q)}.
\end{aligned}
\tag{4.4}
$$

$A_{\mu_1}A_{\mu_2}A_{\mu_3}$ *Vertex:*



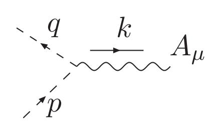
$$
\begin{aligned}
V_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) &= 2g(2\pi)^4\delta^4(p_1+p_2+p_3)\\
&\times e^{[\eta(p_1)+\eta(p_2)+\eta(p_3)]}\sin(\omega(p_1,p_2))\\
&\times[g_{\mu_1\mu_2}(p_1-p_2)_{\mu_3}+g_{\mu_1\mu_3}(p_3-p_1)_{\mu_2}+g_{\mu_3\mu_2}(p_2-p_3)_{\mu_1}].
\end{aligned}
\tag{4.5}
$$

$A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}$ *Vertex:*



$$
\begin{aligned}
V_{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) &= -4ig^2(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)e^{[\eta(p_1)+\eta(p_2)+\eta(p_3)+\eta(p_4)]}\\
&\times\{\sin[\omega(p_1,p_2)]\sin[\omega(p_3,p_4)]\,(g_{\mu_1\mu_3}g_{\mu_2\mu_4}-g_{\mu_1\mu_4}g_{\mu_2\mu_3})\\
&+\ \sin[\omega(p_1,p_3)]\sin[\omega(p_2,p_4)]\,(g_{\mu_1\mu_2}g_{\mu_3\mu_4}-g_{\mu_1\mu_4}g_{\mu_2\mu_3})\\
&+\ \sin[\omega(p_1,p_4)]\sin[\omega(p_2,p_3)]\,(g_{\mu_1\mu_2}g_{\mu_3\mu_4}-g_{\mu_1\mu_3}g_{\mu_2\mu_4})\}\,.
\end{aligned}
\tag{4.6}
$$

$\bar{c}cA_\mu$ *Vertex:*



$$
G_\mu(p,q;k) = 2ig(2\pi)^4\delta^4(p-k-q)p_\mu e^{[\eta(-p)+\eta(k)+\eta(q)]}\sin(\omega(p,q)). \tag{4.7}
$$

According to (2.53), $\eta(p)=\eta_1(p)+i\eta_2(p)$ and $\eta(-p)=\eta^*(p)$. Using the above Feynman rules, the one-loop corrections to fermion and photon propagators and three-point vertex can be computed. The Feynman integral of the one-loop fermion self-energy function [Fig. 1] is given by

$$
\begin{aligned}
-i\Sigma(k) &= -g^2\mu^\epsilon e^{2\eta_1(k)}\int\frac{d^dp}{(2\pi)^d}\\
&\times\frac{\gamma_\mu[\gamma\cdot(k+p)+m]\gamma^\mu}{p^2[(p+k)^2-m^2]}.
\end{aligned}
\tag{4.8}
$$

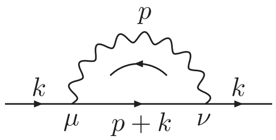


FIG. 1. One-loop fermion self-energy diagram.

In Fig. 2, the one-loop diagrams contributing to one-loop photon self-energy are presented. The corresponding Feynman integrals to Figs. 2(a)–2(d) are given by

$$\begin{aligned} i\Pi^{(a)}_{\mu\nu}(q) &= -g^2\mu^{\epsilon}e^{2\eta_1(q)}\int\frac{d^dp}{(2\pi)^d} \\ &\quad\times\frac{\mathrm{tr}((\gamma\cdot p-m)\gamma_\mu(\gamma\cdot(p+q)+m)\gamma_\nu)}{(p^2-m^2)[(p+q)^2-m^2]}, \\ i\Pi^{(b)}_{\mu\nu}(q) &= \frac{1}{2}(-4g^2\mu^{\epsilon})e^{2\eta_1(q)}\int\frac{d^dp}{(2\pi)^d}\frac{\sin^2(\omega(p,q))N_{\mu\nu}}{p^2(p+q)^2}, \\ i\Pi^{(c)}_{\mu\nu}(q) &= 4g^2\mu^{\epsilon}(d-1)g^{\mu\nu}e^{2\eta_1(q)} \\ &\quad\times\int\frac{d^dp}{(2\pi)^d}\frac{\sin^2(\omega(p,q))}{p^2}, \\ i\Pi^{(d)}_{\mu\nu}(q) &= 4g^2\mu^{\epsilon}e^{2\eta_1(q)}\int\frac{d^dp}{(2\pi)^d}\frac{p_\mu(p+q)_\nu\sin^2(\omega(p,q))}{p^2(p+q)^2}, \end{aligned} \tag{4.9}$$

where in $i\Pi^{(b)}_{\mu\nu}(q)$, $N_{\mu\nu}$ is defined by

$$\begin{aligned} N_{\mu\nu} \equiv{}& (g_{\mu\sigma}(-2q-p)_\rho + g_{\mu\rho}(q-p)_\sigma \\ &+ g_{\sigma\rho}(2p+q)_\mu)(g^{\sigma\rho}(-2p-q)_\nu \\ &+ \delta^{\sigma}{}_{\nu}(2q+p)^\rho + \delta_\nu{}^\rho(p-q)^\sigma). \end{aligned}$$

The one-loop diagrams contributing to the three-point vertex function are presented in Fig. 3, and the corresponding Feynman integrals to Figs. 3(a) and 3(b) are given by

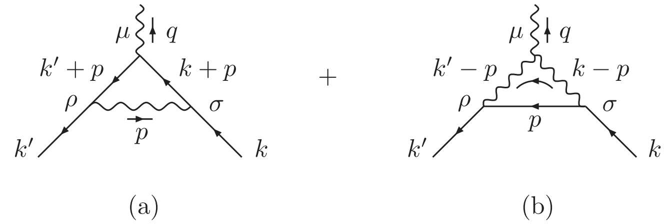


FIG. 3. Diagrams contributing to three-point function.

$$\begin{aligned} V^{(a)}_\mu(k,k';q=k-k') &= g^3\mu^{3\epsilon/2}e^{[\eta(-k)+\eta(k')+\eta(q)]}e^{-i\omega(k,k')} \\ &\quad\times\int\frac{d^dp}{(2\pi)^d}e^{2i\omega(p,q)}\bigg(\gamma_\sigma\frac{1}{[\gamma\cdot(p+k)-m]} \\ &\quad\times\gamma_\mu\frac{1}{[\gamma\cdot(p+k')-m]}\gamma^\sigma\frac{1}{k^2}\bigg), \\ V^{(b)}_\mu(k,k';q=k-k') & \\ &= g^3\mu^{3\epsilon/2}e^{[\eta(-k)+\eta(k')+\eta(q)]}e^{-i\omega(k,k')} \\ &\quad\times\int\frac{d^dp}{(2\pi)^d}(e^{2i[\omega(p,q)+\omega(k,k')]}-1) \\ &\quad\times\frac{\gamma^\sigma(\gamma\cdot p+m)\gamma^\rho}{(p^2-m^2)(k-p)^2(k'-p)^2} \\ &\quad\times(g_{\sigma\rho}(2p-k-k')_\mu + g_{\sigma\mu}(2k-k'-p)_\rho \\ &\quad+ g_{\mu\rho}(2k'-k-p)_\sigma). \end{aligned} \tag{4.10}$$

Comparing (4.8)–(4.10) with the one-loop integrals in commutative $U(1)$ gauge theory, there are phases depending on $\eta(p)$ and $\omega(p,q)$, that arise from the definition of the translational-invariant star product and the form of propagators and vertices. The appearance of momentum dependent phases in the Feynman integrals is indeed a characteristic feature for noncommutative quantum field theory. Here, similar to the ordinary Moyal noncommutative gauge theory, we will classify the Feynman integrals into two categories of planar and nonplanar integrals: The planar integrals involve phases that do not depend on the loop integration momenta. The phases appearing in

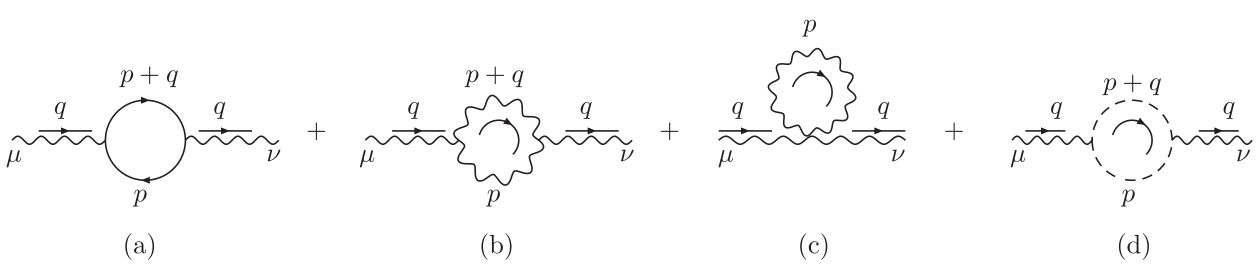


FIG. 2. One-loop photon self-energy diagram.

the nonplanar integrals, however, depend on loop momentum and cause the mixing of UV and IR divergencies in the loop integrations [6]. Comparing the one-loop integrals from (4.8)–(4.10) with their counterparts in the ordinary Moyal noncommutative $U(1)$ gauge theory from, e.g., [17,18], it turns out the loop integrations from (4.8)–(4.10) are the same as the loop integrations of the corresponding diagrams in the Moyal noncommutative case, and the additional phases involving $\eta(p)$ in (4.8)–(4.10) are only functions of the momenta of external legs. We conclude therefore that the UV and IR divergence properties of the above one-loop integrals are similar to the divergence properties of the integrals appearing in ordinary Moyal noncommutative gauge theory.

It is easy to see the cancellation of the phases involving $\eta(\ell) = \eta_1(\ell) + i\eta_2(\ell)$ in the loop integrations over ℓ. The point is that as can be seen from the expressions for the vertices (4.4)–(4.7), each vertex contains a sum $\sum_i \eta_2(p_i)$, with p_i the *outgoing* momenta of the legs of the vertex. Now since in a loop, each internal line of momentum ℓ_i from a vertex matches a single internal line from another vertex's leg with opposite momenta, and also because $\eta_2(\ell_i)$ is odd under $\ell_i \to -\ell_i$, all contributions of $\eta_2(\ell_i)$ with internal loop momenta ℓ_i cancel out for all i. On the other hand, although the contributions of even parity $\eta_1(\ell_i)$ from the vertices on both sides of an internal line of momentum ℓ_i add to $2\eta_1(\ell_i)$, the total contribution of $\eta_1(\ell_i)$ cancels due to the presence of an additional $-2\eta_1(\ell_i)$ from the propagators between two vertices. We are therefore left only with the antisymmetric $\omega(p, q)$ as a function of internal loop integration in the nonplanar loop integrals.

V. PLANAR AND NONPLANAR AXIAL ANOMALIES OF TRANSLATIONAL-INVARIANT $U(1)$ GAUGE THEORY

Quantum anomalies of the Moyal noncommutative gauge theory are widely discussed in the literature [13–15]. In this section, the axial anomalies of the translational-invariant gauge theory corresponding to the covariant current $J_{\mu,5}$ from (3.16) and the invariant current $j_{\mu,5}$ from (3.17) will be determined. We will show, that whereas the axial anomaly corresponding to the covariant current arises from planar integrals and is given by a star modification of the axial anomaly of commutative $U(1)$ gauge theory, the axial anomaly corresponding to the invariant current is affected by the above mentioned UV/IR mixing that arises from the phase factor $\omega(p, q)$ in the nonplanar Feynman loop integrals. The remaining phases involving the functions $\eta_1(p)$ and $\xi(p, q)$ are independent of the loop integration momentum and do not affect the UV and IR behavior of the Feynman integrals. Note that, apart from the appearance of these additional phase factors, the situation is similar to the case of Moyal noncommutativity (see, e.g., in [13,14]).

A. Planar axial anomaly

As we have noted in the Sec. II, the classical equations of motion of the translational-invariant gauge theory (3.12) lead, in the chiral limit $m \to 0$, to the classical continuity equation $D_\mu J^{\mu,5} = 0$, where $D_\mu = \partial_\mu + ig[A_\mu, \cdot]_\star$. It is the purpose of this section to determine the quantum correction to this conservation law by computing the vacuum expectation value $\langle D_\mu J^{\mu,5}\rangle$. To do this, let us consider the following three-point function of one axial vector current and two vector currents:

$$\Gamma_P^{\mu\lambda\nu}(x, y, z) = \langle T(J_5^\mu(x) J^\lambda(y) J^\nu(z))\rangle, \qquad (5.1)$$

and determine $\partial_\mu^x \Gamma_P^{\mu\lambda\nu}(x, y, z)$. The vector currents appearing in (5.1) are given in (3.11). Expressing the currents in terms of fermionic fields and performing the corresponding Wick contractions, it can be shown that two triangle diagrams contribute to (5.1) (see Fig. 4).

The corresponding Feynman integrals are given by

$$\begin{aligned}\Gamma_P^{\mu\lambda\nu}(x, y, z) = &\int \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} e^{-i(k_2+k_3)x} e^{ik_2 y} e^{ik_3 z} \\ &\times \int \frac{d^d \ell}{(2\pi)^D} [\mathrm{Tr}(\gamma^\mu \gamma^5 D^{-1}(\ell + k_3)\gamma^\nu \\ &\times D^{-1}(\ell)\gamma^\lambda D^{-1}(\ell - k_2)) F_a(k_2, k_3) \\ &+ ((k_2, \lambda) \leftrightarrow (k_3, \nu)) F_b(k_2, k_3)], \qquad (5.2)\end{aligned}$$

where $D(\ell) \equiv \gamma \cdot \ell - m$, and

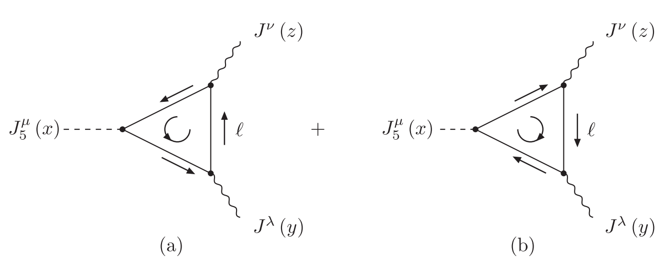


FIG. 4. Triangle diagrams for the anomaly in the axial vector current $J^{\mu(5)}(x)$, indicated by the dashed line.

$$F_a(k_2,k_3) = e^{\alpha(k_2+k_3,\ell+k_3)+\alpha(-k_2,\ell-k_2)+\alpha(-k_3,\ell)-2[\eta_1(\ell-k_2)+\eta_1(\ell)+\eta_1(\ell+k_3)]},$$
$$= \exp(-[\eta_1(k_2+k_3)+\eta_1(k_2)+\eta_1(k_3)] + i[\xi(-k_2,k_3)+\omega(k_2,k_3)]),$$
$$= e^{-2[\eta_1(k_2)+\eta_1(k_3)]+\alpha(k_2+k_3,k_3)}, \qquad (5.3)$$

arises from the contribution of Fig. 4(a). The phase factor appearing on the first line of (5.3) is simplified using the definition of $\alpha(p,q)$ in terms of $\eta_1(p)$, $\xi(p,q)$ and $\omega(p,q)$, as well as the properties of $\omega(p,q)$ and $\xi(p,q)$ (see the Appendix for a list of these properties). The contribution of Fig. 4(b) is given, as is denoted in the second term of (5.2), by replacing $k_2 \leftrightarrow k_3$ as well as $\lambda \leftrightarrow \nu$. The phase factor corresponding to Fig. 4(b), i.e., $F_b(k_2,k_3)$ can be read from (5.3) by replacing k_2 with k_3 and vice versa. As it turns out, both phase factors are independent of the loop momentum ℓ. The Feynman integral appearing in (5.2) are therefore, planar. The planar anomaly is given by $\partial^x_\mu \Gamma^{\mu\lambda\nu}_P$. Taking the partial derivative with respect to x^μ from $\Gamma^{\mu\lambda\nu}_P$ in (5.2) and following the same steps as is described in detail in [13], we arrive first at

$$\partial^x_\mu \Gamma^{\mu\lambda\nu}_P(x,y,z) = -\frac{i}{4\pi^2}\epsilon^{\lambda\nu\alpha\beta}\int \frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d} e^{-i(k_2+k_3)x}e^{ik_2y}e^{ik_3z}e^{-2[\eta_1(k_2)+\eta_1(k_3)]}k_{2\alpha}k_{3\beta}(e^{\alpha(k_2+k_3,k_2)}+e^{\alpha(k_2+k_3,k_3)}). \qquad (5.4)$$

Using now the definition of $\langle J^\mu_5(x)\rangle$ in terms of the three-point function $\Gamma^{\mu\lambda\nu}$,

$$\langle J^\mu_5(x)\rangle = \frac{1}{2}\int d^dyd^dzA_\lambda(y)\star\Gamma^{\mu\lambda\nu}_P(x,y,z)\star A_\nu(z), \qquad (5.5)$$

and the definition of the translational-invariant star product (2.1) and (2.2), we get

$$\langle \partial_\mu J^{\mu,5}(x)\rangle = \frac{i}{4\pi^2}\epsilon^{\lambda\nu\alpha\beta}\partial_\alpha A_\lambda(x)\star\partial_\beta A_\nu(x). \qquad (5.6)$$

After considering the contribution of the square and pentagon diagrams Figs. 5(b) and 5(c), we arrive at

$$\langle D_\mu J^{\mu,5}(x)\rangle = \frac{i}{16\pi^2}F_{\mu\nu}(x)\star\tilde{F}^{\mu\nu}(x), \qquad (5.7)$$

where $\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. Thus, similar to the case of Moyal noncommutativity, the planar (covariant) anomaly corresponding to the covariant current $J_{\mu,5}(x)$ of translational-invariant $U(1)$ gauge theory is given by a star modification of the axial anomaly of commutative $U(1)$ gauge theory.

B. Nonplanar axial anomaly

Let us consider the invariant current $j_{\mu,5}$ from (3.17). It satisfies the classical conservation law $\partial_\mu j^\mu_5 = 0$. This can be shown using the equations of motion of the translational-invariant $U(1)$ gauge theory in the chiral limit, (3.12). The quantum anomaly corresponding to this current can be computed from the definition of $j_{\mu,5}$ in terms of the three-point function $\Gamma^{\mu\lambda\nu}_{NP}$

$$\langle j^\mu_5(x)\rangle = \frac{1}{2}\int d^Dyd^DzA_\lambda(y)\star\Gamma^{\mu\lambda\nu}_{NP}(x,y,z)\star A_\nu(z), \qquad (5.8)$$

where in contrast to (5.1),

$$\Gamma^{\mu\lambda\nu}_{NP}(x,y,z) = \langle T(j^\mu_5(x)J^\lambda(y)J^\nu(z))\rangle, \qquad (5.9)$$

is a time ordered product of one invariant axial current and two covariant vector currents. Similar to the previous case, $\Gamma^{\mu\lambda\nu}_{NP}$ receives a contribution from two triangle diagrams from Fig. 4, where J^μ_5 is replaced by j^μ_5. It is given by

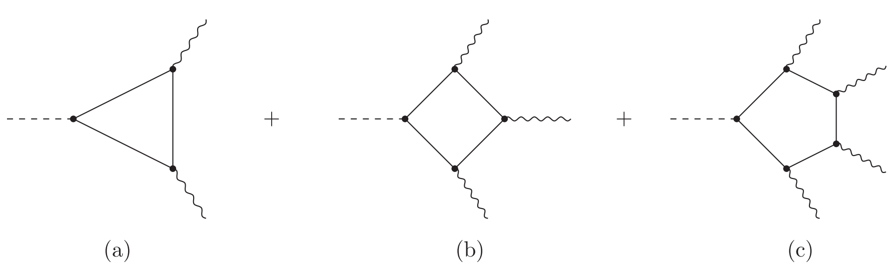


FIG. 5. Triangle, square, and pentagon diagrams contributing to the anomaly in the axial vector current, which is indicated by the dashed line.

$$\Gamma_{NP}^{\mu\lambda\nu}(x,y,z)=\int\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}e^{-i(k_2+k_3)x}e^{ik_2y}e^{ik_3z}\int\frac{d^d\ell}{(2\pi)^d}[\mathrm{Tr}(\gamma^\mu\gamma^5D^{-1}(\ell+k_3)\gamma^\nu D^{-1}(\ell)\gamma^\lambda D^{-1}(\ell-k_2))F_a(\ell;k_2,k_3)$$
$$+((k_2,\lambda)\leftrightarrow(k_3,\nu))F_b(\ell;k_2,k_3)], \tag{5.10}$$

where the first (second) term is the contribution from Fig. 4(a) (4(b)). The phase factor $F_a(\ell,k_2,k_3)$ in (5.10) is given by

$$F_a(\ell;k_2,k_3)=e^{\alpha(k_2+k_3,k_2-\ell)+\alpha(-k_2,\ell-k_2)+\alpha(-k_3,\ell)-2[\eta_1(\ell+k_3)+\eta_1(\ell-k_2)+\eta_1(\ell)]},$$
$$F_b(\ell;k_2,k_3)=e^{\alpha(k_2+k_3,k_3-\ell)+\alpha(-k_3,\ell-k_3)+\alpha(-k_2,\ell)-2[\eta_1(\ell+k_2)+\eta_1(\ell-k_3)+\eta_1(\ell)]}. \tag{5.11}$$

After simple algebraic manipulations, where the definition of $\alpha(p,q)$ in terms of $\eta_1(p)$, $\xi(p,q)$ and $\omega(p,q)$, as well as the properties of $\omega(p,q)$ and $\xi(p,q)$ are used,[5] it can be shown that $F_{a/b}(\ell;k_2,k_3)$ can be separated into an ℓ-independent and an ℓ-dependent part

$$F_a(\ell;k_2,k_3)=\exp(-[\eta_1(k_2+k_3)+\eta_1(k_2)+\eta_1(k_3)]+i\xi(-k_2,k_3)-i\omega(k_2,k_3)+2i[\omega(\ell,k_2)+\omega(\ell,k_3)]),$$
$$F_b(\ell;k_2,k_3)=\exp(-[\eta_1(k_2+k_3)+\eta_1(k_2)+\eta_1(k_3)]+i\xi(-k_3,k_2)+i\omega(k_2,k_3)+2i[\omega(\ell,k_2)+\omega(\ell,k_3)]). \tag{5.12}$$

This is in contrast to the ℓ-independent (planar) phase factor that appears in (5.2). To add the contributions of both graphs, we will use the fact that $\xi(-k_2,k_3)=\xi(-k_3,k_2)$ from (A11) and will separate the ℓ-dependent and ℓ-independent part of $F_{a/b}(\ell;k_2,k_3)$ appropriately. Using further the definition of $\omega(p,q)$ from (2.31), and building $\partial_\mu\Gamma_{NP}^{\mu\lambda\nu}$, we arrive at (see also [14] for notations)

$$\partial_\mu^x\Gamma_{NP}^{\mu\lambda\nu}=\int\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}e^{-ik_1x}e^{ik_2y}e^{ik_3z}e^{-[\eta_1(k_1)+\eta_1(k_2)+\eta_1(k_3)]+i\xi(-k_2,k_3)}[\mathcal{A}^{\lambda\nu}(k_2,k_3)+\mathcal{R}^{\lambda\nu}(k_2,k_3)], \tag{5.13}$$

with $k_1\equiv k_2+k_3$. The anomalous part of $\partial_\mu^x\Gamma_{NP}^{\mu\lambda\nu}$ is given by

$$\mathcal{A}^{\lambda\nu}(k_2,k_3)\equiv-2i\int\frac{d^d\ell}{(2\pi)^d}[\mathrm{Tr}(D^{-1}(\ell-k_2)\gamma^5\ell_\perp\cdot\gamma^\perp D^{-1}(\ell+k_3)\times\gamma^\lambda D^{-1}(\ell)\gamma^\nu)G_a(\ell;k_2,k_3)+((k_2,\lambda)\leftrightarrow(k_3,\nu))G_b(\ell;k_2,k_3)], \tag{5.14}$$

with $\ell_\perp=(\ell_4,\cdots,\ell_{d-1})$, and[6]

$$G_a(\ell;k_2,k_3)=e^{-ik_2\wedge k_3+2i\ell\wedge(k_2+k_3)},$$
$$G_b(\ell;k_2,k_3)=e^{ik_2\wedge k_3+2i\ell\wedge(k_2+k_3)}, \tag{5.15}$$

and the rest term of $\partial_\mu^x\Gamma_{NP}^{\mu\lambda\nu}$ by

$$\mathcal{R}^{\lambda\nu}(k_2,k_3)\equiv i\int\frac{d^d\ell}{(2\pi)^d}[\mathrm{Tr}(D^{-1}(\ell-k_2)\gamma^5\gamma^\nu D^{-1}(\ell)\gamma^\lambda+\gamma^5D^{-1}(\ell+k_3)\gamma^\nu D^{-1}(\ell)\gamma^\lambda)G_a(\ell;k_2,k_3)+((k_2,\lambda)\leftrightarrow(k_3,\nu))G_b(\ell;k_2,k_3)]. \tag{5.16}$$

After performing an appropriate shift of the integration variable, the rest term can be shown to vanish and we are therefore left with the anomalous part, which is the same as appears also in [14] for Moyal noncommutative $U(1)$ case. Simple algebraic manipulations lead to

$$\mathcal{A}^{\lambda\nu}(k_2,k_3)=-16\varepsilon^{\lambda\nu\alpha\beta}k_{2\alpha}k_{3\beta}\int_0^1d\beta_1\int_0^{1-\beta_1}d\beta_2\int\frac{d^d\ell}{(2\pi)^d}\times\frac{\ell_\perp^2F_a(\ell+k_2\beta_1-k_3\beta_2;k_2,k_3)}{(\ell^2+\Delta)^3}, \tag{5.17}$$

with $\Delta\equiv k_3^2\beta_1(1-\beta_1)+k_3^2\beta_2(1-\beta_2)+2k_2k_3\beta_1\beta_2$. Following the same steps as is described in detail in [14], $\mathcal{A}^{\lambda\nu}$ is given by

$$\mathcal{A}^{\lambda\nu}(k_2,k_3)=-\frac{2}{\pi^2}\varepsilon^{\lambda\nu\alpha\beta}k_{2\alpha}k_{3\beta}\int_0^1d\beta_1\int_0^{1-\beta_1}d\beta_2\times\cos[k_2\wedge k_3(1-2\beta_1-2\beta_2)]\frac{1}{\ln\Lambda^2}\times\left(\mathcal{E}_1(k_1,\Delta;\Lambda_{\mathrm{eff}})-\frac{k_1\circ k_1}{8}\mathcal{E}_2(k_1,\Delta;\Lambda_{\mathrm{eff}})\right), \tag{5.18}$$

where $q\circ q\equiv-q_\mu\theta^{\mu\nu}\theta_{\nu\rho}q^\rho$ and $\frac{1}{\Lambda_{\mathrm{eff}}^2}\equiv\frac{1}{\Lambda^2}+\frac{k_1\circ k_1}{4}$. Moreover, we have used

[5] In the Appendix, we have summarized useful relations for $\omega(p,q)$ and $\xi(p,q)$.

[6] Here, we have restricted ourselves to noncommutativity between two space coordinates x_1 and x_2.

$$\mathcal{E}_1(k_1,\Delta;\Lambda_{\rm eff}) = \int_0^\infty \frac{d\rho}{\rho}\exp\left(-\rho\Delta - \frac{1}{\Lambda_{\rm eff^2}\rho}\right) = 2K_0\left(2\sqrt{\frac{\Delta}{\Lambda_{\rm eff}^2}}\right) \overset{\Lambda_{\rm eff}\to\infty}{\simeq} \ln\frac{\Lambda_{\rm eff}^2}{\Delta},$$

$$\mathcal{E}_2(k_1,\Delta;\Lambda_{\rm eff}) = \int_0^\infty \frac{d\rho}{\rho^2}\exp\left(-\rho\Delta - \frac{1}{\Lambda_{\rm eff^2}\rho}\right) = 2\sqrt{\Delta\Lambda_{\rm eff}^2}K_1\left(2\sqrt{\frac{\Delta}{\Lambda_{\rm eff}^2}}\right) \overset{\Lambda_{\rm eff}\to\infty}{\simeq} \Lambda_{\rm eff}^2 - \Delta\ln\frac{\Lambda_{\rm eff}^2}{\Delta}. \tag{5.19}$$

Plugging $\mathcal{A}^{\lambda\nu}$ back in (5.13), and using

$$\langle\partial_\mu j_5^\mu(x)\rangle = \frac{1}{2}\int d^dy d^dz A_\lambda(y)\star\partial_\mu^x\Gamma_{NP}^{\mu\lambda\nu}(x,y,z)\star A_\nu(z), \tag{5.20}$$

we arrive, after performing the integration over y and z and inserting $\mathcal{E}_i(k_1,\Delta;\theta,\Lambda_{\rm eff})$, $i=1,2$, from (5.19), at

$$\begin{aligned}\langle\partial_\mu j_5^\mu(x)\rangle = -\frac{1}{\pi}\varepsilon^{\lambda\nu\alpha\beta}\int\frac{d^dk_2}{(2\pi)^d}k_{2\alpha}\tilde{A}_\lambda(k_2)e^{-ik_2x} &\times\int\frac{d^dk_3}{(2\pi)^d}k_{3\beta}\tilde{A}_\nu(k_3)e^{-ik_3x}e^{\rho(k_2,k_3)}\int_0^1 d\beta_1 \\ &\times\int_0^{1-\beta_1}d\beta_2\cos[k_2\wedge k_3(1-2\beta_1-2\beta_2)] \\ &\times\frac{1}{\ln\Lambda^2}\left[\left(\ln\frac{1}{\frac{1}{\Lambda^2}+\frac{(k_1\circ k_1)}{4}}-\ln\Delta\right)-\frac{(k_1\circ k_1)}{8}\right. \\ &\left.\times\left(\frac{1}{\frac{1}{\Lambda^2}+\frac{(k_1\circ k_1)}{4}}-\Delta\ln\frac{1}{\frac{1}{\Lambda^2}+\frac{(k_1\circ k_1)}{4}}+\Delta\ln\Delta\right)\right],\end{aligned} \tag{5.21}$$

where the exponent $\rho(k_2,k_3)$ on the first line is defined by

$$\rho(k_2,k_3)\equiv-\eta_1(k_2+k_3)+i\xi(-k_2,k_3). \tag{5.22}$$

Comparing to the Moyal noncommutative case an additional factor $e^{\rho(k_2,k_3)}$ appears on the first line of (5.21). The UV/IR behavior of the remaining expression is the same as in the Moyal case. In [14], we have shown that while the above nonplanar anomaly vanishes in the UV limit, $\frac{k_1\circ k_1}{4}\gg\frac{1}{\Lambda^2}$, a finite anomaly arises due to the IR singularity for $\frac{k_1\circ k_1}{4}\ll\frac{1}{\Lambda^2}$. In this limit, all terms proportional to $k_1\circ k_1$ in (5.21) can be neglected, and the finite anomaly arises from the factor $\frac{1}{\ln\Lambda^2}\ln\frac{1}{\frac{1}{\Lambda^2}}\overset{\Lambda\to\infty}{\longrightarrow}1$ on the third line of (5.21). After integrating over β_1 and β_2, it is then given by

$$\begin{aligned}\langle\partial_\mu j_5^\mu(x)\rangle = -\frac{1}{2\pi^2}\varepsilon^{\lambda\nu\alpha\beta}\int_{(k_1\circ k_1/4)\ll(1/\Lambda^2)}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d} &\times\partial_\alpha\tilde{A}_\lambda(k_2)e^{-ik_2x}e^{\rho(k_2,k_3)}\frac{\sin(k_2\wedge k_3)}{k_2\wedge k_3} \\ &\times\partial_\beta\tilde{A}_\nu(k_3)e^{-ik_3x}.\end{aligned} \tag{5.23}$$

Defining, similar to the Moyal noncommutative case [14], a new generalized star product

$$f(x)\star' g(x)\equiv\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\times\tilde{f}(p)e^{\rho(p,q)}\frac{\sin(p\wedge q)}{p\wedge q}\tilde{g}(q)e^{-i(p+q)x}, \tag{5.24}$$

with the symmetric function $\rho(p,q)\equiv-\eta_1(p+q)+i\xi(-p,q)$ and the antisymmetric construction $p\wedge q$ defined in (2.31), the resulting nonplanar anomaly, that arises due to the UV/IR mixing is then given by

$$\langle\partial_\mu j_5^\mu(x)\rangle = -\frac{1}{2\pi^2}\varepsilon^{\lambda\nu\alpha\beta}\int_{(k_1\circ k_1/4)\ll(1/\Lambda^2)}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\times F_{\alpha\lambda}(k_2)e^{-ik_2x}\star' F_{\beta\nu}(k_3)e^{-ik_3x}. \tag{5.25}$$

Here, the contributions from square and pentagon diagrams in Fig. 5 are also added to (5.23).

VI. CONCLUDING REMARKS

According to its definition (2.1) and (2.2), the translational-invariant noncommutative star product is characterized by a function $\alpha(p,q)$, whose dependence on the momenta p and q is mainly restricted by the associativity condition on this product. In the first part of this paper, we have determined the structure of $\alpha(p,q)$, for a general noncommutative case, in terms of an arbitrary real even function $\eta_1(p)$ and two real antisymmetric functions $\xi(p,q)$ and $\omega(p,q)$ that appear in the imaginary part of $\alpha(p,q)$ [see (2.14) for the real part and (2.19) for the imaginary part of $\alpha(p,q)$]. Focusing then on a special two-dimensional noncommutative space, we have derived the general form of $\xi(p,q)$ and $\omega(p,q)$ from a recursive relation arising from the associativity. We have shown that $\omega(p,q)$, as an even antisymmetric function, is given by $\omega(p,q)=p\wedge q$, and $\xi(p,q)$, as an odd antisymmetric function, is given in terms of an arbitrary real odd function $\eta_2(p)$ [see (2.52)]. Combining $\xi(p,q)$ from (2.52) with the real part of $\alpha(p,q)$ appearing in (2.14), we have defined an arbitrary function $\eta(p)=\eta_1(p)+i\eta_2(p)$, with $\eta_1(p)$ an arbitrary even and $\eta_2(p)$ an arbitrary odd function of p, satisfying $\eta_1(0)=\eta_2(0)=0$. The characteristic function $\alpha(p,q)$ is then expressed alternatively in terms of $\eta(p)$ and $\omega(p,q)$, i.e., $\alpha(p,q)=\sigma(p,q)+i\omega(p,q)$. Note that $\sigma(p,q)=\eta(q)-\eta(p)+\eta(p-q)$, from (2.54), and $\omega(p,q)=p\wedge q$ are two distinct and unique solutions for the associativity relation $\alpha(p,q)+\alpha(q,r)=\alpha(p,r)+\alpha(p-r,q-r)$, where $\alpha(p,q)$ is a generic function of two-dimensional momenta p and q. It is interesting to look for the solutions of this characteristic relation for d-dimensional vectors p and q for higher dimensions.

In the second part of the paper, we have explored the effect of functions $\eta(p)$ and $\omega(p,q)$ on the divergence properties of Feynman integrals appearing in the noncommutative $U(1)$ gauge theory including the translational-invariant star product. At the one-loop level, it turned out that $\eta(p)$ appears only as a function of external loop momenta, and only $\omega(p, q)$ is responsible for the UV/IR mixing that appears also in the ordinary Moyal noncommutative field theory. Using the algebraic properties of $\eta(p)$, however, it was shown that $\eta(p)$ cancels out of all internal loop integrations and appears only as a function of external momenta. It cannot therefore affect the divergence properties of the Feynman integrals. The general topological arguments leading to this simple but remarkable result is described in the last paragraph of Sec. IV. Our findings confirm the fact indicated in [5], that the UV behavior of noncommutative theories is in general described by the canonical commutation relation between the coordinates (1.1), which is unchanged between the translational-invariant product and the Moyal as well as Wick-Voros products considered in [5,11].

Finally, the planar and nonplanar anomalies of the above gauge theory were also discussed. As it turned out the nonplanar anomaly, once nonvanishing, is given, in contrast of nonplanar anomaly of ordinary Moyal noncommutativity, as a function of a new generalized star product including the symmetric function $\rho(p, q) = -\eta_1(p + q) + i\xi(-p, q)$ *and* the antisymmetric combination $\omega(p, q) = p \wedge q$. The planar anomaly, however, is given, as in the ordinary Moyal noncommutativity, by the star modification of the well-known Adler-Bell-Jackiw axial anomaly.

In the case of the Moyal product, the noncommutative gauge theory appears in the decoupling limit of string theory, on a brane, where $\omega(p, q)$ is related to the background bulk antisymmetric field B. Here, in the general noncommutative gauge theory, we have, in addition, the function $\eta(p)$ appearing as a profile function for each field in the momentum representation. It is intriguing to explore the string theoretical origin of this factor.

ACKNOWLEDGMENTS

We thank M. Alishahiha for bringing Ref. [11] to our attention, and H. Arfaei for useful discussions.

APPENDIX: USEFUL RELATIONS FOR $\omega(p, q)$ AND $\xi(p, q)$

The antisymmetric functions $\omega(p, q)$ and $\xi(p, q)$ that appear in the imaginary part of $\alpha(p, q)$ satisfy the following relations:

$$\omega(p, p) = \omega(0, p) = \omega(p, 0) = 0, \tag{1}$$

$$\omega(p, q) = -\omega(q, p), \tag{2}$$

$$\omega(-p, -q) = \omega(p, q), \tag{3}$$

$$\omega(p - q, p) = \omega(p, q), \tag{4}$$

$$\omega(-q, p) = \omega(p, q), \tag{5}$$

$$\omega(p - r, q - r) = \omega(p, q) + \omega(q, r) - \omega(p, r), \tag{6}$$

as well as

$$\xi(p, p) = \xi(0, p) = \xi(p, 0) = 0, \tag{7}$$

$$\xi(p, q) = -\xi(q, p), \tag{8}$$

$$\xi(-p, -q) = -\xi(p, q), \tag{9}$$

$$\xi(p - q, p) = -\xi(p, q), \tag{10}$$

$$\xi(-q, p) = \xi(-p, q), \tag{11}$$

$$\xi(p - r, q - r) = \xi(p, q) + \xi(q, r) - \xi(p, r). \tag{12}$$

[1] A. Connes, M. R. Douglas, and A. S. Schwarz, J. High Energy Phys. 02 (1998) 003.

[2] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001); M. R. Douglas and C. M. Hull, J. High Energy Phys. 02 (1998) 008; F. Ardalan, H. Arfaei, and M. M. Sheikh-Jabbari, J. High Energy Phys. 02 (1999) 016; Nucl. Phys. **B576**, 578 (2000); Y. K. Cheung and M. Krogh, Nucl. Phys. **B528**, 185 (1998); C. S. Chu and P. M. Ho, Nucl. Phys. **B550**, 151 (1999); V. Schomerus, J. High Energy Phys. 06 (1999) 030; N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032.

[3] H. J. Grönewald, Physica (Amsterdam) **12**, 405 (1946); J. E. Moyal, Proc. Cambridge Philos. Soc. **45**, 99 (1949).

[4] A. Voros, Phys. Rev. A **40**, 6814 (1989); M. Bordemann and S. Waldmann, arXiv:q-alg/9605012; arXiv:q-alg/9611004.

[5] S. Galluccio, F. Lizzi, and P. Vitale, Phys. Rev. D **78**, 085007 (2008).

[6] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, J. High Energy Phys. 02 (2000) 020.

[7] V. G. Drinfeld, Alg. Anal. **1**, 114 (1989); R. Oeckl, Nucl. Phys. **B581**, 559 (2000).

[8] M. Chaichian, P. P. Kulish, K. Nishijima *et al.*, Phys. Lett. B **604**, 98 (2004); J. Wess, arXiv:hep-th/0408080; P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp, and J. Wess, Classical Quantum Gravity **22**, 3511 (2005).

[9] D. V. Vassilevich, Mod. Phys. Lett. A **21**, 1279 (2006); A. P. Balachandran, G. Mangano, A. Pinzul, and S. Vaidya, Int. J. Mod. Phys. A **21**, 3111 (2006); A. P. Balachandran, A. Pinzul, and B. A. Qureshi, Phys. Lett. B **634**, 434 (2006); A. Tureanu, Phys. Lett. B **638**, 296 (2006); M. Chaichian and A. Tureanu, Phys. Lett. B **637**, 199 (2006); J.-G. Bu, H.-C. Kim, Y. Lee *et al.*, Phys. Rev. D **73**, 125001 (2006); J. Zahn, Phys. Rev. D **73**, 105005 (2006); G. Fiore and J. Wess, Phys. Rev. D **75**, 105022 (2007); G. Fiore, arXiv:0705.1120; M. Chaichian, K. Nishijima, T. Salminen *et al.*, J. High Energy Phys. 06 (2008) 078.

[10] J. Lukierski and M. Woronowicz, Proc. of 2nd Corfu School on Quantum Gravity and Quantum Geometry, 2009, Corfu [special issue of Gen. Relativ. Gravit. (2010)]; M. Daszkiewicz, J. Lukierski, and M. Woronowicz, J. Phys. A **42**, 355201 (2009); Talk presented at Supersymmetry and Quantum Supersymmetry 2007 (SQS'07) Conference (Dubna, 2007) and IVth Central European Seminar ''Commutative and Noncommutative Quantum Fields'' (Vienna, 2007) [To be published in the proceedings of SQS'07 (2008]; Phys. Rev. D **77**, 105007 (2008).

[11] S. Galluccio, F. Lizzi, and P. Vitale, J. High Energy Phys. 09 (2009) 054.

[12] A. Tanasa and P. Vitale, Phys. Rev. D **81**, 065008 (2010); A. Tanasa, SIGMAP bulletin **6**, 047 (2010); S. Galluccio, arXiv:1004.4655.

[13] F. Ardalan and N. Sadooghi, Int. J. Mod. Phys. A **16**, 3151 (2001).

[14] F. Ardalan and N. Sadooghi, Int. J. Mod. Phys. A **17**, 123 (2002).

[15] L. Bonora, M. Schnabl, and A. Tomasiello, Phys. Lett. B **485**, 311 (2000); C. P. Martin, J. Phys. A **34**, 9037 (2001); K. A. Intriligator and J. Kumar, Nucl. Phys. **B620**, 315 (2002); R. Banerjee and S. Ghosh, Phys. Lett. B **533**, 162 (2002); A. Armoni, E. Lopez, and S. Theisen, J. High Energy Phys. 06 (2002) 050; R. Banerjee, Int. J. Mod. Phys. A **19**, 613 (2004); R. Banerjee and K. Kumar, Phys. Rev. D **71**, 045013 (2005); **72**, 085012 (2005); F. Ardalan, H. Arfaei, and N. Sadooghi, Int. J. Mod. Phys. A **21**, 4161 (2006).

[16] M. R. Garousi, Nucl. Phys. **B579**, 209 (2000); H. Liu and J. Michelson, Nucl. Phys. **B614**, 279 (2001); **B614**, 330 (2001); H. Liu and J. Michelson, Nucl. Phys. **B615**, 169 (2001); Phys. Lett. B **518**, 143 (2001).

[17] M. Hayakawa, arXiv:hep-th/9912167; Phys. Lett. B **478**, 394 (2000).

[18] t. B. Nguyen, arXiv:hep-th/0301084.