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Poincaré gauge theory of gravity: Friedman cosmology with even and odd parity modes: Analytic part

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We propose a cosmological model in the framework of the Poincaré gauge theory of gravity (PG). The gravitational Lagrangian is quadratic in both curvature and torsion. In our specific model, the Lagrangian contains (i) the curvature scalar R and the curvature pseudoscalar X linearly and quadratically (including an RX term) and (ii) pieces quadratic in the torsion *vector* $\mathcal{V}$ and the torsion *axial* vector $\mathcal{A}$ (including a $\mathcal{V}\mathcal{A}$ term). We show generally that in quadratic PG models we have nearly the same number of parity conserving terms (''world'') and of parity violating terms (''shadow world''). This offers new perspectives in cosmology for the coupling of gravity to matter and antimatter. Our specific model generalizes the fairly realistic ''torsion cosmologies'' of Shie-Nester-Yo (2008) and Chen *et al.* (2009). With a Friedman type ansatz for an orthonormal coframe and a Lorentz connection, we derive the two field equations of PG in an explicit form and discuss their general structure in detail. In particular, the second field equation can be reduced to first order ordinary differential equations for the curvature pieces $R(t)$ and $X(t)$. Including these along with certain relations obtained from the first field equation and curvature definitions, we present a first order system of equations suitable for numerical evaluation. This is deferred to the second, numerical part of this paper.

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I. INTRODUCTION

In order to accommodate the *local Poincaré group* in spacetime, Sciama [1] and Kibble [2] had to extend the Riemannian spacetime of general relativity (GR) to a Riemann-Cartan spacetime with nonvanishing torsion T^α (for the notation see the end of the Introduction). Thereby the orthonormal coframe ϑ^α and the Lorentz connection $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$ became independent gauge potentials of weak and strong gravity, respectively. The corresponding gauge field strengths are torsion $T^\alpha = D\vartheta^\alpha$ and curvature $R^{\alpha\beta} \sim D\Gamma^{\alpha\beta}$, as spelled out in Sec. II. There we also display the irreducible decompositions of T^α and $R^{\alpha\beta}$.

If one allows in a Yang-Mills manner for a gravitational Lagrangian V that is *quadratic in torsion and curvature*, we speak of a Poincaré gauge theory of gravity (PG) [3–10]. In Sec. III A, we introduce the gravitational excitations $H_\alpha = -\partial V/\partial T^\alpha$ and $H_{\alpha\beta} = -\partial V/\partial R^{\alpha\beta}$ and recapitulate the general form (21) and (22) of the two field equations of gravity.

Then, in Sec. III B, we turn to the conventional parity conserving quadratic Lagrangian V_+, which includes the somewhat degenerate Einstein-Cartan Lagrangian $V_{\rm EC}$. Because of the existence of the Euler 4-form of the curvature, we can show that one curvature square piece is trivial. In Secs. III C and III D, we review parity violating *admixtures* to the EC Lagrangian that have been formulated in the past by different groups, stressing in Sec. III E the importance of the corresponding cosmological models of Shie-Nester-Yo [11] and Chen *et al.* [12].

Having in this way the PG at our disposal, we open for it in Sec. IV a new ''window'' to a ''shadow world''. In Sec. IV A, we show in a systematic way that, besides the parity conserving Lagrangian V_+, there exists an equally important Lagrangian V_- the pieces of which are all *parity violating*. Accordingly, for PG we propose the gravitational Lagrangian $V_\pm = V_+ + V_-$. An equivalent Lagrangian has already been discussed earlier by Obukhov *et al.* [13].

Because of the complexity of this general Lagrangian, we select for further study in Sec. IV B in Eq. (64) the simpler *9-parameter Lagrangian* $V_{\rm BHN}$, which should carry the characteristic features of parity conserving and parity violating effects. In Sec. IV C, a novel method is proposed for diagonalizing the quadratic pieces in $V_{\rm BHN}$. No linearization is involved and the output consists of exact analytic results. Besides the Einstein mode 2^+, we find for the *torsion modes* spin and parity $0^\pm$, $1^\pm$ and for the *curvature modes* $0^\pm$.

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We calculate the gravitational excitations of $V_{\rm BHN}$ (Sec. IV D) and display in Sec. IV E the corresponding field equations explicitly. They turn out to be first order partial differential equations in torsion and curvature, respectively.

We continue by looking closer into the structure of $V_{\rm BHN}$ and its field equations. In Sec. IV F the Nieh-Yan identity is used to show that the coupling constants of $V_{\rm BHN}$ occur only in certain linear combinations in the field equations.

Eventually, in Sec. V, we turn to a cosmological model. In Sec. V A the coframe and the torsion are assumed to be homogeneous and isotropic in accordance with a Friedman-Lemaître-Robertson-Walker (FLRW) type model and in Sec. V B the corresponding irreducible pieces of the curvature are calculated. We define a spinless perfect fluid in Sec. V C and find then, in Sec. V D, the field equations of gravity for this cosmological model. The first field equation yields equations for the density $\rho(t)$ and for the pressure $p(t)$ of the perfect fluid. These equations are subsequently manipulated in order to bring them into a more transparent form. The second field equation has also two independent components, namely, first order ordinary differential equations for $R(t)$ and $X(t)$. We uncouple them and bring them in the very compact form (176) and (177) by introducing certain "frequencies" ω_0, ω_1, ω_2, ω_3.

Now we are able to evaluate our exact results by numerical methods. This will be done in follow up work.

Notation

Our notation is as follows (see [14,15]): We use the formalism of exterior differential forms. We denote the *frame* by e_α, with the anholonomic or frame indices $\alpha, \beta, \ldots = 0, 1, 2, 3$. Decomposed with respect to a *natural frame* ∂_i, we have $e_\alpha = e^i{}_\alpha \partial_i$, where $i, j, \ldots = 0, 1, 2, 3$ are holonomic or coordinate indices. The frame e_α is the vector basis of the tangent space at each point of the 4D spacetime manifold. The symbol $\rfloor$ denotes the interior and $\wedge$ the exterior product. The *coframe* $\vartheta^\beta = e_j{}^\beta dx^j$ is dual to the frame, i.e., $e_\alpha \rfloor \vartheta^\beta = \delta^\beta_\alpha$. The * denotes the Hodge star operator that acts on the quantities on its right, as, for instance, in $^\star(\Sigma_a \wedge \vartheta^\beta)$. If $\vartheta^{\alpha\beta} := \vartheta^\alpha \wedge \vartheta^\beta$, etc., then we can introduce the *eta-basis* by $\eta := {}^\star 1$, $\eta^\alpha := e^\alpha \rfloor \eta = {}^\star\vartheta^\alpha$, $\eta^{\alpha\beta} := e^\beta \rfloor \eta^\alpha = {}^\star\vartheta^{\alpha\beta}$, etc. Parentheses surrounding indices $(\alpha\beta) := (\alpha\beta + \beta\alpha)/2$ denote symmetrization and brackets $[\alpha\beta] := (\alpha\beta - \beta\alpha)/2$ antisymmetrization.

The coframe ϑ^α and the η-system are related by

$$
\begin{aligned}
\vartheta^\alpha \wedge \eta_\beta &= \delta^\alpha_\beta \eta, \\
\vartheta^\alpha \wedge \eta_{\beta\gamma} &= \delta^\alpha_\gamma \eta_\beta - \delta^\alpha_\beta \eta_\gamma, \\
\vartheta^\alpha \wedge \eta_{\beta\gamma\delta} &= \delta^\alpha_\delta \eta_{\beta\gamma} + \delta^\alpha_\gamma \eta_{\delta\beta} + \delta^\alpha_\beta \eta_{\gamma\delta}, \\
\vartheta^\alpha \eta_{\beta\gamma\delta\mu} &= \delta^\alpha_\mu \eta_{\beta\gamma\delta} - \delta^\alpha_\delta \eta_{\beta\gamma\mu} + \delta^\alpha_\gamma \eta_{\beta\delta\mu} - \delta^\alpha_\beta \eta_{\gamma\delta\mu}.
\end{aligned} \tag{1}
$$

Differentiating the η's, we find in a metric-affine space (for a definition see Sec. II B) the relations

$$
\begin{aligned}
D\eta_\alpha &= -2Q \wedge \eta_\alpha + T^\mu \wedge \eta_{\alpha\mu}, \\
D\eta_{\alpha\beta} &= -2Q \wedge \eta_{\alpha\beta} + T^\mu \wedge \eta_{\alpha\beta\mu}, \\
D\eta_{\alpha\beta\gamma} &= -2Q \wedge \eta_{\alpha\beta\gamma} + T^\mu \wedge \eta_{\alpha\beta\gamma\mu}, \\
D\eta_{\alpha\beta\gamma\delta} &= -2Q \wedge \eta_{\alpha\beta\gamma\delta},
\end{aligned} \tag{2}
$$

where Q is the Weyl covector and T^μ the torsion.

We use the abbreviations: GR = general relativity theory (Riemann spacetime), EC = Einstein-Cartan theory of gravity (Riemann-Cartan spacetime with torsion and curvature, gauge Lagrangian linear in curvature), PG = Poincaré gauge theory of gravity (Riemann-Cartan spacetime, gauge Lagrangian arbitrary function of torsion and curvature; often a quadratic function), MAG = metric-affine theory of gravity (metric-affine spacetime, gauge Lagrangian arbitrary function of in torsion, nonmetricity, and curvature; often a quadratic function).

II. SPACETIME OF GAUGE THEORY OF GRAVITY

A. The coframe ϑ^α and weak gravity

One of the fundamental structures in a gauge theory of gravity is the coframe field ϑ^α—and all quantities are referred to it. It is represented by four linearly independent 1-forms ϑ^α, with $\alpha = 0, 1, 2, 3$. They can be decomposed with respect to a natural coframe dx^i according to $\vartheta^\alpha = e_i{}^\alpha dx^i$. Here the $e_i{}^\alpha$ are the coordinate components of ϑ^α, also called tetrad components.

Since a Riemannian *metric* g with Lorentz signature is assumed to exist, the coframe can always be chosen to be orthonormal according to

$$
g = g_{ij} dx^i \otimes dx^j = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta, \tag{3}
$$

with $g_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$. In this way the metric is absorbed by the coframe and has no longer independent physical degrees of freedom.

This choice is convenient and will be kept throughout this article. However, sometimes one may want to choose arbitrary coframes: then the metric emerges explicitly again. Also for that reason, the metric and the coframe, besides the linear connection (see Sec. II B), are treated in the variational principle as independent gauge field variables. However, it eventually turns out that the field equations resulting from the variation of the metric and of the coframe are equivalent. In other words, the orthonormal "gauge" of the coframe, which we use in this article, does not restrict the generality of our considerations.

We call ϑ^α the potential of *weak* gravity of the *Newton-Einstein* type. It couples to matter via the Einstein gravitational constant κ, which has the dimension of a reciprocal force. Basically, ϑ^α represents four gauge boson fields—each of helicity 1—that conspire, at least in linear approximation, to build up the massless spin 2 graviton modes of general relativity (GR), provided the appropriate Hilbert Lagrangian is chosen.

B. The linear connection $\Gamma_\alpha{}^\beta$ and the hypothesis of strong gravity

In classical gauge theories of gravity the linear connection of spacetime $\Gamma_\alpha{}^\beta$, besides the coframe ϑ^α, is assumed to exist as an independent field variable. This is a physical hypothesis that has eventually to be checked by experiment. We call $\Gamma_\alpha{}^\beta$ the potential of *strong* gravity of *Yang-Mills* type. It is represented by $4 \times 4 = 16$ bosonic 1-form fields that can be decomposed according to $\Gamma_\alpha{}^\beta = \Gamma_{i\alpha}{}^\beta dx^i$. They couple to matter in a Yang-Mills like fashion via a hypothetical coupling constant ϱ of the dimension $(\text{action})^{-1}$.

A differential manifold equipped with a metric g and a linear connection $\Gamma_\alpha{}^\beta$ is called a metric-affine space. If the linear connection is unconstrained, then $\Gamma_\alpha{}^\beta$ has values in the Lie algebra of the general linear group $GL(4, R)$. The gauge theory with independent metric and independent connection is called *metric-affine* (gauge theory of) *gravity* (MAG). Since in MAG the connection components $\Gamma_{i\alpha}{}^\beta$ carry three indices, the strong gravity potential can provide additional strong gravity modes of up to spin 3 (see [16]).

C. Beyond general relativity: relaxing the torsion

Let us start from GR and assume the connection to be Riemannian, namely $\tilde{\Gamma}_\alpha{}^\beta$; we will always indicate the Riemannian nature of a quantity by a tilde. Then $\tilde{\Gamma}_\alpha{}^\beta$ does not provide a mode that is independent of the coframe. With a suitable Lagrangian, this corresponds to GR.

If the connection is metric-compatible but non-Riemannian, it carries an independent piece that has values in the Lie algebra of the Lorentz group $SO(1, 3)$. In orthonormal frames, we have for the *Lorentz* (or spin) *connection* the relation $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$. It represents 6 bosonic1-form fields for strong gravity. Its maximum spin is 2. The independent fields, coframe ϑ^α and Lorentz connection $\Gamma^{\alpha\beta}$, represent the gauge potentials of the Poincaré group. The corresponding gauge field theory is called the Poincaré gauge theory of gravity (Poincaré gravity or PG).

In PG, the *contortion*

$$K^{\alpha\beta} := \tilde{\Gamma}^{\alpha\beta} - \Gamma^{\alpha\beta} = -K^{\beta\alpha} \tag{4}$$

measures the difference between the Riemann and the Riemann-Cartan (RC) geometry and, as a difference between two connections, it constitutes a tensor. Alternatively, the deviation from the Riemannian geometry of GR can be described by the *torsion*

$$T^\alpha := \overset{\Gamma}{D}\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta = \frac{1}{2} T_{ij}{}^\alpha dx^i \wedge dx^j. \tag{5}$$

Here $\overset{\Gamma}{D}$ is the exterior covariant derivative with respect to the connection $\Gamma_\alpha{}^\beta$. The newly emerging Lorentz-connection modes reflect themselves also in those of the torsion, since the second term in the torsion, $\Gamma_\beta{}^\alpha \wedge \vartheta^\beta$, depends on $\Gamma_\alpha{}^\beta$. It can be shown (see [14]) that torsion and contortion are related by

$$\begin{aligned} T^\alpha &= \vartheta^\beta \wedge K_\beta{}^\alpha, \\ K_{\alpha\beta} &= e_{[\alpha} \rfloor T_{\beta]} - \frac{1}{2}(e_\alpha \rfloor e_\beta \rfloor T_\gamma)\vartheta^\gamma. \end{aligned} \tag{6}$$

On the level of the gauge *potentials*, we have then in PG the frame ϑ^α and the Lorentz connection $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$. On the level of the newly introduced torsion, the translation gauge *field strength*, we can execute an irreducible decomposition in order to learn more about its structure. It can be decomposed according to $24 = 16 \oplus 4 \oplus 4$ into three pieces: into a second rank tensor piece (tentor), ${}^{(1)}T^\alpha$, into a vector piece, namely, in components, the trace of the torsion (trator),

$${}^{(2)}T^\alpha := -\frac{1}{3}\mathcal{V} \wedge \vartheta^\alpha \quad \text{with} \quad \mathcal{V} := e_\beta \rfloor T^\beta, \tag{7}$$

and into an axial-vector piece (axitor), which corresponds in components to the totally antisymmetric piece of the torsion (the star denotes the Hodge operator):

$${}^{(3)}T^\alpha = \frac{1}{3}{}^\star(\mathcal{A} \wedge \vartheta^\alpha) \quad \text{with} \quad \mathcal{A} := {}^\star(\vartheta_\alpha \wedge T^\alpha). \tag{8}$$

We have then the irreducible decomposition

$$T^\alpha = \underbrace{{}^{(1)}T^\alpha}_{\text{tentor}} + \underbrace{{}^{(2)}T^\alpha}_{\text{trator}} + \underbrace{{}^{(3)}T^\alpha}_{\text{axitor}}. \tag{9}$$

The tensor piece can carry at most spin 2 modes, whereas the vector and the axial-vector pieces are good for at most spin 1 modes.

D. Keeping the nonmetricity to zero

In MAG, besides the torsion T^α, we have a nonvanishing nonmetricity

$$Q_{\alpha\beta} := -\overset{\Gamma}{D} g_{\alpha\beta} = Q_{\beta\alpha} = -dg_{\alpha\beta} + 2\Gamma_{(\alpha\beta)}. \tag{10}$$

Therefore, the nonmetricity $Q_{\alpha\beta}$ of the metric-affine geometry is a measure for the difference between the linear connection $\Gamma_\alpha{}^\beta$ and the Lorentz connection. In an orthonormal coframe ϑ^α the metric referred to the coframe $g_{\alpha\beta}$ is a constant and the modes of the symmetric[1] $\Gamma_{(\alpha\beta)} = \Gamma_{i(\alpha\beta)}dx^i$ are passed through, according to (10), to the nonmetricity $Q_{\alpha\beta} = Q_{i\alpha\beta}dx^i$, with the components $Q_{i\alpha\beta} = Q_{i\beta\alpha}$. There emerge, besides the 6 of the Lorentz connection, 10 more bosonic 1-form fields $Q_{\alpha\beta}$ clearly encompassing strong gravity contributions of spin 0,1,2, and 3.

[1]Not to be confused with the symmetric part of the connection $\Gamma_{(\alpha\beta)\gamma} = e_{(\alpha} \rfloor \Gamma_{\beta)\gamma}$.

In MAG, we have the gravitational potentials ϑ^α and $\Gamma_\alpha{}^\beta$. The gauge field strength attached to the coframe ϑ^α is the torsion (5), that attached to the linear connection $\Gamma_\alpha{}^\beta$ the curvature 2-form

$$R_\alpha{}^\beta := d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta. \tag{11}$$

We can raise the index α and can decompose the curvature into the antisymmetric "rotational" piece and the symmetric "strain" piece according to

$$\begin{aligned} &R^{\alpha\beta} = W^{\alpha\beta} + Z^{\alpha\beta} \quad \text{with} \\ &W^{\alpha\beta} := R^{[\alpha\beta]}, \qquad Z^{\alpha\beta} := R^{(\alpha\beta)}. \end{aligned} \tag{12}$$

Even though we succeeded in relating a quadratic MAG-Lagrangian to a consistent classical field theory of massless spin 3 fields via the trace-free part of the nonmetricity [16], we will restrict ourselves in this article to vanishing nonmetricity, $Q_{\alpha\beta} = 0$; consequently the strain curvature vanishes, too: $Z^{\alpha\beta} = 0$. From a phenomenological point of view the overwhelming importance of the (rigid) Poincaré group in special relativity directs our attention primarily to the gauge theory of the *local* Poincaré group, namely, PG. Accordingly, from now on $Z^{\alpha\beta} = 0$ and $R^{\alpha\beta} = W^{\alpha\beta}$.

E. Irreducible decomposition of the rotational curvature

For the physical interpretation it is significant to understand the different pieces of the curvature. The rotational curvature decomposes irreducibly into six pieces, see [14,17–20], according to

$$\begin{aligned} R_{\alpha\beta} = &\underbrace{{}^{(1)}R_{\alpha\beta}}_{\text{weyl } 10} + \underbrace{{}^{(2)}R_{\alpha\beta}}_{\text{paircom } 9} + \underbrace{{}^{(3)}R_{\alpha\beta}}_{\text{pscalar } 1} + \underbrace{{}^{(4)}R_{\alpha\beta}}_{\text{ricsymf } 9} \\ &+ \underbrace{{}^{(5)}R_{\alpha\beta}}_{\text{ricanti } 6} + \underbrace{{}^{(6)}R_{\alpha\beta}}_{\text{scalar } 1}. \end{aligned} \tag{13}$$

The number of independent components is specified subsequent to the (computer) name of the corresponding irreducible piece. Pseudoscalar and scalar qualify as linear Lagrangians. We take from the literature (remember that $\vartheta_{\alpha\beta} = \vartheta_\alpha \wedge \vartheta_\beta$ and $\eta_{\alpha\beta} = {}^\star\vartheta_{\alpha\beta}$)

$$\begin{aligned} {}^{(3)}R_{\alpha\beta} &= -\frac{1}{12} X \eta_{\alpha\beta}, \\ X &:= e_\alpha \rfloor X^\alpha, \qquad X^\alpha := {}^\star(R^{\beta\alpha} \wedge \vartheta_\beta); \end{aligned} \tag{14}$$

$$\begin{aligned} {}^{(6)}R_{\alpha\beta} &= -\frac{1}{12} R \vartheta_{\alpha\beta}, \\ R &:= e_\alpha \rfloor R^\alpha, \qquad R^\alpha := e_\beta \rfloor R^{\alpha\beta}. \end{aligned} \tag{15}$$

We recognize that the scalar R and the pseudoscalar X play a preferred role. Note that X is purely post-Riemannian, that is, $\tilde{X} \equiv 0$. In components we have

$$X = \eta_{\alpha\beta\gamma\delta} R^{[\alpha\beta\gamma\delta]}/4! \quad \text{and} \quad R = R_{\beta\alpha}{}^{\alpha\beta}, \tag{16}$$

with the decomposition $R^{\alpha\beta} = R_{\gamma\delta}{}^{\alpha\beta} \vartheta^{\gamma\delta}/2$.

III. POINCARÉ GAUGE THEORY OF GRAVITY (PG)

A. Lagrangian and field equations

In PG, we have the gravitational potentials ϑ^α and $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$. The corresponding gauge field strengths are the torsion T^α and the rotational ("Lorentz") curvature

$$R^{\alpha\beta} := d\Gamma^{\alpha\beta} - \Gamma^\alpha{}_\gamma \wedge \Gamma^{\gamma\beta} = -R^{\beta\alpha}. \tag{17}$$

We assume a first order Lagrangian consisting of a gauge and a minimally coupled matter part,

$$L = V(g_{\alpha\beta}, \vartheta^\alpha, T^\alpha, R^{\alpha\beta}) + L_{\text{mat}}(g_{\alpha\beta}, \vartheta^\alpha, \Psi, \overset{\Gamma}{D}\Psi), \tag{18}$$

with the matter field(s) Ψ. Then we can define the translation and the Lorentz *excitations*, respectively,

$$H_\alpha := -\frac{\partial V}{\partial T^\alpha}, \qquad H_{\alpha\beta} := -\frac{\partial V}{\partial R^{\alpha\beta}} = -H_{\beta\alpha}, \tag{19}$$

and the canonical matter currents of *energy-momentum* and *spin* (angular momentum) according to

$$\Sigma_\alpha := \frac{\delta L_{\text{mat}}}{\delta \vartheta^\alpha}, \qquad \tau_{\alpha\beta} := \frac{\delta L_{\text{mat}}}{\delta \Gamma^{\alpha\beta}} = -\tau_{\beta\alpha}; \tag{20}$$

in the case of a minimally coupled matter Lagrangian, as in (18), the variational derivatives degenerate to partial derivatives.

The action principle yields the field equations [3]

$$DH_\alpha - E_\alpha = \Sigma_\alpha \quad \text{(first)}, \tag{21}$$

$$DH_{\alpha\beta} - E_{\alpha\beta} = \tau_{\alpha\beta} \quad \text{(second)}, \tag{22}$$

with the gauge currents of energy-momentum and spin

$$E_\alpha := e_\alpha \rfloor V + (e_\alpha \rfloor T^\beta) \wedge H_\beta + (e_\alpha \rfloor R^{\beta\gamma}) \wedge H_{\beta\gamma}, \tag{23}$$

$$E_{\alpha\beta} := -\vartheta_{[\alpha} \wedge H_{\beta]}. \tag{24}$$

If the gauge Lagrangian V is prescribed *explicitly*, we can compute first the excitations H_α, $H_{\alpha\beta}$ by partial differentiation of V and subsequently the gauge currents E_α, $E_{\alpha\beta}$ by substitution; these quantities are then inserted into the two field equations (21) and (22). As noted already above, the field equation resulting from a variation of the metric $g_{\alpha\beta}$ is equivalent to (21), provided (22) is fulfilled.

The matter currents on the right-hand-side of the field equations (21) and (22) can be understood as those of a spin fluid (see [21–23]). An *approximate* representation of such a spin fluid can be specified as follows: If the fluid moves with the velocity $\mathbf{u} = u^\alpha e_\alpha$, that is, with the flow 3-form $\mathcal{U} := \mathbf{u} \rfloor \eta = u^\alpha \eta_\alpha$, and transports an energy-momentum density p_α and a spin density $s_{\alpha\beta} = -s_{\beta\alpha}$, then a convective Weyssenhoff ansatz for the matter currents reads

$$\Sigma_\alpha = p_\alpha \mathcal{U} \quad \text{and} \quad \tau_{\alpha\beta} = s_{\alpha\beta}\mathcal{U}. \tag{25}$$

B. Quadratic Yang-Mills type Lagrangian with even parity terms

A quadratic PG Lagrangian of the Yang-Mills type has the general structure (λ_0 is the cosmological constant)

$$V \sim \frac{1}{\kappa}(\text{curv} + \text{torsion}^2 + \lambda_0) + \frac{1}{\varrho}\text{curv}^2. \tag{26}$$

Since the coframe ϑ^α and the Lorentz connection $\Gamma^{\alpha\beta}$ are independent variables, such a first order Lagrangian yields second order field equations; *higher derivatives do not emerge*.

The *simplest* nontrivial Lagrangian corresponds to the first term on the right-hand-side of (26). It is of the Hilbert type, i.e., linear in the curvature, namely, the so-called Einstein-Cartan Lagrangian, see (15),

$$V_{\rm EC} = \frac{1}{2\kappa}\eta_{\alpha\beta} \wedge R^{\alpha\beta} = \frac{1}{2\kappa}\eta_{\alpha\beta} \wedge {}^{(6)}R^{\alpha\beta} = \frac{1}{2\kappa}{}^\star R. \tag{27}$$

The corresponding two field excitations, $H_\alpha = 0$ and $H_{\alpha\beta} = -\eta_{\alpha\beta}/(2\kappa)$, if substituted into (21) and (22), yield the field equations [1,2,24]:

$$\frac{1}{2}\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} = \kappa\Sigma_\alpha, \tag{28}$$

$$\frac{1}{2}\eta_{\alpha\beta\gamma} \wedge T^\gamma = \kappa\tau_{\alpha\beta}. \tag{29}$$

The viable Einstein-Cartan(-Sciama-Kibble) theory (EC), as compared to GR, supplies an additional spin-contact interaction of *weak* gravitational origin, since only Einstein's gravitational constant enters (28) and (29). The Lorentz connection $\Gamma^{\alpha\beta}$ cannot propagate and thus EC represents[2] a degenerate PG. In order to enable $\Gamma^{\alpha\beta}$ to propagate, we have to use additionally at least the *quadratic* curvature piece in (26); for discussions on the physical relevance of torsion one should also compare Shapiro [27] and Ni [28].

Esser [29], see also [30], constructed the most general quadratic Lagrangian with even (+) parity pieces (for this notion see Sec. III C). For a RC-spacetime it reads:

$$V_+ = \frac{1}{2\kappa}\Big(-a_0 R^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda_0\eta + T^\alpha \wedge \sum_{I=1}^{3} a_I {}^{\star(I)}T_\alpha\Big) - \frac{1}{2\varrho}R^{\alpha\beta} \wedge \sum_{I=1}^{6} w_I {}^{\star(I)}R_{\alpha\beta}; \tag{30}$$

without restricting the generality of our considerations, we can choose $\varrho > 0$. In a RC-space, Esser found the additional term

$$-\frac{1}{2\varrho}R^{\alpha\beta} \wedge {}^\star[w_7\vartheta_\alpha \wedge (e_\gamma \rfloor {}^{(5)}R^\gamma{}_\beta)]. \tag{31}$$

However, this term can be transformed successively into a pure w_5 term. Let us first consider invariants of the form $R^{\alpha\beta} \wedge {}^\star[\vartheta_\alpha \wedge (e_\gamma \rfloor {}^{(A)}R^\gamma{}_\beta)]$ with $A \in \{1 \cdots 6\}$. For $A = 5$ we find

$$\begin{aligned} R^{\alpha\beta} &\wedge {}^\star[\vartheta_\alpha \wedge (e_\gamma \rfloor {}^{(5)}R^\gamma{}_\beta)] \\ &= {}^{(5)}R^{\alpha\beta} \wedge {}^\star[\vartheta_\alpha \wedge (e_\gamma \rfloor {}^{(5)}R^\gamma{}_\beta)] \\ &= {}^{(5)}R^{\alpha\beta} \wedge {}^{\star(5)}R_{\alpha\beta}. \end{aligned} \tag{32}$$

Thus we can absorb the w_7 term into the w_5 term. Consequently, without restricting the generality of our considerations, we can put $w_7 = 0$.

In our Lagrangian V_+ in (30) not all the constants are independent. In a Riemannian as well as in a RC spacetime, the integrand of the topological Euler 4-form

$$B_{RR^{(*)}} = -\frac{1}{2}R_\alpha{}^\beta \wedge R^{(*)\alpha}_\beta = \frac{1}{4}\eta_{\beta\alpha\delta\gamma}R^{\alpha\beta} \wedge R^{\gamma\delta} = dC_{RR^{(*)}} \tag{33}$$

is exact, with

$$C_{RR^{(*)}} = \frac{1}{4}\eta^\alpha{}_\beta{}^\gamma{}_\delta(R_\alpha{}^\beta \wedge \Gamma_\gamma{}^\delta + \frac{1}{3}\Gamma_\alpha{}^\beta \wedge \Gamma_\gamma{}^\varepsilon \wedge \Gamma_\varepsilon{}^\delta). \tag{34}$$

The dual is here taken with respect to the frame indices α, β of the curvature 2-form $R_\alpha{}^\beta$, it has to be carefully distinguished from the Hodge dual. Then, together with the Bach-Lanczos identity [14], Eq. (A.3.7),

$$R^{(\alpha|\gamma} \wedge R^{(*)|\beta)}{}_\gamma - \frac{1}{4}g^{\alpha\beta}R^{\mu\nu} \wedge R^{(*)}_{\mu\nu} = 0, \tag{35}$$

one can show that only five of the six w_I's are linearly independent.

The excitations can now be calculated by differentiation:

$$H_\alpha = -\frac{1}{\kappa}\sum_{I=1}^{3} a_I {}^{\star(I)}T_\alpha, \tag{36}$$

$$H_{\alpha\beta} = \frac{a_0}{2\kappa}\eta_{\alpha\beta} + \frac{1}{\varrho}\sum_{I=1}^{6} w_I {}^{\star(I)}R_{\alpha\beta}. \tag{37}$$

Because of (15), the last equation can be slightly rewritten as

[2]Recently, an EC model with fermionic matter and its application to the early universe have been discussed by Ribas and Kremer [25] and by Dolan [26].

$$H_{\alpha\beta} = \left(\frac{a_0}{2\kappa} - \frac{w_6}{12\varrho}R\right)\eta_{\alpha\beta} + \frac{1}{\varrho}\sum_{I=1}^{5} w_I {}^{\star(I)}R_{\alpha\beta}. \tag{38}$$

There have been numerous investigations into the properties of the Lagrangian (30). In linear approximation, on a flat Minkowskian background, Eq. (30) encompasses, besides the weak gravity modes of the coframe, propagating strong gravity modes of the Lorentz connection with spin $2^{\pm}$, $1^{\pm}$, and $0^{\pm}$, as shown by Hayashi and Shirafuji [4], by Sezgin and van Nieuwenhuizen [31], and by Kuhfuss and Nitsch [32]. For a model with quadratic curvature Lagrangian in which only the Lorentz connection is dynamic, compare Cho *et al.* [33].

A good dynamic mode transports positive energy at speed $\leq c$. At most three modes can be simultaneously dynamic; all the cases were tabulated; many combinations are satisfactory to linear order. The Hamiltonian analysis, as shown by Blagojević and Nikolić [34,35], revealed the related constraints. In more detailed investigations [36–39] it was concluded that effects due to nonlinearities could be expected to render all of these cases physically unacceptable, with the exception of two "scalar" connection modes with spin 0^+ and spin 0^-.

Before we come back to the mode analysis in Sec. IV C, we want to extend the gravitational Lagrangian such that also odd parity pieces are included.

C. Even parity and odd parity Lagrangians, twisted and untwisted forms

Let us study the spatial reflection or parity transformation; the sign of the time axis will be kept fixed. A (pure) scalar field $\Phi(x)$ remains invariant under the parity transformation or if we transform a right-handed coordinate system x^i into a left-handed one $x^{i'}$: $\Phi(x') = \Phi(x)$. In contrast, a twisted scalar field $\hat{\Phi}(x)$ (also called a pseudoscalar field) changes its sign under those circumstances, that is, the sign of the determinant $J := \det\|\partial x^i/\partial x^{i'}\|$ of the Jacobian transformation matrix enters its transformation law: $\hat{\Phi}(x^{i'}) = (\mathrm{sign}J)\hat{\Phi}(x)$. The analogous behavior characterizes the relation between twisted and untwisted forms; for a mathematical discussion compare Frankel [40].

A *Lagrangian* 4-form L has to be a *twisted* 4-form in order to make its action $W := \int_{\Omega_4} L$ a pure scalar. These Lagrangians are also called even parity Lagrangians. However, in physics we know since the discovery of parity violation in the weak interaction in 1956, see Sozzi [41] for a review, that also odd parity would-be Lagrangians can occur; they have to be multiplied by *pseudoscalar coupling constants* in order to transform them to decent (twisted) Lagrangians, which can be added to the other even parity Lagrangian pieces.

In PG, the field strengths T^α and $R^{\alpha\beta}$ are untwisted 2-forms, similarly the potentials ϑ^α and $\Gamma^{\alpha\beta}$ are untwisted 1-forms. One may compare the case of electrodynamics with the untwisted potential A and the untwisted field strength $F = dA$ (the differential d is untwisted). Consequently, a twisted Lagrangian, according to (19), leads to the excitations H_α and $H_{\alpha\beta}$ being twisted 2-forms and the material currents Σ_α and $\tau_{\alpha\beta}$, see (20), being twisted 3-forms.

The Einstein-Cartan Lagrangian

$$V_{\mathrm{EC}} = \frac{1}{2\kappa}R^{\alpha\beta}\wedge{}^\star(\vartheta_\alpha\wedge\vartheta_\beta) = \frac{1}{2\kappa}{}^\star R \tag{39}$$

is twisted, since the Hodge star in our formalism (see [15], Sec. C.2.8) is twisted, that is, it maps twisted into untwisted forms and vice versa. The Maxwell Lagrangian

$$V_{\mathrm{Max}} = -\frac{Y_0}{2}F\wedge{}^\star F \tag{40}$$

is also twisted, hence of even parity, where Y_0 is the (scalar) vacuum admittance. We recognize that an odd number of stars occurring in a Lagrangian, which is expressed in terms of field strength and potentials, guarantees its standard twisted nature. In contrast, the topological Chern type Lagrangian

$$V_{\mathrm{Max}'} = -\frac{Y_1}{2}F\wedge F \tag{41}$$

is only twisted (even parity), if we declare Y_1 to be a pseudoscalar; the analogous is true in gravity for

$$V_{\mathrm{EC}'} = \frac{1}{2\kappa'}R^{\alpha\beta}\wedge(\vartheta_\alpha\wedge\vartheta_\beta) = \frac{1}{2\kappa'}{}^\star X, \tag{42}$$

with the pseudoscalar constant κ'; note that this Lagrangian vanishes identically in a Riemannian space, since $\tilde{R}^{[\alpha\beta\gamma\delta]} = 0$.

D. Parity violating admixtures to the Einstein-Cartan Lagrangian

Already in 1964, Leitner and Okubo [42] wondered about possible odd parity terms in the gravitational Lagrangian. Related questions were addressed by Hayashi [43] and Hari Dass [44]. The effect of adding the non-Riemannian odd parity pseudoscalar curvature to the Hilbert-Einstein-Cartan scalar curvature was first studied by Hojman, Mukku, and Sayed [45] (for Mukku's recent view see [46]):

$$\begin{aligned} V_{\mathrm{HMS}} &= \frac{1}{2\kappa}(a_0{}^\star R + b_0{}^\star X) \\ &= -\frac{1}{2\kappa}(a_0\eta_{\alpha\beta} + b_0\vartheta_{\alpha\beta})\wedge R^{\alpha\beta}. \end{aligned} \tag{43}$$

Note that on the right-hand-side of this equation the star only enters in $\eta_{\alpha\beta} = {}^\star(\vartheta_\alpha\wedge\vartheta_\beta)$, that is, the second term $\vartheta_{\alpha\beta} = \vartheta_\alpha\wedge\vartheta_\beta$ is of odd parity and thus b_0 is a pseudoscalar. The excitations turn out to be

$$H_\alpha = 0, \qquad H_{\alpha\beta} = \frac{1}{2\kappa}(a_0\eta_{\alpha\beta} + b_0\vartheta_{\alpha\beta}). \tag{44}$$

If we introduce the left-hand-side of (28) as the Einstein 3-form $G_\alpha := \frac{1}{2}\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma}$, the field equations (21) and (22) read

$$a_0 G_\alpha - b_0{}^\star X_\alpha = \kappa\Sigma_\alpha, \tag{45}$$

$$\frac{a_0}{2} T^\mu \wedge \eta_{\alpha\beta\mu} + b_0 T_{[\alpha} \wedge \vartheta_{\beta]} = \kappa\tau_{\alpha\beta}. \tag{46}$$

The situation with respect to the second field equation (46) is similar to the EC theory. For vanishing material spin, $\tau_{\alpha\beta} = 0$, the torsion vanishes, too, $T^\alpha = 0$. This can be shown by substituting the irreducible decomposition of torsion (9) into (46) and using the geometrical identities

$$\begin{aligned} {}^{(1)}T^\mu \wedge \eta_{\alpha\beta\mu} &= 2^{\star(1)}T_{[\alpha} \wedge \vartheta_{\beta]}, \\ {}^{(2)}T^\mu \wedge \eta_{\alpha\beta\mu} &= -4^{\star(2)}T_{[\alpha} \wedge \vartheta_{\beta]}, \\ {}^{(3)}T^\mu \wedge \eta_{\alpha\beta\mu} &= -{}^{\star(3)}T_{[\alpha} \wedge \vartheta_{\beta]}. \end{aligned} \tag{47}$$

To supply nonvanishing torsion we either need material spin or at least (for the vacuum case) field Lagrangians quadratic in the field strengths. In this sense, the Lagrangian (43) is as degenerate as the one of EC and it is natural to turn to quadratic odd parity Lagrangians.

Subsequently questions related to the $V_{\rm HMS}$-Lagrangian, in the realm of classical Riemann-Cartan spacetime, were investigated by Nelson [47], Nieh and Yan [48] (see also Nieh's recent article [49]), and McCrea *et al.* [50,51], see also Refs. [52,53].

In the general context of the Ashtekar formalism or, more generally, of *loop quantum gravity*, compare Kiefer [54] and Rovelli [55], the $V_{\rm HMS}$-Lagrangian was taken up again by Holst [56], Freidel *et al.* [57,58], Khriplovich *et al.* [59], and Bojowald *et al.* [60]; similar parity violating pieces were studied by Mukhopadhyaya *et al.* [61,62], see also the related papers by Mielke [63,64].

In the framework of a *quantum* field theoretical context, Poplawski [65], Randono [66], and Bjorken [67] developed cosmological models with torsion and parity violating pieces that are induced by the vacuum structure.

Jackiw and Pi [68] proposed a specific model with violation of parity and Lorentz invariance in the context of GR. They introduced, in addition to the Hilbert-Einstein Lagrangian, an *external* scalar field β, not to be varied in the action principle, multiplied by the Chern-Simons (CS) term attached to the curvature:

$$V_{\rm GRCS} = \frac{1}{2\kappa}{}^\star\tilde{R} + \frac{\beta}{2\varrho}\tilde{R}_\alpha{}^\beta \wedge \tilde{R}_\beta{}^\alpha. \tag{48}$$

This model was extended to the EC theory by Cantcheff [69],

$$V_{\rm ECCS} = \frac{1}{2\kappa}{}^\star R + \frac{\beta}{2\varrho}R_\alpha{}^\beta \wedge R_\beta{}^\alpha, \tag{49}$$

see also Ertem [70]. Of course, both theories differ in their physical content, as does GR from the EC theory. We turn our attention to the EC version in (49).

We know from geometry that a CS term is an exact form. We have for a RC space, see [14], Eqs. (3.9.3) and (3.9.8),

$$-\frac{1}{2}R_\alpha{}^\beta \wedge R_\beta{}^\alpha = dC_{\rm RR} \tag{50}$$

with

$$C_{\rm RR} := -\frac{1}{2}(\Gamma_\alpha{}^\beta \wedge R_\beta{}^\alpha + \frac{1}{3}\Gamma_\alpha{}^\beta \wedge \Gamma_\beta{}^\gamma \wedge \Gamma_\gamma{}^\alpha). \tag{51}$$

Thus,

$$V_{\rm ECCS} = \frac{1}{2\kappa}{}^\star R - \frac{\beta}{\varrho}dC_{\rm RR}. \tag{52}$$

This Lagrangian contains an odd parity piece quadratic in curvature, see (50). However, it is of a fairly degenerate character. Still, since β is a prescribed field, the field equations are affected by the CS term. We will come back to an explicit evaluation of the curvature square piece below.

Explicit odd parity curvature square pieces were present in some cosmological models with spin 0^+ and spin 0^- modes. These models motivated us for a further search in this direction.

E. Interlude: the impact of the cosmological model of Shie-Nester-Yo on PG

Shie, Nester, and Yo (SNY) [11,71], in the framework of PG and in accordance with [38,39], formulated a new cosmological model. It contains, besides the graviton mode 2^+ of GR, one propagating connection and accordingly one propagating torsion mode of spin 0^+; the $+$ refers to the positive parity. The corresponding Lagrangian reads effectively[3]

$$V_{\rm SNY} = \frac{1}{2\kappa}\left(a_0{}^\star R + \frac{1}{3}a_2\mathcal{V} \wedge {}^\star\mathcal{V}\right) - \frac{1}{24\varrho}w_6 R^2\eta. \tag{53}$$

This model has fairly realistic features and encouraged further developments. Li *et al.* [72,73], following Ref. [11], investigated the cosmological evolution of the SNY model with advanced numerical techniques.

In order to embrace additionally connection modes with spin 0^-, that is with odd parity, in a next step, Chen *et al.* [12] generalized the SNY Lagrangian to

$$\begin{aligned} V_{\rm SNY'} = \frac{1}{2\kappa}(a_0{}^\star R - 2\lambda_0\eta) + \frac{1}{6\kappa}(a_2\mathcal{V} \wedge {}^\star\mathcal{V} \\ - a_3\mathcal{A} \wedge {}^\star\mathcal{A}) - \frac{1}{24\varrho}(w_6R^2 - w_3X^2)\eta. \end{aligned} \tag{54}$$

[3]Their actual Lagrangian [11], Eq. (18) also contains squared pieces of the axial torsion ${}^{(3)}T^\alpha$ and the tensor torsion ${}^{(1)}T^\alpha$. However, they find from the second field equation (for vanishing spin) that ${}^{(1)}T^\alpha = {}^{(3)}T^\alpha = 0$. Accordingly, in the SNY model only the vector piece ${}^{(2)}T^\alpha$ of the torsion is active explicitly. An analogous remark applies to (54).

It contains, as odd parity terms, the *axial-vector torsion* $\mathcal{A}$ and the *pseudoscalar curvature* X, see (8) and (14), respectively. However, these odd parity terms are concealed in an even parity Lagrangian since each of the terms $\mathcal{A}\wedge{}^\star\mathcal{A}$ and $X^2\eta = X\,{}^\star X$ contain one explicit star, respectively. Accordingly, this model with propagating modes 2^+ and $0^\pm$ is of even parity, but contains concealed the odd parity terms $\mathcal{A}$ and X.

It is then tempting to provide further add-ons, namely, the parity violating mixed terms $\mathcal{A}\wedge{}^\star\mathcal{V}$ and $X\,{}^\star R = R\,{}^\star X$. This provides a further extension of the SNY model with respective metric and connection propagating modes of 2^+ and $0^\pm$.

IV. THE SHADOW WORLD OF QUADRATIC PG-LAGRANGIANS WITH ODD PARITY TERMS

A. Constructing systematically quadratic odd parity Lagrangians

As discussed by Sozzi [41], from the validity of the CPT symmetry in nature and from the fact that the C and *CP* symmetries are only broken in weak interaction, but are valid otherwise, one would expect roughly an equal amount of matter and antimatter in the Universe. It appears however, as shown by Steigman [74], that only what we call matter is around in the Universe. Accordingly, we have to face the matter/antimatter asymmetry in nature and may want to approach this question from a gravitational point of view. Can we, in PG, construct odd parity Lagrangians in a natural way so that we can estimate the possible influence of those terms for the evolution of the Universe?[4]

We proved identities of the type (32) under heavy use of the computer algebra system REDUCE, including the package EXCALC for handling directly exterior differential forms, see [75–78]. In this way, we can construct a quadratic odd parity Lagrangian V_- that in its general structure reflects the even parity Lagrangian V_+ in (30). It is the "shadow" of V_+:

$$\begin{aligned} V_- = &-\frac{b_0}{2\kappa}{}^{(3)}R_{\alpha\beta}\wedge\vartheta^{\alpha\beta} + \frac{1}{\kappa}(\sigma_1{}^{(1)}T^\alpha\wedge{}^{(1)}T_\alpha \\ &+ \sigma_2{}^{(2)}T^\alpha\wedge{}^{(3)}T_\alpha) - \frac{1}{2\varrho}(\mu_1{}^{(1)}R^{\alpha\beta}\wedge{}^{(1)}R_{\alpha\beta} \\ &+ \mu_2{}^{(2)}R^{\alpha\beta}\wedge{}^{(4)}R_{\alpha\beta} + \mu_3{}^{(3)}R^{\alpha\beta}\wedge{}^{(6)}R_{\alpha\beta} \\ &+ \mu_4{}^{(5)}R^{\alpha\beta}\wedge{}^{(5)}R_{\alpha\beta}). \end{aligned} \tag{55}$$

All its constants are pseudoscalars. In a Riemannian spacetime such an extended shadow does not exist, since only the μ_1 term survives, because of $T^\alpha = 0$ and ${}^{(2)}R^{\alpha\beta} = {}^{(3)}R^{\alpha\beta} = {}^{(5)}R^{\alpha\beta} = 0$. Therefore, in PG we can free ourselves from the constraint to use even parity Lagrangians with only one odd term; in fact, PG brings the existence of numerous odd parity Lagrangians to light.

The special cases of Jackiw and Pi [68] and Cantcheff [69] can now be straightforwardly evaluated. If we define in four dimensions the (pseudo-)scalar product $\langle A, B\rangle$ for any two tensor-valued 2-forms $A_{\alpha_1\ldots\alpha_r}$ and $B_{\alpha_1\ldots\alpha_r}$ by

$$\langle A, B\rangle := {}^\star(A_{\alpha_1\ldots\alpha_r}\wedge B^{\alpha_1\ldots\alpha_r}), \tag{56}$$

we can write the curvature square piece of our odd Lagrangian (55) as

$$\begin{aligned} \overset{\mathrm{curv}^2}{V_-} = &\frac{1}{2\varrho}{}^\star(\mu_1\langle{}^{(1)}R, {}^{(1)}R\rangle + \mu_2\langle{}^{(2)}R, {}^{(4)}R\rangle \\ &+ \mu_3\langle{}^{(3)}R, {}^{(6)}R\rangle + \mu_4\langle{}^{(5)}R, {}^{(5)}R\rangle). \end{aligned} \tag{57}$$

On the other hand, the Einstein-Cartan-Chern-Simons Lagrangian (49) reads

$$V_{\mathrm{ECCS}} = \frac{1}{2\kappa}{}^\star R + \frac{\beta}{2\varrho}{}^\star\langle R, R\rangle. \tag{58}$$

If we substitute the irreducible pieces of the curvature into the scalar product, we find [see [14], Eq. (B.4.37)]

$$\begin{aligned} \langle R, R\rangle = &\langle{}^{(1)}R, {}^{(1)}R\rangle + 2\langle{}^{(2)}R, {}^{(4)}R\rangle + 2\langle{}^{(3)}R, {}^{(6)}R\rangle \\ &+ \langle{}^{(5)}R, {}^{(5)}R\rangle. \end{aligned} \tag{59}$$

In a Riemannian space of the theory of Jackiw and Pi only the conformally invariant Weyl square piece, the first term on the right-hand-side of (59), is left over, whereas the RC space complicates the structures by additional post-Riemannian pieces. Comparing (57) with (59), we find

$$\overset{\mathrm{curv}^2}{V_-}(\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 1) = -\frac{1}{\varrho}dC_{\mathrm{RR}}. \tag{60}$$

In other words, our odd parity curvature square Lagrangians, for the coupling constants specified, becomes an exact form. Consequently only three of the four μ's can be chosen independently.

The mixed quadratic Lagrangian with even and odd parity is then

$$V_\pm = V_+ + V_-, \tag{61}$$

see also [13].

B. Cosmological model with parity violating terms

If we compare $V_\pm$ with $V_{\mathrm{SNY}'}$, the next step of "minimally" generalizing (54) and hopefully keeping the nice properties of the model is to allow for unconcealed odd parity pieces, but only those odd parity pieces that already occur in (54), namely X and $\mathcal{A}$. Thus, starting with (61) and putting the following constants to zero,

[4] At the Large Hadron Collider (LHC) in Geneva the detector LHCb (the "b" = beauty refers to the "bottom" quark) was constructed particularly for investigations of matter-antimatter interactions, namely, for the *CP* violation in the interaction of b-hadrons. It is hoped that these experiments will provide new insight into the interrelationship between matter and antimatter.

$$\begin{aligned} a_1 &= 0, \qquad \sigma_1 = 0; \\ w_1 &= w_2 = w_4 = w_5 = 0; \\ \mu_1 &= \mu_2 = \mu_4 = 0, \end{aligned} \tag{62}$$

we arrive at the new Lagrangian

$$\begin{aligned} V_{\rm BHN} = \frac{1}{2\kappa}(&-a_0{}^{(6)}R^{\alpha\beta}\wedge\eta_{\alpha\beta} - b_0{}^{(3)}R^{\alpha\beta}\wedge\vartheta_{\alpha\beta} \\ &- 2\lambda_0\eta + a_2{}^{(2)}T^\alpha\wedge{}^{\star(2)}T_\alpha + a_3{}^{(3)}T^\alpha\wedge{}^{\star(3)}T_\alpha \\ &+ 2\sigma_2{}^{(2)}T^\alpha\wedge{}^{(3)}T_\alpha) - \frac{1}{2\varrho}(w_3{}^{(3)}R^{\alpha\beta}\wedge{}^{\star(3)}R_{\alpha\beta} \\ &+ w_6{}^{(6)}R^{\alpha\beta}\wedge{}^{\star(6)}R_{\alpha\beta} + \mu_3{}^{(3)}R^{\alpha\beta}\wedge{}^{(6)}R_{\alpha\beta}). \end{aligned} \tag{63}$$

Substituting the irreducible pieces of the torsion (7) and (8) and of the curvature (14) and (15) into (63), we find the *more compact form of our new gravitational Lagrangian*

$$\begin{aligned} V_{\rm BHN} = &\frac{1}{2\kappa}(a_0{}^\star R + b_0{}^\star X - 2\lambda_0\eta) + \frac{1}{6\kappa}(a_2\mathcal{V}\wedge{}^\star\mathcal{V} \\ &- a_3\mathcal{A}\wedge{}^\star\mathcal{A} - 2\sigma_2\mathcal{V}\wedge{}^\star\mathcal{A}) \\ &- \frac{1}{24\varrho}(w_6R\,{}^\star R - w_3X\,{}^\star X + \mu_3R\,{}^\star X). \end{aligned} \tag{64}$$

The constants b_0, σ_2, μ_3 are *pseudoscalar*, the remaining ones are scalar.[5]

We can read off from the Lagrangian a symmetry between R and X, between w_6 and w_3, and between a_0 and b_0. There is also a symmetry between $\mathcal{V}$ and $\mathcal{A}$ on the one side and R and X on the other side; this implies that a_2, a_3, σ_2 are mirrored in w_6, w_3, μ_3. These symmetries are also reflected in the field equations. We will come back to this in Sec. IV E.

The decomposition of the linear terms in R and X into Riemannian and post-Riemannian pieces, modulo surface terms, yields

$$\begin{aligned} V_{\rm BHN} = &\frac{1}{2\kappa}[a_0{}^\star\tilde{R} - 2\lambda_0\eta + {}^{(1)}T_\alpha\wedge(b_0 + a_0{}^\star){}^{(1)}T^\alpha \\ &- \frac{2}{3}m^+\mathcal{V}\wedge{}^\star\mathcal{V} + \frac{1}{6}m^-\mathcal{A}\wedge{}^\star\mathcal{A} - \frac{2}{3}m^\times\mathcal{V}\wedge{}^\star\mathcal{A}] \\ &- \frac{1}{24\varrho}(w_6R\,{}^\star R - w_3X\,{}^\star X + \mu_3R\,{}^\star X), \end{aligned} \tag{65}$$

with

$$m^+ := a_0 - \frac{a_2}{2}, \qquad m^- := a_0 - 2a_3, \qquad m^\times := b_0 + \sigma_2. \tag{66}$$

We shall see that the m's play a role in the discussion of the second field equation. Note that m^+ and m^- are of even parity, whereas $m^\times$ is odd. A corresponding decomposition of the quadratic curvature terms in (65) does not seem to provide new insight.

[5] A teleparallel Lagrangian with a term with $\sigma_2 \neq 0$ has been considered earlier by Müller-Hoissen and Nitsch [79].

C. Diagonalization of the BHN-Lagrangian

1. Eigenvalues of the kinetic matrix of the translational gauge potential ϑ^α

In the Lagrangian (65) the term $\mathcal{V}\wedge{}^\star\mathcal{A} = \mathcal{A}\wedge{}^\star\mathcal{V} = \mathcal{A}_\mu\mathcal{V}^\mu\eta$ represents an interaction term of two four-vectors of the type *vector* × *axial-vector*. To get some more insight into the dynamics of the fields governed by the Lagrangian (65), we decompose these four-vectors into (1 ⊕ 3) with

$$\mathcal{A}_\mu = (\mathcal{A}_0, \vec{\mathcal{A}}) \quad \text{and} \quad \mathcal{V}_\mu = (\mathcal{V}_0, \vec{\mathcal{V}}). \tag{67}$$

The translational part of the Lagrangian (65), with ${}^{(1)}T^\alpha = 0$, is proportional to the quadratic form

$$\begin{aligned} \mathcal{Q} := &4(\mathcal{V}_0^2 - \mathcal{V}_1^2 - \mathcal{V}_2^2 - \mathcal{V}_3^2)m^+ - (\mathcal{A}_0^2 - \mathcal{A}_1^2 \\ &- \mathcal{A}_2^2 - \mathcal{A}_3^2)m^- + 4(\mathcal{A}_0\mathcal{V}_0 - \mathcal{A}_1\mathcal{V}_1 \\ &- \mathcal{A}_2\mathcal{V}_2 - \mathcal{A}_3\mathcal{V}_3)m^\times. \end{aligned} \tag{68}$$

Expressed in terms of matrices, we have

$$\begin{aligned} \mathcal{Q} &= (\mathcal{V}_0, \vec{\mathcal{V}}, \mathcal{A}_0, \vec{\mathcal{A}})\cdot\begin{pmatrix} -4m^+\mathbf{g} & -2m^\times\mathbf{g} \\ -2m^\times\mathbf{g} & m^-\mathbf{g}\end{pmatrix}\cdot\begin{pmatrix}\mathcal{V}_0 \\ \vec{\mathcal{V}} \\ \mathcal{A}_0 \\ \vec{\mathcal{A}}\end{pmatrix}, \\ &= (\mathcal{V}_0, \vec{\mathcal{V}}, \mathcal{A}_0, \vec{\mathcal{A}})\cdot\mathcal{T}\cdot(\mathcal{V}_0, \vec{\mathcal{V}}, \mathcal{A}_0, \vec{\mathcal{A}})^{\rm T}, \end{aligned} \tag{69}$$

with $\mathbf{g}$ as the four-dimensional Minkowski metric.

With the useful abbreviations

$$x := 4m^+, \qquad y := 2m^\times, \qquad z := m^-, \tag{70}$$

the new matrix $\mathcal{T}$ reads

$$\begin{aligned} \mathcal{T} &= \begin{pmatrix} x & 0 & 0 & 0 & y & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & 0 & -y & 0 & 0 \\ 0 & 0 & -x & 0 & 0 & 0 & -y & 0 \\ 0 & 0 & 0 & -x & 0 & 0 & 0 & -y \\ y & 0 & 0 & 0 & -z & 0 & 0 & 0 \\ 0 & -y & 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & -y & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & -y & 0 & 0 & 0 & z \end{pmatrix} \\ &= \begin{pmatrix} -x\mathbf{g} & -y\mathbf{g} \\ -y\mathbf{g} & z\mathbf{g}\end{pmatrix}. \end{aligned} \tag{71}$$

It has some simple properties:

$$\begin{aligned} \mathcal{T}^{\mathrm{T}} &= \mathcal{T}, \\ \mathrm{trace}\mathcal{T} &= -2(x-z), \\ \det\mathcal{T} &= (xz+y^2)^4 > 0, \\ \mathcal{T}^{-1} &= \frac{1}{xz+y^2}\begin{pmatrix} -z\mathbf{g} & -y\mathbf{g} \\ -y\mathbf{g} & x\mathbf{g} \end{pmatrix}. \end{aligned} \tag{72}$$

In the following we will make use of Dirac's bra-ket notation, see Dirac [80] and Schouten [81]. For this purpose we define (abstract) vectors according to

$$\begin{aligned} <\mathcal{X}| &:= (\mathcal{V}_0, \vec{\mathcal{V}}, \mathcal{A}_0, \vec{\mathcal{A}}) \quad \text{and} \\ |\mathcal{X}> &:= (\mathcal{V}_0, \vec{\mathcal{V}}, \mathcal{A}_0, \vec{\mathcal{A}})^{\mathrm{T}} \end{aligned} \tag{73}$$

such that the quadratic form $\mathcal{Q}$ becomes

$$\mathcal{Q} = <\mathcal{X}|\mathcal{T}|\mathcal{X}>. \tag{74}$$

To diagonalize the form (74), we introduce a new vector $|\mathcal{Y}>$ together with a suitable orthonormal matrix $\mathcal{K}$ such that

$$|\mathcal{X}> =: \mathcal{K}|\mathcal{Y}> \quad \text{and} \quad <\mathcal{X}| = <\mathcal{Y}|\mathcal{K}^{\mathrm{T}}. \tag{75}$$

Substitution of (75) into (74) yields the covariant expression

$$\mathcal{Q} = <\mathcal{X}|\mathcal{T}|\mathcal{X}> = <\mathcal{Y}|\mathcal{K}^{\mathrm{T}}\cdot\mathcal{T}\cdot\mathcal{K}|\mathcal{Y}> = <\mathcal{Y}|\mathcal{D}|\mathcal{Y}>. \tag{76}$$

We will choose the orthonormal matrix $\mathcal{K}$ such that the product $\mathcal{D} := \mathcal{K}^{\mathrm{T}}\cdot\mathcal{T}\cdot\mathcal{K}$ is diagonal and thus contains the eigenvalues of $\mathcal{T}$ as entries.

The eigenvalues of $\mathcal{T}$ turn out to be:

$$\begin{aligned} \Lambda_1 &= \frac{1}{2}(x-z+\sqrt{(x+z)^2+4y^2}), \\ \Lambda_2 &= \Lambda_3 = \Lambda_4 = -\Lambda_1, \\ \Lambda_5 &= \frac{1}{2}(x-z-\sqrt{(x+z)^2+4y^2}), \\ \Lambda_6 &= \Lambda_7 = \Lambda_8 = -\Lambda_5. \end{aligned} \tag{77}$$

According to the Lorentz structure, that is, the $(1\oplus 3)$ decomposition of four-vectors, we have two different eigenvalues for the time components and 2 threefold eigenvalues for the spatial components, respectively.

For the explicit construction of the matrix $\mathcal{K}$ we need the eigenvectors $\vec{u}_{\Lambda_n}$ of $\mathcal{T}$, where n runs from 1 to 8. Their components can be expressed (for $y\neq 0$) in terms of A and B:

$$\begin{aligned} A &:= -\frac{1}{2y}\left(x+z+\sqrt{(x+z)^2+4y^2}\right), \\ B &:= -\frac{1}{2y}\left(x+z-\sqrt{(x+z)^2+4y^2}\right). \end{aligned} \tag{78}$$

We normalize the eigenvectors by means of

$$a := \sqrt{1+A^2} \quad \text{and} \quad b := \sqrt{1+B^2}. \tag{79}$$

The columns of the matrix $\mathcal{K}$ are the normalized eigenvectors of the matrix $\mathcal{T}$. Accordingly, we find $\mathcal{K} =$

$$\begin{pmatrix} -A/a & 0 & 0 & 0 & -B/b & 0 & 0 & 0 \\ 0 & 1/b & 0 & 0 & 0 & -1/a & 0 & 0 \\ 0 & 0 & -A/a & 0 & 0 & 0 & -B/b & 0 \\ 0 & 0 & 0 & 1/b & 0 & 0 & 0 & -1/a \\ 1/a & 0 & 0 & 0 & 1/b & 0 & 0 & 0 \\ 0 & B/b & 0 & 0 & 0 & -A/a & 0 & 0 \\ 0 & 0 & 1/a & 0 & 0 & 0 & 1/b & 0 \\ 0 & 0 & 0 & B/b & 0 & 0 & 0 & -A/a \end{pmatrix}. \tag{80}$$

Because of the identity $BA = -1$ and its consequences $-A/a = 1/b$ and $B/b = 1/a$, there is actually much more symmetry than is apparent in the matrix (80), which in fact depends essentially on only one parameter. This matrix has simple properties like $\det\mathcal{K} = +1$, $\mathcal{K}\cdot\mathcal{K}^{\mathrm{T}} = \mathbf{1}_{8\times 8}$; the eigenvalues are $e^{\pm i\alpha}$, both with multiplicity four and with $\tan\alpha = 1/A$, cf. Eq. (87). Hence, in the eight-dimensional vector space the matrix $\mathcal{K}$ represents a pure rotational matrix.

Thus, the quadratic form $\mathcal{Q}$ assumes the diagonal form

$$\mathcal{Q} = <\mathcal{X}|\mathcal{T}|\mathcal{X}> = <\mathcal{Y}|\mathrm{diag}(\Lambda_1, -\vec{\Lambda}_1, \Lambda_5, -\vec{\Lambda}_5)|\mathcal{Y}>, \tag{81}$$

with the obvious abbreviations $\vec{\Lambda}_1 := \Lambda_1(1,1,1)$ and $\vec{\Lambda}_5 := \Lambda_5(1,1,1)$. The new vector $|\mathcal{Y}>$ turns out to be

$$|\mathcal{Y}> := \begin{pmatrix} \mathbf{V}_0 \\ \vec{\mathbf{V}} \\ \mathbf{A}_0 \\ \vec{\mathbf{A}} \end{pmatrix} = \mathcal{K}^{\mathrm{T}}|\mathcal{X}> = \mathcal{K}^{\mathrm{T}}\cdot\begin{pmatrix} \mathcal{V}_0 \\ \vec{\mathcal{V}} \\ \mathcal{A}_0 \\ \vec{\mathcal{A}} \end{pmatrix}. \tag{82}$$

Accordingly, the quadratic form (68) can now be written in diagonal form as

$$\mathcal{Q} = \Lambda_1({\mathbf{V}_0}^2 - \vec{\mathbf{V}}^2) + \Lambda_5({\mathbf{A}_0}^2 - \vec{\mathbf{A}}^2). \tag{83}$$

We need to pay special attention also to the case $y = 0$, namely, when there is no coupling between the vector and the axial-vector of the torsion. In this case, the matrix $\mathcal{T}$ is diagonal with

$$\mathcal{T}(y=0) = \begin{pmatrix} -x\mathbf{g} & 0 \\ 0 & z\mathbf{g} \end{pmatrix}, \tag{84}$$

and the eigenvectors of (84) correspond to the eight-dimensional unit vector. Thus, for $y \to 0$, the transformation matrix becomes the unit matrix: $\mathcal{K} = \mathbf{1}_{8\times 8}$. The transformation to the diagonal matrix yields

$$\mathcal{K}^{\mathrm{T}}\mathcal{T}\mathcal{K} = \mathcal{D} = \mathrm{diag}(x, -x, -x, -x, -z, z, z, z). \quad (85)$$

In this case, the quadratic form $\mathcal{Q}$ reduces to

$$\begin{aligned}\mathcal{Q}(y \to 0) &= x(\mathbf{V}_0{}^2 - \vec{\mathbf{V}}^2) - z(\mathbf{A}_0{}^2 - \vec{\mathbf{A}}^2)\\ &= 4m^+(\mathbf{V}_0{}^2 - \vec{\mathbf{V}}^2) - m^-(\mathbf{A}_0{}^2 - \vec{\mathbf{A}}^2).\end{aligned} \quad (86)$$

2. Representation of $\mathcal{K}$ in terms of angles

The matrix $\mathcal{K}$ as a rotation matrix can be parametrized by introducing a suitable angle variable. For this purpose we use

$$\sin\alpha = \frac{1}{\sqrt{1 + A^2}} \quad \text{and} \quad \cos\alpha = \frac{A}{\sqrt{1 + A^2}}. \quad (87)$$

In terms of this parameter α, the matrix $\mathcal{K}$ can be displayed in compact form as

$$\mathcal{K} = \begin{pmatrix} \cos\alpha\,\mathbf{I}_{4\times 4} & \sin\alpha\,\mathbf{I}_{4\times 4} \\ -\sin\alpha\,\mathbf{I}_{4\times 4} & \cos\alpha\,\mathbf{I}_{4\times 4}\end{pmatrix}, \quad (88)$$

where $\mathbf{I}_{4\times 4}$ is the 4×4 unit matrix and (x, y, z) are related to the parameter α by

$$\tan\alpha = \frac{x + z - \beta}{2y} = \frac{-2y}{x + z + \beta}, \quad (89)$$

with

$$\beta := \sqrt{(x + z)^2 + 4y^2}. \quad (90)$$

Thus, the matrix product $\mathcal{T} \cdot \mathcal{K}$ reduces to

$$\mathcal{T} \cdot \mathcal{K} = \begin{pmatrix} \cos\alpha\,\mathbf{I}_{4\times 4} & \sin\alpha\,\mathbf{I}_{4\times 4} \\ -\sin\alpha\,\mathbf{I}_{4\times 4} & \cos\alpha\,\mathbf{I}_{4\times 4}\end{pmatrix} \cdot \mathcal{D} = \mathcal{K} \cdot \mathcal{D}, \quad (91)$$

the condition for a similarity transformation.

3. A possible parameter set

We can read off from (77) the relations $\Lambda_1\Lambda_2 < 0$ and $\Lambda_5\Lambda_6 < 0$. As an example, let us consider the case $\Lambda_1 > 0$ and $\Lambda_5 > 0$. With these assumptions we can define

$$\begin{aligned} &\nu_0 := \sqrt{\Lambda_1}\mathbf{V}_0, \qquad \vec{\nu} := \sqrt{\Lambda_1}\vec{\mathbf{V}}, \qquad \Lambda_1 > 0,\\ &\alpha_0 := \sqrt{\Lambda_5}\mathbf{A}_0, \qquad \vec{\alpha} := \sqrt{\Lambda_5}\vec{\mathbf{A}}, \qquad \Lambda_5 > 0.\end{aligned} \quad (92)$$

With (92), the quadratic form $\mathcal{Q}$ (83) assumes its "special relativistic" appearance

$$\mathcal{Q} = (\nu_0^2 - \vec{\nu}^2) + (\alpha_0^2 - \vec{\alpha}^2) = -g_{\mu\nu}(\nu^\mu\nu^\nu + \alpha^\mu\alpha^\nu). \quad (93)$$

This is the sum of the squares of two four-vectors in a suitable orthonormal reference frame and the Lorentz covariance is manifest.

Let us analyze the conditions to be imposed provided one assumes $\Lambda_1 > 0$, $\Lambda_5 > 0$. From (77) we derive the constraints

$$\begin{aligned} x - z > 0 &\Longleftrightarrow 4m^+ - m^- > 0\\ &\Longleftrightarrow 3a_0 - 2(a_2 - a_3) > 0,\end{aligned} \quad (94)$$

and

$$\begin{aligned} xz + y^2 < 0 &\Longleftrightarrow m^+m^- + (m^\times)^2 < 0\\ &\Longleftrightarrow \left(a_0 - \frac{a_2}{2}\right)(a_0 - 2a_3) + (b_0 + \sigma_2)^2 < 0.\end{aligned} \quad (95)$$

On the other hand, assuming instead $\Lambda_1 > 0$, $\Lambda_5 < 0$ leads to the condition

$$\begin{aligned} xz + y^2 > 0 &\Longleftrightarrow m^+m^- + (m^\times)^2 > 0\\ &\Longleftrightarrow \left(a_0 - \frac{a_2}{2}\right)(a_0 - 2a_3) + (b_0 + \sigma_2)^2 > 0.\end{aligned} \quad (96)$$

One could similarly find the parameter conditions associated with the other two cases.

If $\mathbf{V}_\mu$ and $\mathbf{A}_\mu$ are both timelike—as they will turn out to be for the cosmological model which we derive below—every set of parameters fulfilling the inequalities (94) and (95) will lead to a strictly positive kinetic energy matrix for the translational gauge potentials.

4. Eigenvalues of the kinetic matrix of the Lorentz gauge potential $\Gamma^{\alpha\beta}$

Similarly, as in Sec. IV C 1, we consider the quadratic form $\mathcal{C}$ representing the curvature square terms in (65) and hence the kinetic parts of the connection. This quadratic form is given by

$$\begin{aligned}\mathcal{C} &:= w_6R^2 - w_3X^2 + \mu_3RX\\ &= (R, X) \cdot \begin{pmatrix} w_6 & \mu_3/2 \\ \mu_3/2 & -w_3\end{pmatrix} \cdot \begin{pmatrix} R \\ X\end{pmatrix}\\ &= <\mathcal{Z}|\mathcal{B}|\mathcal{Z}>.\end{aligned} \quad (97)$$

We wish to diagonalize the *symmetric* (2×2)-matrix

$$\mathcal{B} := \begin{pmatrix} w_6 & \mu_3/2 \\ \mu_3/2 & -w_3\end{pmatrix}, \quad (98)$$

whose eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}(w_6 - w_3 \pm \sqrt{\chi}), \qquad \chi := (w_6 - w_3)^2 + \Delta, \quad (99)$$

where

$$\Delta := -4\det\mathcal{B} = 4w_3w_6 + \mu_3^2. \quad (100)$$

This diagonalization can be simply accomplished with a rotation matrix whose columns are orthogonal unit eigenvectors; the transformation matrix and the eigenvectors can be parameterized by a single angle γ that can be determined from

$$\tan\gamma = \frac{w_6 + w_3 - \sqrt{X}}{\mu_3} = \frac{-\mu_3}{w_6 + w_3 + \sqrt{X}}. \quad (101)$$

Then,

$$\mathcal{M}^{\mathrm{T}}\mathcal{B}\mathcal{M} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \mathcal{M} := \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix}. \quad (102)$$

Consequently, with

$$(\hat{R}, \hat{X}) = (R, X) \cdot \mathcal{M}, \quad (103)$$

the quadratic form $\mathcal{C}$ can be expressed as

$$\begin{aligned} \mathcal{C} &= <Z|\mathcal{B}|Z> = <Z\mathcal{M}|\mathcal{M}^{\mathrm{T}}\mathcal{B}\mathcal{M}|\mathcal{M}^{\mathrm{T}}Z> \\ &= <\hat{Z}|\mathcal{D}_R|\hat{Z}> = \lambda_1\hat{R}^2 + \lambda_2\hat{X}^2, \end{aligned} \quad (104)$$

where $\mathcal{D}_R := \mathcal{M}^{\mathrm{T}}\mathcal{B}\mathcal{M}$ denotes a diagonal matrix.

The quadratic curvature terms in the Lagrangian can now be rewritten in the form

$$V_{\mathrm{R}^2} = -\frac{1}{24\varrho}(\lambda_1\hat{R}^2 + \lambda_2\hat{X}^2)\eta. \quad (105)$$

As an illustrative example, consider, in particular, the case where λ_1 and λ_2 are both negative. This immediately leads to the following constraints on the coupling constants:

$$w_6 - w_3 < 0, \quad \text{and} \quad \mu_3^2 + 4w_3w_6 < 0, \quad (106)$$

from which we can infer that

$$w_6 < 0, \quad \text{and} \quad w_3 > 0 \quad (107)$$

for the aforementioned case, $\lambda_1 < 0$ and $\lambda_2 < 0$. Then we can rescale the variables and introduce new ones according to

$$\mathbf{R} := \frac{\hat{R}}{\sqrt{|\lambda_1|}} \quad \text{and} \quad \mathbf{X} := \frac{\hat{X}}{\sqrt{|\lambda_2|}} \quad (108)$$

such that the quadratic form $\mathcal{C}$ in this particular case becomes

$$\mathcal{C} = -(\mathbf{R}^2 + \mathbf{X}^2). \quad (109)$$

For the three other cases, namely $(\lambda_1 > 0, \lambda_2 > 0)$, $(\lambda_1 < 0, \lambda_2 > 0)$, $(\lambda_1 > 0, \lambda_2 < 0)$, one can do an analogous rescaling.

5. Partly diagonalized Lagrangian

Thus, the process of diagonalization for the case $\Lambda_1 > 0$, $\Lambda_5 > 0$, $\lambda_1 < 0$, $\lambda_2 < 0$ leads to the following diagonal pieces of the V_{BHN}-Lagrangian,

$$\begin{aligned} V_{\mathrm{T}^2} &= \frac{1}{12\kappa}[(\nu_0^2 - \vec{\nu}^2) + (\alpha_0^2 - \vec{\alpha}^2)]\eta = \frac{1}{12\kappa}\mathcal{Q}\eta, \\ V_{\mathrm{R}^2} &= \frac{1}{24\varrho}(\mathbf{R}^2 + \mathbf{X}^2)\eta = -\frac{1}{24\varrho}\mathcal{C}\eta \end{aligned} \quad (110)$$

(with analogous results for the other sign choice cases). Collecting the results received so far, we can give a new representation of the Lagrangian (65) in the form of

$$\begin{aligned} V_{\mathrm{BHN}} = {}& \frac{1}{2\kappa}(a_0\tilde{R} - 2\lambda_0)\eta + \frac{1}{2\kappa}(a_0{}^{\star(1)}T^\alpha \wedge {}^{(1)}T_\alpha \\ & + b_0{}^{(1)}T^\alpha \wedge {}^{(1)}T_\alpha) + \frac{1}{12\kappa}\mathcal{Q}\eta - \frac{1}{24\varrho}\mathcal{C}\eta. \end{aligned} \quad (111)$$

6. Correspondences of eigenvalues of the kinetic matrices to spin and parity

In this section we considered sufficient conditions for the coefficients of the kinetic energy matrix being positive. We now assume that the trace and the axial-vector pieces of the torsion are both propagating independently. Then we require that the four-vectors ν^μ and α^μ are timelike during the whole evolution. Accordingly, the propagation of independent *massive* modes is characterized by $\nu^\mu\nu_\mu < 0$ and $\alpha^\mu\alpha_\mu < 0$. This is met by the requirement (92). Other choices of the signs of Λ_k's will lead to spacelike four-vectors. The null case will be treated separately in a continuation of this paper.

The Lagrangian (111) admits the introduction of a number of strictly positive functions, that is, functions of Λ_k's, such that for each of its dynamical variables we can associate to each eigenvalue of the kinetic energy matrix the corresponding spin and parity state.

If we decompose the four-dimensional 1-forms $\mathcal{V}$ and $\mathcal{A}$ into $(1 \oplus 3)$, respectively, we are naturally led to their spin contents. Namely, we can introduce (massive) three-dimensional vectors $\vec{\nu}$ and $\vec{\alpha}$ and can associate to each of them a corresponding three-dimensional spin and parity state. This method is not sensitive to a possible occurrence of multiplicities of spin and parity states. For this purpose, we have to investigate the corresponding lower-dimensional subspaces.

In our model, with ${}^{(1)}T^\alpha = 0$, we have only propagating *scalar* and three-dimensional *vector* modes. In a heuristic manner, the diagonalization allows for the following tentative correspondences:

$$\begin{aligned} &\Lambda_1(\mathbf{V}_0) \circ\!\!-\!\!\bullet\ 0^+, \qquad \Lambda_1(\vec{\mathbf{V}}) \circ\!\!-\!\!\bullet\ 1^+, \\ &\Lambda_5(\mathbf{A}_0) \circ\!\!-\!\!\bullet\ 0^-, \qquad \Lambda_5(\vec{\mathbf{A}}) \circ\!\!-\!\!\bullet\ 1^-, \\ &\lambda_1(\hat{R}) \circ\!\!-\!\!\bullet\ 0^+, \qquad \lambda_2(\hat{X}) \circ\!\!-\!\!\bullet\ 0^-, \\ &\qquad\qquad \tilde{R} \circ\!\!-\!\!\bullet\ 2^+. \end{aligned} \quad (112)$$

In the case of a nonvanishing tensor torsion ${}^{(1)}T^\alpha \neq 0$, those terms deliver massive modes of spin state $2^\pm$ that would combine with the corresponding spin 2^+-mode of the Riemannian curvature scalar $\tilde{R}$. Because of the complexity of these results, we will defer their presentation to follow up work.

D. Excitations of the gravitational field

We differentiate the Lagrangian (64) with respect to torsion and curvature. Then, with the help of (19), we find the translational excitation

$$H_\alpha = -\frac{1}{\kappa}(a_2{}^{\star(2)}T_\alpha + a_3{}^{\star(3)}T_\alpha) - \frac{1}{\kappa}\sigma_2({}^{(2)}T_\alpha + {}^{(3)}T_\alpha) \tag{113}$$

and the Lorentz excitation

$$H_{\alpha\beta} = \frac{1}{2\kappa}(a_0\eta_{\alpha\beta} + b_0\vartheta_{\alpha\beta}) + \frac{1}{\varrho}(w_3{}^{\star(3)}R_{\alpha\beta} + w_6{}^{\star(6)}R_{\alpha\beta}) + \frac{1}{2\varrho}\mu_3({}^{(3)}R_{\alpha\beta} + {}^{(6)}R_{\alpha\beta}), \tag{114}$$

respectively. Alternatively, the field excitations can be given in terms of the field strengths in a more suitable and symmetric form ($\mathcal{V} = \mathcal{V}_\alpha\vartheta^\alpha$ and $\mathcal{A} = \mathcal{A}_\alpha\vartheta^\alpha$)

$$H_\alpha = \frac{1}{3\kappa}[(a_2\mathcal{V}_\mu - \sigma_2\mathcal{A}_\mu)\eta^\mu{}_\alpha + (\sigma_2\mathcal{V}_\mu + a_3\mathcal{A}_\mu)\vartheta^\mu{}_\alpha], \tag{115}$$

$$H_{\alpha\beta} = \left(\frac{a_0}{2\kappa} - \frac{w_6}{12\varrho}R - \frac{\mu_3}{24\varrho}X\right)\eta_{\alpha\beta} + \left(\frac{b_0}{2\kappa} + \frac{w_3}{12\varrho}X - \frac{\mu_3}{24\varrho}R\right)\vartheta_{\alpha\beta}. \tag{116}$$

From (115) we find in particular

$$H_\alpha \wedge \vartheta^\alpha = \frac{1}{\kappa}{}^\star(a_2\mathcal{V} - \sigma_2\mathcal{A}). \tag{117}$$

E. Explicit form of field equations for the new Lagrangian

By substituting the excitations (113) and (114), and the Lagrangian (64) into the gauge currents (23) and (24), and then the latter two, together with the excitations (113) and (114), into the field equations (21) and (22), we find the explicit forms of the field equations: the first field equation reads,

$$\begin{aligned}&\left(\frac{a_0}{\kappa} - \frac{w_6}{6\varrho}R - \frac{\mu_3}{12\varrho}X\right)G_\alpha + \frac{\lambda_0}{\kappa}\eta_\alpha - \left(\frac{b_0}{\kappa} + \frac{w_3}{6\varrho}X - \frac{\mu_3}{12\varrho}R\right){}^\star X_\alpha + \frac{1}{24\varrho}(w_3X^2 - w_6R^2 - \mu_3RX)\eta_\alpha\\ &+ \frac{1}{3\kappa}D\{{}^\star[(a_2\mathcal{V} - \sigma_2\mathcal{A})\wedge\vartheta_\alpha] + (a_3\mathcal{A} + \sigma_2\mathcal{V})\wedge\vartheta_\alpha\} + \frac{2a_2}{9\kappa}\left(\mathcal{V}_\alpha\mathcal{V}^\beta - \frac{1}{4}\mathcal{V}^2\delta_\alpha^\beta\right)\eta_\beta\\ &+ \frac{2a_3}{9\kappa}\left(\mathcal{A}_\alpha\mathcal{A}^\beta - \frac{1}{4}\mathcal{A}^2\delta_\alpha^\beta\right)\eta_\beta + \frac{1}{\kappa}(e_\alpha \rfloor {}^{(1)}T^\beta)\wedge[a_2{}^{\star(2)}T_\beta + a_3{}^{\star(3)}T_\beta + \sigma_2({}^{(2)}T_\beta + {}^{(3)}T_\beta)] = \Sigma_\alpha,\end{aligned} \tag{118}$$

and the second field equation,

$$\begin{aligned}&\frac{a_0}{2\kappa}\left(2{}^{\star(1)}T_{[\alpha}\wedge\vartheta_{\beta]} - \frac{2}{3}\mathcal{V}\wedge\eta_{\alpha\beta} + \frac{1}{3}\mathcal{A}\wedge\vartheta_{\alpha\beta}\right) + \frac{b_0}{2\kappa}\left(2{}^{(1)}T_{[\alpha}\wedge\vartheta_{\beta]} - \frac{2}{3}\mathcal{V}\wedge\vartheta_{\alpha\beta} - \frac{1}{3}\mathcal{A}\wedge\eta_{\alpha\beta}\right)\\ &+ \frac{1}{24\varrho}(2w_3dX - \mu_3dR)\wedge\vartheta_{\alpha\beta} - \frac{1}{24\varrho}(2w_6dR + \mu_3dX)\wedge\eta_{\alpha\beta} - \frac{1}{24\varrho}(2w_6R + \mu_3X)T^\gamma\wedge\eta_{\alpha\beta\gamma}\\ &+ \frac{1}{12\varrho}(2w_3X - \mu_3R)T_{[\alpha}\wedge\vartheta_{\beta]} - \frac{1}{3\kappa}[a_2\mathcal{V}_{[\alpha}\eta_{\beta]} - \sigma_2\mathcal{A}_{[\alpha}\eta_{\beta]} + (a_3\mathcal{A} + \sigma_2\mathcal{V})\wedge\vartheta_{\alpha\beta}] = \tau_{\alpha\beta}.\end{aligned} \tag{119}$$

The source of the second field equation is the material spin angular momentum 3-form $\tau_{\alpha\beta}$. According to its definition in (20), it is antisymmetric in α and β. It is related to the source of the first field equation, the canonical energy-momentum 3-form of matter Σ_α, via the angular momentum law $D\tau_{\alpha\beta} + \vartheta_{[\alpha}\wedge\Sigma_{\beta]} = 0$.

It may not be superfluous to look at the structures of the two field equations and at the ways we ordered them. The first line of (118) emerges from the curvature dependent pieces of the gauge energy-momentum E_α. We find, symbolically written, ~Einstein + cosmol.term + pseudoscalar curv + curv^2. If only $a_0 \neq 0$ and $b_0 \neq 0$, we recover the left-hand-side of (45); if only $a_0 \neq 0$, we have just the EC theory. The second line and the first piece of the third line of (118) are of the following structure: d torsion + torsion^2. From the point of view of gauge theory, "d torsion" is the leading term, see DH_α in (21); the remaining "torsion^2" pieces collect the torsion dependent parts of the gauge energy-momentum E_α. Of course, also the frame e_α and the coframe ϑ^α feature in this equation directly or indirectly via $\eta_\alpha = {}^\star\vartheta_\alpha$.

We ordered the second field equation (119) in a similar way. In the first line we have the terms linear in torsion originating from the gauge spin $E_{\alpha\beta}$, see (24); compare also as a special case the left-hand-side of (46). In the second and third line, we have "d curv + curv × torsion." The leading term is "d curv", the rest arises from the differentiation process of "D curv" with the help of (2).

Again the coframe ϑ^α enters explicitly or implicitly via $\vartheta_{\alpha\beta} = \vartheta_\alpha \wedge \vartheta_\beta$, $\eta_{\alpha\beta} = {}^\star(\vartheta_\alpha \wedge \vartheta_\beta)$, and $\eta_{\alpha\beta\gamma} = {}^\star(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma)$.

Accordingly, the two field equations are now expressed in terms of the torsion and the curvature of spacetime. In the sense of gauge theory one may now want to insert the definitions of torsion, Eq. (5), and curvature, Eq. (12) cum (11). Then we would get second order quasilinear partial differential equations in the coframe and Lorentz connection: "$d\,{}^\star d\vartheta$ + lower order $\sim$ energy-momemtum" and "$d\,{}^\star d\Gamma$ + lower order $\sim$ spin ." However, our experience on the search for exact solutions, in particular, for cosmological models, has shown that it is to be preferred to stay with the well-behaved tensor-valued 2-forms of torsion and curvature and not to switch to the proper gauge variables coframe and Lorentz connection.

As we saw already, the Lagrangian (64) has a remarkable symmetry which we will find also on the level of the excitations (115) and (116), the field equations (118) and (119), as well as on the level of the coupling constants. Hence, we can introduce the following tentative correspondences (in four dimensions) between variables and parameters, respectively,

$$\boxed{\mathcal{V} \circ\!\!-\!\!\bullet \mathcal{A}} \qquad \boxed{a_2 \circ\!\!-\!\!\bullet a_3} \tag{120}$$

$$\boxed{\begin{array}{c} R \circ\!\!-\!\!\bullet X \\ G_\alpha \circ\!\!-\!\!\bullet {}^\star X_\alpha \end{array}} \qquad \boxed{\begin{array}{c} w_6 \circ\!\!-\!\!\bullet w_3 \\ a_0 \circ\!\!-\!\!\bullet b_0 \end{array}} \tag{121}$$

$$\boxed{\sigma_2 \circ\!\!-\!\!\bullet \mu_3} \tag{122}$$

F. A consequence of a topological term

There is a subtlety present in the Lagrangian (64) and the corresponding field equations (118) and (119). Because of the Nieh-Yan identity [48]—see also [14], Eqs. (3.9.7) and (B.4.15)—we have

$$d(\vartheta^\alpha \wedge T_\alpha) \equiv T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^{\alpha\beta} = T^\alpha \wedge T_\alpha - {}^\star X. \tag{123}$$

The torsion square term can be expressed in its irreducible components according to

$$\begin{aligned} T^\alpha \wedge T_\alpha &= {}^{(1)}T^\alpha \wedge {}^{(1)}T_\alpha + 2{}^{(2)}T^\alpha \wedge {}^{(3)}T_\alpha \\ &= {}^{(1)}T^\alpha \wedge {}^{(1)}T_\alpha - \frac{2}{3}\mathcal{A} \wedge {}^\star\mathcal{V}. \end{aligned} \tag{124}$$

We substitute (124) into the right-hand-side of the Nieh-Yan identity (123) and find

$$d(\vartheta^\alpha \wedge T_\alpha) = {}^{(1)}T^\alpha \wedge {}^{(1)}T_\alpha - \frac{2}{3}\mathcal{A} \wedge {}^\star\mathcal{V} - X^\star. \tag{125}$$

For the sake of a neater argument let us first extend our parity mixed PG Lagrangian (64) by including also the σ_1-term from (55):

$$\hat{V} = V_{\mathrm{BHN}} + \overset{\sigma_1}{V} = V_{\mathrm{BHN}} + \frac{\sigma_1}{\kappa}{}^{(1)}T^\alpha \wedge {}^{(1)}T_\alpha. \tag{126}$$

If we substitute (64) into (126), then we recognize that we can recover from $\kappa\hat{V}$ the right-hand side (rhs) of the Nieh-Yan identity (125) for the specific coupling constants

$$b_0 = -2, \qquad \sigma_1 = 1, \qquad \sigma_2 = 2, \tag{127}$$

all other constants, apart from κ and ϱ, vanish. Since the left-hand side (lhs) of (125) is an exact form, the choice (127) corresponds to a "null Lagrangian" with vanishing field equations.

By the same token, we can add a multiple (say ϵ/κ) of the exact "topological" form $d(\vartheta^\alpha \wedge T_\alpha)$ to $\hat{V}$. After some simple algebra, we find

$$\begin{aligned} \hat{V} + \frac{\epsilon}{\kappa}d(\vartheta^\alpha \wedge T_\alpha) &= \frac{1}{\kappa}(\sigma_1 + \epsilon){}^{(1)}T^\alpha \wedge {}^{(1)}T_\alpha \\ &\quad + \frac{1}{2\kappa}[a_0{}^\star R + (b_0 - 2\epsilon)\,{}^\star X - 2\lambda_0\eta] \\ &\quad + \frac{1}{6\kappa}[a_2\mathcal{V} \wedge {}^\star\mathcal{V} - a_3\mathcal{A} \wedge {}^\star\mathcal{A} - 2(\sigma_2 + 2\epsilon)\mathcal{A} \wedge {}^\star\mathcal{V}] \\ &\quad - \frac{1}{24\varrho}(w_6 R\,{}^\star R - w_3 X\,{}^\star X + \mu_3 R\,{}^\star X). \end{aligned} \tag{128}$$

This is equivalent to certain changes in the parameters of our action (126), specifically

$$\sigma_1 \rightarrow \sigma_1 + \epsilon, \qquad b_0 \rightarrow b_0 - 2\epsilon, \qquad \sigma_2 \rightarrow \sigma_2 + 2\epsilon. \tag{129}$$

From this we can infer that the field equations cannot depend on the parameters σ_1, b_0, σ_2 by themselves, but rather must depend on these parameters only through certain combinations which are *invariant* under this transformation, such as

$$b_0 + 2\sigma_1, \qquad 2\sigma_1 - \sigma_2, \qquad b_0 + \sigma_2. \tag{130}$$

The two field equations of the Lagrangian $\hat{V}$ are found via $\overset{\sigma_1}{H}_\alpha = -2\sigma_1{}^{(1)}T_\alpha/\kappa$ as

$$\begin{aligned} \text{lhs of (118)} - \frac{2\sigma_1}{\kappa}[D{}^{(1)}T_\alpha \\ + (e_\alpha \rfloor {}^{(1)}T^\beta) \wedge ({}^{(2)}T^\beta + {}^{(3)}T_\beta)] = \Sigma_\alpha, \end{aligned} \tag{131}$$

$$\text{lhs of (119)} - \frac{2\sigma_1}{\kappa}\vartheta_{[\alpha} \wedge {}^{(1)}T_{\beta]} = \tau_{\alpha\beta}. \tag{132}$$

Specifically, using the first Bianchi identity,

$$
{}^\star X_\alpha = R_{\beta\alpha} \wedge \vartheta^\beta \equiv DT_\alpha = D({}^{(1)}T_\alpha + {}^{(2)}T_\alpha + {}^{(3)}T_\alpha), \tag{133}
$$

and the expressions (7) and (8) for the irreducible pieces, one can find that the b_0, σ_1, and σ_2 terms on the left-hand-side of the first field equation (131) add up to

$$
-\frac{1}{\kappa} D[(b_0 + 2\sigma_1)^{(1)}T_\alpha + (b_0 + \sigma_2)({}^{(2)}T_\alpha + {}^{(3)}T_\alpha)] + \frac{1}{\kappa}(2\sigma_1 - \sigma_2)(e_\alpha \rfloor {}^{(1)}T^\beta) \wedge ({}^{(2)}T_\beta + {}^{(3)}T_\beta); \tag{134}
$$

similarly, for the left-hand-side of the second field equation (132) we have

$$
\frac{1}{\kappa}(b_0 + 2\sigma_1)\, {}^{(1)}T_{[\alpha} \wedge \vartheta_{\beta]} - \frac{1}{6\kappa}(b_0 + \sigma_2)(2\mathcal{V} \wedge \vartheta_{\alpha\beta} + \mathcal{A} \wedge \eta_{\alpha\beta}). \tag{135}
$$

There are several points worth noting: (i) As expected, the parameters occur only in the invariant combinations (130). (ii) All the σ_1 terms are proportional to ${}^{(1)}T^\mu$. (iii) For solutions such that ${}^{(1)}T^\mu$ vanishes, the equations contain the parameters only in the combination $b_0 + \sigma_2$.

Having obtained these insights, it is no longer necessary to keep the rather complicated σ_1 term. Exploiting the freedom to choose a suitable ϵ in (129), namely $\epsilon = -\sigma_1$, we can, without loss of generality, drop the σ_1-term altogether and return to our model Lagrangian (64) with the two field equations (118) and (119).

V. FRIEDMAN COSMOLOGIES WITH PROPAGATING MODES OF THE LORENTZ CONNECTION

Since the early 1970s, cosmological models for EC and PG have been developed, see Kopczyński [82], Trautman [83], Tafel [84], Kuchowicz [85,86], Kerlick [87], and others [88,89], to name a few. Minkevich *et al.* [90] developed the subject in a series of papers. A report on the status of the subject was given by Puetzfeld [91].

For our new Lagrangian we follow these procedures and search for FLRW type of cosmological models.

A. Homogeneous and isotropic coframe and torsion

Assuming a homogeneous and isotropic scenario, the *orthonormal coframe* for a Friedman cosmos is

$$
\begin{aligned}
\vartheta^0 &= dt,\\
\vartheta^1 &= \frac{a(t)dr}{\sqrt{1-kr^2}},\\
\vartheta^2 &= a(t) r d\theta,\\
\vartheta^3 &= a(t) r \sin\theta d\phi,
\end{aligned} \tag{136}
$$

with the metric

$$
g = -\vartheta^0 \otimes \vartheta^0 + \sum_{a=1}^{3} \vartheta^a \otimes \vartheta^a = -dt^2 + \frac{a^2(t)}{1-kr^2} dr^2 + a^2(t) r^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{137}
$$

where $a(t)$ is the expansion factor and k the curvature index.

Now we can compute, up to antisymmetry, the non-vanishing components of the Riemannian connection ($a, b, c, \ldots = 1, 2, 3$ are spatial anholonomic (frame) indices):

$$
\begin{aligned}
\tilde{\Gamma}_a{}^0 &= \frac{a'(t)}{a(t)} \vartheta^a, \qquad \tilde{\Gamma}_2{}^1 = -\frac{\sqrt{1-kr^2}}{a(t) r} \vartheta^2,\\
\tilde{\Gamma}_3{}^1 &= -\frac{\sqrt{1-kr^2}}{a(t) r} \vartheta^3, \qquad \tilde{\Gamma}_3{}^2 = -\frac{\cot\theta}{a(t) r} \vartheta^3.
\end{aligned} \tag{138}
$$

As in Einstein's theory, the temporal rate of change of the expansion factor $a(t)$ determines the Hubble function

$$
H(t) := a'(t)/a(t). \tag{139}
$$

By differentiation and elimination of $a'(t)$, we find

$$
H'(t) + H^2(t) = a''(t)/a(t). \tag{140}
$$

This determines the Riemannian sector of spacetime.

The most general torsion compatible with homogeneity and isotropy can be characterized by two independent functions $u(t)$ and $v(t)$, see Goenner and Müller-Hoissen [92] and Baekler [93]. We will choose for the torsion the parametrization

$$
\begin{aligned}
T^0 &= 0,\\
T^1 &= u(t)\vartheta^{01} + v(t)\vartheta^{23},\\
T^2 &= u(t)\vartheta^{02} + v(t)\vartheta^{31},\\
T^3 &= u(t)\vartheta^{03} + v(t)\vartheta^{12}.
\end{aligned} \tag{141}
$$

The irreducible decomposition of T^α implies a vanishing tensor piece, whereas the vector and axial-vector pieces survive:

$$
{}^{(1)}T^\alpha = 0, \qquad {}^{(2)}T^\alpha = u \begin{pmatrix} 0 \\ \vartheta^{01} \\ \vartheta^{02} \\ \vartheta^{03} \end{pmatrix}, \qquad {}^{(3)}T^\alpha = v \begin{pmatrix} 0 \\ \vartheta^{23} \\ \vartheta^{31} \\ \vartheta^{12} \end{pmatrix}. \tag{142}
$$

This yields for the corresponding 1-forms in (7) and (8)

$$
\mathcal{V} = -3u(t)\vartheta^0, \qquad \mathcal{A} = -3v(t)\vartheta^0. \tag{143}
$$

Incidentally, the purely spatial part of the torsion ${}^{(3)}T^\alpha$ corresponds to Cartan's spiral staircase [94,95] with a time dependent pitch of the spiral. As such, one can easily visualize it, see [95].

By simple algebra, we can calculate the contortion $K_{\alpha\beta}$ in terms of the torsion, see [14],

$$K_{\alpha\beta} = e_{[\alpha} \rfloor T_{\beta]} - \frac{1}{2}(e_\alpha \rfloor e_\beta \rfloor T_\gamma)\vartheta^\gamma. \tag{144}$$

We find then the Riemann-Cartan connection according to $\Gamma_{\alpha\beta} = \tilde{\Gamma}_{\alpha\beta} - K_{\alpha\beta}$ or, explicitly,

$$\Gamma_a{}^0 = [H(t) - u(t)]\vartheta^a, \qquad \Gamma_a{}^b = \tilde{\Gamma}_a{}^b + \frac{1}{2}v(t)\epsilon_{ca}{}^b\vartheta^c. \tag{145}$$

B. Irreducible pieces of the curvature

Having now coframe, connection, and torsion at our disposal, we can immediately calculate the different pieces of the curvature 2-form. We introduce the Hubble function $H(t)$ (139) and find straightforwardly

$$^{(1)}R_{\alpha\beta} = 0, \tag{146}$$

$$^{(2)}R_{\alpha\beta} = \frac{1}{4}\{v(t)[H(t) - 2u(t)] - v'(t)\} \times \begin{pmatrix} 0 & \vartheta^{23} & -\vartheta^{13} & \vartheta^{12} \\ \diamond & 0 & -\vartheta^{03} & \vartheta^{02} \\ \diamond & \diamond & 0 & -\vartheta^{01} \\ \diamond & \diamond & \diamond & 0 \end{pmatrix}, \tag{147}$$

$$^{(3)}R_{\alpha\beta} = -\frac{1}{12}X(t)\eta_{\alpha\beta}, \tag{148}$$

$$^{(4)}R_{\alpha\beta} = \frac{1}{2}\left[H'(t) - u'(t) + H(t)u(t) - u^2(t) + \frac{1}{4}v^2(t) - \frac{k}{a^2(t)}\right]\begin{pmatrix} 0 & \vartheta^{01} & \vartheta^{02} & \vartheta^{03} \\ \diamond & 0 & \vartheta^{12} & \vartheta^{13} \\ \diamond & \diamond & 0 & \vartheta^{23} \\ \diamond & \diamond & \diamond & 0 \end{pmatrix}, \tag{149}$$

$$^{(5)}R_{\alpha\beta} = 0, \tag{150}$$

$$^{(6)}R_{\alpha\beta} = -\frac{1}{12}R(t)\vartheta_{\alpha\beta}, \tag{151}$$

with the (pseudo-)scalar functions

$$X(t) = -3\{v'(t) + v(t)[3H(t) - 2u(t)]\}, \tag{152}$$

$$R(t) = 6\left\{[H'(t) - u'(t)] + H(t)[H(t) - u(t)] + [H(t) - u(t)]^2 - \frac{1}{4}v^2(t) + \frac{k}{a^2(t)}\right\}. \tag{153}$$

The matrices in (147) and (149) are antisymmetric, respectively. Therefore, we indicated those matrix elements with a diamond symbol that are, because of this antisymmetry, redundant.

Since we chose as our variables curvature and torsion, the relations between curvature and torsion provided by the nonvanishing irreducible pieces and, in particular, by (152) and (153), are vital information in our search for exact solutions.

C. A spinless perfect fluid as model of matter

The sources of the two field equations are the energy-momentum current Σ_α and the spin current $\tau_{\alpha\beta}$. These 3-forms we represent as tensor-valued 0-forms according to

$$\Sigma_\alpha = \mathcal{T}_\alpha{}^\mu\eta_\mu, \qquad \tau_{\alpha\beta} = S_{\alpha\beta}{}^\mu\eta_\mu; \tag{154}$$

the reciprocal relations read

$$\mathcal{T}_\alpha{}^\beta = {}^\star(\Sigma_\alpha \wedge \vartheta^\beta), \qquad S_{\alpha\beta}{}^\gamma = {}^\star(\tau_{\alpha\beta} \wedge \vartheta^\gamma). \tag{155}$$

In the following we will only consider matter models with vanishing spin, $\tau_{\alpha\beta} = 0$. This simplifying assumption, which may be justified for spherical symmetry, certainly has to be dropped in a more advanced stage of our model building.

Because of the Friedman (or FLRW) symmetry of our cosmological model, the energy-momentum tensor must have the spinless perfect fluid form

$$\mathcal{T}_\alpha{}^\beta = [\rho(t) + p(t)]U_\alpha U^\beta + p(t)\delta_\alpha^\beta, \tag{156}$$

where $\rho = \rho(t)$ is the energy density, $p = p(t)$ the pressure, and U_α the four-velocity of the fluid, with the normalization $U^\alpha U_\alpha = -1$. Because of the symmetry requirements, we only have the dependencies $\rho(t)$ and $p(t)$. In a comoving reference system, we have $U^\alpha = \delta_0^\alpha$, $U^0 = 1$, and

$$\mathcal{T}_\alpha{}^\beta \stackrel{*}{=} \mathrm{diag}(-\rho(t), p(t), p(t), p(t)). \tag{157}$$

From (156) we deduce for the trace

$$\mathcal{T}_\alpha{}^\alpha = -\rho(t) + 3p(t) \tag{158}$$

and for the trace-free part

$$\mathcal{T}^{\dagger\beta}_\alpha = [\rho(t) + p(t)](U_\alpha U^\beta + \frac{1}{4}\delta_\alpha^\beta) \stackrel{*}{=} \frac{1}{4}[\rho(t) + p(t)]\,\mathrm{diag}(-3, 1, 1, 1). \tag{159}$$

D. Differential equations for torsion and curvature

According to the FLRW-symmetry requirements, the first field equation (118) as a vector-valued 3-form, has only two (algebraically) independent components and in the same manner the second field equation (119) as an antisymmetric tensor-valued 3-form also has only two independent components.

1. First field equation

We substitute into the first field equation (118) the coframe (136), the torsion (141), and the expansion factor via (139) and (140), but leave $R(t)$ and $X(t)$ as they are. Then we find as independent nonvanishing equations only the components $\rho(t) = e_1 \rfloor [e_2 \rfloor (e_3 \rfloor \Sigma_0)]$ and $p(t) = e_0 \rfloor [e_2 \rfloor (e_3 \rfloor \Sigma_1)]$:

$$\begin{aligned}\kappa\rho(t) = {}& \frac{1}{2}[a_0 R(t) + b_0 X(t)] + 3a_0\{[u(t) - H(t)]' + H(t)[u(t) - H(t)]\} + \frac{3}{2}b_0[v'(t) + H(t)v(t)] - \frac{3}{2}\{a_2 u(t)[u(t) \\ & - 2H(t)] - a_3 v^2(t)\} + 3\sigma_2 v(t)[u(t) - H(t)] - \frac{\kappa}{4\varrho}[2w_6 R(t) + \mu_3 X(t)]\{[u(t) - H(t)]' + H(t)[u(t) - H(t)]\} \\ & - \frac{\kappa}{8\varrho}[\mu_3 R(t) - 2w_3 X(t)][v'(t) + H(t)v(t)] - \frac{\kappa}{24\varrho}[w_6 R^2(t) + \mu_3 R(t)X(t) - w_3 X^2(t)] - \lambda_0 \end{aligned} \tag{160}$$

and

$$\begin{aligned}\kappa p(t) = {}& -\frac{1}{2}[a_0 R(t) + b_0 X(t)] + a_0\left[H'(t) + 3H^2(t) + \frac{2k}{a^2(t)}\right] - (a_0 + a_2)u'(t) + \frac{1}{2}(2\sigma_2 - b_0)v'(t) \\ & - (5a_0 + 2a_2)H(t)u(t) - \frac{1}{2}(5b_0 - 4\sigma_2)H(t)v(t) + \frac{1}{2}(4a_0 + a_2)u^2(t) + (2b_0 - \sigma_2)u(t)v(t) - \frac{1}{2}(a_0 + a_3)v^2(t) \\ & + \lambda_0 - \frac{\kappa}{12\varrho}[2w_6 R(t) + \mu_3 X(t)]\left[H'(t) + 3H^2(t) + \frac{2k}{a^2(t)} - u'(t) - 5H(t)u(t) + 2u^2(t) - \frac{1}{2}v^2(t)\right] \\ & + \frac{\kappa}{24\varrho}[\mu_3 R(t) - 2w_3 X(t)][v'(t) + 5H(t)v(t) - 4u(t)v(t)] + \frac{\kappa}{24\varrho}[w_6 R^2(t) + \mu_3 R(t)X(t) - w_3 X^2(t)]. \end{aligned} \tag{161}$$

A further relation between the fluid density $\rho(t)$ and the pressure $p(t)$ can be gained by taking the trace $\vartheta^\alpha \wedge \Sigma_\alpha$ of the first field equation or, equivalently, by computing $\rho(t) - 3p(t)$ from (160) and (161). However, in order to find a compact expression, we resolve (152) with respect to $v'(t)$ and (153) with respect to $H'(t)$. If we substitute these expressions, we find[6]

$$\begin{aligned}\kappa[\rho(t) - 3p(t)] &= a_0 R(t) + (b_0 + \sigma_2)X(t) + 3a_2[u'(t) - u^2(t) + 3H(t)u(t)] + 3a_3 v^2(t) - 4\lambda_0 \\ &= \frac{1}{2}(2a_0 - a_2)R(t) + (b_0 + \sigma_2)X(t) + \frac{3}{4}(4a_3 - a_2)v^2(t) + 3a_2\left(H'(t) + 2H^2(t) + \frac{k}{a^2(t)}\right) - 4\lambda_0. \end{aligned} \tag{162}$$

Let us now get back to the first field equation (160) with (161). One strategy is to eliminate the time derivative $H'(t)$ of the Hubble function by a suitable linear combination of (160) and (161). Accordingly, we put the linear combination $\{e_1 \rfloor [e_2 \rfloor (e_3 \rfloor \Sigma_0)] + 3e_0 \rfloor [e_2 \rfloor (e_3 \rfloor \Sigma_1)] - [\rho(t) + 3p(t)]\}$ to zero and, isolating the derivatives of the torsion functions, we find

$$\begin{aligned}-3[a_2 u'(t) - \sigma_2 v'(t)] = {}& \kappa[\rho(t) + 3p(t)] + 3H(t)[a_2 u(t) - \sigma_2 v(t)] + a_0 R(t) + b_0 X(t) + 6b_0 v(t)[H(t) - u(t)] \\ & + \frac{\kappa}{2\varrho}[2w_3 X(t) - \mu_3 R(t)]v(t)[H(t) - u(t)] - 6a_0\left\{[H(t) - u(t)]^2 - \frac{1}{4}v^2(t) + \frac{k}{a^2(t)}\right\} \\ & + \frac{\kappa}{2\varrho}[\mu_3 X(t) + 2w_6 R(t)]\left\{[H(t) - u(t)]^2 - \frac{1}{4}v(t)^2 + \frac{k}{a^2(t)}\right\} - 2\lambda_0 \\ & - \frac{\kappa}{12\varrho}[w_6 R^2(t) + \mu_3 R(t)X(t) - w_3 X^2(t)]. \end{aligned} \tag{163}$$

Alternatively, the last relation (163) can also be expressed as

[6] On the right-hand-side there emerges only a time derivative of the torsion, this can be seen from the general structure of this expression: $\vartheta^\alpha \wedge \Sigma_\alpha = (\rho - 3p)\eta = -d(\vartheta^\alpha \wedge H_\alpha) - 4V - T^\alpha \wedge H_\alpha - 2R_\alpha{}^\beta \wedge H^\alpha{}_\beta$.

$$-\frac{3}{a(t)}\frac{d}{dt}\{a(t)[a_2u(t)-\sigma_2v(t)]\} = \kappa[\rho(t)+3p(t)] + a_0R(t) + b_0X(t) + 6b_0v(t)[H(t)-u(t)] + \frac{\kappa}{2\varrho}(2w_3X(t) - \mu_3R(t))v(t)[H(t)-u(t)] - 6a_0\left\{[H(t)-u(t)]^2 - \frac{1}{4}v^2(t) + \frac{k}{a^2(t)}\right\} + \frac{\kappa}{2\varrho}[\mu_3X(t)+2w_6R(t)]\left\{[H(t)-u(t)]^2 - \frac{1}{4}v(t)^2 + \frac{k}{a^2(t)}\right\} - 2\lambda_0 - \frac{\kappa}{12\varrho}[w_6R^2(t) + \mu_3R(t)X(t) - w_3X^2(t)]. \tag{164}$$

This equation suggests to impose the interrelationship

$$u(t) = \sigma v(t), \quad \text{with} \quad \sigma := \frac{\sigma_2}{a_2}, \tag{165}$$

between the two torsion pieces as a simple special case; a further possible choice could be

$$u(t) - \sigma v(t) = \frac{c}{a(t)}, \qquad c = \text{constant} \neq 0. \tag{166}$$

Then the left-hand-sides of (163) and (164) vanish and we find a purely algebraic equation in the variables $\{v(t)(\text{or } u(t)), R(t), X(t), H(t), \rho(t), p(t)\}$. We will defer the study of these two alternatives to future work.

We can also manipulate the first field equation (160) with (161) in a different way in order to arrive at algebraic relations. We can resolve (153) and (152) with respect to $u'(t)$ and $v'(t)$ and substitute these expressions into (160) and (161), respectively.

After eliminating the derivatives of the torsion we arrive at

$$\kappa\rho(t) = -3m^+u(t)[2H(t)-u(t)] - \frac{3}{4}m^-v^2(t) + 3a_0\left(H^2(t) + \frac{k}{a^2(t)}\right) - 3m^\times[H(t)-u(t)]v(t) - \frac{\kappa}{4\varrho}[\mu_3X(t)+2w_6R(t)]\left\{[H(t)-u(t)]^2 - \frac{1}{4}v^2(t) + \frac{k}{a^2(t)}\right\} + \frac{\kappa}{4\varrho}[\mu_3R(t) - 2w_3X(t)]v(t)[H(t)-u(t)] + \frac{\kappa}{24\varrho}[w_6R^2(t) - w_3X^2(t) + \mu_3R(t)X(t)] - \lambda_0 \tag{167}$$

and

$$\kappa p(t) = -\frac{1}{3}m^+R(t) - \frac{1}{3}m^\times X(t) + 2(m^+ - a_0)H'(t) + m^\times v(t)[u(t)-H(t)] + \frac{1}{4}m^-v^2(t) + a_0\left\{[H(t)-u(t)]^2 - \frac{1}{2}v^2(t) + \frac{k}{a^2(t)}\right\} + 2(m^+ - a_0)\left\{H(t)[2H(t)-u(t)] + \frac{1}{2}u^2(t) - \frac{1}{4}v^2(t) + \frac{k}{a^2(t)}\right\} + \frac{\kappa}{12\varrho}[\mu_3R(t) - 2w_3X(t)]v(t)[H(t)-u(t)] - \frac{\kappa}{12\varrho}[\mu_3X(t)+2w_6R(t)]\left\{[H(t)-u(t)]^2 - \frac{1}{4}v^2(t) + \frac{k}{a^2(t)}\right\} + \frac{\kappa}{72\varrho}[w_6R^2(t) - w_3X^2(t) + \mu_3R(t)X(t)] + \lambda_0. \tag{168}$$

Inspecting the equations there are the following dependencies,

$$\begin{aligned} \rho(t) &= \rho[a(t), H(t), u(t), v(t), R(t), X(t)], \\ p(t) &= p[a(t), H(t), H'(t), u(t), v(t), R(t), X(t)], \end{aligned} \tag{169}$$

that is, the density depends only algebraically on the variables whereas the pressure, besides non linear terms, contains only the derivative of $H(t)$. Thus, also this general case belongs to the class of *descriptor systems*, that is, to the differential-algebraic systems.

2. *Second field equation*

Similarly, also for the second field equation (119), we find only two independent components. Both vanish by the assumption of vanishing matter spin $\tau_{\alpha\beta}$. Thus,

$$\begin{aligned} &\kappa\{2w_6R'(t) + \mu_3X'(t) + [\mu_3v(t) + 4w_6u(t)]R(t) \\ &\quad + 2[\mu_3u(t) - w_3v(t)]X(t)\} - 12\varrho[2m^+u(t) + m^\times v(t)] \\ &= 24\kappa\varrho e_0\rfloor[e_2\rfloor(e_3\rfloor\tau_{01})] = 0, \end{aligned} \tag{170}$$

$$\kappa\{\mu_3 R'(t) - 2w_3 X'(t) + 2[\mu_3 u(t) - w_6 v(t)]R(t) - [4w_3 u(t) + \mu_3 v(t)]X(t)\} - 12\varrho[2m^\times u(t) - m^- v(t)] = 24\kappa\varrho e_0 \lrcorner [e_2 \lrcorner (e_3 \lrcorner \tau_{23})] = 0. \tag{171}$$

These are two ordinary linear differential equations (ODEs) of first order in the curvature components $R(t)$ and $X(t)$. By suitable linear combinations, we can uncouple the first derivatives of these equations. We add $2w_3\times$ Eq. (170) to $\mu_3\times$ Eq. (171) and find, provided $\Delta = \mu_3^2 + 4w_3w_6 \neq 0$, see (100),

$$R'(t) + 2\left(u(t) + \mu_3 \frac{w_3 - w_6}{\Delta} v(t)\right)R(t) - \frac{\mu_3^2 + 4w_3^2}{\Delta} v(t)X(t) - \frac{12\varrho/\kappa}{\Delta}[2(2m^+ w_3 + m^\times \mu_3)u(t) + (-m^- \mu_3 + 2m^\times w_3)v(t)] = 0. \tag{172}$$

Similarly, we add $\mu_3\times$ Eq. (170) to $-2w_6\times$ Eq. (171) and find, provided $\mu_3^2 + 4w_3w_6 \neq 0$,

$$X'(t) + 2\left(u(t) - \mu_3 \frac{w_3 - w_6}{\Delta} v(t)\right)X(t) + \frac{\mu_3^2 + 4w_6^2}{\Delta} v(t)R(t) - \frac{12\varrho/\kappa}{\Delta}[(2m^+ \mu_3 - 4m^\times w_6)u(t) + (2m^- w_6 + m^\times \mu_3)v(t)] = 0. \tag{173}$$

Let us introduce, by also using (66), the following abbreviations

$$\omega_0^2 := \frac{4\varrho}{\kappa}\frac{2m^- w_6 + m^\times \mu_3}{\Delta}, \qquad \omega_1^2 := \frac{4\varrho}{\kappa}\frac{2m^+ w_3 + m^\times \mu_3}{\Delta}, \tag{174}$$

$$\omega_2^2 := \frac{4\varrho}{\kappa}\frac{m^- \mu_3 - 2m^\times w_3}{\Delta}, \qquad \omega_3^2 := \frac{4\varrho}{\kappa}\frac{2m^+ \mu_3 - 4m^\times w_6}{\Delta}. \tag{175}$$

The constants ω_0^2 and ω_1^2 are of even parity, whereas ω_2^2 and ω_3^2 are of odd parity. This allows us to give the components of the second field equation (172) and (173) a more compact and transparent form,

$$R'(t) = 6\omega_1^2 u(t) - 3\omega_2^2 v(t) - 2u(t)R(t) + \frac{v(t)}{\Delta}[(\mu_3^2 + 4w_3^2)X(t) - 2\mu_3(w_3 - w_6)R(t)], \tag{176}$$

$$X'(t) = 3\omega_3^2 u(t) + 3\omega_0^2 v(t) - 2u(t)X(t) - \frac{v(t)}{\Delta}[(\mu_3^2 + 4w_6^2)R(t) - 2\mu_3(w_3 - w_6)X(t)]. \tag{177}$$

The choice of signs in (174) and (175) will be motivated in the next subsection, for the moment they are just a shorthand notation for certain non linear combinations of the fundamental coupling constants of the theory.

It may be a bit more transparent, to put (176) and (177) into a matrix form and to reinsert Δ:

$$\frac{d}{dt}\begin{pmatrix} R(t) \\ X(t) \end{pmatrix} = 3\begin{pmatrix} 2\omega_1^2 & -\omega_2^2 \\ \omega_3^2 & \omega_0^2 \end{pmatrix}\cdot\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - 2\left[u(t) + \mu_3 \frac{w_3 - w_6}{\mu_3^2 + 4w_3w_6} v(t)\right]\begin{pmatrix} R(t) \\ X(t) \end{pmatrix} - \frac{v(t)}{\mu_3^2 + 4w_3w_6}\begin{pmatrix} 0 & \mu_3^2 + 4w_3^2 \\ \mu_3^2 + 4w_6^2 & 0 \end{pmatrix}\cdot\begin{pmatrix} R(t) \\ X(t) \end{pmatrix}. \tag{178}$$

We have eight variables but only seven relations between them. However, we still have to choose an appropriate *equation of state* $p = p(\rho)$. In cosmological models for late times, $p(t) \approx 0$ is a widespread assumption.

E. Rearranging the field equations into first order form

For certain purposes (in particular numerical evolution) it is more convenient to replace the 3 dynamical second order equations for the gauge potentials by 6 first order equations for the observable quantities $\{a, H, u, v, R, X\}$. From (139), (152), and (162) one can obtain the first order set

$$a'(t) = H(t)a(t), \tag{179}$$

$$H'(t) = -2H^2(t) - \frac{k}{a^2(t)} + \frac{1}{3a_2}\left\{\kappa[\rho(t) - 3p(t)] + 4\lambda_0 - \left(a_0 - \frac{a_2}{2}\right)R(t) - (b_0 + \sigma_2)X(t) + \frac{3}{4}(a_2 - 4a_3)v^2(t)\right\}, \tag{180}$$

$$u'(t) = u^2(t) - 3H(t)u(t) + \frac{1}{3a_2}\{\kappa[\rho(t) - 3p(t)] + 4\lambda_0 - a_0R(t) - (b_0 + \sigma_2)X(t) - 3a_3v^2(t)\}, \tag{181}$$

$$v'(t) = -\frac{1}{3}X(t) - v(t)[3H(t) - 2u(t)], \tag{182}$$

along with Eqs. (176) and (177) for $R'(t)$, $X'(t)$. In addition to the dynamical geometric variables, these equations also include the material energy density and the pressure, $\rho(t)$, $p(t)$. The material energy density $\rho(t)$ is related to the dynamical variables by (167)—a relation which could be used to eliminate it from the system. The energy density and pressure are necessarily related by

$$\rho'(t) = -3H(t)[\rho(t) + p(t)]. \tag{183}$$

This relation follows, on the one hand from the basic Noether symmetry conservation law applied to the source, and, on the other hand can be derived directly from (167) using the system of 6 first order Eqs. (176), (177), and (179)–(182).

The above system needs to be supplemented by an appropriate equation specifying a relation for $p(t)$ in terms of suitable dynamical variables. In GR such a relation is often taken in the form of a fluid equation of state $p = p(\rho)$. One could also use such an assumption for our PG model or, more generally, one could consider any specific relation of the form

$$p = p(a, H, u, v, R, X) \tag{184}$$

to reduce the dynamical equations of the model to a closed system of 6 nonlinear coupled first order ordinary differential equations (1st order ODEs) describing the dynamics of 3 geometric degrees of freedom. Alternatively, if one had a relation of the form

$$p'(t) = f[p(t), \rho(t), u(t), v(t), \cdots], \tag{185}$$

this would be sufficient for integrating the system (176), (177), (179)–(183), and (185).

To investigate more general models in the presence of torsion, one cannot just prescribe a simple equation of state. One could consider some explicit source fields and their dynamics; these sources would determine $\rho(t)$, $p(t)$. Also one could relax the assumption of vanishing source spin density. In follow up work some systems will be presented which might be more stable numerically.

The prescription of an (algebraic) equation of state reduces the phase-space to one of 6 dimensions. Systems of ODEs with algebraic constraints are usually called *differential-algebraic equations* (DAEs) or also descriptor systems.[7] For the numerical evaluation we would have a DAE of the form $6 \oplus 1$, that is, 6 ODEs of first order and one algebraic equation which makes the whole system of equations determinate.

Numerical simulations of those systems, including the case $u(t) \sim v(t)$, will be discussed in detail in a continuation of this paper. For the subcase of the Shie-Nester-Yo Lagrangian (53), such computations have been already done by Li, Sun, and Xi [72,73,96].

F. Acceptable choices of signs

Concerning the acceptable choices of signs for the parameters: at the Lagrangian level of analysis we know of only one necessary requirement, namely, in order to satisfy the *principle of least-action* it is absolutely necessary to take kinetic energy terms—here meaning specifically the quadratic-in-time-derivative terms—to have a positive coefficient.

For our model this is sufficient to fix the signs for the quadratic-in-curvature terms, since it turns out that the scalar curvature and pseudoscalar curvatures are each linear in the time derivatives of certain connection coefficients. Hence, physically in Eq. (105) one must take only the case $\lambda_1 < 0$, $\lambda_2 < 0$.

Regarding the quadratic torsion terms, the situation is not so simple. For our cosmological model at least, from (139), (143), and (145), it can be seen that only $\mathcal{V}_0$ contains a time derivative of a gauge potential (specifically, a'). Thus, by the least-action requirement on this quadratic-in-time-derivative kinetic term one must take, from (64), the coefficient $a_2 < 0$. Consequently, since by convention $a_0 > 0$, one should require $m^+ > 0$.

Beyond these considerations, one can ascertain which constraints are physically appropriate for the parameters only from a detailed analysis of the equations of motion. This is left to future work.

VI. DISCUSSION AND CONCLUSION

In this work we introduce systematically the notion of *even and odd parity terms* for the construction of gravitational field Lagrangians in the context of the Poincaré gauge field theory (PG). Exploiting the theory of *algebraic* invariants, a Lagrangian results that is at most quadratic in the field strengths torsion and curvature. Here, we rigorously include interaction terms of even and odd parity form, like those of (vector torsion $\times$ axial vector torsion), that is, $\mathcal{V} \wedge {}^\star\mathcal{A}$, and (scalar curvature $\times$ pseudoscalar curvature), that is, $R \wedge {}^\star X$. To obtain some insight into the dynamics of those "shadows" in a physically realistic model, we constrained ourselves to a model containing only scalar and vector parts and their corresponding axial versions. The $V_{\rm BHN}$ Lagrangian (64) can be viewed as a generalization of the recently presented $V_{\rm SNY'}$ Lagrangian (54) of Chen *et al.* [12]. In light of the difficulties caused by nonlinearities [36–39], our model (aside from adding a couple of unimportant terms quadratic in the nondynamic ${}^{(1)}T^\alpha$ torsion components) may well be the most general PG model that can be expected to have a dynamical connection with well-behaved dynamics.

From a theoretical point of view, the inclusion of additional interaction terms of *odd* parity character could explain some empirical facts we are faced with in cosmology. Besides the usual handling of even parity functionals of the field variables, we treat those of odd parity character on the same footing. This may open the discussion to explain the empirical imbalance of matter and *antimatter* on a cosmological scale and other related questions that are still open.

Empirically, the inclusion of additional parameters beyond those of the model (54) will enhance the capacity to

[7] See the link http://www4.ncsu.edu/eos/users/s/slc/www/RESDESCRIPT/resdescript.html: "Usually the term DAE refers to systems of ordinary differential equations $F(x', x, t) = 0$ with the Jacobian of F with respect to x' being singular."

account for the accelerated universe observations in terms of dynamical geometry—dark energy could be a PG dynamical connection. It is noteworthy that with the new pseudoscalar cross coupling parameters the acceleration of the universe can be more directly influenced by the 0^- mode, see (180), which is known to also couple to fermion spin.

In this work, we present for the first time (as far as we know) the notion of the *diagonalization of a Lagrangian*. This identifies certain special parameter combinations of the primary coupling constants that are expected to play important roles in future studies of the dynamics of our model, and leads to the recognition of certain conditions on the set of primary coupling constants $\{a_2, a_3, w_3, w_6, \mu_3\}$ such that the (diagonalized) kinetic energy matrix $\mathcal{T}$ has strictly positive entries. A working hypothesis is that these conditions are needed to have a well-defined propagation of massive modes.

According to the diagonalization, the irreducible pieces ${}^{(2)}T^\alpha$ and ${}^{(3)}T^\alpha$ can be associated with the two four-vectors ν^μ and α^μ of even and odd parity character. For these vectors to have proper evolution in time, it is expected that certain signature properties (conjectured to be positive) for the corresponding eigenvalues of the kinetic energy matrix are necessary. The irreducible curvature pieces ${}^{(3)}R^{\alpha\beta}$ and ${}^{(6)}R^{\alpha\beta}$ are essentially a scalar R and a pseudoscalar X, respectively. The proper choice of parameters is such that the associated kinetic matrix has positive eigenvalues.

In our model, we noted that in the general PG *weak gravity* sector, mediated by the coframe ϑ^α, the associated field strength, the torsion, could carry modes of spin 2, of spin 1, and of spin 0 (each with even and odd parity). Restricting to the even parity terms, this is similar to Bekenstein's TeVeS (tensor-vector-scalar) theory [97]. However, in our case the different modes are carried by the torsion alone. There is no need for any other scalar or vector fields.

For *strong gravity*, mediated by the Lorentz connection $\Gamma^{\alpha\beta}$, in our model we found spin 0 of both parities. This restriction to zero spin modes is due to our simple Lagrangian (64) in which only the scalar and pseudoscalar pieces of the curvature were allowed. Straightforward generalizations are possible. In metric-affine gravity (MAG), even the inclusion of spin 3 modes, see [16], is possible—and all of this on the basis of a Riemann-Cartan or metric-affine geometry of spacetime, respectively.

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