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Measurable neutrino mass scale in $A_4 \times SU(5)$

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We propose a supersymmetric $A_4 \times SU(5)$ model of quasidegenerate neutrinos which predicts the effective neutrino mass m_{ee} relevant for neutrinoless double beta decay to be proportional to the neutrino mass scale, thereby allowing its determination approximately independently of unknown Majorana phases. Such a natural quasidegeneracy is achieved by using A_4 family symmetry (as an example of a non-Abelian family symmetry with real triplet representations) to enforce a contribution to the neutrino mass matrix proportional to the identity. Tribimaximal neutrino mixing as well as quark CP violation with $\alpha \approx 90°$ and a leptonic CP phase $\delta_{\rm MNS} \approx 90°$ arise from the breaking of the A_4 family symmetry by the vacuum expectation values of four "flavon" fields pointing in specific postulated directions in flavor space.

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I. INTRODUCTION

Over the past dozen years or so our knowledge of the neutrino sector has increased dramatically with the discovery of neutrino mass and mixing in atmospheric and solar neutrino oscillations, followed by the observation of terrestrial neutrino oscillations in long baseline neutrino experiments which have confirmed and refined the earlier results [1]. Yet, despite this progress, which may even be termed a "neutrino revolution," there are many questions about neutrinos which remain unanswered. Perhaps the most pressing of these is the origin, nature, and magnitude of neutrino mass, since neutrino oscillations only provide information about the squared mass differences between neutrino species which are independent of the absolute neutrino mass scale or the nature of the neutrino mass (i.e. Dirac or Majorana). In the absence of any confirmed experimental signal from either beta decay end-point experiments or neutrinoless double beta decay experiments, the most stringent limits on the absolute neutrino mass scale come indirectly from cosmology where one typically obtains the limit on the absolute neutrino mass scale expressed in terms of the lightest neutrino mass as $m_{\rm lightest} \lesssim 0.2$ eV [2]. Thus, there remains the interesting possibility that neutrinos are quasidegenerate, which one may roughly define as $m_{\rm lightest} > 0.05$ eV, where the lower limit is approximately set equal to the square root of the atmospheric neutrino mass squared difference.

The current generation of running or planned neutrinoless double beta decay experiments is capable of discovering quasidegenerate neutrinos, as defined above, within the next years. Such a discovery would herald a new neutrino revolution to rival the last one and would lead to an explosion of interest in theoretical models capable of accounting for quasidegenerate neutrinos. In general having quasidegenerate neutrinos does not lead to a sharp prediction for the neutrinoless double beta decay observable m_{ee} as a function of $m_1 \simeq m_2 \simeq m_3$ due to the presence of unknown phases in the neutrino mass matrix [3]. The general conclusion that unknown phases enter the prediction for m_{ee} also remains valid in models which combine the experimental observation of (at least approximate) tribimaximal (TB) [4] lepton mixing with the possibility of a quasidegenerate neutrino mass spectrum. The reason is that the relevant phases entering m_{ee} are Majorana phases. Allowing for arbitrary Majorana phases and considering a quasidegenerate neutrino mass spectrum and TB mixing, m_{ee} can still be in the approximate interval $m_{ee} \in [m_{\rm lightest}/3, m_{\rm lightest}]$.

What would we learn about the origin of neutrino mass from the discovery of quasidegenerate neutrinos in neutrinoless double beta decay? Clearly this would imply that neutrinos are Majorana, and possibly (but not necessarily) that would indicate that a seesaw mechanism is at work, but what kind of seesaw mechanism; i.e. is it type I or II?[1] There are known examples of type I and type II seesaw models which can lead to quasidegenerate neutrinos as well as TB lepton mixing, so clearly quasidegenerate neutrinos would not distinguish different types of seesaw mechanism. For example, the supersymmetric (SUSY) grand unified theory (GUT) based on $SO(10)$ with family symmetry $PSL(2, 7)$ proposed in [5] is based on the type II seesaw mechanism, leads to TB mixing, and allows quasidegenerate neutrinos. On the other hand, the SUSY A_4 model in [6] based on the type I seesaw mechanism also leads to TB mixing and allows quasidegenerate neutrinos. Interestingly, the SUSY $A_4 \times SU(5)$ model with a type I seesaw mechanism does not favor quasidegenerate neutrinos [7], whereas a related model with a type II seesaw mechanism does allow quasidegenerate neutrinos [8].

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[1]We shall not consider the type III or further types of seesaw mechanism in this paper.

More generally, there is a huge literature on family symmetry models based on A_4 [9] or other symmetries [10]. However, to our knowledge, in all above examples, quasidegenerate neutrinos are subject to the mentioned phase uncertainties in the prediction of m_{ee} as a function of m_1.

It is interesting to ask, in what class of theories would we learn the most about the neutrino mass scale m_{lightest} from the discovery of a measurement of m_{ee} in neutrinoless double beta decay? Clearly the answer would be those theories which predict m_{ee} uniquely as a function of m_{lightest} without ambiguities from unknown phases, but the next question is do such theories exist? Perhaps surprisingly the answer is in the affirmative, and, even more surprisingly, the class of theories which have this property turn out to suggest the way that the neutrino mass matrix is generated, namely, by a usual type I seesaw contribution with two or three right-handed neutrinos, plus an additional contribution proportional to the unit matrix. In [11], two of us proposed a class of theories of exactly this kind which we referred to as a "type II upgrade of type I seesaw models." In this class of models the additional contribution to the neutrino mass matrix was realized by an additional type II seesaw. The type I seesaw part of the neutrino mass matrix, which controls the mass squared differences and mixing angles, was governed by sequential right-handed neutrino dominance [12]. The effect of such an additional unit matrix structure implies that for quasidegenerate neutrino masses the Majorana CP phases are small and thus $m_{ee} \approx m_{\text{lightest}}$. Although the class of models was specified, no realistic type II upgrade model has ever been proposed.

In this paper we shall propose a model following the idea of an additional contribution the neutrino mass matrix proportional to the unit matrix based on A_4 family symmetry with $SU(5)$ grand unification. The model contains tribimaximal neutrino mixing after the A_4 family symmetry is broken as an indirect result of the assumed aligned "neutrino flavons" in the type I seesaw sector via constrained sequential dominance [13]. These neutrino flavons break the A_4 symmetry, being assumed to be aligned along the columns of the TB mixing matrix, but quadratic combinations of the neutrino flavons respect accidentally the neutrino flavor symmetry as discussed in [14]. Further "quark flavons" are assumed to be misaligned compared to the neutrino flavons and are, together with the neutrino flavons, responsible for quark and charged lepton masses and quark mixings. As expected, due to a possibly large type II seesaw contribution, or alternatively due to an additional type I seesaw contribution from an additional triplet representation of right-handed neutrinos, the model can naturally predict the neutrinoless double beta decay mass observable to be approximately equal to the neutrino mass scale. We also make a detailed fit to quark masses and mixing using the misaligned quark flavons and show that a simple ansatz for the phase of one of the misaligned quark flavons leads to successful quark CP violation. In order for radiative corrections not to modify too much the TB mixing for quasidegenerate neutrinos [15], we shall restrict ourselves to low values of the ratio of Higgs vacuum expectation values (VEVs) $\tan\beta < 1.5$. For such low $\tan\beta < 1.5$, a viable GUT scale ratio of y_μ / y_s is achieved within SUSY $SU(5)$ GUTs using a Clebsch factor of $9/2$, as proposed recently by two of us in [16]. For the third generation we use b-τ Yukawa coupling unification $y_\tau / y_b = 1$ at the GUT scale which is viable for low $\tan\beta$ (see, e.g., [17]).

The layout of the remainder of the paper is as follows. In Sec. II we present the model. In Sec. III we perform a numerical fit to the quark and charged lepton masses and quark mixing angles and CP violating phase and discuss the neutrino masses and lepton mixing angles. Section IV summarizes and concludes the paper. In Appendix A we give a renormalizable superpotential and explicit expressions for the effective couplings, and Appendix B contains a possible vacuum alignment.

II. THE MODEL

In this section, we propose and describe a SUSY GUT model based on the unified $SU(5)$ gauge group as well as on the family symmetry A_4 amended by some discrete $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$ symmetries and an $U(1)_R$ symmetry as specified in Table I.

A. Symmetries and field content of the $SU(5)$ GUT model

Let us start introducing the model by specifying the field content and the symmetries. The standard model matter fields fit nicely into the two representations $\bar{\mathbf{5}}$, which we call F, and $\mathbf{10}$, which we call T. Explicitly they are given as

$$F_i = \begin{pmatrix} d_R^c & d_B^c & d_G^c & e & -\nu \end{pmatrix}_i,$$
$$T_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -u_G^c & u_B^c & -u_R & -d_R \\ u_G^c & 0 & -u_R^c & -u_B & -d_B \\ -u_B^c & u_R^c & 0 & -u_G & -d_G \\ u_R & u_B & u_G & 0 & -e^c \\ d_R & d_B & d_G & e^c & 0 \end{pmatrix}_i, \qquad (2.1)$$

where the lower indices R, B, and G denote the quark colors and $i = 1, 2, 3$ is the family index. In our model, we consider that the three generations F_i form a triplet representation $\mathbf{3}$ of an A_4 family symmetry whereas the three generations T_i form singlets $\mathbf{1}$ under A_4.[2] In the following, we suppress the A_4 indices. In addition, we consider two right-handed neutrinos, singlets under $SU(5)$ as well as under A_4, labeled by N_1 and N_2.

[2] We note that in principle any non-Abelian family symmetry with real triplet representations, like, e.g., $SO(3)$, would in principle be suitable for the construction of models with additional contributions to the neutrino mass matrix proportional to the unit matrix. In this paper we focus on A_4 as a specific example.

TABLE I. Representations and charges of the superfields. The subscript i on the fields T_i, N_i, and C_i is a family index. The flavon fields ϕ_i and $\tilde{\phi}_{23}$ can be associated to a family via their charges under $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$. The subscripts on the Higgs fields H and $\bar{H}$ and extra vectorlike matter fields A and $\bar{A}$ denote the transformation properties under $SU(5)$.

	$SU(5)$	A_4	$\mathbb{Z}_2$	$\mathbb{Z}_4'$	$\mathbb{Z}_2'$	$\mathbb{Z}_4$	$U(1)_R$
Chiral matter							
F	$\bar{\mathbf{5}}$	**3**	$+$	0	$+$	0	1
T_1, T_2, T_3	**10**, **10**, **10**	**1**, **1**, **1**	$+, +, -$	0, 1, 0	$+, +, +$	1, 0, 0	1, 1, 1
N_1, N_2	**1**, **1**	**1**, **1**	$+, +$	0, 1	$+, +$	1, 0	1, 1
Flavons and Higgs multiplets							
ϕ_{23}, ϕ_{123}, ϕ_3	**1**	**3**	$+, +, -$	0, 3, 0	$+, +, +$	3, 0, 0	0, 0, 0
$\tilde{\phi}_{23}$	**24**	**3**	$+$	3	$-$	0	0
H_5, $\bar{H}_5$	**5**, $\bar{\mathbf{5}}$	**1**, **1**	$+, +$	0, 0	$+, +$	0, 0	0, 0
H_{15}, $\bar{H}_{15}$	**15**, $\overline{\mathbf{15}}$	**1**, **1**	$+, +$	0, 0	$+, +$	0, 0	0, 0
H_{45}, $\bar{H}_{45}$	**45**, $\overline{\mathbf{45}}$	**1**, **1**	$+, +$	0, 0	$-, -$	0, 0	0, 0
Matterlike messengers							
A_5, $\bar{A}_5$	**5**, $\bar{\mathbf{5}}$	**1**, **1**	$+, +$	1, 3	$-, -$	0, 0	1, 1
A_{10}, $\bar{A}_{10}$	**10**, $\overline{\mathbf{10}}$	**3**, **3**	$+, +$	0, 0	$+, +$	0, 0	1, 1
A_1	**1**	**3**	$+$	0	$+$	0	1
Higgs-like messengers							
B, $\bar{B}$	**5**, $\bar{\mathbf{5}}$	**1**, **1**	$+, +$	2, 2	$+, +$	0, 0	0, 2
C_1, $\bar{C}_1$	**1**, **1**	**1**, **1**	$+, +$	0, 0	$+, +$	2, 2	2, 0
C_2, $\bar{C}_2$	**1**, **1**	**1**, **1**	$+, +$	2, 2	$+, +$	0, 0	2, 0

We furthermore consider 15-dimensional Higgs representations H_{15} and $\bar{H}_{15}$ which contain $SU(2)_L$-triplet Higgs fields that obtain induced VEVs after electroweak symmetry breaking. H_{15} induces in this way a type II seesaw contribution to the neutrino mass matrix which is, to leading order, proportional to the unit matrix and can increase the neutrino mass scale without modifying the values for the leptonic mixing angles.

$SU(5)$ is spontaneously broken by the VEV of the $\tilde{\phi}_{23}$ field, electroweak symmetry is broken by the VEVs of the Higgs fields H_5, $\bar{H}_5$, H_{45}, and $\bar{H}_{45}$, and A_4 is spontaneously broken by the VEVs of the flavon fields, i.e. the family symmetry breaking Higgs fields ϕ_{123}, ϕ_{23}, ϕ_3, and $\tilde{\phi}_{23}$. We comment below on the specific directions in which we assume A_4 to be broken by the flavons.

On top of that, we consider additional "messenger" fields which are heavy and which, after effectively integrating them out of the theory, give rise to higher-dimensional operators generating the Yukawa coupling matrices as well as the mass matrix of the gauge singlet (right-handed) neutrinos N_i.

The field content of our model as well as the symmetries is specified in Table I. We note that it is always possible to replace any product of commuting discrete symmetries by a single Abelian group $U(1)$ with a suitable choice of charges for the fields, so it is possible to replace the $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$ symmetry by a single $U(1)$ symmetry, with an appropriate choice of charges. Indeed many models in the literature use an Abelian $U(1)$ symmetry rather than a product of $\mathbb{Z}_N$ symmetries to control the operators. Although this looks simpler, it should be remarked that, first, an Abelian symmetry has infinitely many more group elements than any discrete symmetry, and, second, one must then confront the question of Goldstone bosons once the assumed global Abelian symmetry is broken. If the Abelian symmetry is gauged, one must further complicate the model by ensuring that it is anomaly-free. Therefore an auxiliary discrete symmetry, even a large one, has definite advantages over an Abelian symmetry. Furthermore discrete symmetries are ubiquitous in string theory constructions. Finally, the auxiliary discrete symmetry used here is rather a simple one consisting of a product of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ parity factors. Thus we regard the use of the discrete $\mathbb{Z}_2$ and $\mathbb{Z}_4$ symmetries as being a well motivated, simple, and attractive alternative to the use of an Abelian $U(1)$ symmetry.

We would like to remark that we do not explicitly consider the full flavor and GUT Higgs sector of the model and just assume that the $SU(5)$ and A_4 breaking VEVs are aligned in the desired directions of field space. We assume that in these sectors issues like doublet-triplet splitting are resolved. Without specifying these sectors, a reliable calculation of the proton decay rate must also be beyond the scope of the present paper. The focus of the present paper is thus to illustrate that quasidegenerate light neutrino masses can be realized together with a type II seesaw in a $SU(5)$ GUT framework.

We would furthermore like to remark that in addition to the type II seesaw contribution there is a possible additional contribution to the neutrino mass matrix proportional to the unit matrix from the messenger field A_1 which is a singlet under $SU(5)$ and a triplet under A_4. When it is integrated out, it also induces a contribution to the neutrino mass operator which is proportional to the unit matrix.

B. The effective $A_4 \times SU(5)$ symmetric superpotential

The renormalizable superpotential resulting from Table I is given in Appendix A. Integrating out the heavy messenger superfields denoted by A, B, and C, the Feynman diagrams in Figs. 1–3 then lead to the effective nonrenormalizable superpotential terms in the $SU(5)$ and A_4 unbroken phase:

$$W_{Y_l} = \frac{\sqrt{2}}{M_{A_{10}}} F(a_1 \phi_{23} T_1 + a_2 \phi_{123} T_2 + a_3 \phi_3 T_3)\bar{H}_5 + \frac{\sqrt{2}\tilde{a}_2}{M_{A_5}} F \tilde{\phi}_{23} T_2 \bar{H}_{45}, \tag{2.2}$$

$$W_{Y_u} = \frac{1}{4}\left(\frac{a_{12}}{M_{A_{10}}^2} T_1 T_2 (\phi_{123} \cdot \phi_{23}) + \frac{a_{13}}{M_{A_{10}}^2} T_1 T_3 (\phi_3 \cdot \phi_{23}) + \frac{a_{23}}{M_{A_{10}}^2} T_2 T_3 (\phi_{123} \cdot \phi_3)\right) H_5 + \frac{1}{4}\left(a_{33} T_3^2 + \frac{a_{22}}{M_{A_{10}}^2} T_2^2 \phi_{123}^2 + \frac{a_{11}}{M_{A_{10}}^2} T_1^2 \phi_{23}^2 + \frac{\tilde{a}_{22}}{M_{A_5}^2} T_2^2 \tilde{\phi}_{23}^2\right) H_5, \tag{2.3}$$

$$W_{Y_\nu} = \frac{1}{M_{A_{10}}} F(a_{\nu_1} \phi_{23} N_1 + a_{\nu_2} \phi_{123} N_2) H_5, \tag{2.4}$$

$$W_\nu^\Delta = y_\Delta H_{15} F F, \tag{2.5}$$

$$W_\nu^{d=5} = \frac{\bar{\kappa}_F^2}{M_{A_1}} F H_5 F H_5, \tag{2.6}$$

$$W_\nu^{M_R} = \frac{a_{R_{11}}}{M_{A_{10}}^2} \phi_{23}^2 N_1^2 + \frac{a_{R_{22}}}{M_{A_{10}}^2} \phi_{123}^2 N_2^2 + \frac{a_{R_{12}}}{M_{A_{10}}^2} (\phi_{123} \cdot \phi_{23}) N_1 N_2. \tag{2.7}$$

After GUT symmetry breaking the $SU(2)_L$ doublet components from H_5 and H_{45}, respectively, $\bar{H}_5$ and $\bar{H}_{45}$ mix and only the light states acquire the $SU(2)_L$ breaking VEVs which give the fermion masses, as discussed in Appendix A where the effective couplings a appearing in the effective superpotential are also explicitly given.

C. Assumed vacuum alignment

In the following we assume that the VEVs of the A_4 breaking flavon fields point in the following directions in field space such that

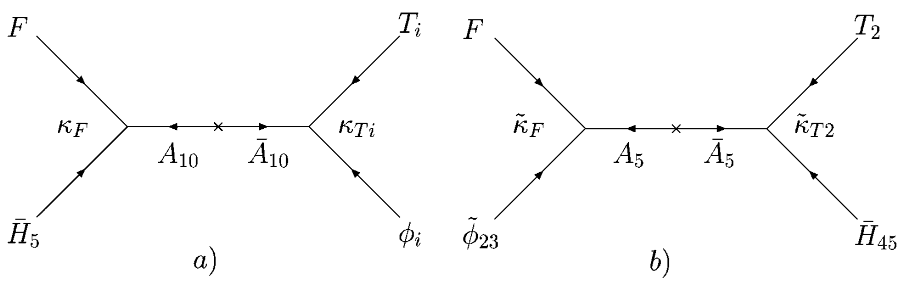

FIG. 1. Supergraph diagrams inducing the effective superpotential operators for the down-type quarks and charged leptons.

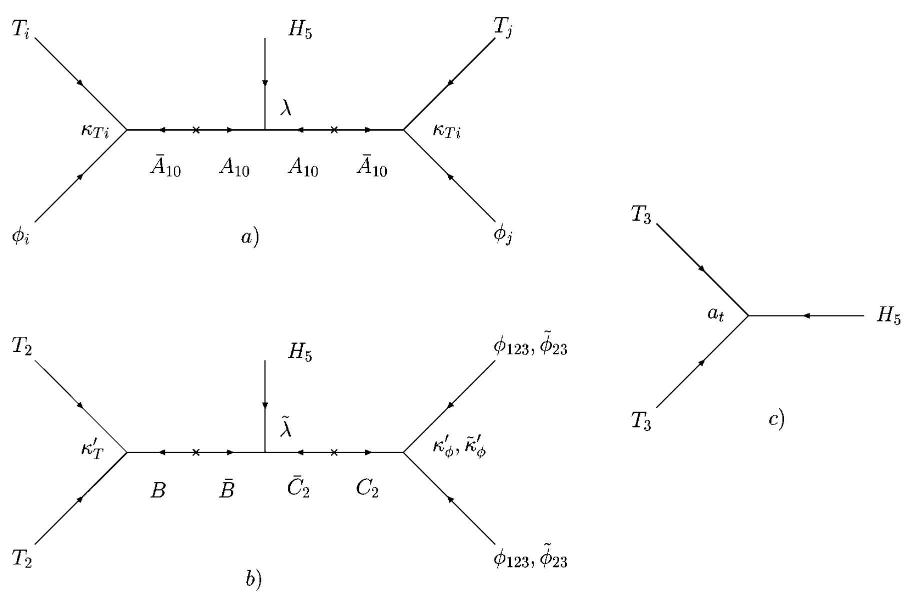

FIG. 2. Supergraph diagrams inducing the effective superpotential operators for the up-type quarks.

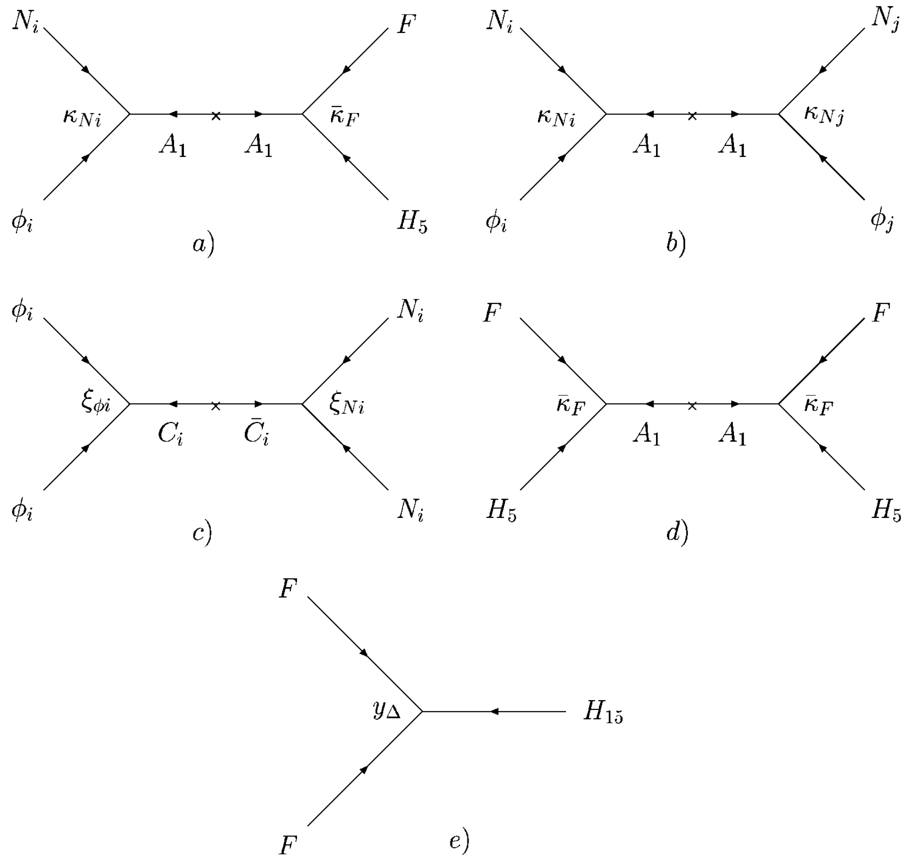

FIG. 3. Supergraph diagrams inducing the effective superpotential operators for the neutrino sector.

$$b_1 \frac{\langle \phi_{23} \rangle}{M_{A_{10}}} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \epsilon_{23}, \qquad b_2 \frac{\langle \phi_{123} \rangle}{M_{A_{10}}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \epsilon_{123},$$
$$b_3 \frac{\langle \phi_3 \rangle}{M_{A_{10}}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \epsilon_3. \tag{2.8}$$

The b_i are defined in Appendix A. These relations also define the quantities ϵ_{123}, ϵ_{23}, and ϵ_3. The breaking of A_4 along the field directions of ϕ_{123} and ϕ_{23} allows us to realize tribimaximal neutrino mixing via constrained sequential dominance [12]. It is also worth noting that the flavon VEVs $\langle \phi_{123} \rangle$ and $\langle \phi_{23} \rangle$ are orthogonal, causing some of the terms in the superpotential to give a vanishing contribution to the mass matrices. In the following, we assume that CP is only broken spontaneously by the VEV of the flavon $\tilde{\phi}_{23}$.

For the flavon $\tilde{\phi}_{23}$ one may suppose *a priori* a less constrained alignment:

$$\tilde{b}_2 \frac{\langle \tilde{\phi}_{23} \rangle}{M_{A_5}} = \begin{pmatrix} 0 \\ v \\ w \end{pmatrix} \tilde{\epsilon}_{23}. \tag{2.9}$$

However *empirically* we find that the numerical fit to quark masses and mixings, in particular, quark CP violation, seems strongly to prefer that the vacuum alignment of the flavon $\tilde{\phi}_{23}$ has its second component along the imaginary direction. To simplify the results of the numerical fit we shall restrict ourselves to the case

$$v = -i. \tag{2.10}$$

In some future more ambitious theory one may attempt to reproduce Eq. (2.10) as a result of some special vacuum alignment, but here we shall simply regard it as a special choice, or ansatz, which leads to a successful fit to quark CP violation.

In Appendix B, we will discuss another possibility, namely, to realize the flavon $\tilde{\phi}_{23}$ effectively by splitting it up into two flavons $\tilde{\phi}_2$ and $\tilde{\phi}_3$, where one gets a purely real and the other a purely imaginary VEV. For the effective superpotential in Eqs. (2.2), (2.3), (2.4), (2.5), (2.6), and (2.7) this would correspond to simply replacing $\tilde{\phi}_{23}^2 \to \tilde{\phi}_2^2 + \tilde{\phi}_3^2$ and $\tilde{\phi}_{23}\bar{H}_{45} \to \tilde{\phi}_2\bar{H}_{45} + \tilde{\phi}_3\bar{H}_5'$ with an additional Higgs field H_5'. The field content of this extended version of the model that now includes also a vacuum alignment sector is presented in Appendix B in Table IV. The predictions of the two model variants are identical at the level of precision discussed here.

III. NUMERICAL FIT TO FERMION MASSES AND MIXINGS

A. The quark and charged lepton sector

We define our conventions for the Yukawa matrices such that the operators of the form $FT\phi\bar{H}$ and $T^2\phi^2 H$ give the following Yukawa terms in the Lagrangian:

$$\mathcal{L}_{\rm Yuk} = -(Y_d^*)_{ij} Q_i \bar{d}_j H_d - (Y_e^*)_{ij} L_i \bar{e}_j H_d - (Y_u^*)_{ij} Q_i \bar{u}_j H_u + \text{H.c.}, \quad (3.1)$$

where the $SU(5)$ relation $Y_d = Y_e^T$ would be fulfilled, if all Clebsch-Gordan factors were one. The convention we use here is the same as the one used by the Particle Data Group [18].

From Eqs. (2.2), (2.3), (2.8), (2.9), and (2.10), the Yukawa matrix coupling the up-type quarks to the light up-type Higgs doublet with the b coefficients as defined in Appendix A is given as

$$Y_u = \begin{pmatrix} 2b_{11}\epsilon_{23}^2 & 0 & b_{13}\epsilon_{23}\epsilon_3 \\ 0 & 3b_{22}\epsilon_{123}^2 + (w^2-1)\tilde{b}_{22}^2\tilde{\epsilon}_{23}^2 & b_{23}\epsilon_{123}\epsilon_3 \\ b_{13}\epsilon_{23}\epsilon_3 & b_{23}\epsilon_{123}\epsilon_3 & b_{33} \end{pmatrix}, \quad (3.2)$$

whereas the Yukawa matrices coupling the down-type quarks and charged leptons to the light down-type Higgs doublet are given as

$$Y_d = \begin{pmatrix} 0 & \epsilon_{23} & -\epsilon_{23} \\ \epsilon_{123} & \epsilon_{123} + i\tilde{\epsilon}_{23} & \epsilon_{123} + w\tilde{\epsilon}_{23} \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \quad (3.3)$$

$$Y_e^T = \begin{pmatrix} 0 & c_{23}\epsilon_{23} & -c_{23}\epsilon_{23} \\ c_{123}\epsilon_{123} & c_{123}\epsilon_{123} + i\tilde{c}_{23}\tilde{\epsilon}_{23} & c_{123}\epsilon_{123} + w\tilde{c}_{23}\tilde{\epsilon}_{23} \\ 0 & 0 & c_3\epsilon_3 \end{pmatrix}, \quad (3.4)$$

where c_3, c_{23}, $\tilde{c}_{23}$, and c_{123} are the Clebsch-Gordan factors arising from GUT symmetry breaking; see, e.g., [16]. We have used the orthogonality of $\langle\phi_{23}\rangle$ and $\langle\phi_{123}\rangle$ and considered the above described notation for the flavon VEVs. We note that in the definition for the Yukawa matrices we have introduced a complex conjugation which here appears as a phase factor of $+i$ in the 2-2 elements of the down-type quark and charged lepton Yukawa matrices.

With the given representations of the flavons, we obtain the following Clebsch-Gordan coefficients:

$$c_{123} = 1, \qquad c_{23} = 1, \qquad c_3 = 1, \qquad \tilde{c}_{23} = 9/2. \quad (3.5)$$

For small values of $\tan\beta$ as we consider, the 1-loop SUSY threshold corrections are small and, taking the actual experimental values of the strange quark and muon masses into account, the GUT scale value of y_μ/y_s prefers $\tilde{c}_{23} = 9/2$, as argued in [16] (see also [19]).

From the charged lepton Yukawa matrix we can derive the following approximate relations for the eigenvalues:

$$y_\tau = c_3\epsilon_3, \qquad y_\mu = |c_{123}\epsilon_{123} + i\tilde{c}_{23}\tilde{\epsilon}_{23}|, \qquad y_e = \frac{c_{23}\epsilon_{23}c_{123}\epsilon_{123}}{y_\mu}. \quad (3.6)$$

Furthermore, since there is no 1–2 mixing from the up sector, the mixing angle θ_{12} is approximately given as

$$\theta_{12}^{\rm CKM} = \left|\frac{\epsilon_{23}}{\epsilon_{123} + i\tilde{\epsilon}_{23}}\right|. \quad (3.7)$$

From those four equations the four ϵ's can be calculated and the relation for the Cabibbo-Kobayashi-Maskawa (CKM) phase gives at the GUT scale

$$|\tan\delta_{\rm CKM}| = \left|\frac{\tilde{\epsilon}_{23}}{\epsilon_{123}}\right| \approx 1.22. \quad (3.8)$$

The renormalization group (RG) evolution of the measured value for $\delta_{\rm CKM}$ gives a GUT scale value of 1.20. So the value for the CKM phase (based on our assumed vacuum alignment) is already remarkably good if we only take the lepton masses and the value for θ_{12} into account which are measured to high accuracy.[3]

For the detailed fit of the model to the data we applied the following procedure: We have taken the GUT scale Yukawa matrices from Eqs. (3.2), (3.3), and (3.4) and calculated their RG evolution down to the scale $m_t(m_t)$ for $\tan\beta = 1.4$[4] and $M_{\rm SUSY} = 500$ GeV with the REAP software package [21]. At the low scale we performed a χ^2 fit to the quark masses and mixing and charged lepton masses depending on the parameters of the GUT scale Yukawa matrices. The fit gave a total χ^2 of about 3.5 where we have assumed a relative error of 1% for the charged lepton masses, and for the other observables we have taken the experimental errors. Since we have 11 parameters and 13 observables this corresponds to a χ^2/dof of about 1.6. This is a good fit since we have neglected theoretical uncertainties like, e.g., threshold corrections which could be treated as additional errors on the data lowering the total χ^2.

The results for the GUT scale parameters are listed in Table II. We would like to remark that these parameters depend on $\tan\beta$ and $M_{\rm SUSY}$ and also are subject to several theoretical uncertainties. For example, we note that the Higgs fields H_{15} and $\bar{H}_{15}$ containing the Higgs triplets of the type II seesaw mechanism have masses at intermediate

[3] We would like to remark that with the assumed spontaneous CP violation, real $\det Y_u$ and $\det Y_d$, and the small $|\tilde{\epsilon}_{23}| = \mathcal{O}(10^{-4})$, the model might also provide a solution to the strong CP problem, along the lines discussed in [20].

[4] We note that in the minimal supersymmetric standard model small values of $\tan\beta$ are somewhat constrained due to bounds on the Higgs mass. However, we emphasize that our model may well be formulated in the context of the nonminimal supersymmetric standard model or other nonminimal SUSY models where $\tan\beta$ of order one can readily be realized without these constraints.

TABLE II. The model parameters for $\tan\beta = 1.4$ and $M_{\mathrm{SUSY}} = 500$ GeV from a fit to the experimental data [18,22].

Parameter	Value
$2b_{11}\epsilon_{23}^2$ in 10^{-6}	9.62
$3b_{22}\epsilon_{123}^2$ in 10^{-4}	−1.10
$(w^2-1)\tilde{b}_{22}\tilde{\epsilon}_{23}^2$ in 10^{-3}	−1.10
$b_{13}\epsilon_{23}\epsilon_3$ in 10^{-3}	−2.92
$b_{23}\epsilon_{123}\epsilon_3$ in 10^{-2}	3.21
b_{33}	2.44
ϵ_{123} in 10^{-5}	5.88
ϵ_{23} in 10^{-5}	4.30
$\tilde{\epsilon}_{23}$ in 10^{-4}	−1.61
ϵ_3 in 10^{-2}	1.12
w	1.44

TABLE III. Fit results for the quark Yukawa couplings and mixing and the charged lepton Yukawa couplings at low energy compared to experimental data [18,22]. A pictorial representation of the agreement between our fit and experiment can also be found in Fig. 4.

Quantity [at $m_t(m_t)$]	Model	Experiment	Deviation
y_τ in 10^{-2}	1.00	1.00	−0.027%
y_μ in 10^{-4}	5.89	5.89	−0.029%
y_e in 10^{-6}	2.79	2.79	−0.130%
y_b in 10^{-2}	1.58	1.58 ± 0.05	0.086σ
y_s in 10^{-4}	2.83	2.99 ± 0.86	-0.184σ
y_d in 10^{-6}	27.6	$15.9^{+6.8}_{-6.6}$	1.723σ
y_t	0.938	0.936 ± 0.016	0.084σ
y_c in 10^{-3}	3.54	3.39 ± 0.46	0.318σ
y_u in 10^{-6}	6.70	$7.01^{+2.76}_{-2.30}$	-0.134σ
$\theta_{12}^{\mathrm{CKM}}$	0.2257	$0.2257^{+0.0009}_{-0.0010}$	-0.022σ
$\theta_{23}^{\mathrm{CKM}}$	0.0413	$0.0415^{+0.0011}_{-0.0012}$	0.004σ
$\theta_{13}^{\mathrm{CKM}}$	0.0036	0.0036 ± 0.0002	-0.157σ
δ_{CKM}	1.1782	$1.2023^{+0.0786}_{-0.0431}$	-0.560σ

energy scale between M_{GUT} and M_{EW}. Their effects are not included in the RG analysis. The effects are small and may be neglected if y_Δ is small, but they could be sizable if y_Δ is large.[5] Because of the additional theoretical uncertainties we do not explicitly give the errors on the high energy parameters or low energy fit results. The important input parameters for us are the charged lepton masses and quark mixing angles which have an experimental error much smaller than these uncertainties.

In Table III the low energy results are shown and compared to experimental data [18,22]. A graphical illustration is given in Fig. 4. They illustrate that our minimal example model, with the assumed vacuum alignment of Eqs. (2.8) and (2.9), can fit well the data. We turn now to the results for the neutrino sector.

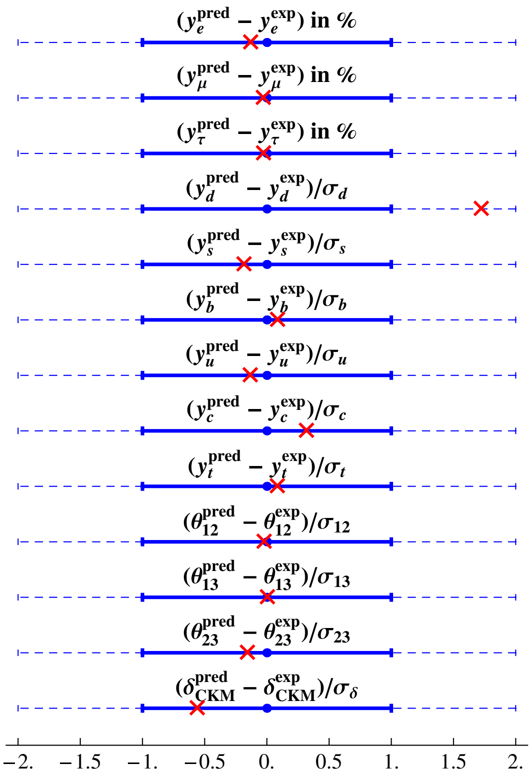

FIG. 4 (color online). Pictorial representation of the deviation of our fit from low energy experimental data [18,22] for the charged lepton Yukawa couplings and quark Yukawa couplings and mixing parameters. The deviations of the charged lepton masses are given in percent while all other deviations are given in units of standard deviations σ. The straight blue lines give the 1% (1σ) bound while the dashed lines give the 2% (2σ) bound. The red crosses denote our fit results.

B. The neutrino sector

The neutrino Yukawa matrix is obtained from Eq. (2.4) as

$$Y_\nu = \begin{pmatrix} 0 & b_{\nu_2}\epsilon_{123} \\ b_{\nu_1}\epsilon_{23} & b_{\nu_2}\epsilon_{123} \\ -b_{\nu_1}\epsilon_{23} & b_{\nu_2}\epsilon_{123} \end{pmatrix}. \tag{3.9}$$

Additionally we have a diagonal mass matrix for the two right-handed neutrinos from Eq. (2.7):

$$M_R = \begin{pmatrix} 2b_{R_1}\epsilon_{23}^2 & 0 \\ 0 & 3b_{R_2}\epsilon_{123}^2 \end{pmatrix}, \tag{3.10}$$

and a contribution proportional to the unit matrix coming from Eqs. (2.5) and (2.6):

[5] Since the coupling y_Δ gives a contribution proportional to the unit matrix, it affects only the RG evolution of the mass eigenvalues but not of the mixing angles. Nevertheless, the possibility of additional RG effects from y_Δ provides a theoretical uncertainty in our setup.

$$M_L = \begin{pmatrix} m_0 & 0 & 0 \\ 0 & m_0 & 0 \\ 0 & 0 & m_0 \end{pmatrix}. \tag{3.11}$$

Using the seesaw relation

$$m_\nu = M_L - v_u^2 Y_\nu M_R^{-1} Y_\nu^T, \tag{3.12}$$

we obtain for the neutrino mass matrix

$$m_\nu = m_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{m_2^I}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_3^I}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \tag{3.13}$$

with

$$m_0 = c_\gamma^2 v_u^2 \frac{y_\Delta \bar{\lambda}_{15}}{\mu_{15}} + c_\gamma^2 v_u^2 \frac{\bar{\kappa}_F^2}{M_{A_1}}, \quad m_2^I = -v_u^2 \frac{b_{\nu_2}^2}{b_{R_2}}, \quad \text{and} \quad m_3^I = -v_u^2 \frac{b_{\nu_1}^2}{b_{R_1}}. \tag{3.14}$$

In our model we therefore identify the neutrino masses as $m_1 = m_0$, $m_2 = m_0 + m_2^I$, and $m_3 = m_0 + m_3^I$, where without loss of generality we can take m_0 to be positive and real while m_2^I, m_3^I are real but can take either sign. With $|m_0| \gg |m_3^I|, |m_2^I|$ a quasidegenerate mass spectrum of the light neutrinos can be explained in a natural way. In the following, we will mainly restrict ourselves to this case.

From the structure of these matrices we see that we obtain tribimaximal mixing in the neutrino sector:

$$\theta_{13}^\nu = 0, \qquad \theta_{23}^\nu = 45°, \qquad \theta_{12}^\nu = \arcsin\frac{1}{\sqrt{3}} \approx 35.3°. \tag{3.15}$$

From the lepton sector we get the additional mixing contributions

$$\theta_{13}^e = 0, \qquad \theta_{23}^e = 0, \qquad |\theta_{12}^e| = \left| \frac{c_{123}\epsilon_{123}}{c_{123}\epsilon_{123} - i\tilde{c}_{23}\tilde{\epsilon}_{23}} \right| \approx 4.6°. \tag{3.16}$$

There is also a complex phase introduced by the charged lepton Yukawa matrix which can be calculated in the same way as in the quark sector:

$$\delta_{12}^e = \arctan\frac{\tilde{c}_{23}\tilde{\epsilon}_{23}}{c_{123}\epsilon_{123}} \approx -85.4°. \tag{3.17}$$

For the approximate calculation of the Maki-Nakagawa-Sakata (MNS) mixing parameters at the GUT scale we can use [23]

$$\begin{aligned} s_{23}^{\text{MNS}} &\approx s_{23}^\nu - \theta_{23}^e, \\ s_{13}^{\text{MNS}} e^{-i\delta_{13}^{\text{MNS}}} &\approx \theta_{13}^\nu - s_{23}^\nu \theta_{12}^e e^{-i\delta_{12}^e}, \\ s_{12}^{\text{MNS}} e^{-i\delta_{12}^{\text{MNS}}} &\approx s_{12}^\nu - c_{23}^\nu c_{12}^\nu \theta_{12}^e e^{-i\delta_{12}^e}, \end{aligned} \tag{3.18}$$

where we have already discarded all trivial phases and RG corrections which we will discuss later. For the total leptonic mixing angles we obtain

$$\theta_{12}^{\text{MNS}} \approx 35.1°, \qquad \theta_{13}^{\text{MNS}} \approx 3.3°, \qquad \theta_{23}^{\text{MNS}} = 45.0°. \tag{3.19}$$

For the phases we have $\delta_{13}^{\text{MNS}} = \pi - \delta_{12}^e \approx 94.6°$, $\delta_{12}^{\text{MNS}} = 4.6°$, and $\delta_{23}^{\text{MNS}} = 0°$ from which the final MNS phases can be calculated according to [23]

$$\begin{aligned} \delta_{\text{MNS}} &= \delta_{13}^{\text{MNS}} - \delta_{12}^{\text{MNS}} \approx 90.0°, \\ \alpha_1 &= 2(\delta_{12}^{\text{MNS}} + \delta_{23}^{\text{MNS}}) = 2\delta_{12}^{\text{MNS}} \approx 9.3°, \\ \alpha_2 &= 2\delta_{23}^{\text{MNS}} = 0°, \end{aligned} \tag{3.20}$$

where α_1 and α_2 are the Majorana phases as in the Particle Data Group parameterization where they are contained in a diagonal matrix diag$(e^{i\alpha_1/2}, e^{i\alpha_2/2}, 1)$.

We note that with the mixing pattern of our model, i.e. tribimaximal mixing produced in the neutrino sector, and charged lepton mixing corrections only from θ_{12}^e, the leptonic mixing angles and the Dirac CP phase δ_{MNS} satisfy the lepton mixing sum rule [24]

$$\theta_{12}^{\text{MNS}} - \theta_{13}^{\text{MNS}} \cos(\delta_{\text{MNS}}) \approx \arcsin(1/\sqrt{3}). \tag{3.21}$$

The approximately maximal CP violation, i.e. $\delta_{\text{MNS}} \approx 90°$, affects that although the charged lepton corrections generate $\theta_{13}^{\text{MNS}} \approx 3.3°$, the solar mixing angle remains very close to its tribimaximal value of $\arcsin(1/\sqrt{3})$. So far, we have discussed the neutrino mixing parameters at the GUT scale. To calculate the low energy values we have to take RG running of the parameters into account.

C. Renormalization group corrections

For a quasidegenerate neutrino mass spectrum, RG corrections to the neutrino parameters can in principle change the high scale values dramatically. However, as has been discussed for type II upgraded seesaw models in [11] and more generally in [21,25], for small $\tan\beta$ and small neutrino Yukawa couplings (in our example model they are much smaller than y_τ) the corrections to the mixing angles and CP phases are under control. Setting the small Majorana phases to zero and with $\delta_{\text{MNS}} \approx 90°$, we can estimate in leading order [21,25]

$$\frac{d\theta_{12}^{\text{MNS}}}{d\ln(\mu/\mu_0)} \approx -\frac{y_\tau^2}{32\pi^2} \sin(2\theta_{12}^{\text{MNS}})(s_{23}^{\text{MNS}})^2 \frac{|m_1 + m_2|^2}{\Delta m_{\text{sol}}^2}, \tag{3.22}$$

$$\frac{d\theta_{13}^{\rm MNS}}{d\ln(\mu/\mu_0)} \approx 0, \tag{3.23}$$

$$\frac{d\theta_{23}^{\rm MNS}}{d\ln(\mu/\mu_0)} \approx -\frac{y_\tau^2}{32\pi^2}\sin(2\theta_{23}^{\rm MNS}) \times \frac{(c_{12}^{\rm MNS})^2|m_2+m_3|^2 + (s_{12}^{\rm MNS})^2|m_1+m_3|^2}{\Delta m_{\rm atm}^2}, \tag{3.24}$$

where μ is the renormalization scale. In the case of quasidegenerate neutrino masses we can further use the approximation $m_3 \approx m_2 \approx m_1 = m_0$. Integrating these equations approximately with the parameters on the right side taken constant and equal to their GUT scale values, and plugging in these numbers, we obtain the estimated low energy values of the mixing angles as

$$\theta_{12}^{\rm MNS}|_{m_t(m_t)} \approx \theta_{12}^{\rm MNS}|_{M_{\rm GUT}} + 0.15^\circ \frac{m_0^2}{(0.1\ {\rm eV})^2}, \tag{3.25}$$

$$\theta_{13}^{\rm MNS}|_{m_t(m_t)} \approx \theta_{13}^{\rm MNS}|_{M_{\rm GUT}}, \tag{3.26}$$

$$\theta_{23}^{\rm MNS}|_{m_t(m_t)} \approx \theta_{23}^{\rm MNS}|_{M_{\rm GUT}} \pm 0.01^\circ \frac{m_0^2}{(0.1\ {\rm eV})^2}. \tag{3.27}$$

In the last equation, the "+" applies for a normal neutrino mass ordering, whereas the "−" applies for an inverse mass ordering, i.e. the case $\Delta m_{\rm atm}^2 < 0$. It is important to note that both mass orderings can be realized in our model. The strong suppression for the RG running of $\theta_{13}^{\rm MNS}$ is caused by the particular values of the *CP* violating phases in our model. For similar reasons the running of the *CP* phases themselves is also suppressed, as can be seen using the analytical results in [21,25]. In summary, RG corrections are under control in our setup and only cause comparatively small corrections to the mixing parameters in the lepton sector.

In summary, the leptonic mixing parameters in our model are compatible with the experimental 1σ ranges at low energy which are $\theta_{12}^{\rm MNS} = (34.5 \pm 1.0)^\circ$, $\theta_{13}^{\rm MNS} = (5.7^{+3.0}_{-3.9})^\circ$, and $\theta_{23}^{\rm MNS} = (42.3^{+5.3}_{-2.8})^\circ$, taken from [26], as long as m_0 is smaller than the cosmological bounds suggest: $m_0 \lesssim 0.2$ eV [2].

The values for the leptonic mixing angles and Dirac *CP* phase $\delta_{\rm MNS}$ resulting from our assumed vacuum alignment and stated in Eqs. (3.19), (3.20), and (3.21) can be tested accurately by ongoing and future precision neutrino oscillation experiments [27].

D. Predictions for beta decay experiments

The effective mass relevant for neutrinoless double beta decay is

$$m_{ee} = |m_1 c_{12}^2 c_{13}^2 e^{i\alpha_1} + m_2 s_{12}^2 c_{13}^2 e^{i\alpha_2} + m_3 s_{13}^2 e^{2i\delta_{\rm MNS}}|, \tag{3.28}$$

while the kinematic mass accessible in the single beta decay end-point experiment KATRIN [28] is

$$m_\beta^2 \equiv m_1^2 c_{12}^2 c_{13}^2 + m_2^2 s_{12}^2 c_{13}^2 + m_3^2 s_{13}^2. \tag{3.29}$$

For quasidegenerate neutrino mass spectrum ($m_0 = m_1 \simeq m_2 \simeq m_3$) we obtain that

$$m_\beta \approx m_0 \tag{3.30}$$

directly gives information about the absolute neutrino mass scale.

On the other hand, due to the phases appearing in Eq. (3.28) there is typically a sizable ambiguity in the relation between m_{ee} and m_0, as long as the Majorana *CP* phases are not predicted. Allowing, for instance, for arbitrary Majorana phases and considering a quasidegenerate neutrino mass spectrum ($m_0 = m_1 \simeq m_2 \simeq m_3$) and with small $\theta_{13}^{\rm MNS}$, m_{ee} can still be in the approximate interval $m_{ee} \in [m_{\rm lightest}/3, m_{\rm lightest}]$.

This ambiguity is resolved in our model, since the large contribution to the neutrino mass matrix proportional to the unit matrix[6] (with $|m_0| \gg |m_3^I|, |m_2^I|$) results in small Majorana *CP* phases, and thus we predict

$$m_{ee} \approx m_0. \tag{3.31}$$

The assumed dominance of m_0 in our model allows us to realize a quasidegenerate neutrino spectrum (with normal or inverse mass ordering) in a natural way, without any tuning of parameters. The possible values for m_{ee} as a function of the lightest neutrino mass $m_{\rm lightest}$ is shown in Fig. 5.

We would like to remark that for smaller m_0 one can also naturally extend the model to hierarchical or inverted hierarchical neutrino masses without changing the leptonic mixing angles. With $|m_0| \approx |m_2^I|$ or $|m_0| \approx |m_3^I|$ or both we then also encounter cases where the Majorana phases are close to π. For a quasidegenerate spectrum the model disfavors these unnatural cases since they would correspond to heavily fine-tuned parameters of the model. Similarly, an inverse strongly hierarchical spectrum would require unnatural tuning between m_0 and m_3^I to make $m_3 = |m_0 - m_3^I|$ very small. By contrast, for a typical parameter choice of the model, a normally ordered hierarchical spectrum simply corresponds to $|m_0| \ll |m_2^I|, |m_3^I|$ and does not require any tuning at all. It is also interesting to note that with the phases in our model there is *no* possibility to have cancellations in Eq. (3.28) that could make m_{ee} vanish exactly.[7] Neutrinoless double beta decay is thus, also for

[6] In our model this part of the neutrino mass matrix is induced by a standard type I (with right-handed neutrinos) and an additional type II seesaw contribution. Nevertheless, the conclusions remain the same as for the pure type II seesaw case since we assume that the type II contribution is dominant.

[7] In fact we find numerically that $m_{ee} \gtrsim 0.007$ eV.

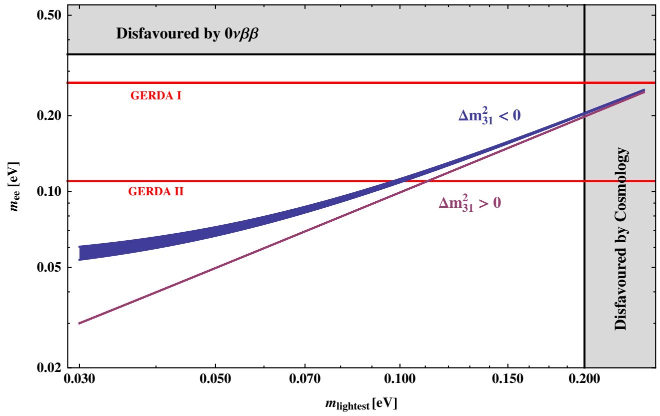

FIG. 5 (color online). The effective mass m_{ee} in our setup relevant for neutrinoless double beta decay as a function of the mass m_{lightest} of the lightest neutrino, for an inverted neutrino mass ordering ($\Delta m^2_{31} < 0$, upper line) and for a normal mass ordering ($\Delta m^2_{31} > 0$, lower line). The bands represent the experimental uncertainties of the mass squared differences. The mass bounds from cosmology [2] and from the Heidelberg-Moscow experiment [32] are displayed as gray shaded regions. The red lines show the expected sensitivities of the GERDA experiment in phase I and II [33].

smaller m_0, an unavoidable consequence in this class of models.

IV. SUMMARY AND CONCLUSIONS

We have proposed a model of quasidegenerate neutrinos which predicts the neutrinoless double beta decay mass observable to be approximately equal to the neutrino mass scale m_{lightest}, thereby allowing its determination approximately independently of unknown Majorana phases. In general such quasidegenerate neutrino masses may be naturally realized if there exists a non-Abelian family symmetry with real triplet representations that enforces an additional contribution to the neutrino mass matrix proportional to the unit matrix, hence determining the neutrino mass scale. In our model, the additional contribution was generated by a type II seesaw or, alternatively, by another type I seesaw contribution from an additional triplet representation of right-handed neutrinos (or neutrino messenger fields). In addition, the standard type I seesaw contribution determines the neutrino mixing angles. Although such a mechanism (called a type II upgrade in [11]) has been known for some time, the model in this paper is the first of its kind to combine this mechanism with tribimaximal mixing arising from A_4 family symmetry together with a $SU(5)$ SUSY GUT.

The SUSY $A_4 \times SU(5)$ model considered here has several attractive features. The full renormalizable superpotential for the coupling of matter to flavons, messenger fields, and Higgs fields is specified, and only discrete auxiliary symmetries are introduced, rather than the more common continuous Abelian symmetry that is typically invoked in such models. In the considered model, the neutrino mass scale can originate either from the induced vacuum expectation value of a $SU(2)_{\text{L}}$ triplet contained in a 15-dimensional representation of $SU(5)$ or from an additional contribution to the neutrino mass operator induced by $SU(5)$ singlet messenger fields in the triplet representation of A_4. Since the type I seesaw contribution involves very hierarchical additional neutrino mass contributions, responsible for the tiny mass splittings between the quasi-degenerate neutrinos, we use the constrained sequential dominance mechanism which is well suited for achieving such strongly hierarchical contributions. In such a model the neutrino flavor symmetry associated with tribimaximal mixing is achieved indirectly from the family symmetry, given the assumed vacuum alignment. We have included renormalization group corrections to the mixing angles and shown that they are under control for small values of $\tan\beta$ in our framework and do not significantly modify the leptonic mixing angles. We have therefore considered small values of $\tan\beta$ throughout our study.

The model has several interesting phenomenological features which emerge from our numerical fit to the quark and lepton masses and quark mixing. In addition to b-τ unification that may be viable for small $\tan\beta$, the model realizes the GUT scale relation $y_\mu / y_s \approx 9/2$ proposed in [16], which is favorable for small values of $\tan\beta$. In the quark sector, we observe that the correct δ_{CKM} (corresponding to the right unitarity triangle with $\alpha \approx 90°$) can be realized for the simple ansatz in Eq. (2.10). We remark that this simple ansatz leads to an interesting alternative texture (different from [29]) that gives rise to a right-angled unitarity triangle with $\alpha \approx 90°$. For the leptonic mixing angles and CP phases we find $\theta_{13}^{\text{MNS}} \approx 3.3°$, $\theta_{12}^{\text{MNS}} \approx 35.1°$, $\theta_{23}^{\text{MNS}} \approx 45.0°$, and $\delta_{\text{MNS}} \approx 90°$. The leptonic mixing angles satisfy the "lepton mixing sum rule" proposed in [24] with only small theoretical errors since the 1–3 mixing in the charged lepton mass matrix is very small and the 2–3 mixing vanishes. The Majorana CP phases are small for quasidegenerate neutrino masses via a large additional contribution proportional to the unit matrix, as expected, and the model thus predicts the neutrinoless double beta decay mass observable to be approximately equal to the neutrino mass scale, or lightest neutrino mass, i.e. $m_{ee} \approx m_{\text{lightest}}$.

In conclusion, if neutrinoless double beta decay were observed in the near future, then this would herald another neutrino revolution in which neutrino masses would be quasidegenerate. Among the many possible models of quasidegenerate neutrinos, the model with an additional contribution to the neutrino mass matrix proportional to the unit matrix, as considered here, is distinguished by the prediction that the neutrinoless double beta decay mass observable is approximately equal to the neutrino mass scale m_{lightest}. We have proposed the first realistic model of this kind involving A_4 and SUSY $SU(5)$ GUTs.

The A_4 family symmetry has the dual effect of enforcing, on the one hand, quasidegeneracy via the additional unit matrix contribution to the neutrino mass matrix and, on the other hand, tribimaximal mixing via the type I seesaw mechanism and constrained sequential dominance. A numerical fit to quark masses and mixing angles reveals that a simple ansatz describing quark CP violation with $\alpha \approx 90°$ also leads to the leptonic phase $\delta_{\mathrm{MNS}} \approx 90°$. In such models the absolute neutrino masses could be directly measurable by experiment quite soon.

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APPENDIX A: THE RENORMALIZABLE SUPERPOTENTIAL AND EFFECTIVE COUPLINGS

With the field content and symmetries specified in Table I the superpotential contains the following renormalizable terms:

$$W_H = \mu_5 H_5 \bar{H}_5 + \mu_{15} H_{15} \bar{H}_{15} + \mu_{45} H_{45} \bar{H}_{45} + \bar{\lambda}_{15} \bar{H}_{15} H_5 H_5 + \lambda_{15} H_{15} \bar{H}_5 \bar{H}_5, \tag{A1}$$

$$W_A = M_{A_{10}} A_{10} \bar{A}_{10} + M_{A_5} A_5 \bar{A}_5 + M_{A_1} A_1^2 + M_B B \bar{B} + M_{C_1} \bar{C}_1 C_1 + M_{C_2} \bar{C}_2 C_2, \tag{A2}$$

$$\begin{aligned} W_{\mathrm{int}} = {} & \kappa_F F \bar{H}_5 A_{10} + \tilde{\kappa}_F F \tilde{\phi}_{23} A_5 + \kappa_{Ti} T_i \phi_i \bar{A}_{10} \\ & + \tilde{\kappa}_{T2} T_2 \bar{H}_{45} \bar{A}_5 + y_\Delta H_{15} F F + \lambda H_5 A_{10} \bar{A}_{10} \\ & + \tilde{\lambda} H_5 \bar{C}_2 \bar{B} + \kappa'_T B T_2^2 + \kappa'_\phi C_2 \phi_{123}^2 + \tilde{\kappa}'_\phi C_2 \tilde{\phi}_{23}^2 \\ & + a_t H_5 T_3^2 + \bar{\kappa}_F F \bar{H}_5 A_1 + \kappa_{Ni} N_i \phi_i A_1 + \xi_{\phi i} C_i \phi_i^2 \\ & + \xi_{Ni} \bar{C}_i N_i^2. \end{aligned} \tag{A3}$$

As discussed in the main text, after GUT symmetry breaking the $SU(2)_L$ doublet components from H_5 and H_{45}, respectively, $\bar{H}_5$ and $\bar{H}_{45}$ mix and only the light states acquire the $SU(2)_L$ breaking VEVs which give the fermion masses. We parameterize the Higgs mixing with the mixing angles γ, respectively, $\bar{\gamma}$:

$$\begin{pmatrix} H_5 \\ H_{45} \end{pmatrix} = \begin{pmatrix} c_\gamma & -s_\gamma \\ s_\gamma & c_\gamma \end{pmatrix} \begin{pmatrix} H_l \\ H_h \end{pmatrix}, \quad \begin{pmatrix} \bar{H}_5 \\ \bar{H}_{45} \end{pmatrix} = \begin{pmatrix} c_{\bar{\gamma}} & -s_{\bar{\gamma}} \\ s_{\bar{\gamma}} & c_{\bar{\gamma}} \end{pmatrix} \begin{pmatrix} \bar{H}_l \\ \bar{H}_h \end{pmatrix}, \tag{A4}$$

where we have used the common abbreviation $c_\gamma \equiv \cos\gamma$ and similar for the sine and the other angle $\bar{\gamma}$. The light Higgs doublets are denoted with an index l while the heavy Higgs doublets are denoted with an index h.

The effective couplings a appearing in the effective superpotential in the main text can be expressed in term of the fundamental couplings from Eq. (A3) and the messenger masses from (A2):

$$a_i = \kappa_F \kappa_{Ti}, \qquad \tilde{a}_2 = \tilde{\kappa}_F \tilde{\kappa}_{Ti}, \tag{A5}$$

$$\begin{aligned} a_{11} &= \lambda \kappa_{T1}^2, \\ a_{22} &= \lambda \kappa_{T2}^2 + \tilde{\lambda} \kappa'_T \kappa'_\phi \frac{M_{A_{10}}^2}{M_B M_{C_2}}, \\ a_{33} &= a_t + \frac{\lambda \kappa_{T3}^2}{M_{A_{10}}^2}, \end{aligned} \tag{A6}$$

$$\begin{aligned} \tilde{a}_{22} &= \tilde{\lambda} \kappa'_T \tilde{\kappa}'_\phi \frac{M_{A_5}^2}{M_B M_{C_2}}, \\ a_{ij} &= \lambda \kappa_{Ti} \kappa_{Tj}, \quad \text{for } i \neq j, \\ a_{\nu_i} &= \kappa_{Ni} \bar{\kappa}_F \frac{M_{A_{10}}}{M_{A_1}}, \end{aligned} \tag{A7}$$

$$a_{R_i} = \kappa_{Ni}^2 \frac{M_{A_{10}}^2}{M_{A_1}} + \xi_{\phi_i} \xi_{Ni} \frac{M_{A_{10}}^2}{M_{C_i}}, \qquad a_{R_{12}} = \kappa_{Ni} \kappa_{Nj} \frac{M_{A_{10}}^2}{M_{A_1}}. \tag{A8}$$

The effective couplings b appearing in the Yukawa couplings can be expressed in terms of the couplings a and the Higgs mixing angles as

$$b_i = c_{\bar{\gamma}} a_i, \qquad \tilde{b}_2 = s_{\bar{\gamma}} \tilde{a}_2, \tag{A9}$$

$$b_{ij} = \frac{c_\gamma}{c_{\bar{\gamma}}^2} \frac{a_{ij}}{a_i a_j}, \qquad \tilde{b}_{22} = \frac{c_\gamma}{s_{\bar{\gamma}}^2} \frac{\tilde{a}_{22}}{\tilde{a}_2^2}, \qquad b_{33} = c_\gamma a_t + \frac{c_\gamma}{c_{\bar{\gamma}}^2} \frac{\epsilon_3^2}{a_3^2}, \tag{A10}$$

$$b_{\nu_i} = \frac{c_\gamma}{c_{\bar{\gamma}}} \frac{a_{\nu_i}}{a_i}, \tag{A11}$$

$$b_{R_i} = \frac{1}{c_{\bar{\gamma}}^2} \frac{a_{R_i}}{a_i^2}, \qquad b_{R_{12}} = \frac{1}{c_{\bar{\gamma}}^2} \frac{a_{R_{12}}}{a_1 a_2}. \tag{A12}$$

APPENDIX B: A POSSIBLE VACUUM ALIGNMENT

In this appendix, we will discuss a possibility to extend the model in the main part by a viable vacuum alignment.

This possibility is based on realizing the flavon $\tilde{\phi}_{23}$ effectively by splitting it up into two flavons $\tilde{\phi}_2$ and $\tilde{\phi}_3$, which get purely real or purely imaginary VEVs by a vacuum alignment as described below, i.e.

$$\langle\tilde{\phi}_{23}\rangle = \langle\tilde{\phi}_2\rangle + \langle\tilde{\phi}_3\rangle, \quad \text{where} \quad \langle\tilde{\phi}_2\rangle = \begin{pmatrix} 0 \\ -i \\ 0 \end{pmatrix}\tilde{\epsilon}_{23} \quad \text{and} \quad \langle\tilde{\phi}_3\rangle = \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix}\tilde{\epsilon}_{23}. \tag{B1}$$

For the effective superpotential in Eqs. (2.2), (2.3), (2.4), (2.5), (2.6), and (2.7) this would mean to simply replace $\tilde{\phi}_{23}^2 \to \tilde{\phi}_2^2 + \tilde{\phi}_3^2$ and $\tilde{\phi}_{23}\bar{H}_{45} \to \tilde{\phi}_2\bar{H}_{45} + \tilde{\phi}_3\bar{H}'_5$ with an additional Higgs field H'_5. The field content and the symmetries of this extended version of the model is given in Table IV. We note that, at the level of precision discussed here, the model predictions are the same as in the main part of the paper.

We now turn to the discussion of the vacuum alignment sector with the desired purely real or imaginary alignment (see also [30]). At the effective theory level the new operators involving the "driving fields" P_i, D_i, and O_i that are generated by the fields and symmetries specified in Table IV are given by [dropping $\mathcal{O}(1)$ coupling constants]

$$W_{\tilde{\phi}} = \tilde{P}_i\left(\frac{\tilde{\phi}_i^4}{M_{C_i}^2} - \tilde{M}_i^2\right) + \tilde{D}_i(\tilde{\phi}_i \star \tilde{\phi}_i) + \tilde{O}_{ij}\tilde{\phi}_i\tilde{\phi}_j, \tag{B2}$$

$$W_{\phi_{123}} = D_{123}(\tilde{\xi}\phi_{123} + \phi_{123} \star \phi_{123}) + P_{123}\left(\frac{\phi_{123}^4}{M_{C_{123}}^2} + \frac{\tilde{\xi}^4}{M_{C_{123}}^2} + \frac{\phi_{123}^2\tilde{\xi}^2}{M_{C_{123}}^2} - M_{123}^2\right), \tag{B3}$$

TABLE IV. Representations and charges of the superfields. The subscript i on the fields T_i and N_i is a family index, while for $\tilde{P}_i$ and P_i the i matches the corresponding flavon field. The flavon fields ϕ_i and $\tilde{\phi}_{23}$ can be associated to a family via their charges under $\mathbb{Z}_2 \times \mathbb{Z}_4^4$. The subscripts on the Higgs fields H and $\bar{H}$ and extra vectorlike matter fields A and $\bar{A}$ denote the transformation properties under $SU(5)$.

	$SU(5)$	A_4	$\mathbb{Z}_2$	$\mathbb{Z}_4^{(1)}$	$\mathbb{Z}_4^{(2)}$	$\mathbb{Z}_4^{(3)}$	$\mathbb{Z}_4^{(4)}$	$U(1)_R$
Chiral matter								
F	$\bar{\mathbf{5}}$	**3**	+	0	0	0	0	1
T_1, T_2, T_3	**10, 10, 10**	**1, 1, 1**	+, +, −	0, 1, 0	0, 0, 0	0, 0, 0	1, 0, 0	1, 1, 1
N_1, N_2	**1, 1**	**1, 1**	+, +	0, 1	0, 0	0, 0	1, 0	1, 1
Flavons								
$\phi_{23}, \phi_{123}, \phi_3$	**1, 1, 1**	**3, 3, 3**	+, +, −	0, 3, 0	0, 0, 0	0, 0, 0	3, 0, 0	0, 0, 0
$\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3$	**1, 24, 1**	**3, 3, 3**	−, +, +	3, 3, 3	3, 3, 0	3, 0, 3	0, 0, 0	0, 0, 0
$\tilde{\xi}$	**1, 1**	**1, 1**	+	1, 3	0, 0	0, 0	0, 0	0, 0
Higgs multiplets								
$H_5, \bar{H}_5$	**5**, $\bar{\mathbf{5}}$	**1, 1**	+, +	0, 0	0, 0	0, 0	0, 0	0, 0
$H'_5, \bar{H}'_5$	**5**, $\bar{\mathbf{5}}$	**1, 1**	+, +	0, 0	0, 0	3, 1	0, 0	0, 0
$H_{15}, \bar{H}_{15}$	**15**, $\overline{\mathbf{15}}$	**1, 1**	+, +	0, 0	0, 0	0, 0	0, 0	0, 0
$H_{45}, \bar{H}_{45}$	**45**, $\overline{\mathbf{45}}$	**1, 1**	+, +	0, 0	3, 1	0, 0	0, 0	0, 0
Matterlike messengers								
$A_5, \bar{A}_5$	**5**, $\bar{\mathbf{5}}$	**1, 1**	+, +	1, 3	1, 3	0, 0	0, 0	1, 1
$A'_5, \bar{A}'_5$	**5**, $\bar{\mathbf{5}}$	**1, 1**	+, +	1, 3	0, 0	1, 3	0, 0	1, 1
$A_{10}, \bar{A}_{10}$	**10**, $\overline{\mathbf{10}}$	**3, 3**	+, +	0, 0	0, 0	0, 0	0, 0	1, 1
A_1	**1**	**3**	+	0	0	0	0	1
Higgs-like messengers								
$B, \bar{B}$	**5**, $\bar{\mathbf{5}}$	**1, 1**	+, +	2, 2	0, 0	0, 0	0, 0	0, 2
$C_{23}, \bar{C}_{23}$	**1, 1**	**1, 1**	+, +	0, 0	0, 0	0, 0	2, 2	2, 0
$C_{123}, \bar{C}_{123}$	**1, 1**	**1, 1**	+, +	2, 2	0, 0	0, 0	0, 0	2, 0
$C_1, \bar{C}_1$	**1, 1**	**1, 1**	+, +	2, 2	2, 2	2, 2	0, 0	2, 0
$C_2, \bar{C}_2$	**1, 1**	**1, 1**	+, +	2, 2	2, 2	0, 0	0, 0	2, 0
$C_3, \bar{C}_3$	**1, 1**	**1, 1**	+, +	2, 2	0, 0	2, 2	0, 0	2, 0
Driving fields								
$\tilde{P}_i, P_i$	**1, 1**	**1, 1**	+, +	0, 0	0, 0	0, 0	0, 0	2, 2
D_{123}, D_3	**1, 1**	**3, 3**	+, +	2, 0	0, 0	0, 0	0, 0	2, 2
$\tilde{D}_1, \tilde{D}_2, \tilde{D}_3$	**1, 1, 1**	**3, 3, 3**	+, +, +	2, 2, 2	2, 2, 0	2, 0, 2	0, 0, 0	2, 2, 2
$\tilde{O}_{12}, \tilde{O}_{13}, \tilde{O}_{23}$	**24, 1, 24**	**1, 1, 1**	−, −, +	2, 2, 2	2, 1, 1	1, 2, 1	0, 0, 0	2, 2, 2
O_{13}, O_{23}	**1, 24**	**1, 1**	−, −	1, 1	1, 1	1, 0	0, 0	2, 2
$O_{1;23}, O_{123;23}$	**1, 1**	**1, 1**	−, +	1, 1	1, 0	1, 0	0, 1	2, 2

$$W_{\phi_3} = P_3(\phi_3^2 - M_3^2) + D_3(\phi_3 \star \phi_3) + O_{13}\tilde{\phi}_1\phi_3 + O_{23}\tilde{\phi}_2\phi_3, \tag{B4}$$

$$W_{\phi_{23}} = P_{23}\left(\frac{\phi_{23}^4}{M_{C_{23}}^2} - M_{23}^2\right) + O_{1;23}\tilde{\phi}_1\phi_{23} + O_{123;23}\phi_{123}\phi_{23}. \tag{B5}$$

The minimization of the resulting F-term potential leads to the desired vacuum alignment for all flavons. We note that the new flavons $\tilde{\xi}$ and $\tilde{\phi}_1$ are only "auxiliary" in the sense that they do not couple to the matter sector but are only relevant for the flavon alignment. The flavon $\tilde{\xi}$ leads to an additional term contributing to the charm mass; however, its effect can be absorbed in the messenger masses such that the predictions of the model are unchanged. The four parts $W_{\tilde{\phi}}$, $W_{\phi_{123}}$, W_{ϕ_3}, and $W_{\phi_{23}}$ have the following meaning:

(i) The terms in $W_{\tilde{\phi}}$ enforce nonvanishing VEVs of the three flavons $\tilde{\phi}_1$, $\tilde{\phi}_2$, and $\tilde{\phi}_3$ that are orthogonal to each other and have only one nonzero element, due to the effects of the terms with the driving fields $\tilde{D}_i$. Here the $\star$ denotes the symmetric triplet combination of A_4 (see, e.g., [31]). With real M_i due to the assumed spontaneous CP violation $\tilde{\phi}_i^4$ are forced to be real which means that the VEVs $\langle\tilde{\phi}_i\rangle$ are either purely real or imaginary. We will choose a vacuum where $\tilde{\phi}_2$ is imaginary and $\tilde{\phi}_3$ is real. The VEV of $\tilde{\phi}_1$ can be either real or imaginary since its phase does not affect the results.

(ii) The part $W_{\phi_{123}}$ leads to the desired alignment for the flavon VEV $\langle\phi_{123}\rangle$ along the $(\pm 1, \pm 1, \pm 1)$ direction (see also [6]) in the minima of the potential with $\langle\phi_{123}\rangle \neq 0$ and $\langle\tilde{\xi}\rangle \neq 0$. Again, due to the ϕ_{123}^4 in the term with P_{123} the VEV can be either real or imaginary, and we choose the real vacuum in the $(1, 1, 1)$ direction.

(iii) The terms in W_{ϕ_3} and $W_{\phi_{23}}$ finally provide the alignment for the flavons ϕ_3 and ϕ_{23}. The terms with the driving fields O_i force the VEV of ϕ_3 to be orthogonal to $\langle\tilde{\phi}_1\rangle$ and $\langle\tilde{\phi}_2\rangle$ and the VEV of ϕ_{23} to be orthogonal to $\langle\tilde{\phi}_1\rangle$ and $\langle\phi_{123}\rangle$, leading to the alignments in the desired direction in flavor space. We can again choose the purely real vacuum as explained above.

We note that the driving fields P_i and $\tilde{P}_i$ have the same quantum numbers and therefore, *a priori*, a linear combination of such fields couples to each of the above terms $\tilde{\phi}_i^4/M_{C_i}^2 - \tilde{M}_i^2$ and $\phi_i^4/M_{C_i}^2 - M_{23}^2$. Without loss of generality we have written the driving fields here already in the field basis where only one driving field couples to one of these terms. This can always be achieved by a proper field redefinition.

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