

**Cosmological perturbations in  $f(T)$  gravity**Shih-Hung Chen,<sup>1,\*</sup> James B. Dent,<sup>1,†</sup> Sourish Dutta,<sup>2,‡</sup> and Emmanuel N. Saridakis<sup>3,§</sup><sup>1</sup>*Department of Physics and School of Earth and Space Exploration, Arizona State University, Tempe, Arizona 85287-1404, USA*<sup>2</sup>*Department of Physics and Astronomy, Vanderbilt University, Nashville, Tennessee 37235, USA*<sup>3</sup>*College of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Chongqing 400065, People's Republic of China*

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We investigate the cosmological perturbations in  $f(T)$  gravity. Examining the pure gravitational perturbations in the scalar sector using a diagonal vierbein, we extract the corresponding dispersion relation, which provides a constraint on the  $f(T)$  Ansätze that lead to a theory free of instabilities. Additionally, upon inclusion of the matter perturbations, we derive the fully perturbed equations of motion, and we study the growth of matter overdensities. We show that  $f(T)$  gravity with  $f(T)$  constant coincides with General Relativity, both at the background as well as at the first-order perturbation level. Applying our formalism to the power-law model we find that on large subhorizon scales ( $\mathcal{O}(100)$  Mpc or larger), the evolution of matter overdensity will differ from  $\Lambda$ CDM cosmology. Finally, examining the linear perturbations of the vector and tensor sectors, we find that (for the standard choice of vierbein)  $f(T)$  gravity is free of massive gravitons.

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**I. INTRODUCTION**

Cosmological observations of the last decade [1] indicate that the observable universe experiences an accelerated expansion. Although the simplest way to explain this behavior is the consideration of a cosmological constant [2], the difficulties associated with the fine-tuning problem have led some authors to postulate more radical alternatives. A first direction is to consider the dark energy paradigm, which, at least at an effective level, can originate from various fields, such as a canonical scalar field (quintessence) [3], a phantom field, that is a scalar field with a negative sign of the kinetic term [4], or the combination of quintessence and phantom in a unified model named quintom [5]. A second direction is to modify gravity itself, using a function  $f(R)$  of the curvature scalar [6], higher derivatives in the action [7], braneworld extensions [8], string-inspired constructions [9], holographic properties [10], UV modifications [11], etc.

Recently, a new alternative approach appeared in the literature [12–14]. It is based on the old idea of the “teleparallel” equivalent of General Relativity (TEGR) [15,16], which, instead of using the curvature defined via the Levi-Civita connection, uses the Weitzenböck connection that has no curvature but only torsion. The dynamical objects in such a framework are the four linearly independent vierbeins (these are *parallel* vector fields which is what is implied by the appellations “teleparallel” or “absolute parallelism”). The advantage of this framework is that the torsion tensor is formed solely from products of first de-

rivatives of the tetrad. As described in [16], the Lagrangian density,  $T$ , can then be constructed from this torsion tensor under the assumptions of invariance under general coordinate transformations, global Lorentz transformations, and the parity operation, along with requiring the Lagrangian density to be second order in the torsion tensor. However, instead of using the torsion scalar  $T$ , the authors of [13,14] generalized the above formalism to a modified  $f(T)$  version, thus making the Lagrangian density a function of  $T$ , similar to the well-known extension of  $f(R)$  Einstein-Hilbert action.

In comparison with  $f(R)$  gravity, whose fourth-order equations may lead to pathologies,  $f(T)$  gravity has the significant advantage of possessing second-order field equations. This feature has led to a rapidly increasing interest in the literature, and apart from obtaining acceleration [13,14], one can reconstruct a variety of cosmological evolutions [17,18], add a scalar field [19], use observational data in order to constrain the model parameters [20] and examine the dynamical behavior of the scenario [21].

All the previous investigations on  $f(T)$  gravity have focused on the background evolution. However, in order to reveal the full scope and physical implications of the theory, one must delve into the perturbative framework. Thus, one must first investigate the perturbations of the pure gravitational sector, and extract the corresponding dispersion relations, since such an analysis is necessary in order to determine the stability (or lack thereof) of the theory. Additionally, one should examine the complete set of gravitational plus matter perturbations, since they are related to the structure growth, and therefore their evolution could constrain  $f(T)$  gravity via comparison with the wealth of precision data from observations. Both of the

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above investigations are the main goals of this work. In particular, the present investigation is interested in the first-order perturbations of a Friedmann-Robertson-Walker universe under  $f(T)$  gravity.

The plan of the work is as follows. In Sec. II, we present the cosmological scenario of  $f(T)$  gravity at the background level. In Sec. III, we examine the first-order perturbed equations for the scalar sector, we extract the corresponding dispersion relations, we derive the evolution equation for matter overdensities, and we evolve perturbations for a specific model as an example. In Sec. IV, we extend our analysis in the vector and tensor perturbations at linear order. Finally, Sec. V is devoted to the summary of our results. The calculations in the preceding sections are performed in the Newtonian gauge; however, for completeness we include the Appendix with the perturbed equations for the scalar modes in the synchronous gauge.

## II. THE COSMOLOGICAL BACKGROUND IN $f(T)$ GRAVITY

In this section, we study the cosmology of a universe governed by  $f(T)$  gravity. In this manuscript, our notation is as follows: Greek indices  $\mu, \nu, \dots$  run over all coordinate space-time 0, 1, 2, 3, lower case Latin indices (from the middle of the alphabet)  $i, j, \dots$  run over spatial coordinates 1, 2, 3, capital Latin indices  $A, B, \dots$  run over the tangent space-time 0, 1, 2, 3, and lower case Latin indices (from the beginning of the alphabet)  $a, b, \dots$  will run over the tangent space spatial coordinates 1, 2, 3.

As we stated in the Introduction, the dynamical variable of the old teleparallel gravity, as well as its  $f(T)$  extension, is the vierbein field  $\mathbf{e}_A(x^\mu)$ . This forms an orthonormal basis for the tangent space at each point  $x^\mu$  of the manifold, that is  $\mathbf{e}_A \cdot \mathbf{e}_B = \eta_{AB}$ , where  $\eta_{AB} = \text{diag}(1, -1, -1, -1)$ . Furthermore, the vector  $\mathbf{e}_A$  can be analyzed with the use of its components  $e_A^\mu$  in a coordinate basis, that is  $\mathbf{e}_A = e_A^\mu \partial_\mu$ .

In such a construction, the metric tensor is obtained from the dual vierbein as

$$g_{\mu\nu}(x) = \eta_{AB} e_A^\mu(x) e_B^\nu(x). \quad (1)$$

Contrary to General Relativity, which uses the torsionless Levi-Civita connection, in the present formalism one uses the curvatureless Weitzenböck connection [22],  $\overset{w}{\Gamma}_{\nu\mu}^\lambda \equiv e_A^\lambda \partial_\nu e_\mu^A$ , whose torsion tensor reads

$$T_{\mu\nu}^\lambda = \overset{w}{\Gamma}_{\nu\mu}^\lambda - \overset{w}{\Gamma}_{\nu\mu}^\lambda = e_A^\lambda (\partial_\mu e_\nu^A - \partial_\nu e_\mu^A). \quad (2)$$

Moreover, the contorsion tensor, which equals the difference between the Weitzenböck and Levi-Civita connections, is defined as

$$K^{\mu\nu}{}_\rho = -\frac{1}{2}(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}). \quad (3)$$

Finally, it proves useful to define

$$S_\rho{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}{}_\rho + \delta_\rho^\mu T^{\alpha\nu}{}_\alpha - \delta_\rho^\nu T^{\alpha\mu}{}_\alpha). \quad (4)$$

Note the antisymmetric relations  $T^\lambda{}_{\mu\nu} = -T^\lambda{}_{\nu\mu}$  and  $S_\rho{}^{\mu\nu} = -S_\rho{}^{\nu\mu}$ , as can be easily verified. Using these quantities, one can define the so-called ‘‘teleparallel Lagrangian’’ as [16,23,24]

$$L_T \equiv S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}. \quad (5)$$

In summary, in the present formalism, all the information concerning the gravitational field is included in the torsion tensor  $T^\lambda{}_{\mu\nu}$ , and the teleparallel Lagrangian  $L_T$  gives rise to the dynamical equations for the vierbein, which imply the Einstein equations for the metric.

From the above discussion one can deduce that the teleparallel Lagrangian arises from the torsion tensor, similar to the way the curvature scalar arises from the curvature (Riemann) tensor. Thus, one can simplify the notation by replacing the symbol  $L_T$  by the symbol  $T$ , which is the torsion scalar [14].

While in teleparallel gravity the action is constructed by the teleparallel Lagrangian  $L_T = T$ , the idea of  $f(T)$  gravity is to generalize  $T$  to a function  $T + f(T)$ , which is similar in spirit to the generalization of the Ricci scalar  $R$  in the Einstein-Hilbert action to a function  $f(R)$ . In particular, the action in a universe governed by  $f(T)$  gravity reads

$$I = \frac{1}{16\pi G} \int d^4x e [T + f(T) + L_m], \quad (6)$$

where  $e = \det(e_A^\mu) = \sqrt{-g}$  and  $L_m$  stands for the matter Lagrangian. Variation of the action with respect to the vierbein gives the equations of motion

$$\begin{aligned} e^{-1} \partial_\mu (e S_A{}^{\mu\nu}) [1 + f'(T)] - e_A^\lambda T^\rho{}_{\mu\lambda} S_\rho{}^{\nu\mu} [1 + f'(T)] \\ + S_A{}^{\mu\nu} \partial_\mu (T) f''(T) - \frac{1}{4} e_A^\nu [T + f(T)] \\ = 4\pi G e_A^\rho \overset{\text{em}}{T}{}_\rho{}^\nu, \end{aligned} \quad (7)$$

where a prime denotes the derivative with respect to  $T$  and the mixed indices are used as in  $S_A{}^{\mu\nu} = e_A^\rho S_\rho{}^{\mu\nu}$ . Note that the tensor  $\overset{\text{em}}{T}{}_\rho{}^\nu$  on the right-hand side is the usual energy-momentum tensor, in which we added an overset label in order to avoid confusion with the torsion tensor.

If we assume the background to be a perfect fluid, then the energy-momentum tensor takes the form

$$\overset{\text{em}}{T}{}_{\mu\nu} = p g_{\mu\nu} - (\rho + p) u_\mu u_\nu, \quad (8)$$

where  $u^\mu$  is the fluid four velocity. Note that we are following the conventions of [25], but with an opposite signature metric. In the following sections, we will be interested in the perturbed energy-momentum tensor, and the signs of the perturbed terms are not affected due to the indices mixing (one upper and one lower).

Let us now focus on cosmological scenarios in a universe governed by  $f(T)$  gravity. Thus, throughout the work we consider the common choice for the form of the vierbein, namely

$$e_{\mu}^A = \text{diag}(1, a, a, a), \quad (9)$$

which leads to a flat Friedmann-Robertson-Walker (FRW) background geometry with metric

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j, \quad (10)$$

where  $a(t)$  is the scale factor. It has recently been shown [26] that  $f(T)$  gravity does not preserve local Lorentz invariance. In principle then, one should study the cosmology resulting from a more general vierbein Ansatz. In this paper, we specialize our attention to the diagonal form, leaving the more general case for future investigation.

Using this vierbein, together with the above fluid description of matter, one sees that Eqs. (7) lead to the background (Friedmann) equations

$$H^2 = \frac{8\pi G}{3}\rho_m - \frac{f(T)}{6} - 2f'(T)H^2 \quad (11)$$

$$\dot{H} = -\frac{4\pi G(\rho_m + p_m)}{1 + f'(T) - 12H^2 f''(T)}. \quad (12)$$

In these expressions, we have introduced the Hubble parameter  $H \equiv \dot{a}/a$ , where a dot denotes a derivative with respect to coordinate time  $t$ . Moreover,  $\rho_m$  and  $p_m$  stand, respectively, for the energy density and pressure of the matter content of the universe, with equation-of-state parameter  $w_m = p_m/\rho_m$ . Finally, we have employed the very useful relation

$$T = -6H^2, \quad (13)$$

which straightforwardly arises from evaluation of (5) for the unperturbed vierbein (9). Lastly, note that General Relativity is recovered by setting  $f(T)$  to a constant (which will play the role of a cosmological constant), as expected.

### III. SCALAR PERTURBATIONS IN $f(T)$ GRAVITY

One of the most decisive tests for the reliability of a gravitational theory is the examination of its perturbations. This investigation reveals some of the deep features of the theory, determining its stability and the growth of matter overdensities. In this section, we analyze in detail the linear-order scalar perturbations of  $f(T)$  gravity, leaving the vector and tensor ones for the next section. In particular, we extract the full set of gravitational and energy-momentum tensor perturbations in Sec. III A, and in Sec. III B we examine the stability of the theory. Finally, in Sec. III C we examine the growth of matter overdensities. For simplicity, we perform the calculations in the Newtonian gauge (for completeness, the Appendix displays the scalar perturbations in the synchronous gauge).

#### A. Matter and scalar vierbein perturbations up to linear order

We are interested in examining how the scalar vierbein perturbations affect the equations of motion. We mention that while in General Relativity the fundamental object is the metric, which is then perturbed, in  $f(T)$  gravity, as well as in the old teleparallel equivalent of General Relativity, the fundamental object is the vierbein. Thus, the starting point is the vierbein perturbation, which will then give rise to the perturbed metric.

Using the symbol  $e_{\mu}^A$  for the perturbed vierbein and  $\bar{e}_{\mu}^A$  for the unperturbed one, the scalar perturbation reads

$$e_{\mu}^A = \bar{e}_{\mu}^A + t_{\mu}^A, \quad (14)$$

where

$$\bar{e}_{\mu}^0 = \delta_{\mu}^0 \quad \bar{e}_{\mu}^a = \delta_{\mu}^a \quad a\bar{e}_0^{\mu} = \delta_0^{\mu} \quad \bar{e}_a^{\mu} = \frac{\delta_a^{\mu}}{a} \quad (15)$$

$$t_{\mu}^0 = \delta_{\mu}^0 \psi \quad t_{\mu}^a = -\delta_{\mu}^a a \phi \quad t_0^{\mu} = -\delta_0^{\mu} \psi \quad t_a^{\mu} = \frac{\delta_a^{\mu}}{a} \phi, \quad (16)$$

with indicial notation as stated at the beginning of Sec. II. Here we have made the simplifying assumption that the scalar perturbations  $t_{\mu}^A$  are diagonal, a procedure which is sufficient to extract qualitative results about the stability of the theory. As usual, we have kept terms up to first order in the perturbations. Moreover, unless otherwise indicated, subscripts zero and one will generally denote, respectively, zeroth and linear order values of quantities.

In the above perturbations, we have introduced the scalar modes  $\phi$  and  $\psi$ , which are functions of  $\mathbf{x}$  and  $t$ . These symbols, as well as the various coefficients, have been conveniently chosen in order for the vierbein perturbation to induce a metric perturbation of the known form in Newtonian gauge with signature  $(+ - - -)$ , namely

$$ds^2 = (1 + 2\psi)dt^2 - a^2(1 - 2\phi)\delta_{ij}dx^i dx^j. \quad (17)$$

Thus, the determinant becomes

$$e = \det(e_{\mu}^A) = a^3(1 + \psi - 3\phi). \quad (18)$$

Proceeding forward, we calculate  $T^{\lambda}_{\mu\nu}$  and  $S_{\lambda}^{\mu\nu}$  to first order under the perturbations (15) and (16). The torsion reads

$$T^{\lambda}_{\mu\nu} = (\bar{e}_{\lambda}^{\alpha} + t_{\lambda}^{\alpha})[\partial_{\mu}(\bar{e}_{\nu}^{\alpha} + t_{\nu}^{\alpha}) - \partial_{\nu}(\bar{e}_{\mu}^{\alpha} + t_{\mu}^{\alpha})], \quad (19)$$

and thus the second is easily calculable using (4). After some algebra, we find (indices are not summed over)

$$\begin{aligned}
T^0_{\mu\nu} &= \partial_\mu \psi \delta_\nu^0 - \partial_\nu \psi \delta_\mu^0; \\
T^i_{0i} &= H - \dot{\phi} S_0^{0i} = \frac{\partial_i \dot{\phi}}{a^2}; \\
S_i^{0i} &= -H + \dot{\phi} + 2H\psi \quad T^i_{ij} = \partial_j \phi; \\
S_i^{ij} &= \frac{1}{2a^2} \partial_j (\phi - \psi).
\end{aligned} \tag{20}$$

Additionally, up to first order the torsion scalar defined in (5) is found to be

$$T \equiv S_\rho^{\mu\nu} T^\rho_{\mu\nu} = T_0 + T_1, \tag{21}$$

where

$$T_0 = -6H^2 \tag{22}$$

is the previously seen zeroth order result and

$$T_1 = 12H(H\psi + \dot{\phi}) \tag{23}$$

is the first order one. Thus, we can easily express  $f(T)$  and its derivatives up to first order as:

$$\begin{aligned}
f(T) &= f(T_0) + T_1 \frac{df(T)}{dT} \Big|_{T=T_0} \equiv f_0 + f_1 \\
f'(T) &= \frac{df(T)}{dT} \Big|_{T=T_0} + T_1 \frac{d^2f(T)}{dT^2} \Big|_{T=T_0} \equiv f'_0 + f'_1 \\
f''(T) &= \frac{d^2f(T)}{dT^2} \Big|_{T=T_0} + T_1 \frac{d^3f(T)}{dT^3} \Big|_{T=T_0} \equiv f''_0 + f''_1,
\end{aligned} \tag{24}$$

that is the right-hand sides of these equations are functions of  $T_0$  and linear functions of  $T_1$ .

Let us now consider the perturbations of the energy-momentum tensor. The perturbations are then expressed as

$$\delta T_0^{\text{em } 0} = -\delta\rho_m \tag{25}$$

$$\delta T_0^{\text{em } i} = a^{-2}(\rho_m + p_m)(-\partial_i \delta u) \tag{26}$$

$$\delta T_i^{\text{em } 0} = (\rho_m + p_m)(\partial_i \delta u) \tag{27}$$

$$\delta T_i^{\text{em } j} = \delta_{ij} \delta p_m + \partial_i \partial_j \pi^S. \tag{28}$$

where  $\pi^S$  is the scalar component of the anisotropic stress. Inserting everything in (7), we finally obtain

$$\begin{aligned}
E_0^0 &\equiv (1 + f'_0)(\nabla^2 \phi) + 6(1 + f'_0)H\dot{\phi} + 6(1 + f'_0)H^2\psi \\
&\quad - 3f'_1 H^2 - \frac{T_1 + f_1}{4} = -4\pi G \delta\rho_m,
\end{aligned} \tag{29}$$

$$\begin{aligned}
E_0^i &\equiv (1 + f'_0)\partial_i \dot{\phi} + (1 + f'_0)H\partial_i \psi - 12H\dot{H}f''_0 \partial_i \phi \\
&\quad = -4\pi G(\rho_m + p_m)\partial_i \delta u,
\end{aligned} \tag{30}$$

$$\begin{aligned}
E_a^0 &\equiv 12H^2 \partial_i \delta_a^i (\dot{\phi} + H\psi) f''_0 - (1 + f'_0) \partial_i \delta_a^i (\dot{\phi} + H\psi) \\
&\quad = 4\pi G(\rho_m + p_m) \partial_i \delta_a^i \delta u,
\end{aligned} \tag{31}$$

$$\begin{aligned}
E_a^i \delta_i^a &\equiv \frac{f'_1}{a} (-3H^2 - \dot{H}) + \frac{f''_1}{a} (12H^2 \dot{H}) \\
&\quad - \frac{(1 + f'_0)}{2a} \sum_{b \neq a} \partial^j \delta_j^b \partial_i \delta_b^i (\psi - \phi) - \frac{\phi(T_0 + f_0)}{4a} \\
&\quad - \frac{T_1 + f_1}{4a} + \frac{(1 + f'_0)}{a} [6H\dot{\phi} + 6H^2\psi - 3H^2\phi \\
&\quad + \ddot{\phi} + \dot{H}(2\psi - \phi) + H\dot{\psi}] \\
&\quad + \frac{f''_0}{a} (-24H\dot{H}\dot{\phi} - 48\psi H^2 \dot{H} - 12H^2 \ddot{\phi} \\
&\quad - 12H^3 \dot{\psi} + 12H^2 \dot{H}\phi) \\
&\quad = \frac{4\pi G}{a} (p_m \phi + \delta p_m),
\end{aligned} \tag{32}$$

$$\begin{aligned}
E_{b;b \neq a}^i \delta_i^a &\equiv \frac{(1 + f'_0)}{2} \partial_j \delta_b^j \partial^i \delta_i^a (\phi - \psi) \\
&\quad = 4\pi G a^2 \partial_j \delta_b^j \partial^i \delta_i^a \pi^S,
\end{aligned} \tag{33}$$

where we have used the definition  $\nabla^2 = \sum_i \partial_i \partial^i$  and indices are summed over only when explicitly shown with the  $\sum$  symbol.

The above equations are perfectly general. In what follows, we set the scalar anisotropic stress  $\pi^S$  to zero for simplicity, a choice which precludes the possibility of anisotropic expansion of the Universe. However, the cosmological consequences of dark energy models with anisotropic stress have been previously considered in the literature [27], and a detailed study of the effects of anisotropic stress in  $f(T)$  cosmology would be an interesting avenue for future research.

The zero-anisotropic-stress assumption, which according to (33) implies  $\phi = \psi$ , along with  $\delta p_m = 0$ , may lead the system of perturbation equations becoming overdetermined. Once the vanishing of the anisotropic stress is implemented, we then have four equations determining the three remaining perturbation variables  $\delta\rho_m$ ,  $\phi$  and  $\delta u$ . However, in the limit  $f''(T) \simeq 0$ , Eqs. (30) and (31) become identical, removing the overdetermination. We therefore conclude that the requirement of no anisotropic stress imposes another constraint on  $f(T)$  models, namely, that  $f''(T) \simeq 0$ . However, note that this requirement on  $f(T)$  might be relaxed for more general choices of vierbein than (9).

## B. Stability

In the previous subsection, we derived the linear equations of motion for the perturbations. In this subsection, we will use them to examine the stability of  $f(T)$  gravity and extract the dispersion relation for the pure gravitational perturbations. Since we are concerned with the stability of



the theory, in this subsection we ignore the matter sector in the set of Eqs. (29)–(33). This is because if the gravitational sector itself is unstable, the matter content cannot cure the instability.

Working in Fourier space, we introduce the mode-expansion of  $\phi$  as

$$\phi(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \tilde{\phi}_k(t) e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (34)$$

Therefore, inserting this into (23), we obtain the modes of the Fourier transformed  $T_1$  as

$$\tilde{T}_{1k} = 12H^2 \tilde{\phi}_k + 12H \dot{\tilde{\phi}}_k, \quad (35)$$

while (24) gives the modes of the Fourier transformed  $f_1$  as

$$\tilde{f}_{1k} = \tilde{T}_{1k} \frac{df(T)}{dT} \Big|_{T=T_0} = \tilde{T}_{1k} f'_0, \quad (36)$$

the modes of the Fourier transformed  $f'_1$  as

$$\tilde{f}'_{1k} = \tilde{T}_{1k} \frac{d^2f(T)}{dT^2} \Big|_{T=T_0} = \tilde{T}_{1k} f''_0, \quad (37)$$

and the modes of the Fourier transformed  $f''_1$  as

$$\tilde{f}''_{1k} = \tilde{T}_{1k} \frac{d^3f(T)}{dT^3} \Big|_{T=T_0} = \tilde{T}_{1k} f'''_0, \quad (38)$$

where obviously  $f_0$ ,  $f'_0$  and  $f''_0$  do not depend on  $\phi$ .

Inserting Eqs. (34)–(38) into (32), we obtain

$$\ddot{\tilde{\phi}}_k + \Gamma \dot{\tilde{\phi}}_k + \omega^2 \tilde{\phi}_k = 0, \quad (39)$$

where  $\omega^2$  and  $\Gamma$  are, respectively, given by

$$\omega^2 = \frac{\frac{3H^2}{2} + \dot{H} - \frac{\dot{f}_0}{4} + \dot{H}f'_0 - (36H^4 - 48H^2\dot{H})f''_0 + 144H^4\dot{H}f'''_0}{1 + f'_0 - 12H^2f''_0} \quad (40)$$

$$\Gamma = \frac{4H[1 + f'_0 - (12H^2 + 9\dot{H})f''_0 + 36H^2\dot{H}f'''_0]}{1 + f'_0 - 12H^2f''_0}. \quad (41)$$

Equation (39) allows one to study the stability of any given model. In particular, a model for which  $\omega^2$  is negative will clearly be unstable. Note that as mentioned before, it is sufficient to consider a scenario without any matter content. In such a pure gravitational case, the background Eq. (12) (that is, with  $\rho_m = 0$ ,  $p_m = 0$ ) leads to a constant  $H$ . Therefore, Eq. (40) reduces to the simple form

$$\omega^2 = \frac{\frac{3H^2}{2} - \frac{\dot{f}_0}{4} - 36H^4f''_0}{1 + f'_0 - 12H^2f''_0}. \quad (42)$$

One can use relation (42) in order to determine if a specific  $f(T)$  model is free of instabilities. For instance, we can insert the power-law Ansatz  $f(T) = \alpha(-T)^n$ , with  $\alpha = (6H_0^2)^{1-n}/(2n-1)$  in the absence of matter ( $H_0$  is the present Hubble parameter), and the exponential Ansatz  $f(T) = -\alpha T(1 - e^{pT_0/T})$ , with  $\alpha = 1/[1 - (1-2p)e^p]$  in the absence of matter, considered in [14], and then we can straightforwardly determine the allowed values for the Ansatz-parameters numerically. It is easy to check that in both these cases the stability condition is satisfied for the phenomenologically relevant ranges of parameters, that is  $0 < n < 1$  for the power-law model and  $0 < p < 1$  for the exponential model.

### C. Growth of perturbations

Having examined the stability of  $f(T)$  gravity, in this subsection we switch on the matter sector, and examine the fluctuations about the FRW background in the presence

of matter. This is a crucial subject in every cosmological scenario, and thus it is the starting point for a thorough investigation of the cosmology in the  $f(T)$  framework.

In order to study the growth of perturbations, we assume for simplicity a matter-only universe, that is we impose  $p_m = 0 = \delta p_m$ . As usual, we define the matter overdensity  $\delta$  as

$$\delta \equiv \frac{\delta\rho_m}{\rho_m}. \quad (43)$$

We now write Eq. (29) as

$$3H(1 + f'_0 - 12H^2f''_0)\dot{\tilde{\phi}}_k + [(3H^2 + k^2/a^2)(1 + f'_0) - 36H^4f''_0]\tilde{\phi}_k + 4\pi G\rho\tilde{\delta}_k = 0. \quad (44)$$

Note that this is the relativistic version of the Poisson equation in  $f(T)$  gravity. In summary, Eqs. (39) and (44) can be used in order to evolve  $\delta$  for a given  $f(T)$  model, which will allow for a comparison to observational data.

Before proceeding to the investigation of Eq. (44), let us make a comment concerning the relation to General Relativity. As we observe, a very interesting feature that emerges from the above analysis, both of the pure gravitational sector of the previous subsection as well as of the matter inclusive sector of the present subsection, is that the linear perturbations in  $f(T)$  gravity reduce to those of General Relativity in the limit where  $f(T)$  is constant. For example, if  $f(T) = \text{const} = -2\Lambda$ , where  $\Lambda$  is the cosmological constant, then one can immediately see that Eqs. (29)–(33) reduce to the well-known first-order equations of General Relativity [25,28], and furthermore Eqs. (39) and (44) reduce to the well-known equations

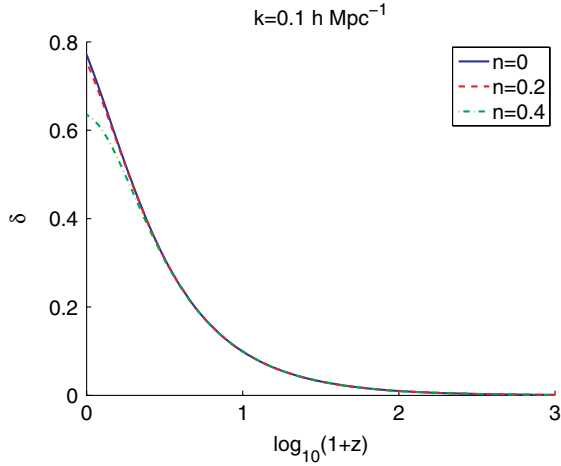


FIG. 1 (color online). The evolution of the matter overdensity  $\delta$  as a function of the redshift  $z$ , on a scale of  $k = 0.1h \text{ Mpc}^{-1}$ , for three choices of  $n$ , for the power-law model given by (45).

for the growth of perturbations in  $\Lambda$ CDM cosmology (see, e.g., [29]).

Obtaining General Relativity as a limiting behavior at the level of both the background and the perturbations, is not a standard feature of a gravitational theory. In particular, any covariant modified gravitational theory contains extra degrees of freedom at the perturbational level, unless this theory can be conformally transformed back to the standard General Relativity. These degrees of freedom may have no impact at the background level (under certain symmetries), but the perturbative modes could in principle leave their imprints at first order, even if one imposes the General Relativity limit. This is the case, for example, for Hořava-Lifshitz gravity [11] where, although the theory in the Infra-Red coincides with General Relativity, at the perturbative level one obtains the known strong coupling problem, even in the IR [30].

However, as we see from the analysis of the present work,  $f(T)$  gravity seems to coincide with General Relativity when  $f(T)$  is constant, both at the background level and at the first-order perturbation level, leaving possible differences to arise at second order or beyond. Such a behavior is seen also in  $f(R)$  gravity, both in its metric and Palatini formulation [31,32], and it is a significant feature of  $f(T)$  gravity and a robust self-consistency test.

We now proceed to the investigation of the physical implications of a nontrivial  $f(T)$ -Ansatz, studying the growth of the overdensity for a specific model. We choose the power-law model suggested in [14]:

$$f(T) = \alpha(-T)^n, \quad (45)$$

where  $\alpha = (6H_0^2)^{(1-n)}(1 - \Omega_{m0})/(2n - 1)$  and  $H_0$  and  $\Omega_{m0}$  refer to the Hubble parameter and the matter density parameter at present.

Our results for the growth of perturbations, arising from a numerical elaboration, are presented in Figs. 1–3. In these

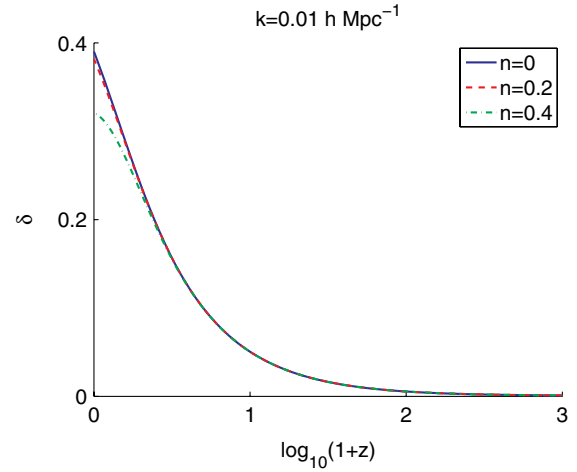


FIG. 2 (color online). The evolution of the matter overdensity  $\delta$  as a function of the redshift  $z$ , on a scale of  $k = 0.01h \text{ Mpc}^{-1}$ , for three choices of  $n$ , for the power-law model given by (45).

figures, we follow the growth of the matter overdensity  $\delta$ , from the time of last scattering to the present one, for different choices of  $n$ , for three different  $k$ -scales. In Fig. 4, we depict the evolution of  $\delta$  for fixed  $n = 0.2$ , but for three different scales. As usual, we use the redshift  $z$  as the independent variable, defined as  $1 + z = a_0/a$  with  $a_0$  the present scale-factor value.

As expected, the  $n = 0$  case is identical to  $\Lambda$ CDM scenario [29]. However, as  $n$  increases we find that there is a suppression of growth at smaller redshifts, which can act as a clear distinguishing feature of these models. We also notice that larger scales are more strongly affected than smaller ones. Therefore, observations which target subhorizon scales close to the horizon could, in principle, constrain these models.

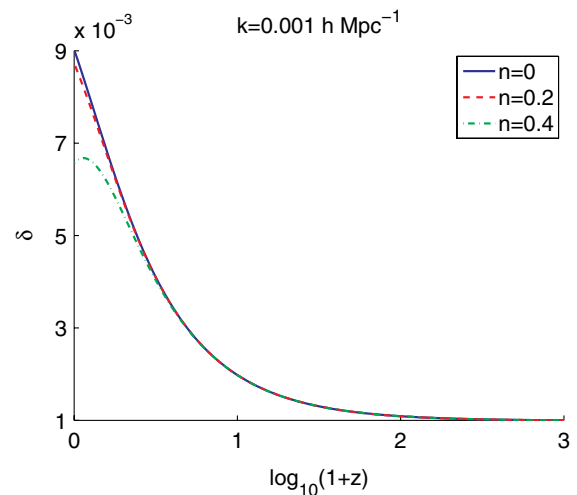


FIG. 3 (color online). The evolution of the matter overdensity  $\delta$  as a function of the redshift  $z$ , on a scale of  $k = 0.001h \text{ Mpc}^{-1}$ , for three choices of  $n$ , for the power-law model given by (45).

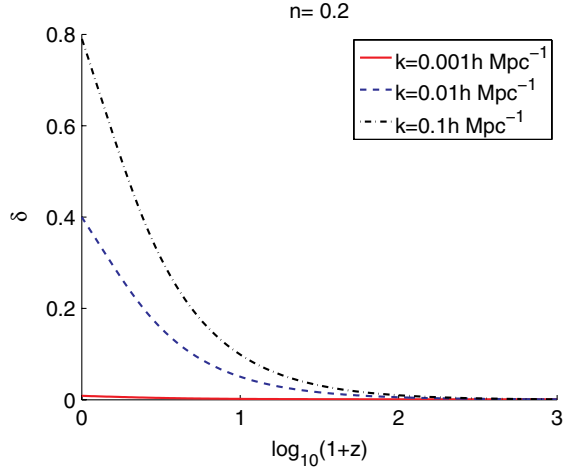


FIG. 4 (color online). The evolution of the matter overdensity  $\delta$  as a function of the redshift  $z$ , for a fixed  $n$  for the power-law model given by (45), for three different scales.

Finally, note that, as we mentioned in the end of Sec. III A, in order for our scenario to be free of over-determination (since we have imposed the zero-anisotropic-stress assumption which reduces the degrees of freedom) we must have  $f''(T) \simeq 0$ . This condition in the case of the power-law Ansatz requires  $n \ll 1$ , which is what we have used for the numerical analysis. Interestingly enough, it is exactly the same condition that is needed in order to acquire an observationally compatible dark-energy and Newton constant phenomenology at the background level [14] (for the power-law Ansatz  $n \simeq 1$  also gives  $f''(T) \simeq 0$  but we do not consider this case due to the aforementioned phenomenological reason).

#### IV. VECTOR AND TENSOR PERTURBATIONS IN $f(T)$ GRAVITY

In the previous section we focused on the scalar perturbations of  $f(T)$  gravity, since they are sufficient to reveal the basic features of the theory, allowing for a discussion of the growth of matter overdensities. For completeness, in this section we extend our analysis in order to include the vector and tensor sectors of the theory in the absence of matter.

The general perturbed vierbein at linear order reads as

$$\begin{aligned}
 e_{\mu}^0 &= \delta_{\mu}^0(1 + \psi) + a(G_i + \partial_i F)\delta_{\mu}^i \\
 e_{\mu}^a &= a\delta_{\mu}^a(1 - \phi) + a(h_i^a + \partial_i \partial^a B + \partial_i C^a + \partial^a C_i)\delta_{\mu}^i \\
 e_0^{\mu} &= \delta_0^{\mu}(1 - \psi) \\
 e_a^{\mu} &= \frac{1}{a}[\delta_a^{\mu}(1 + \phi) + (h_a^{\mu} + \partial^{\mu} \partial_a B + \partial^{\mu} C_a + \partial_a C^{\mu})\delta_0^{\mu}] \\
 &\quad - (G_i + \partial_i F)\delta_a^i \delta_0^{\mu}.
 \end{aligned} \tag{46}$$

In these expressions, apart from the scalar modes  $\phi$  and  $\psi$  of the previous section, we have introduced the transverse vector modes  $G_i$  and  $C_i$ , the transverse traceless tensor mode  $h_i^a$ , and the scalar modes  $F$  and  $B$ , the divergence of which will also contribute to the vector sector. Similarly to the simple scalar case, the coefficients on the above expressions have been chosen in order for this vierbein perturbation to give rise to a perturbed metric of the familiar form:

$$\begin{aligned}
 g_{00} &= 1 + 2\psi \\
 g_{i0} &= a[\partial_i F + G_i] \\
 g_{ij} &= -a^2[(1 - 2\phi)\delta_{ij} + h_{ij} + \partial_i \partial_j B + \partial_j C_i + \partial_i C_j].
 \end{aligned} \tag{47}$$

Let us make a comment here concerning the number of degrees of freedom of the perturbed theory. As may be deduced straightaway,  $T_{\mu\nu}^{\lambda}$ ,  $K_{\mu\nu}^{\rho}$  and  $S^{\rho\mu\nu}$  are space-time tensors under an infinitesimal coordinate transformation of the form

$$x^{\mu} \rightarrow x^{\mu} + \epsilon^{\mu}. \tag{48}$$

This implies that the torsion scalar  $T$  is a generally covariant scalar, and thus actions of the form of (6) are generally covariant as well as invariant under (48). As a result, for our choice of vierbein (9), the number of degrees of freedom (DOF) is identical to General Relativity. In particular, in 3 + 1 space-time dimensions, the metric, being symmetric, has 10 independent DOF. This is reflected in (46), which comprises

- (i) 4 scalar DOF  $\psi$ ,  $\phi$ ,  $F$  and  $B$
- (ii) 4 vector DOF, 2 associated with each of the divergenceless vectors  $G_i$  and  $C_i$
- (iii) 2 tensor DOF associated with the transverse, traceless and symmetric tensor  $h_{ij}$

However, not all of these DOF are independent as there exist 3 + 1 DOF associated with the coordinate transformation (48) (the temporal part of  $\epsilon^{\mu}$  is the scalar  $\epsilon_0$ , and its spatial part can be decomposed into the gradient of a scalar plus a divergenceless vector:  $\partial_i \epsilon^S + \epsilon_i^V$ , leading to a total of 2 scalar and 2 vector DOF). Subtracting these, we are left with a total of 6 DOF: 2 scalar, 2 vector, and 2 tensor, just as in the case of General Relativity.

We can therefore work in the Newtonian gauge, setting  $F$  and  $B$  to zero. This is easily understood since under (48), the gauge transformation of  $B$  is  $-(2\epsilon^S)/a^2$ , while that of  $F$  is  $(1/a)(-\epsilon_0 - \dot{\epsilon}^S + 2H\epsilon^S)$ . Therefore,  $\epsilon^S$  can be chosen in order to give rise to  $B = 0$ , and similarly an accompanying choice of  $\epsilon_0$  will lead to  $F = 0$ .

From the above analysis, it is clear that there are no extra modes in  $f(T)$  theories, which often show up in theories with less symmetry than General Relativity. For example as discussed in [33], in the case of Hořava gravity, which is invariant under spatial diffeomorphisms and

space-independent time reparametrizations but not under space-dependent time reparametrizations, one can no longer choose the longitudinal gauge, and thus an extra scalar DOF remains. However, note that extra DOF can still arise from more general choices of vierbein than (9). The possible existence and phenomenology of these extra modes is beyond the scope of this work and warrants further study.

Additionally, we choose a gauge where the vector mode  $C_i$  vanishes (through an appropriate choice of  $\epsilon_i^V$ ). As usual, the vector modes are transverse, while the tensor mode is transverse and traceless, namely

$$\partial_i C^i = \partial_i G^i = 0; \quad \partial_i h^{ij} = \delta^{ij} h_{ij} = 0. \quad (49)$$

Finally, we easily deduce the following relation between the tensor perturbations in the vierbein and inverse vierbein:

$$h_{\rho=1}{}^{a=1} = -h_{a=1}{}^{\rho=1}. \quad (50)$$

Using the above relations, the perturbed torsion tensor (19) becomes

$$\begin{aligned} T^0{}_{\mu\nu} &= \partial_\mu \psi \delta_\nu^0 - \partial_\nu \psi \delta_\mu^0 + a(\partial_\mu G_\nu - \partial_\nu G_\mu) \\ T^i{}_{0i} &= H - \dot{\phi} + \dot{h}_i{}^c \delta_c^i \\ T^i{}_{ij} &= \partial_j \phi + \partial_i h_j{}^c \delta_c^i - \partial_j h_i{}^c \delta_c^i, \end{aligned} \quad (51)$$

where for notational compactness we have introduced a  $G_0$  part of  $G_i$ , which is zero. This torsion tensor inserted into (4) leads to

$$\begin{aligned} S_0{}^{0i} &= \frac{1}{a^2} \partial_i \phi + \frac{1}{2a^2} \sum_j \partial_j h_i{}^a \delta_a^j + \frac{H}{a} G_i \\ S_0{}^{ij} &= \frac{1}{4a^3} (\partial_i G_j - \partial_j G_i) \\ S_i{}^{0i} &= -H + \dot{\phi} + 2H\psi + \frac{1}{2} \dot{h}_i{}^a \delta_a^i \\ S_i{}^{j0} &= \frac{1}{4a} (\partial_i G_j - \partial_j G_i) \\ S_i{}^{ij} &= \frac{1}{2a^2} \partial_j (\phi - \psi) + \frac{1}{2a^2} (\partial_k h_j{}^a - \partial_j h_k{}^a) \delta_a^k \\ &\quad + \frac{H}{a} G_j + \frac{1}{2a} \dot{G}_j. \end{aligned} \quad (52)$$

However, the torsion scalar is unaffected by the vector and tensor modes, and thus it remains

$$T \equiv T_0 + T_1 = -6H^2 + 12H^2 \psi + 12H \dot{\phi}, \quad (53)$$

and likewise the determinant  $e$  is still given by

$$e = a^3(1 + \psi - 3\phi). \quad (54)$$

We now have all the necessary machinery in order to extract the equations of motion for the vector and tensor sectors. Following the steps of the previous section, we can similarly decompose the energy-momentum tensor into its vector and tensor components and ignore the vector and tensor anisotropic stresses. We finally obtain

$$[1 + f'(T)] \nabla^2 G_j = 0 \quad (55)$$

for the vector mode. Since the quantity in square brackets is zero only for the unphysical model  $f(T) = -T$  (for which the action (6) does not describe the gravitational sector anymore), we can eliminate it, resulting in

$$\nabla^2 G_j = 0. \quad (56)$$

Therefore, the vector modes in  $f(T)$  gravity decay as  $1/a^2$ , that is similar to the General Relativity case.

For the tensor mode, we obtain

$$\begin{aligned} \left\{ [1 + f'(T)] \left( \frac{\ddot{h}_i{}^a}{2a} - \frac{\nabla^2 h_i{}^a}{2a} + \frac{3H \dot{h}_i{}^a}{2a} \right) - \frac{6H \dot{H} f''(T) \dot{h}_i{}^a}{a} \right\} \delta_a^j \\ = 0. \end{aligned} \quad (57)$$

Finally, similar to the scalar case, we can Taylor-expand the derivatives of  $f(T)$  using (24), and Fourier-expand the vector and tensor modes, in order to obtain the corresponding dispersion relations. Moreover, we can split the tensor sector into left-handed and right-handed polarizations. However, such a detailed analysis of the gravitational wave spectrum lies beyond the scope of this work. Here, we retain only the simple forms (56) and (57), since they are adequate in order to reveal the basic features of the behavior.

Concerning the tensor Eq. (57), although there is a new friction term, there are no new mass terms, which is a behavior similar to the scalar case of the previous section. Therefore, we can safely conclude that, in general,  $f(T)$  theories do not introduce massive gravitons; thus when  $f(T)$  tends to a constant we do not obtain the typical problems of massive gravity, which is a significant advantage of  $f(T)$  gravity. Additionally, note that similar to the scalar case, in the limit where  $f(T)$  tends to a constant we do recover the behavior of General Relativity at linear order, which is a self-consistency test of the construction. Lastly, from the equations of motion for scalar, vector and tensor perturbations presented above, it is clear that these three classes of perturbations decouple from one another in  $f(T)$  gravity, just as they do in case of General Relativity.

## V. CONCLUSIONS

In this work we investigated the recently developed  $f(T)$  gravity, going beyond the simple background level.  $f(T)$  gravity is the extension of the teleparallel equivalent of General Relativity, which uses the zero curvature Weitzenböck connection instead of the torsionless



Levi-Civita connection, in the same lines as  $f(R)$  gravity is the extension of standard General Relativity. In particular, we analyzed the first order perturbations of  $f(T)$  gravity.

Examining the scalar perturbations of  $f(T)$  gravity in the Newtonian gauge, we derived the perturbed equations of motion and extracted the corresponding dispersion relation. Therefore, in constructing a realistic  $f(T)$ -gravitational scenario, one should use an  $f(T)$  Ansatz that leads to non-negative  $\omega^2$  in (42), in order to obtain a theory free of instabilities. Moreover, we showed that for the assumption of no scalar anisotropic stress to be consistent, one needs the constraint  $f''(T) \simeq 0$ .

Additionally, we found that  $f(T)$  gravity with  $f(T)$  set to a constant coincides with General Relativity, not only at the level of the background but also for the first-order perturbations. This is a significant advantage of the theory as compared to other modified gravity paradigms.

Furthermore, as an example of an application of our formalism, we followed the growth of perturbations in a specific  $f(T)$  model, namely, the power-law Ansatz proposed in [14]. For this model we found that on large subhorizon scales ( $\mathcal{O}(100)$  Mpc) or larger), the evolution of the matter overdensity differs from  $\Lambda$ CDM. Therefore, future precise observational data on these scales could be used to constrain or rule out such models.

Finally, we investigated the vector and tensor perturbations at linear order, extracting the corresponding equations of motion for the vector and tensor modes. We showed that  $f(T)$  gravity does not introduce massive gravitons, which is a significant advantage. Lastly, we verified again, as in the scalar sector, that in the limit where  $f(T)$  tends to a constant the theory tends to General Relativity, both at the background as well as at the linear perturbation level.

Clearly  $f(T)$  cosmology presents a very rich behavior and deserves further investigation.

## ACKNOWLEDGMENTS

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## APPENDIX: SCALAR GRAVITATIONAL AND MATTER PERTURBATIONS OF $f(T)$ COSMOLOGY IN THE SYNCHRONOUS GAUGE

Let us analyze the scalar gravitational and matter perturbations in the synchronous gauge. The perturbed vierbein read

$$e^i{}_\mu = \bar{e}^i{}_\mu + t^i{}_\mu, \quad (\text{A1})$$

with

$$\begin{aligned} \bar{e}^0{}_\mu &= \delta^0{}_\mu; & \bar{e}^a{}_\mu &= \delta^a{}_\mu a; & \bar{e}^\mu{}_0 &= \delta^\mu{}_0; & \bar{e}^\mu{}_a &= \frac{\delta^\mu{}_a}{a} \\ t^0{}_\mu &= 0; & t^a{}_\mu &= \delta^b{}_\mu \frac{a}{2} (\delta^a{}_b A - a^2 \partial_b \partial^a B) \\ t^\mu{}_0 &= 0; & t^\mu{}_a &= -\delta^b{}_\mu \frac{1}{2a} (\delta^b{}_a A - a^2 \partial^b \partial_a B) \\ e &= a^3 \left( 1 + \frac{3}{2} A + \frac{1}{2} \sum_a \partial_i \partial_i B \right). \end{aligned} \quad (\text{A2})$$

These symbols, as well as the various coefficients, have been conveniently chosen in order for the vierbein perturbation to induce a metric perturbation of the known form in synchronous gauge, namely

$$ds^2 = dt^2 - a^2 [\delta_{ij} (1 + A) + \partial_i \partial_j B] dx^i dx^j. \quad (\text{A3})$$

Inserting these into the perturbed torsion tensor (19), and then to (4) we obtain

$$\begin{aligned} T^i{}_{0\nu} &= \delta^i{}_\nu \left( H + \frac{\dot{A}}{2} \right) - \frac{1}{2} \partial_\nu (a^2 \partial^i \partial_b B) \delta^b{}_\nu \\ T^i{}_{ji} &= \frac{1}{2} \partial_j A \\ S_0{}^{0i} &= \frac{1}{2} \partial^i A \\ S_i{}^{ij} &= \frac{1}{4} \partial^j A \\ S_i{}^{0j} &= -\frac{1}{4} \partial_i (a^2 \partial_j B) \\ S_i{}^{0i} &= -\left( H + \frac{\dot{A}}{2} \right) + \frac{1}{4} \partial_i \left( a^2 \sum_{b \neq a} \partial^j \delta_j^b \partial_i \delta_b^i B \right) \\ &= -\left( H + \frac{\dot{A}}{2} \right) + \frac{1}{4} \partial_i (a^2 \nabla^2 B - a^2 \partial_i \partial^i B). \end{aligned} \quad (\text{A4})$$

Furthermore, up to second-order the torsion scalar defined in (5) is found to be

$$T = -6H^2 - \sum_i \frac{1}{2} \partial^i A \partial_i A - 6H\dot{A} + 2H \partial_i (a^2 \nabla^2 B). \quad (\text{A5})$$

Finally, concerning the first-order matter perturbations, we use the results (25)–(28).

Inserting these results into (7), we extract the perturbed equations of motion as

$$\begin{aligned} E_0^0 &= -\frac{(1 + f'_0)}{2} \nabla^2 A - 3H^2 f'_1(T) - 3H\dot{A}(1 + f'_0) \\ &\quad + H \partial_i (a^2 \nabla^2 B) (1 + f'_0) - \frac{T_1 + f_1(T)}{4} \\ &= 4\pi G \delta T_0^{\text{em} 0}, \end{aligned} \quad (\text{A6})$$

$$E_0^i = \frac{1}{2} \partial^i \dot{A} (1 + f_0') + H \partial^i A (1 + f_0') - 6H\dot{H}f_0''(T) \partial^i A = 4\pi G \delta T_0^{\text{em } i}, \quad (\text{A7})$$

$$E_a^0 = \frac{(1 + f_0')}{2a} \partial_i \delta_a^i \dot{A} + \frac{Hf_0''(T)}{a} \partial_i \delta_a^i \left[ -\frac{1}{2} \sum_b \partial^j \delta_b^j A \partial_i \delta_b^i A - 6H\dot{A} + 2H\partial_i (a^2 \nabla^2 B) \right] = \frac{4\pi G}{a} \delta T_i^{\text{em } 0} \delta_a^i, \quad (\text{A8})$$

$$E_b^i = -\frac{(1 + f_0')}{4a^3} \partial_i [a^2 \partial_t (a^2 \partial_j \delta_b^j \partial^i B)] + \frac{(1 + f_0')}{4a} \partial_j \delta_b^j \partial^i A + \frac{H(1 + f_0')}{4a} \partial_t (a^2 \partial_j \delta_b^j \partial^i B) - \frac{(1 + f_0')}{2a^3} \partial_i (H a^4 \partial^j \partial_j \delta_b^j B) \\ + 3 \frac{f_0''(T)}{a} H\dot{H} \partial_i (a^2 \partial_j \delta_b^j \partial^i B) + 6aH^2 \dot{H} f_0''(T) \partial^i \partial_j \delta_b^j B - \frac{a}{8} \partial_j \delta_b^j \partial^i B (T_0 + f_0) = \frac{4\pi G}{a} \delta T_j^{\text{em } i} \delta_b^j, \quad (\text{A9})$$

$$E_a^i \delta_i^a = -\frac{f_1'}{a} [3H^2 + \dot{H}] + (1 + f_0') \left\{ \frac{1}{a^3} \left[ \partial_t (-H a^2) \left( -\frac{3A}{2} + a^2 \nabla^2 B \right) + \partial_t \left[ a^2 \left[ -HA - \frac{\dot{A}}{2} + \frac{1}{4} \partial_i (a^2 \nabla^2 B - a^2 \partial_i \partial^i B) \right. \right. \right. \right. \\ \left. \left. \left. + \frac{H}{2} (a^2 \nabla^2 B - a^2 \partial_i \partial^i B) \right] \right] \right\} - \frac{1}{4a} (\nabla^2 A - \partial_i \partial^i A) - H\dot{A} + \frac{H}{4} \partial_i [a^2 (\nabla^2 B + \partial_i \partial^i B)] + \frac{f_1''(T) 12H^2 \dot{H}}{a} \\ + \frac{f_0''(T)}{a} \left[ -\frac{H}{2} \partial_t \left( \sum_a \partial_i A \partial^i A \right) - 6H^2 \ddot{A} - 6H^2 \dot{H} A - H\dot{H} \partial_t (a^2 \nabla^2 B) + 2H^2 \partial_t^2 (a^2 \nabla^2 B) + 6H^2 \dot{H} a^2 \partial_i \partial^i B \right. \\ \left. + 3H\dot{H} \partial_t (a^2 \partial_i \partial^i B) \right] - \frac{(T_1 + f_1)}{4a} + (T_0 + f_0) \left( \frac{1}{8a} A - \frac{a}{8} \partial_i \partial^i B \right) 4\pi G \delta T_1^{\text{em } 1} - 4\pi G p_m \frac{1}{2a} A + 4\pi G p_m \frac{a}{2} \partial_i \partial^i B, \quad (\text{A10})$$

where we have used the definition  $\nabla^2 = \sum_i \partial_i \partial^i$ , and indices are summed over only when explicitly shown with the  $\sum$  symbol.

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