Conserved charges for black holes in Einstein-Gauss-Bonnet gravity coupled to nonlinear electrodynamics in AdS space

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Motivated by possible applications within the framework of anti-de Sitter gravity/conformal field theory correspondence, charged black holes with AdS asymptotics, which are solutions to Einstein-Gauss-Bonnet gravity in D dimensions, and whose electric field is described by nonlinear electrodynamics are studied. For a topological static black hole ansatz, the field equations are exactly solved in terms of the electromagnetic stress tensor for an arbitrary nonlinear electrodynamic Lagrangian in any dimension D and for arbitrary positive values of Gauss-Bonnet coupling. In particular, this procedure reproduces the black hole metric in Born-Infeld and conformally invariant electrodynamics previously found in the literature. Altogether, it extends to $D > 4$ the four-dimensional solution obtained by Soleng in logarithmic electrodynamics, which comes from vacuum polarization effects. Falloff conditions for the electromagnetic field that ensure the finiteness of the electric charge are also discussed. The black hole mass and vacuum energy as conserved quantities associated to an asymptotic timelike Killing vector are computed using a background-independent regularization of the gravitational action based on the addition of counterterms which are a given polynomial in the intrinsic and extrinsic curvatures.

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I. INTRODUCTION

Gauge theories which are described by a nonlinear action for Abelian or non-Abelian fields have become standard in the context of superstring theory. Indeed, it was proposed in Ref. [1] that all order loop corrections to gravity should be summed up as a Born-Infeld (BI) type Lagrangian [2]. Furthermore, the dynamics of D-branes is given in terms of a non-Abelian Born-Infeld action [3].

On the other hand, coupling nonlinear electrodynamics (NED) to gravity has been considered in the literature as a plausible mechanism to obtain regular black hole solutions (see, for instance, Ref. [4]). In this respect, the metric for static, spherically symmetric black holes for the BI theory minimally coupled to Einstein gravity was derived in a number of papers [5,6]. Other gravitating NED models supporting electrically charged black hole solutions have been also investigated, e.g., in Ref. [7] for the Euler-Heisenberg effective Lagrangian of QED, in Ref. [8] for a logarithmic Lagrangian, and in Ref. [9] for a Lagrangian defined as powers of the Maxwell term. In the same spirit, as an example of lower-dimensional models, it is worth mentioning the study of black holes generated by Coulomb-like fields in (2 + 1) dimensions [10], and a similar treatment which includes torsion in Ref. [11].

Within the framework of anti-de Sitter/conformal field theory (AdS/CFT) correspondence, higher-derivative corrections to either gravitational or electromagnetic action in AdS space are expected to modify the dynamics of the strongly coupled dual theory. In particular, in hydrodynamic models, the addition of R^2 terms changes the ratio of shear viscosity over entropy density [12], violating the universal bound $1/4\pi$ proposed in Ref. [13]. In turn, it has been proven that higher-derivative terms for Abelian fields in the form of NED do not affect this ratio [14] (for hydrodynamic models dual to R-charged black holes, see, e.g., Ref. [15]). Also, in applications of the AdS/CFT conjecture to high T_c superconductivity, higher-curvature terms violate a universal relation between the critical temperature of the superconductor and its energy gap [16,17]. While the Gauss-Bonnet term makes the condensation easier, the inclusion of Born-Infeld electrodynamics produces the opposite effect [18].

Motivated by the recent results mentioned above, we study black hole solutions in Einstein-Gauss-Bonnet gravity with negative cosmological constant coupled to an arbitrary NED theory. As it is required in the context of AdS/CFT, we provide definitions for the conserved quantities following a background-independent regularization procedure.

II. ACTION AND EQUATIONS OF MOTION

We consider a fully-interacting theory of gravity minimally coupled to nonlinear electrodynamics in a

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D-dimensional manifold $\mathcal{M}$, which comes from the action

$$I_0 = \int_{\mathcal{M}} d^D x \sqrt{-g}\, \mathcal{L}_0 = I_{\rm grav} + I_{\rm NED}. \tag{1}$$

The pure gravity part of the bulk action with the metric $g_{\mu\nu}(x)$ as the dynamic field is given by

$$I_{\rm grav} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^D x \sqrt{-g}\, [R - 2\Lambda + \alpha(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma})], \tag{2}$$

which contains the Einstein-Hilbert (EH) action (linear in the curvature of spacetime), a cosmological term, and a quadratic-curvature correction given by the Gauss-Bonnet (GB) term. The cosmological constant Λ is expressed in terms of the AdS radius ℓ as $\Lambda = -(D-1)(D-2)/2\ell^2$ and G is the gravitational constant. The GB coupling constant α is of dimension [length^2], which takes only positive values and it is related to the Regge slope parameter or string scale.

The matter and its interaction with gravity are described by an electrodynamics action which is nonlinear in the quadratic term $F^2 = g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho}$, where $F_{\mu\nu}(x)$ is the Abelian field strength associated to the gauge connection $A_\mu(x)$ as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We shall assume an action for nonlinear electrodynamics of the form

$$I_{\rm NED} = \int_{\mathcal{M}} d^D x \sqrt{-g}\, \mathcal{L}(F^2), \tag{3}$$

where the Lagrangian density $\mathcal{L}(F^2)$ is an arbitrary function of F^2.

We will consider the spacetimes whose dimension is $D > 4$. The case $D = 4$ is special because the Euler-Gauss-Bonnet term becomes a topological invariant that does not contribute to the equations of motion. In that sense, bulk dynamics in $D = 4$ leaves the GB coupling as completely arbitrary. It is expected, however, that the GB term would modify the boundary dynamics of the theory and the value of the Euclidean continuation of the action. Indeed, in four-dimensional AdS gravity, the only consistent way of achieving the finiteness of both the conserved current and the Euclidean action is setting $\alpha = \ell^2/4$. Furthermore, nonlinear electrodynamics in four dimensions is somewhat particular, because one can consider a Lagrangian that depends additionally on another quadratic invariant $F^*F = \frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}$, which by itself is a topological term. For a recent discussion on electrostatic configurations in four-dimensional gravitating NED, see Ref. [19]. This type of Lagrangian clearly cannot be generalized to the higher-dimensional cases we are interested in.

In order to find the equations of motion of Einstein-Gauss-Bonnet (EGB) gravity, we first note that the gravitational action can be rearranged as

$$I_{\rm grav} = \frac{1}{16\pi G(D-2)(D-3)} \int_{\mathcal{M}} d^D x \sqrt{-g}\, \delta^{[\mu_1\cdots\mu_4]}_{[\nu_1\cdots\nu_4]} \times \left(\frac{1}{2} R^{\nu_1\nu_2}_{\mu_1\mu_2}\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \frac{D-2}{D\ell^2}\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2}\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \frac{\alpha(D-2)(D-3)}{4} R^{\nu_1\nu_2}_{\mu_1\mu_2}R^{\nu_3\nu_4}_{\mu_3\mu_4}\right), \tag{4}$$

where the tensor $\delta^{[\mu_1\cdots\mu_p]}_{[\nu_1\cdots\nu_p]}$ denotes the totally antisymmetric product of p Kronecker deltas (see Appendix A) and we have used the identity

$$R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} = \frac{1}{4}\delta^{[\mu_1\cdots\mu_4]}_{[\nu_1\cdots\nu_4]} R^{\nu_1\nu_2}_{\mu_1\mu_2}R^{\nu_3\nu_4}_{\mu_3\mu_4}. \tag{5}$$

This is a convenient form to take the variation of the Riemann tensor as

$$\delta R^{\mu}{}_{\nu\alpha\beta} = \nabla_\alpha(\delta\Gamma^\mu_{\nu\beta}) - \nabla_\beta(\delta\Gamma^\mu_{\nu\alpha})$$

in terms of the Christoffel symbol. In addition, using the Bianchi identity for the Riemann curvature,

$$\nabla_{[\mu}R^{\alpha\beta}_{\nu\lambda]} = \nabla_\mu R^{\alpha\beta}_{\nu\lambda} + \nabla_\lambda R^{\alpha\beta}_{\mu\nu} + \nabla_\nu R^{\alpha\beta}_{\lambda\mu} = 0,$$

one can show that the gravitational action changes under an arbitrary variation of the metric as

$$\delta I_{\rm grav} = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^D x \sqrt{-g}(g^{-1}\delta g)^\nu_\mu (G^\mu_\nu + H^\mu_\nu) + \int_{\partial\mathcal{M}} d^{D-1}x\, \Theta_{\rm grav}(\delta g, \delta\Gamma), \tag{6}$$

where G^μ_ν is the Einstein tensor with cosmological constant

$$G^\mu_\nu = R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R + \Lambda\delta^\mu_\nu, \tag{7}$$

and the contribution of the GB term to the variation of the bulk action is expressed in terms of the Lanczos tensor

$$H^\mu_\nu = -\frac{\alpha}{8}\delta^{[\mu\mu_1\cdots\mu_4]}_{[\nu\nu_1\cdots\nu_4]} R^{\nu_1\nu_2}_{\mu_1\mu_2}R^{\nu_3\nu_4}_{\mu_3\mu_4} \tag{8}$$

$$= -\frac{\alpha}{2}\delta^\mu_\nu(R^2 - 4R^{\alpha\beta}R_{\alpha\beta} + R^{\alpha\beta\lambda\sigma}R_{\alpha\beta\lambda\sigma}) + 2\alpha(RR^\mu_\nu - 2R^{\mu\lambda}R_{\lambda\nu} - 2R^\mu_{\lambda\nu\sigma}R^{\lambda\sigma} + R^{\mu\alpha\lambda\sigma}R_{\nu\alpha\lambda\sigma}). \tag{9}$$

The boundary term in (6) that appears from the variation of the bulk action reads

$$\int_{\partial\mathcal{M}} d^{D-1}x\Theta_{\rm grav} = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^D x\partial_\mu \times\left[\sqrt{-g}\delta^{[\mu\mu_1\mu_2\mu_3]}_{[\nu\nu_1\nu_2\nu_3]}g^{\nu\alpha}\delta\Gamma^{\nu_1}_{\mu_1\alpha} \times\left(\alpha R^{\nu_2\nu_3}_{\mu_2\mu_3}+\frac{1}{(D-2)(D-3)}\delta^{\nu_2}_{\mu_2}\delta^{\nu_3}_{\mu_3}\right)\right]. \tag{10}$$

On the other hand, arbitrary variations of the metric and the gauge field A_μ in the NED action produce

$$\delta I_{\rm NED} = \int_{\mathcal{M}} d^D x\sqrt{-g}\left[\frac{1}{2}T^\mu_\nu(g^{-1}\delta g)^\nu_\mu - 4\nabla_\mu\left(\frac{d\mathcal{L}}{dF^2}F^{\mu\nu}\right)\delta A_\nu\right] + \int_{\partial\mathcal{M}} d^{D-1}x\Theta_{\rm NED}(\delta A), \tag{11}$$

upon a suitable use of the Bianchi identity for the field strength, $\partial_{[\mu}F_{\nu\lambda]} = \partial_\mu F_{\nu\lambda} + \partial_\mu F_{\nu\lambda} + \partial_\mu F_{\nu\lambda} = 0$. The energy-momentum tensor for the matter content, $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta I_{\rm NED}}{\delta g_{\mu\nu}}$, has the form

$$T^\mu_\nu = \delta^\mu_\nu\mathcal{L} - 4\frac{d\mathcal{L}}{dF^2}F^{\mu\lambda}F_{\nu\lambda}, \tag{12}$$

and the surface term of the electromagnetic part is

$$\int_{\partial\mathcal{M}} d^{D-1}x\Theta_{\rm NED} = 4\int_{\mathcal{M}} d^D x\partial_\mu\left(\sqrt{-g}\frac{d\mathcal{L}}{dF^2}F^{\mu\nu}\delta A_\nu\right). \tag{13}$$

The variation of the total action (1) leads to the field equations plus a surface term

$$\delta I_0 = -\int_{\mathcal{M}} d^D x\sqrt{-g}\left[\frac{1}{16\pi G}\mathcal{E}^\mu_\nu(g^{-1}\delta g)^\nu_\mu + 4\mathcal{E}^\mu\delta A_\mu\right] + \int_{\partial\mathcal{M}} d^{D-1}x\Theta_0(\delta g, \delta\Gamma, \delta A), \tag{14}$$

where Θ_0 is the total boundary term coming from the variation of the bulk action, i.e., $\Theta_0 = \Theta_{\rm grav} + \Theta_{\rm NED}$.

The equations of motion are then obtained as $\delta I_0/\delta g_{\mu\nu} = 0$ and $\delta I_0/\delta A_\mu = 0$, that is,

$$\mathcal{E}^\mu_\nu \equiv G^\mu_\nu + H^\mu_\nu - 8\pi G T^\mu_\nu = 0, \tag{15}$$

$$\mathcal{E}^\mu \equiv \nabla_\nu\left(F^{\mu\nu}\frac{d\mathcal{L}}{dF^2}\right) = 0. \tag{16}$$

In general, the extremization of the action for the fully-interacting theory does not only require the equation of motion to be satisfied, but also the vanishing of the surface term for given boundary conditions. Therefore, a well-posed action principle leads to supplementing the Lagrangian by suitable boundary terms, which will be discussed below.

The Einstein tensor G^μ_ν can be conveniently rewritten in terms of the AdS radius as

$$G^\mu_\nu = -\frac{1}{4}\delta^{[\mu\mu_1\mu_2]}_{[\nu\nu_1\nu_2]}\left(R^{\nu_1\nu_2}_{\mu_1\mu_2} + \frac{1}{\ell^2}\delta^{[\nu_1\nu_2]}_{[\mu_1\mu_2]}\right). \tag{17}$$

Written in this compact form, the total equation of motion (15) is

$$\mathcal{E}^\mu_\nu = -\frac{1}{8}\delta^{[\mu\mu_1\cdots\mu_4]}_{[\nu\nu_1\cdots\nu_4]}\left[\alpha R^{\nu_1\nu_2}_{\mu_1\mu_2}R^{\nu_3\nu_4}_{\mu_3\mu_4} + \frac{1}{(D-3)(D-4)} \times\left(R^{\nu_1\nu_2}_{\mu_1\mu_2}\delta^{[\nu_3\nu_4]}_{[\mu_3\mu_4]} + \frac{1}{\ell^2}\delta^{[\nu_1\nu_2]}_{[\mu_1\mu_2]}\delta^{[\nu_3\nu_4]}_{[\mu_3\mu_4]}\right)\right] - 8\pi G T^\mu_\nu. \tag{18}$$

The GB contribution H^μ_ν given by (8) modifies the cosmological constant in G^μ_ν and therefore, the asymptotic behavior of the solutions. This is particularly evident in absence of matter fields, by taking the condition of maximally symmetric spacetimes with an effective AdS radius $\ell_{\rm eff}$, i.e.,

$$R^{\alpha\beta}_{\mu\nu} = -\frac{1}{\ell^2_{\rm eff}}\delta^{[\alpha\beta]}_{[\mu\nu]}. \tag{19}$$

The vacua of the theory are then solutions of global constant curvature, where $\ell^2_{\rm eff}$ is a root of the quadratic equation

$$\alpha(D-3)(D-4)\frac{1}{\ell^4_{\rm eff}} - \frac{1}{\ell^2_{\rm eff}} + \frac{1}{\ell^2} = 0, \tag{20}$$

so that

$$\ell^{(\pm)2}_{\rm eff} = \frac{2\alpha(D-3)(D-4)}{1\pm\sqrt{1-\frac{4\alpha}{\ell^2}(D-3)(D-4)}}, \qquad \alpha \le \frac{\ell^2}{4(D-3)(D-4)}. \tag{21}$$

The GB term, therefore, sets the equations of motion in the quadratic-curvature form

$$-\frac{\alpha}{8}\delta^{[\mu\mu_1\cdots\mu_4]}_{[\nu\nu_1\cdots\nu_4]}\left(R^{\nu_1\nu_2}_{\mu_1\mu_2} + \frac{1}{\ell^{(+)2}_{\rm eff}}\delta^{[\nu_1\nu_2]}_{[\mu_1\mu_2]}\right) \times\left(R^{\nu_3\nu_4}_{\mu_3\mu_4} + \frac{1}{\ell^{(-)2}_{\rm eff}}\delta^{[\nu_3\nu_4]}_{[\mu_3\mu_4]}\right) = 8\pi G T^\mu_\nu. \tag{22}$$

For the discussion of the present paper, we shall consider solutions that satisfy the condition (19) in the asymptotic region, i.e., tend asymptotically to a constant-curvature spacetime.

However, for different roots $\ell^{(+)2}_{\rm eff} \neq \ell^{(-)2}_{\rm eff}$, there is only one branch of the theory of physical interest. This is because the corresponding AdS radii can be expanded as

$$\ell^{(+)2}_{\rm eff} = \alpha(D-3)(D-4) + \mathcal{O}(\alpha^2), \tag{23}$$

$$\ell_{\rm eff}^{(-)2} = \ell^2 + \mathcal{O}(\alpha), \tag{24}$$

and, thus, $\ell_{\rm eff}^{(-)2}$ reduces to the original AdS radius for vanishing GB coupling, whereas $\ell_{\rm eff}^{(+)2}$ vanishes if the GB term goes to zero.

EGB AdS gravity possesses a unique AdS vacuum when both effective AdS radii are equal, $\ell_{\rm eff}^{(+)2} = \ell_{\rm eff}^{(-)2} = \ell^2/2$, case that corresponds to a GB coupling given by $\alpha = \ell^2/4(D-3)(D-4)$. In five dimensions, at that particular coupling value, the action features a group symmetry enhancement from local Lorentz to AdS_5, and it can be expressed as a Chern-Simons density for the latter group. This gravity theory has particular dynamical features that will not be discussed here [20,21].

III. GENERIC TOPOLOGICAL STATIC BLACK HOLE SOLUTION

A static black hole ansatz for the metric $g_{\mu\nu}$ in the coordinate set $x^\mu = (t, r, \varphi^m)$ is given by

$$\begin{aligned} ds^2 &= g_{\mu\nu}(x)dx^\mu dx^\nu \\ &= -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2\gamma_{mn}(\varphi)d\varphi^m d\varphi^n. \end{aligned} \tag{25}$$

The boundary $\partial\mathcal{M}$ is located at radial infinity ($r\to\infty$), and it is parametrized by $x^i = (t, \varphi^m)$. The metric γ_{nm} with local coordinates φ^m describes a $(D-2)$-dimensional Riemann space Γ_{D-2} with constant curvature, that is,

$$\tilde{\mathcal{R}}_{m_1m_2n_1n_2}(\gamma) = k(\gamma_{m_1n_1}\gamma_{m_2n_2} - \gamma_{m_1n_2}\gamma_{m_2n_1}), \tag{26}$$

where $k = 0$, $+1$ or -1, that corresponds to flat, spherical or hyperbolic transversal section, respectively.

We will consider that the solution possesses an event horizon, defined as the largest root of the equation $f(r_+) = 0$. The nonvanishing components of the Riemann curvature $R^{\mu\nu}_{\lambda\rho}$ are

$$\begin{aligned} R^{tr}_{tr} &= -\frac{1}{2}(f^2)'', \\ R^{tn}_{tm} &= R^{rn}_{rm} = -\frac{1}{2r}(f^2)'\delta^n_m, \\ R^{mn}_{kl} &= \frac{1}{r^2}(k-f^2)\delta^{[mn]}_{[kl]}, \end{aligned} \tag{27}$$

where prime denotes radial derivative. The Ricci tensor $R^\mu_\nu = R^{\mu\lambda}_{\nu\lambda}$ has the components

$$\begin{aligned} R^t_t &= R^r_r = -\frac{1}{2r}[r(f^2)'' + (D-2)(f^2)'], \\ R^n_m &= -\frac{1}{r^2}\delta^n_m[r(f^2)' + (D-3)(f^2-k)], \end{aligned} \tag{28}$$

and the Ricci scalar $R = R^{\mu\nu}_{\mu\nu}$ is

$$\begin{aligned} R = -\frac{1}{r^2}[&r^2(f^2)'' + 2(D-2)r(f^2)' \\ &+ (D-2)(D-3)(f^2-k)]. \end{aligned} \tag{29}$$

For a static solution with a topology equal to the one of the transversal section, we assume an ansatz for the gauge field in the form

$$A_\mu = \phi(r)\delta^t_\mu, \tag{30}$$

with the associated field strength

$$F_{\mu\nu} = E(r)(\delta^t_\mu\delta^r_\nu - \delta^t_\nu\delta^r_\mu), \tag{31}$$

where the electric field is given by

$$E(r) = -\phi'(r). \tag{32}$$

We solve the electric potential in the static ansatz (23) and (31), where $F^2 = -2E^2$, using the only nonvanishing component of the Maxwell-type Eq. (16),

$$\mathcal{E}^t = -\frac{d}{dr}\left(r^{D-2}E\frac{d\mathcal{L}}{dF^2}\bigg|_{F^2=-2E^2}\right) = 0, \tag{33}$$

which leads to the generalized Gauss' law

$$E\frac{d\mathcal{L}}{dF^2}\bigg|_{F^2=-2E^2} = -\frac{q}{r^{D-2}}. \tag{34}$$

Here, q is an integration constant related to the electric charge. Notice that the first integral of Eq. (34) does not depend explicitly on the metric, but only on the function $E(r)$. The algebraic equation in E can be solved as long as the explicit form of NED action is given, and implies that the electric field should vanish for $q = 0$.

We define the electric potential at infinity measured with respect to the event horizon r_+ as $\Phi = \phi(\infty) - \phi(r_+)$.

On the other hand, integrating out Eq. (32) one obtains the electric potential at the distance r measured with respect to radial infinity,

$$\phi(r) = -\int_\infty^r dv E(v), \tag{35}$$

such that the quantity of physical interest Φ is the potential evaluated at the horizon,

$$\Phi = -\phi(r_+). \tag{36}$$

In order to solve the function $f^2(r)$ in the metric, we write the only independent components of the Einstein and Lanczos tensors,

$$\begin{aligned} G^t_t &= G^r_r \\ &= \frac{D-2}{2r^2}\left[r(f^2)' + (D-3)(f^2-k) - (D-1)\frac{r^2}{\ell^2}\right], \\ H^t_t &= H^r_r \\ &= \alpha(D-2)(D-3)(D-4)\frac{k-f^2}{r^3} \\ &\quad\times\left[(f^2)' - (D-5)\frac{k-f^2}{2r}\right]. \end{aligned} \tag{37}$$

A necessary and sufficient condition on the NED Lagrangian density is the weak energy condition on the symmetric energy-momentum tensor

$$T_{\mu\nu}u^{\mu}u^{\nu} \leq 0, \tag{38}$$

that ensures that an observer measures a non-negative energy density $\rho_{\rm NED} = -T_{\mu\nu}u^{\mu}u^{\nu}$ for a timelike vector u^{μ}. For charged static black holes, the electromagnetic stress tensor satisfies $T_t^t = T_r^r$, such that the weak energy condition is equivalent to

$$T_t^t = T_r^r = \mathcal{L} + 4E^2 \frac{d\mathcal{L}}{dF^2} \geq 0, \tag{39}$$

where the Lagrangian $\mathcal{L}$ and its derivatives are evaluated at $F^2 = -2E^2$.

The above inequality restricts the function $\mathcal{L}$, but not its derivative. Indeed, in the asymptotic region the generalized Gauss' law implies $E\frac{d\mathcal{L}}{dF^2} \simeq 0$ and, assuming that electric field vanishes asymptotically, the weak energy condition leads to $\mathcal{L} \geq 0$ for large r. On the other hand, the asymptotic behavior of $\frac{d\mathcal{L}}{dF^2}$ remains arbitrary. Indeed, for Maxwell electrodynamics and Born-Infeld-like Lagrangians, the expression $\frac{d\mathcal{L}}{dF^2}$ is finite for $r \to \infty$. Also, for the Lagrangians of the type $(F^2)^p$, the derivative vanishes when $p > 1$, and it is divergent if $p < 1$. Additionally, one may demand the finiteness of the total energy, that can be expressed as

$$\int_0^{\infty} dr r^{D-2} T_r^r(r) < \infty. \tag{40}$$

Note that the above requirement on the EM energy, applied to black hole solutions, also includes the interior region protected by the horizon [22].

The equations of motion $\mathcal{E}_t^t = \mathcal{E}_r^r = 0$ read

$$\begin{aligned}\frac{16\pi G r^2}{D-2} T_r^r &= r(f^2)' + (D-3)(f^2-k) - (D-1)\frac{r^2}{\ell^2} \\ &\quad + 2\alpha(D-3)(D-4)\frac{k-f^2}{r} \\ &\quad \times \left[(f^2)' - (D-5)\frac{k-f^2}{2r}\right].\end{aligned} \tag{41}$$

One can show, using Eqs. (33) and (41), that $\mathcal{E}_n^m = 0$ is identically satisfied.

The differential Eq. (41) is integrable, because it can be cast in the form

$$\begin{aligned}&\left[r^{D-3}(f^2-k)\left(1-\alpha(D-3)(D-4)\frac{f^2-k}{r^2}\right)\right]' \\ &\quad = \frac{D-1}{\ell^2} r^{D-2} + \frac{16\pi G}{D-2} r^{D-2} T_r^r,\end{aligned} \tag{42}$$

which leads to the general solution

$$\begin{aligned}&(f^2-k)\left(1-\alpha(D-3)(D-4)\frac{f^2-k}{r^2}\right) \\ &\quad = \frac{r^2}{\ell^2} - \frac{\mu}{r^{D-3}} + \frac{16\pi G \mathcal{T}(q,r)}{(D-2)r^{D-3}},\end{aligned} \tag{43}$$

where μ is an integration constant of dimension [mass $\times$ $16\pi G$], and the function $\mathcal{T}(q,r)$ for an arbitrary NED Lagrangian is given by

$$\begin{aligned}\mathcal{T}(q,r) &= \int_{\infty}^{r} dv v^{D-2} T_r^r(v) \\ &= \int_{\infty}^{r} dv (v^{D-2}\mathcal{L}(v) - 4qE(v)) \\ &= \frac{1}{D-1}(r^{D-1}\mathcal{L} - qrE + (D-2)4q\phi)|_{\infty}^{r}.\end{aligned} \tag{44}$$

The Gauss law (34) has been used to eliminate $d\mathcal{L}/dF^2$ from the integral, so that $\mathcal{T}$ depends on the integration constant q. For a general procedure for Lovelock gravity coupled to NED see, e.g., [23].

Electromagnetism does not deform the asymptotic region since the relation $T(q,\infty) = 0$ is identically satisfied according to Eq. (44).

Then, the metric function in the static solution of EGB gravity coupled to NED is obtained solving the quadratic Eq. (43) in f^2. The existence of a real root is ensured by the condition

$$\mathcal{T}(q,r) \leq \frac{(D-2)r^{D-1}}{16\pi G}\left(\frac{1}{4\alpha(D-3)(D-4)} - \frac{1}{\ell^2} + \frac{\mu}{r^{D-1}}\right), \tag{45}$$

that is proven to be satisfied for sufficiently large r, as the r.h.s. is always positive [see the inequality in Eq. (21)]. Thus, the metric possesses two branches,

$$f_{\pm}^2(r) = k + \frac{r^2}{2\alpha(D-3)(D-4)}\left[1 \pm \sqrt{1 - 4\alpha(D-3)(D-4)\left(\frac{1}{\ell^2} - \frac{\mu}{r^{D-1}} + \frac{16\pi G \mathcal{T}(q,r)}{(D-2)r^{D-1}}\right)}\right]. \tag{46}$$

The ground state $\mu = 0$, $q = 0$ corresponds to two AdS vacua,

$$f_{\pm}^2(r)_{\rm vac} = k + \frac{r^2}{\ell_{\rm eff}^{(\pm)2}}. \tag{47}$$

However, it has been shown in [24] that the vacuum $f_+^2(r)_{\rm vac}$ is unstable and the graviton has negative mass, while the solution $f_-^2(r)_{\rm vac}$ is stable and is free of ghosts. For a general solution, from (46) in the weak limit of GB coupling, we have

$$f_+^2(r) = k + r^2\left(\frac{1}{\alpha(D-3)(D-4)} - \frac{1}{\ell^2}\right) + \frac{\mu}{r^{D-3}} - \frac{16\pi G\mathcal{T}(q,r)}{(D-2)r^{D-3}} + \mathcal{O}(\alpha), \quad (48)$$

$$f_-^2(r) = k + \frac{r^2}{\ell^2} - \frac{\mu}{r^{D-3}} + \frac{16\pi G\mathcal{T}(q,r)}{(D-2)r^{D-3}} + \mathcal{O}(\alpha), \quad (49)$$

because $\mathcal{T}$ does not depend on the constant α. The opposite sign in the mass parameter μ in $f_+^2(r)$ indicates instabilities of the graviton so that it is not of physical interest for our discussion below.

On the other hand, the function $f_-^2(r)$ in the limit $\alpha \to 0$ describes static black holes of Einstein-Hilbert AdS gravity coupled to NED. Because of this reason, henceforth, we consider only the negative branch of the metric, $f(r) \equiv f_-(r)$,

$$f^2(r) = k + \frac{r^2}{2\alpha(D-3)(D-4)}\left[1 - \sqrt{1 - 4\alpha(D-3)(D-4)\left(\frac{1}{\ell^2} - \frac{\mu}{r^{D-1}} + \frac{16\pi G\mathcal{T}(q,r)}{(D-2)r^{D-1}}\right)}\right]. \quad (50)$$

When NED Lagrangian corresponds to the one of Maxwell electromagnetism $\mathcal{L}_{\mathrm{Maxwell}}(F^2) = -F^2$, the function $\mathcal{T}(q,r)$ in Eq. (50) becomes $\mathcal{T}_{\mathrm{Maxwell}} = \frac{2q^2}{(D-3)r^{D-3}}$, which reproduces the charged black hole solution first found in [25]. Expanding f^2 for large r, one can notice that the electromagnetic part possesses the same falloff as in Reissner-Nordstrom case.

In general, the contribution of NED to f^2 is smaller than the one of the mass term and can therefore be neglected for large r. Indeed, using Eq. (21), one can prove that, in the asymptotic region, the metric function and its radial derivative behave as

$$f^2 = k + \frac{r^2}{\ell_{\mathrm{eff}}^2} - \frac{\mu}{1 - \frac{2\alpha}{\ell_{\mathrm{eff}}^2}(D-3)(D-4)}\frac{1}{r^{D-3}} + \mathcal{O}\left(\frac{1}{r^{2D-6}}\right), \quad (51)$$

$$(f^2)' = \frac{2r}{\ell_{\mathrm{eff}}^2} + \frac{(D-3)\mu}{1 - \frac{2\alpha}{\ell_{\mathrm{eff}}^2}(D-3)(D-4)}\frac{1}{r^{D-2}} + \mathcal{O}\left(\frac{1}{r^{2D-5}}\right). \quad (52)$$

This fact will make evident that the NED term $\mathcal{T}(q,r)$ in Eq. (50) does not produce additional contributions to the energy of the system, as we shall discuss in Sec. VI B.

In absence of electromagnetic fields, we have that $\mathcal{T}(0,r) = 0$, which means that the solution (50) reduces to the topological version of Boulware-Deser black holes in AdS spaces [24,26,27].

Different NED models have been proposed which possess particlelike solutions whose both electromagnetic and gravitational fields are regular everywhere. However, this does not imply that there are no curvature singularities.

The interior of the black hole is described by the metric function obtained from Eq. (42) as

$$f_{\mathrm{in}}^2(r) = k + \frac{r^2}{2\alpha(D-3)(D-4)}\left[1 \pm \sqrt{1 - 4\alpha(D-3)(D-4)\left(\frac{1}{\ell^2} - \frac{c}{r^{D-1}} + \frac{16\pi G\int_0^r dv\, v^{D-2}T_r^r(v)}{(D-2)r^{D-1}}\right)}\right], \quad (53)$$

where c is the integration constant. In consequence, when one imposes the finiteness condition on the energy-momentum tensor at the origin,

$$\lim_{r\to 0}\frac{1}{r^{D-1}}\int_0^r dv\, v^{D-2}T_r^r(v) < \infty, \quad (54)$$

the metric function takes the value $f_{\mathrm{in}}^2(0) = k \pm \sqrt{\frac{c}{\alpha(D-3)(D-4)r^{D-5}}}$. For $c \neq 0$, this is finite only in five dimensions, otherwise c must vanish. Further analysis is needed to relate c to the asymptotic mass parameter μ, which would imply new conditions in order to remove the conical singularity at the origin. One may also demand $\mathcal{L}$ to be single-valued, continuous, and differentiable. For a more detailed discussion on these issues for particular cases see, e.g., Refs. [19,22].

So far, we have seen that for any nonlinear electrodynamics theory coupled to EGB AdS gravity, both the metric (50) and the electric potential (35) can be determined from the explicit form of the Lagrangian $\mathcal{L}(F^2)$. We illustrate this with a few examples in the next section.

IV. CHARGED BLACK HOLES IN PARTICULAR NED THEORIES

A. Born-Infeld electrodynamics

Born-Infeld electrodynamics [2] is described by the Lagrangian density

$$\mathcal{L}_{\mathrm{BI}}(F^2) = 4b^2\left(1 - \sqrt{1 + \frac{F^2}{2b^2}}\right), \quad (55)$$

where the coupling parameter b (with dimension of mass) is related to the string tension α' as $b = 1/2\pi\alpha'$. This Lagrangian reduces to the Maxwell case in the weak-coupling limit $b \to \infty$. Generally speaking, when a density $\mathcal{L}(F^2)$ recovers the Maxwell theory in weak-coupling limit, i.e., $\mathcal{L}(F^2) = -F^2 + \mathcal{O}(1/b^2)$, it is said to be Born-Infeld-type.

The BI energy-momentum tensor has the form

$$T^{\mu}_{\nu} = 4b^2\delta^{\mu}_{\nu}\left(1 - \sqrt{1 + \frac{F^2}{2b^2}}\right) + \frac{4F^{\mu\lambda}F_{\nu\lambda}}{\sqrt{1 + \frac{F^2}{2b^2}}}, \tag{56}$$

and it generates the electric field

$$E(r) = \frac{q}{\sqrt{\frac{q^2}{b^2} + r^{2D-4}}}. \tag{57}$$

The corresponding electric potential is given by the formula (35). Performing a variable change in the integral, $u = (r/v)^{2D-4}$, it can be expressed in terms of the hypergeometric function $\mathcal{F}(q, r) = {}_2F_1(\frac{1}{2}, \frac{D-3}{2D-4}; \frac{3D-7}{2D-4}; -\frac{q^2}{b^2 r^{2D-4}})$ (see Appendix B), and the solution for the potential is

$$\phi(r) = \frac{q}{(D-3)r^{D-3}}\mathcal{F}(q, r). \tag{58}$$

Then, the integration constant $\Phi = -\phi(r_+)$ reads

$$\Phi = -\frac{q}{(D-3)r_+^{D-3}}\mathcal{F}(q, r_+). \tag{59}$$

In order to find the metric for the black hole with Born-Infeld electric charge, we solve explicitly the integral (44) as

$$\mathcal{T}_{\rm BI}(q, r) = \frac{4b^2 r^{D-1}}{D-1}\left(1 - \sqrt{1 + \frac{q^2}{b^2 r^{2D-4}}}\right) + \frac{4(D-2)q^2}{(D-1)(D-3)r^{D-3}}\mathcal{F}(q, r), \tag{60}$$

and replacing in Eq. (50), we obtain

$$f^2(r) = k + \frac{r^2}{2\alpha(D-3)(D-4)} \times \left\{1 - \left[1 - 4\alpha(D-3)(D-4)\left(\frac{1}{\ell^2} - \frac{\mu}{r^{D-1}} + \frac{64\pi G b^2}{(D-1)(D-2)}\left(1 - \sqrt{1 + \frac{q^2}{b^2 r^{2D-4}}}\right) + \frac{64\pi G q^2 \mathcal{F}(q, r)}{(D-1)(D-3)r^{2D-4}}\right)\right]^{1/2}\right\}. \tag{61}$$

This class of black holes has been discussed in Ref. [25]. The generalization to non-Abelian gauge fields has been studied in Ref. [28]. In the limit of vanishing GB coupling, the metric reduces to the one of topological Einstein-BI black holes in AdS spaces [29–31].

B. Conformally invariant electrodynamics

Born-Infeld Lagrangian in higher dimensions is a physically sensible extension of four-dimensional Maxwell electrodynamics. However, if one is interested in a generalization of the conformal invariance property of four-dimensional (4D) Maxwell theory, there exist NED actions given as power-law functions of the form

$$\mathcal{L}_{\rm CED}(F^2) = -2\chi F^{2p}, \tag{62}$$

where χ is a positive coupling constant [32]. Then the conformal invariance $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, $A_\mu \to A_\mu$ is realized for the power $p = D/4$.

The energy-momentum tensor for A_μ reads

$$T^{\mu}_{\nu} = -2\chi\left(\delta^{\mu}_{\nu} - 4p\frac{F^{\mu\lambda}F_{\nu\lambda}}{F^2}\right)F^{2p}, \tag{63}$$

and it produces the electric field

$$E(r) = \frac{\tilde{q}}{r^\beta}, \tag{64}$$

where $\beta = \frac{D-2}{2p-1}$ and $\tilde{q} = (\frac{(-1)^{p+1}q}{2^p p\chi})^{\beta/(D-2)}$. When one demands conformal invariance ($p = D/4$), the electric field takes the 4D Maxwell's form, $E = \tilde{q}/r^2$, in any dimension.

Then, one can calculate explicitly the function (44) in the metric,

$$\mathcal{T}_{\rm CED}(q, r) = -\frac{2(D-2)(-2)^p\tilde{q}^{2p}\chi}{\beta(\beta-1)}\frac{1}{r^{\beta-1}}, \tag{65}$$

that, plugged in Eq. (50), produces a line element which matches the form of the black holes found in Ref. [9] for EGB AdS gravity.

C. Logarithmic electrodynamics

NED Lagrangians that contain logarithmic terms in the electromagnetic field strength appear in the description of vacuum polarization effects. These terms were obtained as exact 1-loop corrections for electrons in a uniform electromagnetic field background by Euler and Heisenberg [7], and therefore are a typical feature of quantum electrodynamics effective actions.

Furthermore, logarithmic ED Lagrangians come as a realization of the old idea of removing singularities in the gravitational field, in a similar way as the BI electrodynamics removes divergences in the electric field. They have also been used to describe an equation of state of radiation in an alternative mechanism for inflation [33].

A simple example of a BI-like Lagrangian with a logarithmic term, that can be added as a correction to the original BI one, was discussed in Ref. [8] in asymptotically flat Einstein gravity in $D = 4$. This model does not cancel the curvature singularity for small r, but makes the

Kretschmann invariant behave as $1/r^4$, which is a weaker singularity than in, e.g., Schwarzschild or Reissner-Nordström black holes.

In an arbitrary dimension, the logarithmic ED lagrangian has the form

$$\mathcal{L}_{\text{Log}}(F^2) = -8b^2 \ln\left(1 + \frac{F^2}{8b^2}\right). \tag{66}$$

It can be shown from Eq. (34) that the electric field has two branches, but only one features the Maxwell limit ($b \to \infty$),

$$E(r) = \frac{2b^2}{q}\left(r^{D-2} - \sqrt{r^{2D-4} + \frac{q^2}{b^2}}\right). \tag{67}$$

Considering this, the electric potential reads

$$\phi(r) = -\frac{2b^2 r^{D-1}}{q(D-1)}\left(1 - \sqrt{1 + \frac{q^2}{b^2 r^{2D-4}}}\right) - \frac{2q(D-2)\mathcal{F}(q,r)}{(D-1)(D-3)r^{D-3}}, \tag{68}$$

where $\mathcal{F}(q,r) = {}_2F_1(\frac{1}{2}, \frac{D-3}{2D-4}; \frac{3D-7}{2D-4}; -\frac{q^2}{b^2 r^{2D-4}})$.

The electromagnetic contribution to the metric is then given by

$$\begin{aligned}\mathcal{T}_{\text{Log}}(q,r) &= -\frac{8b^2 r^{D-1}}{D-1}\ln\left[\frac{2b^2 r^{D-2}}{q^2}\left(\sqrt{r^{2D-4} + \frac{q^2}{b^2}} - r^{D-2}\right)\right] \\ &\quad + \frac{8b^2(2D-3)r}{(D-1)^2}\left(r^{D-2} - \sqrt{r^{2D-4} + \frac{q^2}{b^2}}\right) \\ &\quad + \frac{8q^2(D-2)^2\mathcal{F}(q,r)}{(D-1)^2(D-3)r^{D-3}}. \end{aligned} \tag{69}$$

Using the fact that $\mathcal{F}(0,r) = 1 = \mathcal{F}(q,\infty)$, one can show explicitly that electromagnetism vanishes asymptotically, that is, $\mathcal{T}_{\text{Log}}(q,\infty) = 0$. Also, in the zero charge limit, $\mathcal{T}_{\text{Log}}(0,r)$ vanishes, as expected.

It is straightforward to write down the metric function $f^2(r)$ by plugging in the above expression into EGB metric in Eq. (50). This general solution reduces to the one of Ref. [8] in 4D Einstein gravity without cosmological constant.

A more realistic version of logarithmic NED action is given by the Hoffmann-Infeld model [34], that do remove singularities in both gravitational and electric fields for static solutions. This theory is described by the Lagrangian

$$\mathcal{L}_{\text{HI}} = 4b^2(1 - \eta(F^2) - \log\eta(F^2)), \tag{70}$$

where $\eta(F^2) = -\frac{F^2}{4b^2}(1 - \sqrt{1 + \frac{F^2}{2b^2}})^{-1}$. It can be easily checked from the expansion $\eta(F^2) = 1 + \frac{F^2}{8b^2} + \mathcal{O}(1/b^4)$ that $\mathcal{L}_{\text{HI}}(F^2)$ is also a BI-like Lagrangian in the weak-coupling limit. In $D = 5$, a solution to this model was discussed in Ref. [22].

V. VARIATIONAL PRINCIPLE AND BOUNDARY TERMS

Any gravity theory is not defined only by its equations of motion in the bulk, but also by the set of boundary conditions that guarantees that the action is truly stationary. In general, this implies that the original bulk action must be supplemented by a boundary term β,

$$\tilde{I} = I_0 + \int_{\partial\mathcal{M}} d^{D-1}x\beta, \tag{71}$$

such that the problem of a well-posed action principle reduces to the on-shell cancelation of the total surface term of the theory, that is,

$$\delta\tilde{I} = \int_{\partial\mathcal{M}} d^{D-1}x(\Theta_0 + \delta\beta) = 0. \tag{72}$$

In our case, the term Θ_0 is given as the sum of Eqs. (10) and (13) and, in principle, β can be split in two parts, namely, $\beta = \beta_{\text{grav}} + \beta_{\text{NED}}$.

A gravitational action whose variation vanishes for a Dirichlet condition on the metric requires the addition of (generalized) Gibbons-Hawking terms. This is particularly easy to see in Gauss-normal coordinates

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = N^2(r)dr^2 + h_{ij}(r,x)dx^i dx^j. \tag{73}$$

We will consider a manifold with a single boundary $\partial\mathcal{M}$ at $r = \infty$, parametrized by the coordinates x^i, and such that h_{ij} is the induced metric on it. The extrinsic properties of the boundary are given in terms of the outward-pointing normal $n_\mu = (n_r, n_i) = (N, \vec{0})$. In particular, we define the extrinsic curvature as the Lie derivative of the induced metric along this normal,

$$K_{ij} = -\frac{1}{2}\mathcal{L}_n h_{ij} = -\frac{1}{2N}h'_{ij}. \tag{74}$$

As it is written in Appendix C, different components of the Christoffel symbol can be expressed in terms of the extrinsic curvature. In doing so, the surface term $\Theta_0 = \Theta_{\text{grav}} + \Theta_{\text{NED}}$ has the form

$$\begin{aligned}\Theta_0 = {}& \frac{1}{8\pi G(D-2)(D-3)}\sqrt{-h}\delta^{[jj_1j_2]}_{[ii_1i_2]} \\ &\times\left[\frac{1}{2}(h^{-1}\delta h)^i_k K^k_j + \delta K^i_j\right]\left[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + 2\alpha(D-2)(D-3)\right. \\ &\left.\times\left(\frac{1}{2}\mathcal{R}^{i_1i_2}_{j_1j_2} - K^{i_1}_{j_1}K^{i_2}_{j_2}\right)\right] + 4\sqrt{-h}\frac{d\mathcal{L}}{dF^2}NF^{ri}\delta A_i, \end{aligned} \tag{75}$$

where the determinant of the metric satisfies $\sqrt{-g} = N\sqrt{-h}$ and $\mathcal{R}^{ij}_{kl}(h)$ is the intrinsic curvature of the boundary, which is related to the spacetime Riemann tensor by $R^{ij}_{kl} = \mathcal{R}^{ij}_{kl} - K^i_k K^j_l + K^i_l K^j_k$ (see Appendix C).

In order to cancel Θ_{NED} part of the surface term, it is a sufficient condition to take $\delta A_i = 0$ at $\partial\mathcal{M}$. This means that β does not depend on the electromagnetic field, i.e., $\beta = \beta_{\rm grav}$. On the other hand, there is a systematic construction of generalized Gibbons-Hawking terms for Gauss-Bonnet and, in general, any Lovelock theory [35,36], which for the present case gives

$$\beta = \frac{\sqrt{-h}}{8\pi G(D-2)(D-3)} \delta^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]} K^{i_1}_{j_1} \times \left[\delta^{i_2}_{j_2} \delta^{i_3}_{j_3} + 2\alpha(D-2)(D-3)\left(\frac{1}{2}\mathcal{R}^{i_2 i_3}_{j_2 j_3} - \frac{1}{3} K^{i_2}_{j_2} K^{i_3}_{j_3}\right)\right], \tag{76}$$

or, in the form which is commonly found in the literature,

$$\beta = \frac{\sqrt{-h}}{8\pi G}\left[K + 2\alpha\left(K\left(K^{ij}K_{ij} - \frac{1}{3}K^2\right) - \frac{2}{3}K^i_k K^k_j K^j_i - 2\mathcal{G}^{ij}K_{ij}\right)\right],$$

where $\mathcal{G}^{ij} = \mathcal{R}^{ij} - \frac{1}{2}\mathcal{R}h^{ij}$ is the Einstein tensor associated to the boundary metric.

In doing so, the corresponding Dirichlet variation of the action is

$$\Theta_0 + \delta\beta = \frac{\sqrt{-h}}{16\pi G(D-3)(D-4)} \delta^{[j j_1 j_2 j_3]}_{[i i_1 i_2 i_3]} (h^{-1}\delta h)^i_j K^{i_1}_{j_1} \times \left[\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + 2\alpha(D-3)(D-4) \times \left(\frac{1}{2}\mathcal{R}^{i_2 i_3}_{j_2 j_3} - \frac{1}{3}K^{i_2}_{j_2}K^{i_3}_{j_3}\right)\right] + 4\sqrt{-h}\frac{d\mathcal{L}}{dF^2} N F^{ri}\delta A_i. \tag{77}$$

Notice that for a radial foliation of the spacetime, the on-shell variation of the action can be cast in the form

$$\delta\tilde{I} = \int_{\partial\mathcal{M}} d^{D-1}x\sqrt{-h}\left(\frac{1}{2}\pi^{ij}\delta h_{ij} + \pi^i \delta A_i\right), \tag{78}$$

where π^{ij} and π^i are the canonical momenta conjugate to h_{ij} and A_i, respectively. If one uses π^{ij} as a quasilocal stress tensor in AdS gravity, the conserved quantities derived from it are divergent in the asymptotic region. In other words, a well-posed variational principle is not necessarily linked to the problem of finiteness of the charges and action.

In the context of AdS/CFT correspondence, the standard way to deal with the regularization problem in a background-independent way is the addition of local counterterms at the boundary, which are constructed using holographic normalization. However, the inclusion of higher-curvature terms in the action turns this procedure considerably more complicated. A practical method to circumvent this obstacle in EGB AdS gravity is to assume the same form of the counterterms as in the EH case, but with arbitrary coefficients [37–39]. The coefficients are then fixed requiring the convergence of the action for particular solutions of the theory. It is clear from this construction that the series cannot be obtained for an arbitrary dimension.

The fact that in AdS gravity the leading order of the asymptotic expansion of the extrinsic curvature is proportional to the one of the boundary metric opens the possibility to consider counterterms which depends on the extrinsic curvature, as well. In this alternative scheme (known as Kounterterm regularization), the boundary terms are related to either topological invariants or Chern-Simons forms in the corresponding dimensions. In this way, it is possible to skip the technicalities of holographic procedures and to write down a general expression for them in any dimension,

$$I = I_0 + c_{D-1}\int_{\partial\mathcal{M}} d^{D-1}x B_{D-1}, \tag{79}$$

where c_{D-1} is a given constant. For EH AdS gravity, the Kounterterm series was shown in Refs. [40,41] as a given polynomial of the extrinsic and intrinsic curvatures, which defines a well-posed action principle. In general, the action (79) varies as

$$\delta I = \int_{\partial\mathcal{M}} d^{D-1}x\Theta = \int_{\partial\mathcal{M}} d^{D-1}x(\Theta_{\rm grav} + \Theta_{\rm NED} + c_{D-1}\delta B_{D-1}), \tag{80}$$

such that the boundary term in (79) makes the action to have an extremum on-shell and solves the regularization problem, as well.

For a given dimension, the series B_{D-1} possesses the remarkable property of preserving its form for EGB AdS gravity [42] and, in general, any theory of the Lovelock type [43]. In what follows, we use the explicit form of the boundary terms to construct the general variation of the action in tensorial notation.

A. Even dimensions ($D = 2n$)

In even dimensions $D = 2n > 4$, the boundary term B_{2n-1} in (79) is given by the nth Chern form [42]

$$B_{2n-1} = 2n\sqrt{-h}\int_0^1 dt\,\delta^{[j_1\cdots j_{2n-1}]}_{[i_1\cdots i_{2n-1}]} K^{i_1}_{j_1}\left(\frac{1}{2}\mathcal{R}^{i_2 i_3}_{j_2 j_3} - t^2 K^{i_2}_{j_2}K^{i_3}_{j_3}\right)\cdots \times\left(\frac{1}{2}\mathcal{R}^{i_{2n-2}i_{2n-1}}_{j_{2n-2}j_{2n-1}} - t^2 K^{i_{2n-2}}_{j_{2n-2}}K^{i_{2n-1}}_{j_{2n-1}}\right), \tag{81}$$

that is the scalar density whose derivative is locally equivalent to the Euler invariant [globally they differ by the Euler characteristic of the manifold, $\chi(\mathcal{M})$]. The integration in the continuous parameter t generates the coefficients when the boundary term is expanded as a polynomial. The constant c_{2n-1} in front of the boundary term B_{2n-1} which produces a well-defined variational principle is given in terms of the effective AdS radius as

$$c_{2n-1} = -\frac{1}{16\pi G}\frac{(-\ell_{\rm eff}^2)^{n-1}}{n(2n-2)!}\times\left(1-\frac{2\alpha}{\ell_{\rm eff}^2}(2n-2)(2n-3)\right). \tag{82}$$

It can been proven that the same choice of c_{2n-1} ensures the convergence of the Euclidean action. The total surface term Θ can be read off from the on-shell variation of the action (80), that in this case is

$$\delta I_{2n} = \frac{1}{16\pi G(2n-2)!2^{n-1}}\int_{\partial\mathcal{M}} d^{2n-1}x\sqrt{-h}\delta^{[j_1\cdots j_{2n-1}]}_{[i_1\cdots i_{2n-1}]}[(h^{-1}\delta h)^{i_1}_k K^k_{j_1} + 2\delta K^{i_1}_{j_1}]\times\Big[(\delta^{[i_2i_3]}_{[j_2j_3]} + 2\alpha(2n-2)(2n-3)R^{i_2i_3}_{j_2j_3})\delta^{[i_4i_5]}_{[j_4j_5]}\cdots\delta^{[i_{2n-2}i_{2n-1}]}_{[j_{2n-2}j_{2n-1}]} - (-\ell_{\rm eff}^2)^{n-1}\left(1-\frac{2\alpha}{\ell_{\rm eff}^2}(2n-2)(2n-3)\right)R^{i_2i_3}_{j_2j_3}\cdots R^{i_{2n-2}i_{2n-1}}_{j_{2n-2}j_{2n-1}}\Big] + 4\int_{\partial\mathcal{M}} d^{2n-1}x\sqrt{-h}\frac{d\mathcal{L}}{dF^2}NF^{ri}\delta A_i. \tag{83}$$

The reader can easily check that imposing the asymptotically locally AdS condition for the spacetime, i.e.,

$$R^{\alpha\beta}_{\mu\nu} + \frac{1}{\ell_{\rm eff}^2}\delta^{[\alpha\beta]}_{[\mu\nu]} = 0, \quad \text{at } \partial\mathcal{M}, \tag{84}$$

identically cancels the leading-order divergences in the gravitational part of the above variation. As a remarkable feature of the addition of Kounterterms, all other divergent terms in (83) are exactly cancelled out. In this way, the finite contribution is coupled to the conformal metric that is kept fixed at the boundary. The NED part of the surface term vanishes for a Dirichlet boundary condition for the transversal components of A_μ,

$$\delta A_i = 0, \quad \text{at } \partial\mathcal{M}. \tag{85}$$

B. Odd dimensions ($D = 2n + 1$)

The extrinsic regularization developed for odd-dimensional Einstein-Hilbert AdS gravity [40] can be mimicked for EGB AdS theory, just replacing the AdS radius ℓ by the effective one $\ell_{\rm eff}$ in the boundary terms. Thus, the Kounterterms series is given in terms of the parametric integrations

$$B_{2n} = 2n\sqrt{-h}\int_0^1 dt\int_0^t ds\,\delta^{[j_1\cdots j_{2n}]}_{[i_1\cdots i_{2n}]}K^{i_1}_{j_1}\delta^{i_2}_{j_2}\times\left(\frac{1}{2}\mathcal{R}^{i_3i_4}_{j_3j_4} - t^2K^{i_3}_{j_3}K^{i_4}_{j_4} + \frac{s^2}{\ell_{\rm eff}^2}\delta^{i_3}_{j_3}\delta^{i_4}_{j_4}\right)\times\cdots\times\left(\frac{1}{2}\mathcal{R}^{i_{2n-1}i_{2n}}_{j_{2n-1}j_{2n}} - t^2K^{i_{2n-1}}_{j_{2n-1}}K^{i_{2n}}_{j_{2n}} + \frac{s^2}{\ell_{\rm eff}^2}\delta^{i_{2n-1}}_{j_{2n-1}}\delta^{i_{2n}}_{j_{2n}}\right). \tag{86}$$

The corresponding constant for this case incorporates the information of the theory through the GB coupling in the form

$$c_{2n} = -\frac{1}{16\pi G}\frac{(-\ell_{\rm eff}^2)^{n-1}}{n(2n-1)!}\times\left(1-\frac{2\alpha(2n-1)(2n-2)}{\ell_{\rm eff}^2}\right)\left[\int_0^1 dt(1-t^2)^{n-1}\right]^{-1} = -\frac{1}{16\pi G}\frac{2(-\ell_{\rm eff}^2)^{n-1}}{n(2n-1)!\beta(n,\frac{1}{2})}\left(1-\frac{2\alpha(2n-1)(2n-2)}{\ell_{\rm eff}^2}\right), \tag{87}$$

where $\beta(n,\frac{1}{2}) = \frac{2^{2n-1}(n-1)!^2}{(2n-1)!}$ is the Beta function for those arguments.

The total action varies on-shell as

$$\delta I_{2n+1} = \frac{1}{2^{n-1}16\pi G(2n-1)!}\int_{\partial\mathcal{M}} d^{2n}x\sqrt{-h}\delta^{[j_1\cdots j_{2n}]}_{[i_1\cdots i_{2n}]}[(h^{-1}\delta h)^{i_1}_k K^k_{j_1} + 2\delta K^{i_1}_{j_1}]\delta^{i_2}_{j_2}\times\Big[(\delta^{[i_3i_4]}_{[j_3j_4]} + 2\alpha(2n-1)(2n-2)R^{i_3i_4}_{j_3j_4})\delta^{[i_5i_6]}_{[j_5j_6]}\cdots\delta^{[i_{2n-1}i_{2n}]}_{[j_{2n-1}j_{2n}]} + 16\pi G(2n-1)!nc_{2n}\int_0^1 dt\left(R^{i_3i_4}_{j_3j_4} + \frac{t^2}{\ell_{\rm eff}^2}\delta^{[i_3i_4]}_{[j_3j_4]}\right)\cdots\left(R^{i_{2n-1}i_{2n}}_{j_{2n-1}j_{2n}} + \frac{t^2}{\ell_{\rm eff}^2}\delta^{[i_{2n-1}i_{2n}]}_{[j_{2n-1}j_{2n}]}\right)\Big] + nc_{2n}\int_{\partial\mathcal{M}} d^{2n}x\sqrt{-h}\int_0^1 dt\,t\,\delta^{[j_1\cdots j_{2n}]}_{[i_1\cdots i_{2n}]}[(h^{-1}\delta h)^{i_1}_k(K^k_{j_1}\delta^{i_2}_{j_2} - \delta^k_{j_1}K^{i_2}_{j_2}) + 2\delta^{i_2}_{j_2}\delta K^{i_2}_{j_2}]\times\left(\frac{1}{2}\mathcal{R}^{i_3i_4}_{j_3j_4} - t^2K^{i_3}_{j_3}K^{i_4}_{j_4} + \frac{t^2}{\ell_{\rm eff}^2}\delta^{i_3}_{j_3}\delta^{i_4}_{j_4}\right)\cdots\left(\frac{1}{2}\mathcal{R}^{i_{2n-1}i_{2n}}_{j_{2n-1}j_{2n}} - t^2K^{i_{2n-1}}_{j_{2n-1}}K^{i_{2n}}_{j_{2n}} + \frac{t^2}{\ell_{\rm eff}^2}\delta^{i_{2n-1}}_{j_{2n-1}}\delta^{i_{2n}}_{j_{2n}}\right) + 4\int_{\partial\mathcal{M}} d^{2n}x\sqrt{-h}\frac{d\mathcal{L}}{dF^2}NF^{ri}\delta A_i. \tag{88}$$

Then, the surface term from the electromagnetic part vanishes when fixing the gauge potential at the boundary, Eq. (85).

Checking explicitly the cancellation of the leading-order divergences in the above action proves to be slightly more complicated than in the even-dimensional case, but one may reason as follows: the second and third lines cancel out when taking the condition on the asymptotic curvature (84) for the particular value of c_{2n} given by Eq. (87). On the other hand, for any asymptotically AdS spacetime, the extrinsic curvature K^i_j has a regular expansion in the asymptotic region, $K^i_j = \frac{1}{\ell_{\rm eff}}\delta^i_j + \mathcal{O}(1/r)$. This means that variations of the extrinsic curvature vanishes at the leading order in the vicinity of $\partial\mathcal{M}$. These conditions guarantee a well-posed action principle for odd-dimensional EGB AdS gravity, issue that was discussed previously in Ref. [42].

VI. CONSERVED QUANTITIES

A. Electric charge

We will first derive the electric charge Q as a conserved quantity associated to $U(1)$ gauge symmetry $\delta_\lambda A_\mu = \partial_\mu\lambda$, $\delta_\lambda g_{\mu\nu} = 0$, as its computation does not depend on the spacetime dimension. The gravitational part of the surface term in Eq. (80) is gauge-invariant, such that it implies the conservation of the Noether current

$$\delta_\lambda I = \int_{\mathcal{M}} d^Dx\partial_\mu J^\mu(\lambda) = 4\int_{\mathcal{M}} d^Dx\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\frac{d\mathcal{L}}{dF^2}\partial_\nu\lambda\right), \tag{89}$$

where the current J^μ transforms as a vector density of weight $+1$. In the radial foliation (73), the electric charge is then the normal component of the above current

$$Q[\lambda] = \int_{\partial\mathcal{M}} d^{D-1}x\frac{1}{N}n_\mu J^\mu(\lambda), \tag{90}$$

which, using the fact that n_μ is covariantly constant, can be rewritten as

$$\sqrt{-h}n_\mu\frac{d\mathcal{L}}{dF^2}F^{\mu\nu}\partial_\nu\lambda = \partial_i\left(\lambda\sqrt{-h}n_\mu F^{\mu i}\frac{d\mathcal{L}}{dF^2}\right) - \lambda n_\mu\sqrt{-h}\mathcal{E}^\mu. \tag{91}$$

As a consequence, since $\mathcal{E}^\mu = 0$, we are able to write down the integrand in Eq. (90) as a total derivative. In order to use the Stokes' theorem, we take a timelike Arnowitt-Deser-Misner foliation for the line element on $\partial\mathcal{M}$ with the coordinates $x^i = (t, y^m)$, as

$$h_{ij}dx^idx^j = -\tilde{N}^2(t)dt^2 + \sigma_{mn}(dy^m + \tilde{N}^m dt)(dy^n + \tilde{N}^n dt),\quad \sqrt{-h} = \tilde{N}\sqrt{\sigma}, \tag{92}$$

that is generated by the timelike normal vector $u_i = (u_t, u_m) = (-\tilde{N}, \vec{0})$. The metric σ_{mn} describes the geometry of the boundary of spatial section at constant time Σ_∞.

Setting $\lambda = 1$, the $U(1)$ charge reads

$$Q = 4\int_{\Sigma_\infty} d^{D-2}y\sqrt{\sigma}u_iNF^{ri}\frac{d\mathcal{L}}{dF^2}. \tag{93}$$

For the static black hole metric (25) (where $N = 1/f^2$ and $\tilde{N} = f^2$) and the electromagnetic field strength (31), one obtains a general formula for NED electric charge

$$Q = -4\,\mathrm{Vol}(\Gamma_{D-2})\lim_{r\to\infty}\left(r^{D-2}E\frac{d\mathcal{L}}{dF^2}\right). \tag{94}$$

Finally, using the generalized Gauss law (34), it is possible to define a finite electric charge of the black hole

$$Q = 4\,\mathrm{Vol}(\Gamma_{D-2})q, \tag{95}$$

for an arbitrary NED Lagrangian. However, this definition does not guarantee only by itself that the electric field is well-behaved in the asymptotic region.

B. Black hole mass

In order to calculate the conserved quantities associated to global isometries of the spacetime, we first consider the action of diffeomorphisms $\delta x^\mu = \xi^\mu(x)$ on the fields $g_{\mu\nu}$ and A_μ in terms of the Lie derivative,

$$\delta_\xi g_{\mu\nu} = \pounds_\xi g_{\mu\nu} \equiv -(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu),\quad \delta_\xi A_\mu = \pounds_\xi A_\mu \equiv -\partial_\mu(\xi^\nu A_\nu) + \xi^\nu F_{\mu\nu}, \tag{96}$$

which implies the transformation rule of the Christoffel symbol

$$\pounds_\xi\Gamma^\alpha_{\mu\nu} = \frac{1}{2}(R^\alpha{}_{\mu\nu\beta} + R^\alpha{}_{\nu\mu\beta})\xi^\beta - \frac{1}{2}(\nabla_\mu\nabla_\nu\xi^\alpha + \nabla_\nu\nabla_\mu\xi^\alpha). \tag{97}$$

This leads to the transformation of the volume element, Jacobian and Lagrangian density $\mathcal{L}_0$ defined by Eq. (1) as

$$\delta_\xi(d^Dx) = d^Dx\partial_\mu\xi^\mu,\quad \delta_\xi\sqrt{-g} = -\sqrt{-g}\nabla_\mu\xi^\mu,\quad \delta_\xi\mathcal{L}_0 = \frac{\partial\mathcal{L}_0}{\partial g_{\mu\nu}}\pounds_\xi g_{\mu\nu} + \frac{\partial\mathcal{L}_0}{\partial\Gamma^\beta_{\mu\nu}}\pounds_\xi\Gamma^\beta_{\mu\nu} + \frac{\partial\mathcal{L}_0}{\partial A_\mu}\pounds_\xi A_\mu + \xi^\mu\partial_\mu\mathcal{L}_0. \tag{98}$$

Then the total action (79) transforms under diffeomorphisms as

$$\delta_\xi I = \int_{\mathcal{M}} d^D x[\mathcal{L}_\xi(\sqrt{-g}\mathcal{L}_0) + \partial_\mu(\sqrt{-g}\xi^\mu\mathcal{L}_0)] + c_{D-1}\int_{\partial\mathcal{M}} d^{D-1}x[\mathcal{L}_\xi B_{D-1} + \partial_i(\xi^i B_{D-1})]$$
$$= \int_{\partial\mathcal{M}} d^{D-1}x n_\mu\Big(\frac{1}{N}\Theta^\mu(\xi) + \sqrt{-h}\xi^\mu\mathcal{L}_0 + c_{D-1}n^\mu\partial_i(\xi^i B_{D-1})\Big) + - \int_{\mathcal{M}} d^D x\sqrt{-g}\Big(\frac{1}{16\pi G}\mathcal{E}^{\mu\nu}\mathcal{L}_\xi g_{\mu\nu} + 4\mathcal{E}^\mu\mathcal{L}_\xi A_\mu\Big), \tag{99}$$

where $\Theta(\xi) = \frac{1}{N}n_\mu\Theta^\mu(\xi)$ is the surface term in Eq. (80) evaluated in the corresponding Lie derivative of the fields.

The Noether current derived from the diffeormorphic invariance, $\delta_\xi I = \int_{\mathcal{M}} d^D x\partial_\mu J^\mu(\xi) = 0$ is, therefore,

$$J^\mu(\xi) = \Theta^\mu(\xi) + \sqrt{-g}\xi^\mu\mathcal{L}_0 + c_{D-1}Nn^\mu\partial_i(\xi^i B_{D-1}). \tag{100}$$

The conservation law $\partial_\mu J^\mu = 0$ implies the existence of a conserved quantity, which corresponds to the normal component of the current J^μ,

$$Q[\xi] = \int_{\partial\mathcal{M}} d^{D-1}x\frac{1}{N}n_\mu J^\mu(\xi). \tag{101}$$

In general, it is not guaranteed that the Noether charge can be written as surface integral in $(D-2)$ dimensions. However, for the action I, the radial component $J^r = \frac{1}{N}n_\mu J^\mu$ in the foliation (73) is globally a total derivative on $\partial\mathcal{M}$, i.e.,

$$J^r = \partial_j(\sqrt{-h}\xi^i(q_i^j + q_{(0)i}^j)). \tag{102}$$

The splitting in the above integrand is justified as follows: q_i^j produces the mass and other conserved quantities for black hole solutions. As we will show below, this part of the charge identically vanishes for the vacuum states of the theory. The term $q_{(0)i}^j$ gives rise to a vacuum energy, which is present only in odd dimensions.

Therefore, the conserved charges $Q[\xi]$ of the theory for a given set of asymptotic Killing vectors $\{\xi\}$ are expressed as integrals on Σ_∞ [whose metric has been defined in Eq. (92)],

$$Q[\xi] = \int_{\Sigma_\infty} d^{D-2}y\sqrt{\sigma}u_j\xi^i(q_i^j + q_{(0)i}^j). \tag{103}$$

1. Even dimensions

In even dimensions, the expression for the surface term $\Theta(\xi)$ is obtained from (83) by replacing the variations by the corresponding Lie derivative of the fields,

$$\frac{1}{N}n_\mu\Theta^\mu(\xi) = \frac{\sqrt{-h}}{16\pi G(2n-2)!2^{n-1}}\delta^{[j_1\cdots j_{2n-1}]}_{[i_1\cdots i_{2n-1}]}[(h^{-1}\mathcal{L}_\xi h)^{i_1}_k K^k_{j_1} + 2\mathcal{L}_\xi K^{i_1}_{j_1}]$$
$$\times\Big[(\delta^{[i_2i_3]}_{[j_2j_3]} + 2\alpha(2n-2)(2n-3)R^{i_2i_3}_{j_2j_3})\delta^{[i_4i_5]}_{[j_4j_5]}\cdots\delta^{[i_{2n-2}i_{2n-1}]}_{[j_{2n-2}j_{2n-1}]} - (-\ell_{\text{eff}}^2)^{n-1}$$
$$\times\Big(1 - \frac{2\alpha}{\ell_{\text{eff}}^2}(2n-2)(2n-3)\Big)R^{i_2i_3}_{j_2j_3}\cdots R^{i_{2n-2}i_{2n-1}}_{j_{2n-2}j_{2n-1}}\Big] + 4\sqrt{-h}\frac{d\mathcal{L}}{dF^2}NF^{ri}\mathcal{L}_\xi A_i. \tag{104}$$

As a result of the Noether procedure, the integrand in the conserved charge (103) is

$$q_i^j = \frac{1}{16\pi G(2n-2)!2^{n-2}}\delta^{[jj_2\cdots j_{2n-1}]}_{[i_1\cdots i_{2n-1}]}K_i^{i_1}$$
$$\times\Big[(\delta^{[i_2i_3]}_{[j_2j_3]} + 2\alpha(2n-2)(2n-3)$$
$$\times R^{i_2i_3}_{j_2j_3})\delta^{[i_4i_5]}_{[j_4j_5]}\cdots\delta^{[i_{2n-2}i_{2n-1}]}_{[j_{2n-2}j_{2n-1}]} - (-\ell_{\text{eff}}^2)^{n-1}$$
$$\times\Big(1 - \frac{2\alpha}{\ell_{\text{eff}}^2}(2n-2)(2n-3)\Big)R^{i_2i_3}_{j_2j_3}\cdots R^{i_{2n-2}i_{2n-1}}_{j_{2n-2}j_{2n-1}}\Big], \tag{105}$$

plus a NED contribution due to the last line in Eq. (104), which vanishes for black hole solutions, as shown below. At the same time, $q_{(0)i}^j = 0$ for even dimensions.

The second and third lines in the expression (105) can be seen as a polynomial of rank $(n-1)$ in the Riemann tensor and the Kronecker delta $\frac{1}{\ell_{\text{eff}}^2}\delta^{[i_2i_3]}_{[j_2j_3]}$, which can be factorized by $(R^{i_2i_3}_{j_2j_3} + \frac{1}{\ell_{\text{eff}}^2}\delta^{[i_2i_3]}_{[j_2j_3]})$. As a consequence of the fact that for any maximally symmetric spacetime this factor vanishes, any conserved quantity defined on it will be identically zero in even dimensions.

The energy of black hole solution to EGB AdS gravity coupled to NED (25) is computed evaluating the formula (103) for the Killing vector $\xi^i = (1, \vec{0})$ and the unit normal $u_i = (-f, \vec{0})$ which defines a constant time slice,

$$\begin{aligned} M &\equiv Q[\partial_t] \\ &= -\frac{1}{16\pi G(2n-2)!2^{n-2}}\int_{\Gamma_{2n-2}} d^{2n-2}\varphi\sqrt{\gamma}f r^{2n-2}\delta^{[m_1\cdots m_{2n-2}]}_{[n_1\cdots n_{2n-2}]}K^t_t \\ &\quad\times\Big[(\delta^{[n_1n_2]}_{[m_1m_2]}+2\alpha(2n-2)(2n-3)R^{n_1n_2}_{m_1m_2})\delta^{[n_3n_4]}_{[m_3m_4]}\cdots\delta^{[n_{2n-3}n_{2n-2}]}_{[m_{2n-3}m_{2n-2}]} \\ &\quad-(-\ell^2_{\rm eff})^{n-1}\Big(1-\frac{2\alpha}{\ell^2_{\rm eff}}(2n-2)(2n-3)\Big)R^{n_1n_2}_{m_1m_2}\cdots R^{n_{2n-3}n_{2n-2}}_{m_{2n-3}m_{2n-2}}\Big]. \end{aligned} \tag{106}$$

From the explicit form of the extrinsic curvature

$$K^i_j = -\frac{1}{2N}h^{ik}h'_{kj} = \begin{pmatrix} -f' & 0 \\ 0 & -\frac{f}{r}\delta^m_n \end{pmatrix}, \tag{107}$$

and the boundary components of the Riemann tensor in Eq. (27), one obtains a general formula for the mass in even dimensions,

$$\begin{aligned} M = &\frac{\mathrm{Vol}(\Gamma_{2n-2})}{16\pi G}\lim_{r\to\infty} r^{2n-2}(f^2)'\Big[1-2\alpha(2n-2)(2n-3) \\ &\times\frac{f^2-k}{r^2}-\Big(1-\frac{2\alpha}{\ell^2_{\rm eff}}(2n-2)(2n-3)\Big) \\ &\times\ell^{2n-2}_{\rm eff}\Big(\frac{f^2-k}{r^2}\Big)^{n-1}\Big]. \end{aligned} \tag{108}$$

In order to relate the above expression to the integration constant μ, one must consider the asymptotic expansion of the metric function (51) in the following way:

$$\frac{f^2-k}{r^2} = \frac{1}{\ell^2_{\rm eff}} - \frac{\mu}{1-\frac{2\alpha}{\ell^2_{\rm eff}}(2n-3)(2n-4)}\frac{1}{r^{2n-1}} + \mathcal{O}\Big(\frac{1}{r^{4n-4}}\Big), \tag{109}$$

$$\begin{aligned} \Big(\frac{f^2-k}{r^2}\Big)^{n-1} = &\frac{1}{\ell^{2n-2}_{\rm eff}} - \frac{(n-1)\mu}{1-\frac{2\alpha}{\ell^2_{\rm eff}}(2n-3)(2n-4)} \\ &\times\frac{1}{\ell^{2n-4}_{\rm eff}r^{2n-1}} + \mathcal{O}\Big(\frac{1}{r^{4n-4}}\Big), \end{aligned} \tag{110}$$

and its derivative (52). When expanded, Eq. (108) might contain divergences of order r^{2n-1}. It is then a remarkable fact that the divergent terms cancel out for the particular value of c_{2n-1} in Eq. (82), which leaves a finite result for the energy

$$M = \frac{(2n-2)\mathrm{Vol}(\Gamma_{2n-2})\mu}{16\pi G}, \tag{111}$$

in agreement with the expression found in, e.g., Ref. [44].

Now, we turn our attention to the NED contribution to the diffeomorphic transformation of the action, that is, the last line in Eq. (104). This part of the surface term produces, by virtue of the Noether theorem, an additional piece with respect to the charge formula given by Eq. (105), which is written in any dimension as

$$Q_{\rm NED}[\xi] = -4\int_{\Sigma_\infty} d^{D-2}y\sqrt{\sigma}u_j\frac{d\mathcal{L}}{dF^2}NF^{rj}(\xi^iA_i). \tag{112}$$

However, when we evaluate Eq. (112) for the Killing vector $\xi = \partial_t$ and the static black hole metric, we notice that

$$Q_{\rm NED}[\partial_t] = -4q\,\mathrm{Vol}(\Gamma_{D-2})\phi(\infty) = 0, \tag{113}$$

as anticipated in the discussion following the deduction of the charge formula.

2. Odd dimensions

The form of the surface term $\Theta(\xi)$ in odd dimensions ($D = 2n+1$) follows from the on-shell variation of the action, Eq. (88). Its expression is slightly more complicated than in the even-dimensional case

$$\begin{aligned} \frac{1}{N}n_\mu\Theta^\mu(\xi) = &\frac{\sqrt{-h}}{16\pi G(2n-1)!2^{n-1}}\delta^{[j_1\cdots j_{2n}]}_{[i_1\cdots i_{2n}]}[(h^{-1}\mathcal{L}_\xi h)^{i_1}_k K^k_{j_1} + 2\mathcal{L}_\xi K^{i_1}_{j_1}]\delta^{i_2}_{j_2} \\ &\times\Big[(\delta^{[i_3i_4]}_{[j_3j_4]}+2\alpha(2n-1)(2n-2)R^{i_3i_4}_{j_3j_4})\delta^{[i_5i_6]}_{[j_5j_6]}\cdots\delta^{[i_{2n-1}i_{2n}]}_{[j_{2n-1}j_{2n}]} + 16\pi G(2n-1)!nc_{2n} \\ &\times\int_0^1 dt\Big(R^{i_3i_4}_{j_3j_4}+\frac{t^2}{\ell^2_{\rm eff}}\delta^{[i_3i_4]}_{[j_3j_4]}\Big)\cdots\Big(R^{i_{2n-1}i_{2n}}_{j2n-1j_{2n}}+\frac{t^2}{\ell^2_{\rm eff}}\delta^{[i_{2n-1}i_{2n}]}_{[j_{2n-1}j_{2n}]}\Big)\Big] + nc_{2n}\sqrt{-h} \\ &\times\int_0^1 dtt\delta^{[j_1\cdots j_{2n}]}_{[i_1\cdots i_{2n}]}[(h^{-1}\delta h)^{i_1}_k(K^k_{j_1}\delta^{i_2}_{j_2}-\delta^k_{j_1}K^{i_2}_{j_2}) + 2\delta^{i_2}_{j_2}\delta K^{i_2}_{j_2}]\times\Big(\frac{1}{2}\mathcal{R}^{i_3i_4}_{j_3j_4} - t^2K^{i_3}_{j_3}K^{i_4}_{j_4} + \frac{t^2}{\ell^2_{\rm eff}}\delta^{i_3}_{j_3}\delta^{i_4}_{j_4}\Big)\cdots \\ &\times\Big(\frac{1}{2}\mathcal{R}^{i_{2n-1}i_{2n}}_{j_{2n-1}j_{2n}} - t^2K^{i_{2n-1}}_{j_{2n-1}}K^{i_{2n}}_{j_{2n}} + \frac{t^2}{\ell^2_{\rm eff}}\delta^{i_{2n-1}}_{j_{2n-1}}\delta^{i_{2n}}_{j_{2n}}\Big) + 4\sqrt{-h}\frac{d\mathcal{L}}{dF^2}NF^{ri}\mathcal{L}_\xi A_i, \end{aligned} \tag{114}$$

where, for shortness' sake, we have chosen not to use the explicit form of c_{2n} given by Eq. (87).

In odd dimensions, the Noether charge appears as the sum of two parts, since $q^j_{(0)i}$ in Eq. (103) is no longer vanishing. The first part takes the form

$$q^j_i = \frac{1}{16\pi G(2n-1)!2^{n-2}}\delta^{[jj_2\cdots j_{2n}]}_{[i_1\cdots i_{2n}]}K^{i_1}_i\delta^{i_2}_{j_2}\times\Big[(\delta^{[i_3i_4]}_{[j_3j_4]}+2\alpha(2n-1)(2n-2)R^{i_3i_4}_{j_3j_4})\delta^{[i_5i_6]}_{[j_5j_6]}\cdots\delta^{[i_{2n-1}i_{2n}]}_{[j_{2n-1}j_{2n}]}$$
$$+16\pi G(2n-1)!nc_{2n}\int_0^1 dt\Big(R^{i_3i_4}_{j_3j_4}+\frac{t^2}{\ell^2_{\rm eff}}\delta^{[i_3i_4]}_{[j_3j_4]}\Big)\cdots\Big(R^{i_{2n-1}i_{2n}}_{j_{2n-1}j_{2n}}+\frac{t^2}{\ell^2_{\rm eff}}\delta^{[i_{2n-1}i_{2n}]}_{[j_{2n-1}j_{2n}]}\Big)\Big]; \tag{115}$$

whereas, the second one is given by

$$q^j_{(0)i} = nc_{2n}\int_0^1 dt\,t\,\delta^{[jj_2\cdots j_{2n}]}_{[ki_2\cdots i_{2n}]}(K^k_i\delta^{i_2}_{j_2}+K^k_{j_2}\delta^{i_2}_i)$$
$$\times\Big(\frac{1}{2}\mathcal{R}^{i_3i_4}_{j_3j_4}-t^2K^{i_3}_{j_3}K^{i_4}_{j_4}+\frac{t^2}{\ell^2_{\rm eff}}\delta^{i_3}_{j_3}\delta^{i_4}_{j_4}\Big)\times\cdots$$
$$\times\Big(\frac{1}{2}\mathcal{R}^{i_{2n-1}i_{2n}}_{j_{2n-1}j_{2n}}-t^2K^{i_{2n-1}}_{j_{2n-1}}K^{i_{2n}}_{j_{2n}}+\frac{t^2}{\ell^2_{\rm eff}}\delta^{i_{2n-1}}_{j_{2n-1}}\delta^{i_{2n}}_{j_{2n}}\Big). \tag{116}$$

We recall the fact that the constant c_{2n} was chosen to cancel at least the leading-order divergence in the variation of the action (88). Thus, it can be readily checked that q^j_i is identically zero for global AdS spacetime which satisfies (19) in the bulk. This means that the second and third lines in the expression (105) are again a polynomial of rank $(n-1)$ in the Riemann tensor and the Kronecker delta $\frac{1}{\ell^2_{\rm eff}}\delta^{[i_2i_3]}_{[j_2j_3]}$, where $R^{i_2i_3}_{j_2j_3}=-\frac{1}{\ell^2_{\rm eff}}\delta^{[i_2i_3]}_{[j_2j_3]}$ is a root of it. Therefore, any maximally symmetric spacetime will have vanishing mass and angular momentum due to the fact that $q^j_i=0$, such that all the contributions to the vacuum energy will come necessarily from Eq. (116), as shown below. On the other hand, the presence of c_{2n} in the formula of vacuum energy reflects the fact that its existence is entirely due to the addition of the Kounterterm series (86).

Proceeding as in the even-dimensional case, we compute the black hole mass evaluating the first term in the formula (103),

$$M = \int_{\Sigma_\infty} d^{D-2}y\sqrt{\sigma}u_t\xi^t q^t_t$$
$$= -\frac{1}{16\pi G(2n-1)!2^{n-2}}\lim_{r\to\infty}\int_{\Gamma_{2n-2}} d^{2n-2}\varphi\sqrt{\gamma}fr^{2n-1}\delta^{[m_1\cdots m_{2n-1}]}_{[n_1\cdots n_{2n-1}]}K^t_t\delta^{n_1}_{m_1}$$
$$\times\Big[\Big(\delta^{[n_2n_3]}_{[m_2m_3]}+2\alpha(2n-1)(2n-2)R^{n_2n_3}_{m_2m_3}\Big)\delta^{[n_4n_5]}_{[m_4m_5]}\cdots\delta^{[n_{2n-2}n_{2n-1}]}_{[m_{2n-2}m_{2n-1}]}$$
$$+16\pi G(2n-1)!nc_{2n}\int_0^1 dt\Big(R^{n_2n_3}_{m_2m_3}+\frac{t^2}{\ell^2_{\rm eff}}\delta^{[n_2n_3]}_{[m_2m_3]}\Big)\cdots\Big(R^{n_{2n-2}n_{2n-1}}_{m_{2n-2}m_{2n-1}}+\frac{t^2}{\ell^2_{\rm eff}}\delta^{[n_{2n-2}n_{2n-1}]}_{[m_{2n-2}m_{2n-1}]}\Big)\Big].$$

Using the Riemann tensor in Eq. (27) and the extrinsic curvature for the generic black hole metric given by Eq. (107), the above formula reduces to

$$M = \frac{{\rm Vol}(\Gamma_{2n-1})}{16\pi G}\lim_{r\to\infty}r^{2n-1}(f^2)'$$
$$\times\Big[1-2\alpha(2n-1)(2n-2)\frac{f^2-k}{r^2}$$
$$+16\pi G(2n-1)!nc_{2n}\int_0^1 dt\Big(\frac{k-f^2}{r^2}+\frac{t^2}{\ell^2_{\rm eff}}\Big)^{n-1}\Big]. \tag{117}$$

It is straightforward to express the mass M in terms of the constant μ in the metric, by means of the expansion of the metric function in Eq. (109), its derivative (52) and the last line in the above relation,

$$\int_0^1 dt\Big(\frac{k-f^2}{r^2}+\frac{t^2}{\ell^2_{\rm eff}}\Big)^{n-1} = -\frac{1}{16\pi G(2n-1)!nc_{2n}}$$
$$\times\Big(1-\frac{2\alpha}{\ell^2_{\rm eff}}(2n-1)(2n-2)\Big)$$
$$\times\Big(1-\frac{\ell^2_{\rm eff}}{2}\frac{(2n-1)\mu}{1-\frac{4\alpha(2n-2)(2n-3)}{\ell^2_{\rm eff}}}\frac{1}{r^{2n}}\Big)+\mathcal{O}\Big(\frac{1}{r^{4n-3}}\Big). \tag{118}$$

Unless the constant c_{2n} is fixed as in Eq. (87), the formula (117) contains divergences of order r^{2n}. Therefore, the boundary term $c_{2n}B_{2n}$ plays a double role: it cancels out the divergences in the Noether charge, but also contributes with a finite piece to give the correct result for the mass

$$M = \frac{(2n-1){\rm Vol}(\Gamma_{2n-1})\mu}{16\pi G}, \tag{119}$$

which matches the one in Ref. [44]. In turn, the vacuum energy for asymptotically AdS (AAdS) black holes is reflected in the formula (116), that in the black hole ansatz (25) adopts the form

$$E_{\rm vac} = \int_{\Sigma_\infty} d^{D-2}y\sqrt{\sigma}u_t\xi^t q^t_{(0)t}$$
$$= 2nc_{2n}\lim_{r\to\infty}\int_{\Gamma_{2n-1}} d^{2n-1}\varphi\sqrt{\gamma}r^{2n-1}f\delta^{[m_1\cdots m_{2n-1}]}_{[n_1\cdots n_{2n-1}]}$$
$$\times (K^t_t\delta^{n_1}_{m_1} - K^{n_1}_{m_1})\times \tag{120}$$

$$\int_0^1 dtt\left(\frac{1}{2}\mathcal{R}^{n_2n_3}_{m_2m_3} - t^2K^{n_2}_{m_2}K^{n_3}_{m_3} + \frac{t^2}{\ell^2}\delta^{n_2}_{m_2}\delta^{n_3}_{m_3}\right)\times\cdots$$
$$\times\left(\frac{1}{2}\mathcal{R}^{n_{2n-2}n_{2n-1}}_{m_{2n-2}m_{2n-1}} - t^2K^{n_{2n-2}}_{m_{2n-2}}K^{n_{2n-1}}_{m_{2n-1}} + \frac{t^2}{\ell^2}\delta^{n_{2n-2}}_{m_{2n-2}}\delta^{n_{2n-1}}_{m_{2n-1}}\right). \tag{121}$$

More explicitly, plugging in the components of the boundary curvature,

$$\mathcal{R}^{\;n_1n_2}_{\;m_1m_2} = \frac{k}{r^2}\delta^{[n_1n_2]}_{[m_1m_2]}, \qquad \mathcal{R}^{tn}_{tm} = 0, \tag{122}$$

the zero-point energy of the system is

$$E_{\rm vac} = 2n(2n-1)!c_{2n}\,{\rm Vol}(\Gamma_{2n-1})\lim_{r\to\infty}\int_0^1 dtt$$
$$\times\left(f^2 - \frac{r(f^2)'}{2}\right)\left[k + \left(\frac{r^2}{\ell^2_{\rm eff}} - f^2\right)t^2\right]^{n-1}. \tag{123}$$

As the metric function and its derivative can be expanded as in Eqs. (51) and (52), we notice that all the terms that depend on the parameter μ vanish in the limit $r\to\infty$. As expected, the vacuum energy depends only on the topological parameter k, the effective AdS radius and GB coupling, that is,

$$E_{\rm vac} = (2n-1)!c_{2n}\,{\rm Vol}(\Gamma_{2n-1})k^n$$
$$= (-k)^n\frac{{\rm Vol}(\Gamma_{2n-1})}{8\pi G}\ell^{2n-2}_{\rm eff}\frac{(2n-1)!!^2}{(2n)!}$$
$$\times\left(1 - \frac{2\alpha}{\ell^2_{\rm eff}}(2n-1)(2n-2)\right). \tag{124}$$

The above formula matches the vacuum energy in EGB gravity obtained in Ref. [42] by means of Kounterterm regularization. This implies that for an arbitrary NED Lagrangian the falloff of the electromagnetic field is always such that it does not contribute to the total energy of the gravitational configuration.

VII. CONCLUSIONS

We have used counterterms for Einstein-Gauss-Bonnet gravity coupled to nonlinear electrodynamics in the form of polynomials in the extrinsic and intrinsic curvatures of the boundary in order to regularize the conserved charges in the AdS sector of the theory. It has been shown that this regularization scheme (also known as Kounterterm method) provides finite values for the mass for charged static black holes with spherical, locally flat and hyperbolic transversal section in all dimensions, and the correct vacuum energy in odd dimensions.

We have also analyzed the falloff conditions that ensure the finiteness of the electric charge for an arbitrary NED Lagrangian $\mathcal{L}(F^2)$, which do not produce additional contributions to the mass of black hole in Einstein-Gauss-Bonnet AdS gravity.

It is well-known that a vacuum energy for global AdS spacetime in odd dimensions appears only in background-independent methods to compute conserved quantities. This is particularly important from the semiclassical point of view in order to interpret the Noether charges as thermodynamic variables, and to consistently incorporate the vacuum energy in the definition of internal energy of the system [45], in a similar fashion as in Einstein-BI system [46] (for a thermodynamic analysis of the same system using a background-subtraction method, see Ref. [47]). The addition of a series of intrinsic counterterms in pure EGB AdS gravity (see, e.g., Refs. [37–39]) presents the advantage of obtaining the conserved quantities from a boundary stress tensor, that is, as holographic charges. However, the explicit form of such series does not exist for a high enough dimension. On the contrary, an expression for the Kounterterms is given by Eqs. (81) and (86) in all dimensions. In that respect, one would like to see the above charges as coming from a quasilocal stress (Brown-York) tensor. There are good reasons that make us think that this could be possible, despite the fact that the on-shell variation of the action takes the form

$$\delta I_D = \int_{\partial\mathcal{M}} d^{D-1}x\sqrt{-h}$$
$$\times\left(\frac{1}{2}\tau^j_i(h^{-1}\delta h)^i_j + \Delta^j_i\delta K^i_j + \Omega^i\delta A_i\right), \tag{125}$$

where one cannot directly define a quasilocal stress tensor as $T^{ij} = (2/\sqrt{-h})(\delta I_D/\delta h_{ij})$.

Indeed, there are gravity theories where the surface term in δI contains variations of the extrinsic curvature δK^i_j, which cannot be eliminated by the addition of a generalized Gibbons-Hawking term, and where a holographic stress tensor for AAdS spacetimes can be still read off from the variation of the action. One example featuring this property is Topologically Massive Gravity in three-dimensional, where the surface term coming from the variation of the gravitational Chern-Simons term contains δK^i_j. It is known that there is no term that can be added to the action to trade it off by a piece along δh_{ij}. However, it can be shown that in the asymptotically AdS sector of the theory, there is a contribution from the gravitational Chern-Simons term to the holographic stress tensor which couples to the conformal structure $g_{(0)ij}$, even though a quasilocal stress tensor associated to δh_{ij} cannot be defined [48]. This follows from the fact that, for AAdS spaces, the leading order in the expansion of the boundary metric is the same as the leading order of the extrinsic curvature. A quasilocal stress tensor cannot be identified either in 4D AdS gravity when one adds the (topological) Gauss-Bonnet term to the

Einstein-Hilbert action. In this case, the Gauss-Bonnet term does not change the field equations in the bulk but, as expected, it modifies the surface term in the variation of the action. In this case, δI also adopts the form of Eq. (125). However, the second term in (76)—which in $D>4$ sets a well-defined action principle when the metric is held fixed at the boundary—cannot be used for the same purpose in four dimensions. One can show that the variation of the action produces a boundary stress tensor τ_i^j for AdS gravity (upon a suitable choice of the GB coupling) which is finite and the same as the one prescribed by holographic renormalization [49]. This is a consequence of the fact that the contribution $\sqrt{-h}\Delta_i^j\delta K_j^i$ vanishes identically when one performs an asymptotic expansion of the fields.

The above examples give some indication on what should be the pattern in higher-dimensional Einstein-Hilbert and Einstein-Gauss-Bonnet AdS case: in $D=2n$ dimensions, the term that contains δK_j^i should always vanish as we approach to the asymptotic region, such that the quasilocal stress tensor can be read off directly from Eq. (125). On the other hand, in odd dimensions, $\Delta_i^j\delta K_j^i$ should contribute with a finite piece to the holographic stress tensor which does not modify the Weyl anomaly. We expect to provide a proof of the above claim elsewhere.

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APPENDIX A: KRONECKER DELTA OF RANK p

The totally antisymmetric Kronecker delta of rank p is defined as the determinant

$$\delta^{[\nu_1\cdots\nu_p]}_{[\mu_1\cdots\mu_p]} := \begin{vmatrix} \delta^{\nu_1}_{\mu_1} & \delta^{\nu_2}_{\mu_1} & \cdots & \delta^{\nu_p}_{\mu_1} \\ \delta^{\nu_1}_{\mu_2} & \delta^{\nu_2}_{\mu_2} & & \delta^{\nu_p}_{\mu_2} \\ \vdots & & \ddots & \\ \delta^{\nu_1}_{\mu_p} & \delta^{\nu_2}_{\mu_p} & \cdots & \delta^{\nu_p}_{\mu_p} \end{vmatrix}. \tag{A1}$$

A contraction of $k\le p$ indices in the Kronecker delta of rank p produces a delta of rank $p-k$,

$$\delta^{[\nu_1\cdots\nu_k\cdots\nu_p]}_{[\mu_1\cdots\mu_k\cdots\mu_p]}\delta^{\mu_1}_{\nu_1}\cdots\delta^{\mu_k}_{\nu_k} = \frac{(N-p+k)!}{(N-p)!}\delta^{[\nu_{k+1}\cdots\nu_p]}_{[\mu_{k+1}\cdots\mu_p]}, \tag{A2}$$

where N is the range of indices.

APPENDIX B: HYPERGEOMETRIC FUNCTION

We use an integral representation of the Gauss' hypergeometric function,

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 du\,\frac{u^{b-1}(1-u)^{c-b-1}}{(1-zu)^a}, \tag{B1}$$

where c is not a negative integer and either $|z|<1$, or $|z|=1$ with $\Re e(c-a-b)>0$. In particular, the following integral is solved in the text,

$$\int_0^1 du\,\frac{u^{b-1}}{\sqrt{1+zu}} = \frac{1}{b}\,{}_2F_1\left(\frac{1}{2},b;b+1;-z\right), \qquad b>0. \tag{B2}$$

The first derivative of the hypergeometric function is

$$\frac{d}{dz}\,{}_2F_1(a,b;c;z) = \frac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;z), \tag{B3}$$

and it expands for small z as

$${}_2F_1(a,b;c;z) = 1+\frac{ab}{c}z+\frac{a(a+1)b(b+1)}{2c(c+1)}z^2+\mathcal{O}(z^3). \tag{B4}$$

APPENDIX C: GAUSS-NORMAL COORDINATE FRAME

In Gaussian coordinates (73), the only relevant components of the connection $\Gamma^\alpha_{\mu\nu}$ are expressed in terms of the extrinsic curvature $K_{ij}=-\frac{1}{2N}h'_{ij}$ as

$$\Gamma^r_{ij} = \frac{1}{N}K_{ij}, \qquad \Gamma^i_{rj} = -NK^i_j, \qquad \Gamma^r_{rr} = \frac{N'}{N}. \tag{C1}$$

The radial foliation (73) implies the Gauss-Codazzi relations for the spacetime curvature, as well,

$$R^{ir}_{kl} = \frac{1}{N}(\nabla_l K^i_k - \nabla_k K^i_l), \tag{C2}$$

$$R^{ir}_{kr} = \frac{1}{N}(K^i_k)' - K^i_l K^l_k, \tag{C3}$$

$$R^{ij}_{kl} = \mathcal{R}^{ij}_{kl}(h) - K^i_k K^j_l + K^i_l K^j_k \equiv \mathcal{R}^{ij}_{kl} - K^{[i}_{[k}K^{j]}_{l]}, \tag{C4}$$

where $\nabla_i=\nabla_i(h)$ is the covariant derivative defined in the Christoffel symbol of the boundary $\Gamma^k_{ij}(g)=\Gamma^k_{ij}(h)$ and $\mathcal{R}^{ij}_{kl}(h)$ is the intrinsic curvature of the boundary.

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