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Instability of black hole formation in gravitational collapse

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We consider here the classic scenario given by Oppenheimer, Snyder, and Datt, for the gravitational collapse of a massive matter cloud, and examine its stability under the introduction of small tangential stresses. We show, by offering an explicit class of physically valid tangential stress perturbations, that an introduction of tangential pressure, however small, can qualitatively change the final fate of collapse from a black hole final state to a naked singularity. This shows instability of black hole formation in collapse and sheds important light on the nature of cosmic censorship hypothesis and its possible formulations. The key effect of these perturbations is to alter the trapped surface formation pattern within the collapsing cloud and the apparent horizon structure. This allows the singularity to be visible, and implications are discussed.

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The continual gravitational collapse of a massive matter cloud within the framework of general relativity was investigated for the first time by the classic works of Oppenheimer and Snyder, and Datt (OSD) [1]. Such a treatment of dynamical collapse would be essential to determine the final fate of a massive collapsing star which shrinks catastrophically under the force of its own gravity when its internal nuclear fuel is exhausted.

The outcome in the above case is seen to be a black hole developing in the spacetime. As the gravitational collapse progresses, an event horizon forms within the collapsing cloud and from the region within the horizon no material particles or light rays can escape, thus forming a black hole. The continually collapsing star enters the horizon and finally ends up forming a spacetime singularity, which is hidden inside the black hole and which is unseen to all the outside observers in the Universe. The matter and energy densities, spacetime curvatures, and all physical quantities blow up and take extreme values in the limit of approach to such a spacetime singularity.

This classic picture became the foundation of an extensive theory and astrophysical applications of modern day black hole physics, further to the suggestion that *all* realistic massive stars undergoing a continual gravitational collapse would have the same qualitative behavior. This means that, while the general theory of relativity necessarily implies the formation of a spacetime singularity as the end state for a massive collapsing star, such a singularity will always be necessarily hidden within a black hole. Such an assumption is known as the cosmic censorship hypothesis [2], and taking it to be valid, the theory and applications of black hole physics have developed extensively in past many decades.

The cosmic censorship has, however, remained an unproved conjecture as yet in gravitation theory, despite numerous attempts to establish the same. Therefore, in past many years, much effort has also been devoted towards understanding and analyzing the final fate of a physically realistic dynamical gravitational collapse scenario. The current status is, despite much work in studying the censorship and its implications, the issue of final fate of a complete gravitational collapse of a massive star remains far from being fully resolved. In particular, we need to formulate in a precise manner the conditions in gravitational collapse that would lead to the formation of black holes necessarily. We now know that under a wide variety of physically realistic situations, the collapse ends in a black hole or a naked singularity, depending on the initial conditions from which the collapse develops and the dynamical evolutions as allowed by the Einstein equations (see e.g. [3,4] and references therein). It is now clear that naked singularities are to be considered as a general feature of general relativistic physics and that they may develop as the end state of collapse in a broad variety of physical collapse situations.

It follows that a careful and extensive study of gravitational collapse phenomena in general relativity is the key to put the theory of black holes and their astrophysical implications on a firm footing.

From such a perspective, we investigated here the effect of introducing small stress perturbations in the collapse dynamics of the classic Oppenheimer-Snyder-Datt gravitational collapse, an idealized model assuming zero pressure, which terminates in a black hole final fate. Our key purpose here is to study the stability of the OSD black hole under introduction of small tangential pressures. Clearly, stresses within a massive collapsing star are very important physical forces to be taken into account while considering its dynamical evolution and the final fate of collapse (see for example [5]).

We show here explicitly the existence of classes of stress perturbations such that the introduction of a smallest tangential pressure within the collapsing OSD cloud changes

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the end state of collapse to formation of a naked singularity, rather than a black hole. It follows that the OSD black hole is not stable under small stress perturbations within the collapsing cloud. As we point out below, this can also be viewed as perturbing the spacetime metric of the cloud in a small way. Our work thus clarifies the role played by tangential stresses in a well-known gravitational collapse scenario. The class of stress perturbations considered here, although specific, is physically reasonable and generic enough so as to provide a good insight into the stability of the OSD black hole. Clearly, such a result provides an important insight into the structure of the censorship principle which as yet remains to be properly understood. This has also implications towards the physical consequences of final outcomes of a continual collapse, some of which are indicated in the concluding remarks.

The general spherically symmetric line element describing the collapsing matter cloud can be written as

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R(t,r)^2 d\Omega^2, \qquad (1)$$

with the stress-energy tensor for a generic matter source given by $T_t^t = -\rho$, $T_r^r = p_r$, $T_\theta^\theta = T_\phi^\phi = p_\theta$. The above is a general scenario, in that it involves no assumptions on the form of the matter or the equation of state.

In order to decide on the stability or otherwise of the OSD model under the injection of small stress perturbations, we need to consider the dynamical development of the collapsing cloud, as governed by the Einstein equations. The visibility or otherwise of the final singularity is determined by the behavior of apparent horizon in the spacetime, which is the boundary of the trapped surface region that develops as the collapse progresses. First, we define a scaling function $v(r,t)$ by the relation $R = rv$ [6]. The Einstein equations for the above spacetime geometry can then be written as

$$p_r = -\frac{\dot{F}}{R^2\dot{R}}, \qquad \rho = \frac{F'}{R^2R'}, \qquad (2)$$

$$\nu' = 2\frac{p_\theta - p_r}{\rho + p_r}\frac{R'}{R} - \frac{p_r'}{\rho + p_r}, \qquad (3)$$

$$2\dot{R}' = R'\frac{\dot{G}}{G} + \dot{R}\frac{H'}{H}, \qquad (4)$$

$$F = R(1 - G + H), \qquad (5)$$

where the functions H and G are defined as $H = e^{-2\nu(r,v)}\dot{R}^2$, $G = e^{-2\psi(r,v)}R'^2$. The above are five equations in seven unknowns, namely, ρ, p_r, p_θ, R, F, G, H. Here ρ is the mass-energy density, p_r and p_θ are the radial and tangential stresses, respectively, R is the physical radius for the matter cloud, and F is the Misner-Sharp mass function.

With the above definitions of v, H and G, we can substitute the unknowns R, H with v, ν. Without loss of generality, the scaling function v can be set $v(t_i, r) = 1$ at the initial time $t_i = 0$ when the collapse commences. It then goes to zero at the spacetime singularity t_s, which corresponds to $R = 0$, i.e. we have $v(t_s, r) = 0$. The above amounts to the scaling $R = r$ at the initial epoch, which is an allowed freedom. The collapse condition here is $\dot{R} < 0$ throughout the evolution, which is equivalent to $\dot{v} < 0$.

We can integrate (4) by defining a suitably regular function $A(r,v)$ by $\nu' \equiv A_{,v}(r,v)R'$ (the function A is defined in full generality here, while often the restriction to the class $\nu = \nu(R)$, implying $A(R) = \nu(R)$, is made, see e.g. [7]). This gives $G(r,t) = b(r)e^{2rA(r,v)}$. The arbitrary function of integration $b(r)$ can be interpreted following the analogy with dust collapse models, where pressures vanish. It turns out to be related to the velocity of the collapsing shells, and once we write it as $b(r) = 1 + r^2 b_0(r)$, we can see that values $b_0 =$ const in the dust limit correspond to the open ($b_0 < 0$), closed ($b_0 > 0$) or flat ($b_0 = 0$) Friedmann-Robertson-Walker models. The radial stress p_r and the energy density ρ are obtained from Eqs. (2), once a specific choice for the mass function $F(r,t)$ is made. The function ν can be taken as the second free function for the system so that once a particular form of ν is specified, Eq. (3) provides the tangential stress profile p_θ. Finally, from the equation of motion (5), we can integrate to obtain $v(r,t)$, thus solving the system of Einstein equations.

We can also invert the function $v(r,t)$, which is monotonically decreasing in t, to obtain the time needed by the matter shell at any radial value r to reach the event with a particular value v. We write the function $t(r,v)$ from Eq. (5) as

$$t(r,v) = \int_v^1 \frac{e^{-\nu}}{\sqrt{\frac{F}{r^3\tilde{v}} + \frac{be^{2rA}-1}{r^2}}} d\tilde{v}. \qquad (6)$$

The time taken by the shell at r to reach the spacetime singularity at $v = 0$ is then $t_s(r) = t(r,0)$.

Since $t(r,v)$ is in general at least C^2 everywhere in the spacetime (because of the regularity of the functions involved), and is continuous at the center, we can write it as

$$t(r,v) = t(0,v) + r\chi(v) + O(r^2). \qquad (7)$$

When $t(r,v)$ is differentiable, we can make a Taylor expansion near the center $r = 0$. Here, $t(0,v)$ is the above integral evaluated at $r = 0$ and $\chi(v) = \frac{dt}{dr}|_{r=0}$. As we point out below, the quantity $\chi(0)$ plays an important role towards determining the nature of the final singularity of collapse. We consider collapse from a regular initial data, and so the Einstein Eq. (5) implies that the Misner-Sharp mass $F(r,v)$ must go as r^3 near the center $r = 0$ in order for the density to be regular at the center, and also to have $t(0,v)$ well defined. Therefore, in general, F must have the form, $F(r,v) = r^3M(r,v)$, where M is a suitably regular function. Then, by continuity, the time for the shell located at any r close to the center to reach the singularity is given as $t_s(r) = t_s(0) + r\chi(0) + O(r^2)$. Basically, this means that the singularity curve should have a well-defined tangent at the center. Regularity at the center also implies that the metric function ν cannot have constant or linear terms in r in a close neighborhood of $r = 0$, and it must go as $\nu \sim r^2$ near the center. Therefore the most general choice of the free function ν is

$$\nu(r, v) = r^2 g(r, v). \tag{8}$$

Since $g(r, v)$ is a regular function (at least C^2), it can be written near $r = 0$ as

$$g(r, v) = g_0(v) + g_1(v)r + g_2(v)r^2 + \ldots. \tag{9}$$

We would now like to investigate how the OSD gravitational collapse scenario, which is a homogeneous pressureless dust cloud collapsing to give rise to a black hole, gets altered when small stress perturbations are introduced in the dynamical evolution of collapse.

To that end, we first note that the dust scenario is obtained if $p_r = p_\theta = 0$ in the above. In that case, from Eq. (3) it follows that $\nu' = 0$, and that together with the condition $\nu(0) = 0$, gives $\nu = 0$ identically. These models have been widely studied in the literature, and it is seen that for generic dust collapse the final outcome can be either a black hole or a naked singularity, depending on the nature of the initial density and velocity profiles of the collapsing matter shells [8]. In the OSD collapse to a black hole, the trapped surfaces or the apparent horizon in the spacetime develop much earlier before the formation of the final singularity of collapse. On the other hand, when inhomogeneities are allowed in the initial density profile, such as a higher density at the center of the star, then the trapped surface formation is delayed in a natural manner within the collapsing cloud and the final singularity becomes visible to faraway observers in the Universe [9].

The OSD case is obtained from above when we further assume that the collapsing dust is necessarily homogeneous at all epochs of collapse. This is of course an idealized scenario because realistic stars would have typically higher densities at the center, which slowly falls off with increasing radius, and they also would have nonzero internal stresses. Specifically, the conditions that must be imposed to obtain the OSD case from the above are given by

(a) $M = M_0$,
(b) $v = v(t)$,
(c) $b_0(r) = k$.

Then we have $F' = 3M_0 r^2$, $R' = v$, and the energy density is homogeneous throughout the collapse with $\rho = \rho(t) = 3M_0/v^3$. The spacetime geometry then becomes the Oppenheimer-Snyder metric,

$$ds^2 = -dt^2 + \frac{v^2}{1 + kr^2} dr^2 + r^2 v^2 d\Omega^2, \tag{10}$$

where the function $v(t)$ is solution of the equation of motion, $\frac{dv}{dt} = \sqrt{(M_0/v) + k}$, obtained from Einstein Eq. (5). In this case we get $\chi(0) = 0$ identically. All the matter shells then collapse into a simultaneous singularity (due to condition (b)), which is necessarily covered by the event horizon that developed in the spacetime at an earlier time, thus giving rise to a black hole.

To examine the effect of introducing stress perturbations in the above scenario and to study the models thus obtained which are close to the Oppenheimer-Snyder in this sense, we need to relax and perturb one or more of the above conditions (a), (b) or (c).

If the collapse outcome were not to be a black hole, the final singularity of collapse cannot be simultaneous. We are thus led to relax condition (b) above, allowing $v = v(t, r)$, rather than $v = v(t)$ only. At the same time, in order not to depart too much from the OSD model, we keep (a) and (c) unchanged. This also brings out more clearly the role played by the stress perturbations in the model.

In terms of the spacetime metric (1), while the metric function $\nu(t, r)$ must be identically vanishing for the dust case, the above amounts to allowing for small perturbations in ν, and allowing it to be nonzero now. This is equivalent to introducing small stress perturbations in the model, and we show below how that affects the apparent horizon developing in the collapsing cloud.

We note immediately that taking $M = M_0$ leads to $F = r^3 M_0$. We have $R' = v + rv' \to v$ for $r \to 0$ and therefore we get $A_{,v} = \nu'/v$. With the expansion near $r = 0$ for both A and g we get the relation between the coefficients of the expansion of g and those for the expansion of A. Integrating (4) in the small r limit we thus obtain $G(r, t) = b(r)e^{2\nu(r,v)}$. The radial stress p_r vanishes in this case as $\dot{F} = 0$, while the tangential pressure, obtained from Eq. (3), has the form $p_\theta = p_1 r^2 + p_2 r^3 + \ldots$, where p_1, p_2 are naturally evaluated in terms of the coefficients of M, g, and R and its derivatives:

$$p_\theta = 3\frac{M_0 g_0}{vR'^2} r^2 + \frac{9}{2}\frac{M_0 g_1}{vR'^2} r^3 + \ldots. \tag{11}$$

Here the choice of sign of the functions g_0 and g_1 is enough to ensure positivity or negativity of p_θ.

We note that scenarios with vanishing radial stresses but nonvanishing tangential stresses have been considered in past, with the most physically significant model (though not the only relevant one) being the so called "Einstein cluster" (see [10]), which describes a cloud of collapsing counter rotating particles. Naked singularities and black holes are found to arise as the end state of such models, depending on the initial density, velocity and stress configurations [11].

The first order coefficient χ in equation of the time curve of the singularity $t_s(r)$ is now obtained as

$$\chi(0) = -\int_0^1 \frac{v^{3/2} g_1(v)}{(M_0 + vk + 2vg_0(v))^{3/2}} dv. \tag{12}$$

As mentioned above, it is $\chi(0)$ that governs the nature of the singularity curve, and whether it is increasing or decreasing away from the center. Clearly, it is the matter initial data in terms of density and stress profiles, the velocity of the collapsing shells, and the allowed dynamical evolutions that govern and fix the value of $\chi(0)$.

The quantity $\chi(0)$ also governs the behavior of apparent horizon and trapped surface formation, as we show below, which in turn governs the nakedness or otherwise of the singularity. The equation for the apparent horizon is given by $F/R = 1$. It is analogous to that of the dust case since $F/R = rM/v$ in both cases [9]. So the apparent horizon curve $r_{ah}(t)$ is given by $r_{ah}^2 = \frac{v_{ah}}{M_0}$, with $v_{ah} = v(r_{ah}(t), t)$, which can also be inverted as a time curve $t_{ah}(r)$. The visibility of the singularity at the center of the collapsing cloud to faraway observers is determined by the nature of this apparent horizon curve which is given by

$$t_{ah}(r) = t_s(r) - \int_0^{v_{ah}} \frac{e^{-\nu}}{\sqrt{\frac{M_0}{v} + \frac{be^{2\nu}-1}{r^2}}} dv \qquad (13)$$

where $t_s(r)$ is the singularity time curve, whose initial point is $t_0 = t_s(0)$. Near $r = 0$ the Eq. (13) becomes

$$t_{ah}(r) = t_0 + \chi(0)r + o(r^2). \qquad (14)$$

From the above, it is now easy to see how the stress perturbation affects the time of formation of the apparent horizon, and therefore the formation of a black hole or naked singularity. A naked singularity typically occurs as a collapse end state when a comoving observer at fixed r does not encounter any trapped surfaces till the time of singularity formation. For a black hole to form, trapped surfaces develop before the singularity, so it is needed that

$$t_{ah}(r) \le t_0 \quad \text{for} \quad r > 0, \qquad \text{near } r = 0. \qquad (15)$$

It is clear that for all functions $g_1(v)$ for which $\chi(0)$ is positive, this condition is violated and the apparent horizon is forced to appear after the formation of the central singularity. The apparent horizon curve then initiates at the central singularity $r = 0$ at $t = t_0$ and increases with increasing r, moving to the future, i.e. $t_{ah} > t_0$ for $r > 0$ near the center. The behavior of outgoing families of null geodesics has been analyzed in detail in such a case when $\chi(0) > 0$ and we know that geodesics terminate at the singularity in the past. Thus timelike and null geodesics come out from the singularity, making it visible to external observers [12].

It follows that g_1 is the term in the stresses p_θ which decides the black hole or naked singularity final fate. We can choose it to be arbitrarily small, and we now see how introducing a generic tangential stress perturbation in the model would change drastically the final outcome of collapse. For all nonvanishing tangential stresses with $g_0 = 0$ and $g_1 < 0$, even the slightest perturbation of the Oppenheimer-Snyder-Datt scenario, injecting a small tangential stress would result in a naked singularity. The space of all functions g_1 that make $\chi(0)$ positive, which includes all the strictly negative functions g_1, causes the collapse to end in a naked singularity. We note that while this is an explicit example, by no means this is the only class.

The remarkable feature of this class is that it corresponds to a collapse model for a simple and straightforward perturbation of the Oppenheimer-Snyder-Datt spacetime metric, where the geometry near the center can be written as

$$ds^2 = -(1 - 2g_1 r^3)dt^2 + \frac{(v + rv')^2}{1 + kr^2 - 2g_1 r^3} dr^2 + r^2 v^2 d\Omega^2. \qquad (16)$$

The metric above satisfies Einstein equations in the neighborhood of the center of the cloud when the function $g_1(v)$ is small and bounded. We could take, for example, $0 < |g_1(v)| < \epsilon$, so that the smaller we take the parameter ϵ the bigger will be the radius where the approximation is valid. The function $v(r, t)$ above is governed by the equation of motion (5) which in the small r limit becomes $dv/dt = (1 - g_1(v)r^3)(\frac{M_0}{v} + k - 2g_1(v)r)^{1/2}$. Finally, $\chi(0)$ in this case is given by Eq. (12) with $g_0 = 0$, and in certain cases can also be integrated.

We note that any realistic matter model must satisfy some energy conditions ensuring the positivity of mass and energy density. In general, the weak energy condition implies restrictions on the density and pressure profiles. The energy density as given by the second of Eqs. (2) must be positive. Since R is positive, to ensure positivity of ρ we require $F > 0$ and $R' > 0$. The choice of positive $M(r)$ (which obviously holds for $M_0 > 0$ and is physically reasonable) ensures positivity of the mass function. Then $R' > 0$ is a sufficient condition for the avoidance of shell crossing singularities. The tangential stress can be written from (3) where $p_r = 0$, and is given by $p_\theta = \frac{1}{2}\frac{R}{R'}\rho\nu'$. So the sign of the function ν' determines the sign of p_θ. Positivity of $\rho + p_\theta$ is then ensured for small values of r throughout collapse for any form of p_θ. In fact, regardless of the values taken by M and g, there will always be a neighborhood of $r = 0$ for which $|p_\theta| < \rho$ and therefore $\rho + p_\theta \ge 0$.

The black hole and naked singularity outcomes of gravitational collapse are very different from each other physically, and would have quite different observational signatures. In the naked singularity case we have the possibility to observe the physical effects happening in the vicinity of the ultra dense regions that form in the very final stages of collapse. However, in a black hole scenario, such regions are necessarily hidden within the event horizon. The fact that a slightest stress perturbation of the OSD collapse could change the outcome drastically, taking it from a black hole to naked singularity formation, means that the naked singularity final state for a collapsing star must be studied carefully to deduce its physical consequences which are not well understood so far.

The existence of subspaces of collapse solutions as we have shown here, that go to a naked singularity final state rather than a black hole, in the arbitrary vicinity of the OSD black hole, presents an intriguing scenario. It gives an idea of the richness of the structure present in gravitation theory and the complex solution space of Einstein equations which are a complicated set of highly nonlinear partial differential equations. What we see here is there are classes of stress perturbations such that an arbitrarily small change from the OSD model is a solution going to naked singularity. In this sense, this manifests an instability in the black hole formation process in gravitational collapse. This also provides an intriguing insight into the nature of cosmic censorship, namely, that the collapse must be properly fine-tuned necessarily if it is to produce a black hole only as the final end state.

Traditionally it was believed that the presence of stresses or pressures in the collapsing matter cloud would increase the chance of black hole formation, thereby ruling out dust models that were found to lead to a naked singularity as collapse end state. That is no longer the case. The model described here not only provides a new class of collapses ending in a naked singularity, but more importantly, shows how the bifurcation line that separates the phase space of ''black hole formation'' from that of the ''naked singularity formation'' runs directly over the simplest and most studied of black hole scenarios such as the OSD model, thus making it unstable under perturbations.

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