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Form invariance and symmetry in the neutrino mass matrix

E. I. Lashin,[1,2,*] S. Nasri,[3,†] E. Malkawi,[3,‡] and N. Chamoun[4,§]

[1]*Department of Physics and Astronomy, College of Science, King Saud University, Riyadh, Saudi Arabia*
[2]*Ain Shams University, Faculty of Science, Cairo 11566, Egypt*
[3]*Department of Physics, UAE University, P.O. Box 17551, Al-Ain, United Arab Emirates*
[4]*Physics Department, HIAST, P.O. Box 31983, Damascus, Syria*
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We present the general form of the unitary matrices keeping invariant the Majorana neutrino mass matrix of specific texture suitable for explaining oscillation data. In the case of the tri-bimaximal pattern with two degenerate masses, we give a specific realization of the underlying $U(1)$ symmetry which can be uplifted to a symmetry in a complete theory including charged leptons. For this, we present a model with three light SM-like Higgs doublets and one heavy Higgs triplet and find that one can accommodate the hierarchy of the charged-lepton masses. The lepton mass spectrum can also be achieved in another model extending the SM with three SM-singlet scalars transforming nontrivially under the flavor symmetry. We discuss how such a model has room for generating enough baryon asymmetry through leptogenesis in the framework of type-I and -II seesaw mechanisms.

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I. INTRODUCTION

The atmospheric, solar, and reactor neutrino oscillations [1–3] have provided robust evidence that neutrinos are massive and lepton flavors are mixed. Moreover, a number of phenomenological Ansätze of lepton flavor mixing with two large rotation angles [4,5] have been proposed and discussed [6], in particular, the tri-bimaximal flavor mixing pattern [7] describes approximately well the oscillation data. The starting point is usually the assumption that there are only three neutrinos and that they are Majorana fermions. The most general neutrino mass matrix, in the flavor basis where the charged-lepton mass matrix is diagonal, is then a symmetric 3×3 matrix M_ν. Any model of neutrino mass always ends up with a simplification of M_ν, thereby reducing the number of independent parameters, and the results are fitted in order to be consistent with the experimental data.

The approach of form invariance proposed by Ma [8], substitutes this *ad hoc* procedure by the symmetry argument that the neutrino mass matrix is invariant when expressed in the flavor basis and another basis related to the former by a specific unitary transformation S:

$$S^T M_\nu S = M_\nu. \tag{1}$$

This, for certain S, imposes a particular form on M_ν which might be able to accommodate the data. The set of these S's might form discrete or continuous symmetry groups, depending on the mass spectrum, and then one can impose this symmetry on the setup so to become the underlying symmetry for the desired form of M_ν.

In this work, we seek the most general symmetry S satisfying the form invariance [Eq. (1)], and we find it by examining the invariance implication on the diagonalized neutrino mass matrix $M_\nu^{\rm diag}$ since the analysis in the latter case is simpler.

The method is applied to the phenomenologically successful tri-bimaximal pattern [7] and its realization in tripartite model [9]. When the three neutrino masses are distinct, the form of S is quite limited and has a well-defined $(Z_2)^3$ symmetry [10]. However, in the special case when two neutrino masses are almost degenerate, the symmetry is *a priori* isomorphic to the Abelian group $U(1)$ corresponding to a rotation in the degenerate mass eigenspace. We find a realization of this approximate $U(1)$ symmetry and deduce the general form of the matrix S characterizing the tripartite model with two degenerate masses, of which the Z_3 symmetry reported in [9] is a special case. Moreover, if a symmetry is behind the observed pattern of M_ν, then it must also apply to the charged-lepton mass matrix M_l. Following [9], we first introduce three Higgs scalar doublets at the electroweak scale, and one heavy Higgs triplet and find that the conditions on the Yukawa couplings necessary to accommodate the Z_3 symmetry of [9] are sufficient to enforce the approximate $U(1)$ continuous symmetry, of which we characterize the conserved current. We introduce later another model with only one Higgs doublet, but extending the standard model (SM) by three SM-singlet scalars transforming nontrivially under the flavor symmetry. Like the first model, all patterns of charged-lepton masses can be accommodated, but moreover, we examine the possibility of the model to produce enough baryogenesis, via leptogenesis, in the framework of seesaw mechanisms.

The plan of the paper is as follows. We start by some basics defining the notations in Sec. II. In Sec. III we

*elashin@ictp.it
†snasri@uaeu.ac.ae
‡EMalkawi@uaeu.ac.ae
§nidal.chamoun@hiast.edu.sy

explain the method for finding the form invariance symmetry, and we apply it to the tripartite model. We treat in Sec. IV the case of almost two degenerate masses. In Sec. V we implement the symmetry in a setup including the charged leptons with many Higgs doublets, and study the current associated with this continuous symmetry. In Sec. VI we introduce another model with additional SM-singlet scalars and study the charged-lepton mass spectrum. In Secs. VII and VIII we treat, within the model, the problems of generating the neutrino mass hierarchies and the lepton and baryon asymmetries in the framework of seesaw mechanisms. We end up by summarizing our results in Sec. IX.

II. BASIC NOTATIONS

In the standard model (SM) of particle interactions, there are three lepton families. The charged-lepton mass matrix linking left handed (e, μ, τ) to their right-handed counterparts is in general arbitrary, but may always be diagonalized by a biunitary transformation:

$$M_l = U_L^l \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} (U_R^l)^\dagger. \tag{2}$$

Similarly, the neutrino mass matrix may also be diagonalized by a biunitary transformation if it is Dirac:

$$M_\nu^D = U_L^\nu \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U_R^\nu)^\dagger, \tag{3}$$

or by one unitary transformation if it is Majorana:

$$M_\nu^M = U_L^\nu \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U_L^\nu)^T. \tag{4}$$

The observed neutrino mixing matrix is the mismatch between U_L^l and U_L^ν, i.e.,

$$U_{l\nu} = (U_L^l)^\dagger U_L^\nu \simeq \begin{pmatrix} 0.83 & 0.56 & <0.2 \\ -0.39 & 0.59 & 0.71 \\ 0.39 & -0.59 & 0.71 \end{pmatrix} \simeq \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \tag{5}$$

This approximate pattern has been dubbed tri-bimaximal by Harrison, Perkins, and Scott [7].

If we work in the flavor basis where M_l is diagonal, thus $U_L^l = \mathbf{1}$ be a unity matrix, and assume the neutrinos are of Majorana type, then the flavor mixing matrix is simplified to $V = U_L^\nu$, and so, with $M_\nu^{\rm diag} = {\rm Diag}(m_1, m_2, m_3)$, we have

$$M_\nu = V M_\nu^{\rm diag} V^T. \tag{6}$$

The tri-bimaximal neutrino mixing pattern can be obtained as follows. First, we consider the product of two Euler rotation matrices:

$$R_{12}(\theta_x) = \begin{pmatrix} c_x & s_x & 0 \\ -s_x & c_x & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{23}(\theta_y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_y & s_y \\ 0 & -s_y & c_y \end{pmatrix} \tag{7}$$

(with $s_x \equiv \sin\theta_x$, $c_y \equiv \cos\theta_y$, and so on). We then fix θ_y to be equal to the "maximal mixing" angle $\theta_y = 45°$, getting the mixing matrix:

$$V = R_{23}\left(\theta_y = \frac{\pi}{4}\right) \quad R_{12}(\theta_x) = \begin{pmatrix} c_x & s_x & 0 \\ -\frac{s_x}{\sqrt{2}} & \frac{c_x}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s_x}{\sqrt{2}} & -\frac{c_x}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{8}$$

The neutrino mass matrix takes then the form

$$M_\nu = \begin{pmatrix} A'_\nu - B'_\nu + C'_\nu & (1/4)\sqrt{2}\tan(2\theta_x)C'_\nu & -(1/4)\sqrt{2}\tan(2\theta_x)C'_\nu \\ (1/4)\sqrt{2}\tan(2\theta_x)C'_\nu & A'_\nu + C'_\nu & B'_\nu - C'_\nu \\ -(1/4)\sqrt{2}\tan(2\theta_x)C'_\nu & B'_\nu - C'_\nu & A'_\nu + C'_\nu \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned} A'_\nu &= -(3/4)\cos(2\theta_x)(m_2 - m_1) + (1/4)(m_2 + m_1) + (1/2)m_3, \\ B'_\nu &= -(1/4)(m_2 + m_1) + (3/4)\cos(2\theta_x)(m_2 - m_1) + (1/2)m_3 \\ C'_\nu &= \cos(2\theta_x)(m_2 - m_1). \end{aligned} \tag{10}$$

In consequence, any "measurable" mixing angle θ_x can be obtained in this way; however, the experimentally measured x-mixing angle in the tri-bimaximal pattern can be characterized as being the mixing angle which makes the terms involving C'_ν in Eq. (9), proportional to the mass difference $m_2 - m_1$, constitute a "democratic" perturbation on the form of the mass matrix when $m_1 = m_2$. This happens when $\theta_x = \arctan(1/\sqrt{2}) \approx 35.3°$ leading to

$$V_0 = R_{23}\left(\theta_y = \frac{\pi}{4}\right)$$
$$R_{12}\left(\theta_x = \arctan\left(\frac{1}{\sqrt{2}}\right)\right) = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (11)$$

The vanishing of the (1, 3) element in V_0 assures an exact decoupling between solar ($\nu_e \rightarrow \nu_\mu$) and atmospheric ($\nu_\mu \rightarrow \nu_\tau$) neutrino oscillations, and the neutrino mass matrix of Eq. (9) takes the form

$$M_\nu = V_0 \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} V_0^{\mathrm{T}}$$
$$= \begin{pmatrix} A_\nu - B_\nu + C_\nu & C_\nu & -C_\nu \\ C_\nu & A_\nu + C_\nu & B_\nu - C_\nu \\ -C_\nu & B_\nu - C_\nu & A_\nu + C_\nu \end{pmatrix}, \quad (12)$$

with

$$A_\nu = \frac{m_3 + m_1}{2}, \qquad B_\nu = \frac{m_3 - m_1}{2}, \qquad C_\nu = \frac{m_2 - m_1}{3}. \quad (13)$$

The form of Eq. (12) is, thus, phenomenologically desirable and the question arises as to whether or not there is a guiding principle, say a symmetry, leading to it. One of the ways to have M_ν of a given form is to impose a form invariance condition [Eq. (1)] for certain unitary S, and our aim is to find the most general form for these unitary matrices S, which can then be uplifted to symmetries underlying the specific form of M_ν.

III. DETERMINING THE FORM INVARIANCE SYMMETRY FOR THE TRI-BIMAXIMAL PATTERN—METHOD

In order to find the symmetry that imposes the form invariance property on a given mass matrix M_ν, we see that Eq. (1), using Eq. (6), is equivalent to

$$U^T M_\nu^{\mathrm{diag}} U = M_\nu^{\mathrm{diag}}, \quad (14)$$

where U is a unitary matrix related to S by

$$S = V_0^* U V_0^T. \quad (15)$$

Thus any "symmetry" U for the diagonalized form can appear as a symmetry S in the flavor basis. Writing Eq. (14) as

$$\left[\sqrt{M_\nu^{\mathrm{diag}}}\, U \left(\sqrt{M_\nu^{\mathrm{diag}}}\right)^{-1}\right]^T \left[\sqrt{M_\nu^{\mathrm{diag}}}\, U \left(\sqrt{M_\nu^{\mathrm{diag}}}\right)^{-1}\right] = 1, \quad (16)$$

where

$$\sqrt{M_\nu^{\mathrm{diag}}} = \begin{pmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{pmatrix},$$

we see that the general form of U is

$$U = \left(\sqrt{M_\nu^d}\right)^{-1} O \sqrt{M_\nu^d}, \quad (17)$$

where O is any 3×3 orthogonal matrix. The set of matrices U defined in Eq. (17) forms a group under matrix multiplication. However, the unitarity condition on U imposes, for real matrices O, the following "vanishing commutator" condition on O:

$$[O, M_\nu^{\mathrm{diag}}] = 0. \quad (18)$$

This condition does eliminate most members of the orthogonal group $O(3)$, except few discrete subgroups such as $(Z_2)^3$ for the case of nondegenerate neutrino mass spectrum, as was shown in [10]. However, in the case of degenerate spectrum there is room for few continuous subgroups to remain as we shall see now.

IV. APPLICATION TO THE TRIPARTITE MODEL WITH TWO DEGENERATE MASSES

Let us consider here the case of an almost degenerate mass spectrum $m_1 \simeq m_2$, where their actual difference [the C_ν part in Eq. (12)] can be treated as a perturbation originating from higher order operators, then M_ν is of the form

$$M_\nu = \begin{pmatrix} A_\nu - B_\nu & 0 & 0 \\ 0 & A_\nu & B_\nu \\ 0 & B_\nu & A_\nu \end{pmatrix}, \quad (19)$$

with $A_\nu = \frac{m_3+m_1}{2}$, $B_\nu = \frac{m_3-m_1}{2}$. Any rotation V corresponding to θ_y being fixed at $\frac{\pi}{4}$ and θ_x arbitrary [Eq. (8)] will diagonalize M_ν.

In [9], a symmetry ($Z_3 \times Z_2$) for the form (19), which determines it uniquely, was given:

$$S_B = \begin{pmatrix} -1/2 & -\sqrt{3/8} & \sqrt{3/8} \\ \sqrt{3/8} & 1/4 & 3/4 \\ -\sqrt{3/8} & 3/4 & 1/4 \end{pmatrix} : S_B^3 = \mathbf{1},$$
$$S_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} : S_2^2 = 1. \quad (20)$$

However, from Sec. II we find that the general symmetry enforcing the form (19), which corresponds to two degenerate masses m_1 and m_2, would correspond, provided m_3 is different from the common degenerate mass in accordance with the experimental data, to an orthogonal matrix O, in Eq. (17) and satisfying the condition (18), of the form

$$O = \begin{pmatrix} O_2 & & 0 \\ & & 0 \\ 0 & 0 & \pm 1 \end{pmatrix},$$

where O_2 is an isometry in the x-y plane. To fix the ideas, we restrict our symmetry here to the connected component of the unity, which are the rotations in the x-y plane, and get:[1]

$$U(\theta) = \begin{pmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (21)$$

We get thus the general symmetry

$$S_\theta = V(\theta_x)U(\theta)V(\theta_x)^T = \begin{pmatrix} c_\theta & \frac{s_\theta}{\sqrt{2}} & -\frac{s_\theta}{\sqrt{2}} \\ -\frac{s_\theta}{\sqrt{2}} & \frac{1}{2}(1+c_\theta) & \frac{1}{2}(1-c_\theta) \\ \frac{s_\theta}{\sqrt{2}} & \frac{1}{2}(1-c_\theta) & \frac{1}{2}(1+c_\theta) \end{pmatrix}. \quad (22)$$

Note that the mixing angle θ_x can be taken arbitrary here since it is not determined in the degenerate masses case. We can check that S_θ determines uniquely the form (19), i.e. a necessary and sufficient condition for a matrix to be of the form [Eq. (19)] is to be symmetric and invariant under the symmetry S_θ for all angles θ:

$$[(M = M^T) \wedge (\forall\ \theta, S_\theta^T M S_\theta = M)] \Leftrightarrow \left[\exists\ A, B{:}M = \begin{pmatrix} A-B & 0 & 0 \\ 0 & A & B \\ 0 & B & A \end{pmatrix}\right]. \quad (23)$$

We see also that the Z_2 and Z_3 of [9] are particular subgroups of this $U(1)$ symmetry, for $S_{-(2\pi/3)} = S_B$ and $S_\pi = S_2$. The S_θ is a three-dimensional representation, albeit "reducible," of the group $U(1)$ in that $S_{\theta_1+\theta_2} = S_{\theta_1}S_{\theta_2}$.

Here, some remarks are in order. First, had we dropped the requirement of the matrix M being symmetric in the form invariance condition, then the symmetry S_θ would impose the following form on M:

$$[(\forall\ \theta, S_\theta^T M S_\theta = M)] \Leftrightarrow \left[\exists\ A, B, C{:}M = \begin{pmatrix} A-B & -C & C \\ C & A & B \\ -C & B & A \end{pmatrix}\right]. \quad (24)$$

Second, we have neglected the Majorana phases, on which there are no experimental bounds, in the discussion so far. By introducing the mixing angles $(\theta_x, \theta_y, \theta_z)$ and the phases (δ, ρ, σ), the unitary matrix V, which diagonalizes the symmetric neutrino mass matrix M_ν in the flavor basis, can be parametrized as follows:

$$V = UP \quad (25)$$

$$P = \mathrm{diag}(e^{i\rho}, e^{i\sigma}, 1),$$

$$U = \begin{pmatrix} c_x c_z & s_x c_z & s_z \\ -c_x s_y s_z - s_x c_y e^{-i\delta} & -s_x s_y s_z + c_x c_y e^{-i\delta} & s_y c_z \\ -c_x c_y s_z + s_x s_y e^{-i\delta} & -s_x c_y s_z - c_x s_y e^{-i\delta} & c_y c_z \end{pmatrix} \quad (26)$$

and Eq. (6) gives the following matrix elements:

$$\begin{aligned}
M_{\nu 11} &= m_1 c_x^2 c_z^2 e^{2i\rho} + m_2 s_x^2 c_z^2 e^{2i\sigma} + m_3 s_z^2, \\
M_{\nu 12} &= m_1(-c_z s_z c_x^2 s_y e^{2i\rho} - c_z c_x s_x c_y e^{i(2\rho-\delta)}) + m_2(-c_z s_z s_x^2 s_y e^{2i\sigma} + c_z c_x s_x c_y e^{i(2\sigma-\delta)}) + m_3 c_z s_z s_y, \\
M_{\nu 13} &= m_1(-c_z s_z c_x^2 s_y e^{2i\rho} + c_z c_x s_x s_y e^{i(2\rho-\delta)}) + m_2(-c_z s_z s_x^2 c_y e^{2i\sigma} - c_z c_x s_x s_y e^{i(2\sigma-\delta)}) + m_3 c_z s_z c_y, \\
M_{\nu 22} &= m_1(c_x s_z s_y e^{i\rho} + c_y s_x e^{i(\rho-\delta)})^2 + m_2(s_x s_z s_y e^{i\sigma} - c_y c_x e^{i(\sigma-\delta)})^2 + m_3 c_z^2 s_y^2, \\
M_{\nu 33} &= m_1(c_x s_z c_y e^{i\rho} - s_y s_x e^{i(\rho-\delta)})^2 + m_2(s_x s_z c_y e^{i\sigma} + s_y c_x e^{i(\sigma-\delta)})^2 + m_3 c_z^2 c_y^2, \\
M_{\nu 23} &= m_1(c_x^2 c_y s_y s_z^2 e^{2i\rho} + s_z c_x s_x (c_y^2 - s_y^2) e^{i(2\rho-\delta)} - c_y s_y s_x^2 e^{2i(\rho-\delta)}) + m_2(s_x^2 c_y s_y s_z^2 e^{2i\sigma} + s_z c_x s_x (s_y^2 - c_y^2) e^{i(2\sigma-\delta)} \\
&\quad - c_y s_y c_x^2 e^{2i(\sigma-\delta)}) + m_3 s_y c_y c_z^2.
\end{aligned} \quad (27)$$

One can see now that the S_θ symmetry imposing the tripartite model with two degenerate masses ($m_1 = m_2$) can only accommodate the special case of two equal Majorana phases ($\rho = \sigma$) and a vanishing Dirac phase ($\delta = 0$). In fact, as we saw in Eq. (23), the S_θ symmetry imposes the form of Eq. (19) with A_ν and B_ν complex in general. We can diagonalize this matrix by writing

$$\begin{pmatrix} A_\nu - B_\nu & 0 & 0 \\ 0 & A_\nu & B_\nu \\ 0 & B_\nu & A_\nu \end{pmatrix} = V \begin{pmatrix} |A_\nu - B_\nu| & 0 & 0 \\ 0 & |A_\nu - B_\nu| & 0 \\ 0 & 0 & |A_\nu + B_\nu| \end{pmatrix} V^T \quad (28)$$

[1]In general, the group U would be generated by the rotations and reflections: $U = \langle R_{12}(\theta), I_x, I_y, I_z \rangle$.

$$V = UP, \qquad P = \mathrm{diag}(e^{i\alpha}, e^{i\alpha}, e^{i\beta}),$$
$$U = R_{23}\left(\theta_y = \frac{\pi}{4}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{29}$$

$$2\alpha = \mathrm{Arg}(A_\nu - B_\nu), \qquad 2\beta = \mathrm{Arg}(A_\nu + B_\nu). \tag{30}$$

We can set $\beta = 0$ by an "unphysical" global phase shift of the neutrino fields ($\psi_i \rightarrow e^{-i\beta}\psi_i$) in the neutrino mass term $[(M_\nu)_{ij}\psi_i\psi_j]$, so that to end up, when comparing with Eq. (26), with the case ($\theta_y = \frac{\pi}{4}$, $\theta_z = 0$, $\rho = \sigma$, $\delta = 0$).

Conversely, one can see directly from the expressions in Eq. (27) that if we restrict to the degenerate case ($m_1 = m_2$), and impose $\theta_y = \frac{\pi}{4}$ (suggested by atmospheric mixing data [1,2]) and $\theta_z = 0$ (suggested by reactor data [3]), while leaving θ_x free, we get the expressions

$$\begin{aligned} M_{\nu 11} &= m_1(c_x^2 e^{2i\rho} + s_x^2 e^{2i\sigma}), \\ M_{\nu 12} &= \frac{m_1 c_x s_x e^{-i\delta}}{\sqrt{2}}(-e^{2i\rho} + e^{2i\sigma}), \\ M_{\nu 13} &= \frac{m_1 c_x s_x e^{-i\delta}}{\sqrt{2}}(e^{2i\rho} - e^{2i\sigma}), \\ M_{\nu 22} &= \frac{m_1 e^{-2i\delta}}{2}(s_x^2 e^{2i\rho} + c_x^2 e^{2i\sigma}) + \frac{m_3}{2}, \\ M_{\nu 33} &= \frac{m_1 e^{-2i\delta}}{2}(s_x^2 e^{2i\rho} + c_x^2 e^{2i\sigma}) + \frac{m_3}{2}, \\ M_{\nu 23} &= \frac{m_1 e^{-2i\delta}}{2}(-s_x^2 e^{2i\rho} - c_x^2 e^{2i\sigma}) + \frac{m_3}{2}. \end{aligned} \tag{31}$$

It is clear now that we have the S_θ-symmetry form of Eq. (19) if and only if we have ($\rho = \sigma$ and $\delta = 0$), as claimed.

V. LEPTON FAMILY SYMMETRY IN PRESENCE OF MANY HIGGS DOUBLETS

Any symmetry defined in the basis (ν_e, ν_μ, ν_τ) is automatically applicable to (e, μ, τ) in the complete Lagrangian, and one should verify that the symmetry S_θ can be imposed in a complete theory including the charged leptons. We follow the approach of [9] and extend the standard model of particle interactions to include three scalar doublets (ϕ_i^0, ϕ_i^-), playing the role of the ordinary SM-Higgs field, and one very heavy triplet (ξ^{++}, ξ^+, ξ^0). The leptonic Yukawa Lagrangian is given by

$$\begin{aligned} \mathcal{L}_Y = h_{ij}[\xi^0\nu_i\nu_j - \xi^+(\nu_i l_j + l_i\nu_j)/\sqrt{2} + \xi^{++} l_i l_j] \\ + f_{ij}^k(l_i\phi_j^0 - \nu_i\phi_j^-)l_k^c + \mathrm{H.c.}, \end{aligned} \tag{32}$$

where, under the S_θ transformation,

$$(\nu, l)_i \rightarrow (S_\theta)_{ij}(\nu, l)_j, \qquad l_k^c \rightarrow l_k^c, \tag{33}$$

$$\begin{aligned} (\phi^0, \phi^-)_i \rightarrow (S_\theta)_{ij}(\phi^0, \phi^-)_j, \\ (\xi^{++}, \xi^+, \xi^0) \rightarrow (\xi^{++}, \xi^+, \xi^0). \end{aligned} \tag{34}$$

This means

$$S_\theta^T h S_\theta = h \tag{35}$$

$$S_\theta^T f^k S_\theta = f^k. \tag{36}$$

Thus, this Lagrangian has the global symmetry $U(1)_L \otimes S_\theta$, where $U(1)_L$ is associated with total lepton number.[2] However, in order to avoid having Goldstone bosons (majorons) in the theory, when ξ^0 gets a vacuum expectation value (vev) breaking spontaneously the $U(1)_L$ symmetry, we add the following soft symmetry breaking term:

$$\delta\mathcal{L}_{\mathcal{Y}} = \frac{\mu_{ij}}{2}\phi_i^T\xi^\dagger i\tau_2\phi_j + \mathrm{H.c.}, \tag{37}$$

where μ_{ij} is not proportional to the identity δ_{ij}, so that the $U(1)$-symmetry S_θ symmetry is broken explicitly as well in order not to have a corresponding "majoron" when the ϕ's take a vev. Assuming that the triplet mass square (M_ξ^2) is positive, then the minimization of the potential with respect to the field ξ gives

$$\langle\xi\rangle = \frac{-\mu_i\mu_j v_i v_j}{M_\xi^2} \tag{38}$$

which can be naturally in the electron volt range for $\mu_{ij} \sim M_\xi \sim 10^{12}$ GeV [11]. The coexistence of two types of final states for ξ^{++}, $l_i^+ l_i^+$ from $\mathcal{L}_Y$ and $\phi_i^+\phi_i^+$ from $\delta\mathcal{L}_{\mathcal{Y}}$, indicates a nonconservation of lepton number. However, one needs also to impose CP violation in out-of-thermal equilibrium decays to ensure that the lepton asymmetry generated by ξ^{++} is not neutralized by the decays of ξ^{--}.

Now, since h is a symmetric matrix then the relations [Eqs. (23) and (35)] lead to

$$h = \begin{pmatrix} a-b & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix}. \tag{39}$$

As to Eq. (36), it has a general solution for f^k in the form [see Eq. (24)]

$$f^k = \begin{pmatrix} a_k - b_k & c_k & -c_k \\ -c_k & a_k & b_k \\ c_k & b_k & a_k \end{pmatrix}, \tag{40}$$

where the coefficients a_k, b_k, and c_k can be any complex numbers. It is noteworthy that the solution in Eq. (40)

[2] We assign a zero lepton number to the doublets ϕ_i, and a lepton number of two units to the heavy triplet ξ.

represents also the general solution of any invariant matrix f^k under Z_3 (i.e. $S_B^T f^k S_B = f^k$). We thus conclude that the underlying symmetry of the Lagrangian [Eq. (32)] presented in [9] is the $U(1)$ symmetry S_θ, and that the phenomenological analysis therein assuming $Z_3 \times Z_2$ symmetry does apply herein with the $U(1)$ symmetry. In fact, when the Higgses get vevs we have the neutrino and charged-lepton "gauge" mass matrices as follows:

$$(M_\nu)_{ij} = \langle \xi^0 \rangle h_{ij} \tag{41}$$

and

$$M_l = \begin{pmatrix} (a_1 - b_1)v_1 + c_1(v_2 - v_3) & (a_2 - b_2)v_1 + c_2(v_2 - v_3) & (a_3 - b_3)v_1 + c_3(v_2 - v_3) \\ -c_1 v_1 + a_1 v_2 + b_1 v_3 & -c_2 v_1 + a_2 v_2 + b_2 v_3 & -c_3 v_1 + a_3 v_2 + b_3 v_3 \\ c_1 v_1 + b_1 v_2 + a_1 v_3 & c_2 v_1 + b_2 v_2 + a_2 v_3 & c_3 v_1 + b_3 v_2 + a_3 v_3 \end{pmatrix}, \tag{42}$$

where $v_i \equiv \langle \phi_i^0 \rangle$. The neutrino mass matrix is proportional to a single vev and this translates the $U(1)$ symmetry S_θ from the Yukawa couplings to the neutrino mass matrix. One can arrange for the vevs and the Yukawa couplings such that M_l, after suitably rotating the charged right-handed singlet leptons l^c, is the charged-lepton mass matrix in the flavor space, where $(M_l M_l^\dagger)$ is diagonal. For example, if $v_{1,2} \ll v_3$ then we have

$$M_l \approx v_3 \begin{pmatrix} -c_1 & -c_2 & -c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{pmatrix}. \tag{43}$$

As the determinant of this M_l is proportional to $v_3^2 \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, where $\mathbf{a}$ is the complex vector of components a_i (similarly for $\mathbf{b}$, $\mathbf{c}$), we conclude that a nonsingular lepton mass matrix should correspond to noncoplanar vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. We get then

$$M_l M_l^\dagger \approx v_3^2 \begin{pmatrix} \|\mathbf{c}\|^2 & -\mathbf{c}\cdot\mathbf{b}^* & -\mathbf{c}\cdot\mathbf{a}^* \\ -\mathbf{b}\cdot\mathbf{c}^* & \|\mathbf{b}\|^2 & \mathbf{b}\cdot\mathbf{a}^* \\ -\mathbf{a}\cdot\mathbf{c}^* & \mathbf{a}\cdot\mathbf{b}^* & \|\mathbf{a}\|^2 \end{pmatrix}, \tag{44}$$

where $\|\mathbf{c}\|^2 = \mathbf{c}\cdot\mathbf{c}^*$ (idem for $\mathbf{a}$, $\mathbf{b}$) and the usual Hermitian product of two complex vectors ($\mathbf{a}$ and $\mathbf{b}$) is defined as $\mathbf{a}\cdot\mathbf{b}^* \equiv \sum_{i=1}^3 a_i b_i^*$.

In order to show that M_l can naturally represent the lepton mass matrix in the flavor space, let us just assume the magnitudes of the three vectors coming in ratios comparable to the lepton mass ratios:

$$\begin{aligned} \frac{\|\mathbf{c}\|}{\|\mathbf{a}\|} &= \lambda_e \equiv \frac{m_e}{m_\tau} \sim 3 \times 10^{-4}, \\ \frac{\|\mathbf{b}\|}{\|\mathbf{a}\|} &= \lambda_\mu \equiv \frac{m_\mu}{m_\tau} \sim 6 \times 10^{-2}. \end{aligned} \tag{45}$$

This yields the squared mass matrix to be written as

$$Q_\lambda \equiv M_l M_l^\dagger \approx v_3^2 \|\mathbf{a}\|^2 \begin{pmatrix} \lambda_e^2 & -\lambda_e \lambda_\mu \cos\psi e^{i\alpha} & -\lambda_e \cos\phi e^{i\beta} \\ -\lambda_e \lambda_\mu \cos\psi e^{-i\alpha} & \lambda_\mu^2 & \lambda_\mu \cos\theta e^{i\gamma} \\ -\lambda_e \cos\phi e^{-i\beta} & \lambda_\mu \cos\theta e^{-i\gamma} & 1 \end{pmatrix}, \tag{46}$$

where θ, ϕ, and ψ are the "angles" between the pairs of complex vectors $(\mathbf{b}, \mathbf{a})$, $(\mathbf{c}, \mathbf{a})$, and $(\mathbf{c}, \mathbf{b})$, respectively, whereas γ, β, and α are the phases of the corresponding Hermitian products.[3] The diagonalization of $M_l M_l^\dagger$ by means of an infinitesimal "rotation" amounts to seeking an anti-Hermitian matrix

$$I_\epsilon = \begin{pmatrix} 0 & \epsilon_1 & \epsilon_2 \\ -\epsilon_1^* & 0 & \epsilon_3 \\ -\epsilon_2^* & -\epsilon_3^* & 0 \end{pmatrix}, \tag{47}$$

with small parameters ϵ's, satisfying

$$(Q_\lambda + [Q_\lambda, I_\epsilon])_{ij} = 0, \qquad i \neq j. \tag{48}$$

If we solve this equation analytically to express the ϵ's in terms of $(\lambda_{e,\mu}, \cos(\psi, \phi, \theta), \alpha, \beta, \gamma)$, we find, apart from "fine-tuned" situations corresponding to coplanar vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, that we get $|\epsilon_3| \sim \lambda_\mu$, $|\epsilon_2| \sim \lambda_e$, and $|\epsilon_1| \sim \lambda_e/\lambda_\mu$, which points to a consistent solution diagonalizing Q_λ close to the identity matrix given by $U_L^l = e^{I_\epsilon} \approx I + I_\epsilon$. For the above numerical values and a common value $\pi/3$ for the angles with representative phases as $(\alpha = \frac{\pi}{3},\ \beta = \frac{\pi}{4},\ \text{and}\ \gamma = \frac{\pi}{5})$, we get $m_e^2{:}m_\mu^2{:}m_\tau^2 = 6 \times 10^{-8}{:}3 \times 10^{-3}{:}1$, with the "exact" unitary diagonalizing matrix given by

[3]The "angle" θ between two complex vectors $\mathbf{b}$ and $\mathbf{a}$ is defined, following Cauchy-Schwartz inequality, as $|\mathbf{b}\cdot\mathbf{a}^*| = \|\mathbf{b}\|\cdot\|\mathbf{a}\|\cdot\cos\theta$, so we have $\mathbf{b}\cdot\mathbf{a}^* = \|\mathbf{b}\|\cdot\|\mathbf{a}\|\cdot\cos\theta\cdot e^{i\mathrm{Arg}(\mathbf{b}\cdot\mathbf{a}^*)}$.

$$U_L^l \sim \begin{pmatrix} 1 & 2.6\times 10^{-3}\exp(-0.502i\pi) & 1.5\times 10^{-4}\exp(-0.75i\pi) \\ -2.6\times 10^{-3}\exp(0.502i\pi) & 1 & 3\times 10^{-2}\exp(0.20i\pi) \\ -1.5\times 10^{-4}\exp(0.75i\pi) & -3\times 10^{-2}\exp(-0.20i\pi) & 1 \end{pmatrix}. \qquad (49)$$

The deviations due to the rotations are generally small, but could interpret measuring a nonzero small value of U_{e3} which is restricted by the reactor data [12] to be less than 0.16 in magnitude. Furthermore, the phases present in U_L^l, as given in Eq. (49), could contribute to Dirac and Majorana phases.

As we said above, the phenomenological features of [9] assuming $Z_3 \times Z_2$ symmetry, in particular, the leptonic flavor changing decays through ϕ exchange, can be repeated here with the underlying $U(1)$ symmetry. However, in contrast to discrete symmetries, the existence of a continuous symmetry leads to a conserved current, which we investigate now.

For illustration, let us restrict the discussion to the neutrino part. The invariance, under S_θ, of the "current-relevant" part of the Lagrangian depending on the field derivative:

$$K_\nu = i\bar{\nu}_k\gamma^\mu\partial_\mu\nu_k \qquad (50)$$

leads to the current:

$$J_\nu^\mu \equiv -i\frac{\partial K_\nu}{\partial(\partial_\mu\nu_j)}T_{jk}\nu_k = T_{jk}\bar{\nu}_j\gamma^\mu\nu_k, \qquad (51)$$

where T_{jk} is the generator of the $U(1)$ symmetry S_θ:

$$T = \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 0 \end{pmatrix}. \qquad (52)$$

In fact, $S(\theta)$ [Eq. (22)], as a three-dimensional representation of the commutative $U(1)$ group, should be reduced to three one-dimensional irreducible representations obtained by diagonalizing the matrix $S(\theta)$ to get

$$S(\theta) = L\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}L^\dagger \qquad (53)$$

$$L = \begin{pmatrix} 0 & \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} V_0 & V_- & V_+ \end{pmatrix}. \qquad (54)$$

The "neutrino" eigenvectors V_0, V_-, V_+ (forming the columns of the matrix L) represent the neutrino fields with definite S charges, respectively, equal to $0, -1, 1$. Writing the neutrino gauge fields $\nu^{\mathbf{g}} = (\nu_{\mathbf{e}}, \nu_\mu, \nu_\tau)$ in terms of these definite S-charge fields, one can see that the "neutrino" current [Eq. (51)] expresses explicitly the conservation of the S charge, in that we have

$$J_\nu^\mu = (0\bar{V}_0\gamma^\mu V_0 - 1\bar{V}_-\gamma^\mu V_- + 1\bar{V}_+\gamma^\mu V_+). \qquad (55)$$

We have here a conserved current associated with a global continuous symmetry with no gauge fields coupled to it. This is similar to the case of $U(1)$ baryon number conservation in the SM.

Using the tri-bimaximal matrix $U_{l\nu}$ to move from the neutrino gauge states $\nu^{\mathbf{g}}_{e,\mu,\tau}$ to neutrino "mass" states $\nu^{\mathbf{m}}_{1,2,3}$, we can express the definite S-charge neutrino fields $\mathbf{V} = (V_0, V_-, V_+)$ in terms of the mass eigenstates: $\mathbf{V} = L^T \cdot \nu^{\mathbf{g}} = L^T \cdot V_0 \cdot \nu^{\mathbf{m}}$, which gives

$$\begin{aligned} V_0 &= \nu_3, \\ V_- &= \Big(-\frac{i}{\sqrt{3}} + \frac{1}{\sqrt{6}}\Big)\nu_1 + \Big(-\frac{i}{\sqrt{6}} - \frac{1}{3}\Big)\nu_2, \\ V_+ &= \Big(\frac{i}{\sqrt{3}} + \frac{1}{\sqrt{6}}\Big)\nu_1 + \Big(\frac{i}{\sqrt{6}} - \frac{1}{3}\Big)\nu_2. \end{aligned} \qquad (56)$$

One can see directly that the particular combination of mass eigenstates in [Eq. (56)] never mixes under free time evolution provided ν_1 and ν_2 have degenerate mass. This degeneracy has already been shown to be a consequence of $U(1)$ symmetry S_θ. The same conclusion still holds if one thinks of the underlying symmetry, in the degenerate two masses case, as $Z_3 \times Z_2$ [see Eq. (20)], due to the compatibility of both $S(\theta)$ and Z_2 in that they commute and have common eigenstates.

VI. LEPTON FAMILY SYMMETRY IN PRESENCE OF MANY HEAVY SINGLET SCALARS

The many Higgs doublets in the previous model were introduced to accommodate the charged-lepton mass spectrum, but at the cost of inducing dangerous flavor changing neutral currents [13], which are difficult to be controlled. To remedy this situation, we introduce in the present model three heavy SM-singlet scalars transforming nontrivially under the flavor symmetry, and keep the SM-Higgs Φ intact. However, we enlarge the flavor symmetry so as to include an inversion in the flavor space, which means that the underlying flavor symmetry, call it S^I, assumes now the form

$$S^I = S \times \langle I\rangle \cong U(1) \times Z_2 \qquad (57)$$

with S given in Eq. (22), and $I = \mathrm{Diag}(-1, -1, -1)$.[4]

[4]More precisely, the group U in Eq. (21) is now a direct product of two commuting groups: $U \equiv \langle R_{12}(\theta), I\rangle \cong SO(2) \times Z_2$.

We assume the SM Higgs Φ and the charged right-handed leptons l_j^c to be singlets under the S^I symmetry, whereas the lepton left doublets transform componentwise faithfully:

$$L_i \rightarrow S^I_{ij} L_j, \tag{58}$$

with $i, j = 1, 2, 3$. The normal SM mass term for charged lepton,

$$\mathcal{L}_1 = Y_{ij} \bar{L}_i \Phi l^c_j, \tag{59}$$

should vanish now since the invariance under S^I restricts the Yukawa couplings to satisfy the matrix equation:

$$(S^I)^T \cdot Y = Y, \tag{60}$$

which cannot be met for an S^I matrix with determinant equal to -1. It is noteworthy that had we chosen not to enlarge the flavor symmetry then this mass term would have been allowed.

In order to generate lepton masses, we introduce three SM-singlet scalar fields, Δ_k, one for each family (the indices $k = 1, 2, 3$ refer also to the flavors e, μ, and τ, respectively), and they are coupled to the corresponding lepton left doublet

$$L_k = \begin{pmatrix} \nu_k \\ l_k \end{pmatrix}$$

via the dimension 5 operator:

$$\mathcal{L}_2 = \frac{f_{ikr}}{\Lambda} \bar{L}_i \Phi \Delta_k l^c_r, \tag{61}$$

where Λ is a heavy mass scale. As said earlier, this *ad hoc* assumption of the coupling of charged leptons with the additional Higgs fields via higher operators, and not through SM-like Yukawa terms, is suitable to reduce the effects of flavor changing neutral currents. We assume the new scalars Δ_k and the lepton left doublets transform similarly under S^I, i.e.,

$$\Delta_i \rightarrow S^I_{ij} \Delta_j. \tag{62}$$

Invariance of the Lagrangian under the symmetry implies

$$S^I_{i\alpha} S^I_{k\beta} f_{ikr} = f_{\alpha\beta r}, \tag{63}$$

which is written, in matrix form, as

$$(S^I)^T f_r S^I = f_r, \tag{64}$$

where f_r, for fixed r, is the matrix whose (i, j) entry is f_{ijr}. Noting that I enters Eq. (64) quadratically, and thus cancels out, then Eq. (24) imposes the form

$$f_r = \begin{pmatrix} A_r - B_r & C_r & -C_r \\ -C_r & A_r & B_r \\ C_r & B_r & A_r \end{pmatrix}. \tag{65}$$

When the fields Δ_k and ϕ° take the vacuum expectation values (vevs) $\langle\Delta_k\rangle = \delta_k$ and $\langle\phi^\circ\rangle = v$, the charged-lepton mass matrix originating from $\mathcal{L}_2$ becomes

$$(\mathcal{M}_l)_{ir} = \frac{v f_{ikr}}{\Lambda} \delta_k. \tag{66}$$

As we are concentrating on the neutrino sector without stating explicitly the Δ_k potential and since the S^I symmetry is broken by "soft" terms in the Higgs sector, we may assume a Δ_3-dominated pattern: $\delta_1, \delta_2 \ll \delta_3$, so to get the charged-lepton mass matrix

$$M_l \approx \frac{v\delta_3}{\Lambda} \begin{pmatrix} -C_1 & -C_2 & -C_3 \\ B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \end{pmatrix}. \tag{67}$$

The determinant of M_l is proportional to the mixed product, $(\frac{v\delta_3}{\Lambda})^3 \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$, where $\mathbf{A}$ is the "complex" vector of components A_i (similarly for $\mathbf{B}$, $\mathbf{C}$), which means that these three vectors should not be coplanar in order to have a nonsingular lepton mass matrix. We get then

$$M_l M_l^\dagger \approx \frac{v^2\delta_3^2}{\Lambda^2} \begin{pmatrix} \mathbf{C}\cdot\mathbf{C}^* & -\mathbf{C}\cdot\mathbf{B}^* & -\mathbf{C}\cdot\mathbf{A}^* \\ -\mathbf{B}\cdot\mathbf{C}^* & \mathbf{B}\cdot\mathbf{B}^* & \mathbf{B}\cdot\mathbf{A}^* \\ -\mathbf{A}\cdot\mathbf{C}^* & \mathbf{A}\cdot\mathbf{B}^* & \mathbf{A}\cdot\mathbf{A}^* \end{pmatrix}. \tag{68}$$

In a similar way to the analysis in the previous many Higgs doublet model, we see that assuming the magnitudes of the three vectors to come in ratios comparable to the lepton mass ratios,

$$\|\mathbf{C}\|^2 : \|\mathbf{B}\|^2 : \|\mathbf{A}\|^2 \sim m_e^2 : m_\mu^2 : m_\tau^2, \tag{69}$$

would imply that the mixing U^l_L, making $U^l_L M_l M_l^\dagger U^{l\dagger}_L$ diagonal, will be naturally very close to the identity matrix with off-diagonal terms of order ($m_e/m_\mu \sim 5 \times 10^{-3}$, $m_e/m_\tau \sim 3 \times 10^{-4}$, $m_\mu/m_\tau \sim 6 \times 10^{-2}$). This would mean again that our basis is the flavor basis to a very good approximation and that the hierarchical charged-lepton masses can be obtained from a hierarchy on the *a priori* arbitrary Yukawa couplings ($\|\mathbf{C}\|^2 \ll \|\mathbf{B}\|^2 \ll \|\mathbf{A}\|^2$).

VII. THE NEUTRINO MASS MATRIX AND TYPE-I SEESAW SCENARIO

In this scenario the effective light left neutrino mass matrix is generated through seesaw mechanism as

$$M_\nu = -M^D_\nu M_R^{-1} (M^D_\nu)^T, \tag{70}$$

where M_R is the heavy Majorana right-handed neutrinos mass matrix, whereas the Dirac neutrino mass matrix comes from the Yukawa term

$$g_{ij} \bar{L}_i \tilde{\Phi} \nu_{Rj}, \tag{71}$$

with $\tilde{\Phi} = i\tau_2 \Phi^*$. As to the right neutrino, we will assume that it transforms faithfully as

$$\nu_{Rj} \rightarrow S^I_{j\gamma} \nu_{R\gamma}, \tag{72}$$

since, as we shall see, this assumption will put constraints on the right Majorana mass matrix. The invariance of the Lagrangian under S^I implies in matrix form:

$$(S^I)^T g S^I = g. \tag{73}$$

Again, noting that when I enters here it does so quadratically, Eq. (24) forces the form

$$M_\nu^D = v\begin{pmatrix} A_D - B_D & C_D & -C_D \\ -C_D & A_D & B_D \\ C_D & B_D & A_D \end{pmatrix}. \tag{74}$$

As to the right-handed Majorana mass matrix, it originates from the term

$$\tfrac{1}{2}\nu_{iR}^T C^{-1}(M_R)_{ij}\nu_{jR}, \tag{75}$$

where C is the charge conjugation matrix. The invariance under S^I implies

$$(S^I)^T M_R S^I = M_R, \tag{76}$$

and thus the symmetric M_R has the form [Eq. (23)]

$$M_R = \Lambda_R\begin{pmatrix} A_R - B_R & 0 & 0 \\ 0 & A_R & B_R \\ 0 & B_R & A_R \end{pmatrix}. \tag{77}$$

Using Eqs. (70), (74), and (77), we have the effective neutrino mass matrix:

$$M_\nu = -\frac{v^2}{\Lambda_R}\begin{pmatrix} A_\nu - B_\nu & 0 & 0 \\ 0 & A_\nu & B_\nu \\ 0 & B_\nu & A_\nu \end{pmatrix}, \tag{78}$$

where

$$\begin{aligned} A_\nu &= \frac{A_R(A_D^2 + B_D^2 + C_D^2) + B_R(C_D^2 - 2A_D B_D)}{(A_R - B_R)(A_R + B_R)}, \\ B_\nu &= \frac{-B_R(A_D^2 + B_D^2 + C_D^2) - A_R(C_D^2 - 2A_D B_D)}{(A_R - B_R)(A_R + B_R)}. \end{aligned} \tag{79}$$

Diagonalizing M_ν we get the neutrino mass eigenvalues:

$$\frac{v^2}{\Lambda_R}(A_\nu - B_\nu, A_\nu - B_\nu, A_\nu + B_\nu). \tag{80}$$

We see here that all possible different patterns of the two degenerate neutrino masses spectrum can be obtained as follows.

(i) *Normal hierarchy* ($m_1 = m_2 \ll m_3$).—It suffices to have

$$0 \le A_{R,D} \simeq B_{R,D}, C_D \ll B_D, \tag{81}$$

for getting a normal hierarchy with

$$A_\nu \simeq \frac{A_D^2}{A_R}, \qquad B_\nu \simeq \frac{A_D^2}{B_R}. \tag{82}$$

We see that one can arrange the Yukawa couplings to enforce $A_\nu \simeq B_\nu$, so that to make the smallest neutrino mass $m_1 = m_2$ as tiny as one wishes.

(ii) *Inverted hierarchy* ($m_1 = m_2 \gg m_3$).—It is sufficient to have

$$0 \le A_{R,D} \simeq -B_{R,D}, \qquad C_D \ll B_D, \tag{83}$$

so that one gets an inverted hierarchy with

$$A_\nu \simeq \frac{2A_D^2}{A_R - B_R}, \qquad B_\nu \simeq \frac{-2B_D^2}{A_R - B_R}. \tag{84}$$

One can arrange the Yukawa couplings to enforce $A_\nu \simeq -B_\nu$, so that to make the tiniest neutrino mass m_3 small at will.

(iii) *Degenerate case* ($m_1 = m_2 \approx m_3$).—If we have

$$A_{R,D} \gg B_{R,D}, \qquad B_D \gg C_D, \tag{85}$$

then we get

$$A_\nu \simeq \frac{A_D^2}{A_R}, \qquad B_\nu \simeq \frac{2A_D B_D}{A_R}, \tag{86}$$

which implies $A_\nu \gg B_\nu$, so that we have a degenerate spectrum.

Thus, we see that any pattern occurring in both the Dirac and the right-handed Majorana mass matrices can reappear in the effective neutrino mass matrix.

The right-handed (RH) neutrino mass term violates lepton number by two units, and the out of equilibrium decay of the lightest RH neutrino to SM particles can be a natural source of lepton asymmetry [14]. This leptogenesis parameter is given by

$$\epsilon \simeq \frac{3}{16\pi v^2}\frac{1}{(\tilde{M}_\nu^{D\dagger}\tilde{M}_\nu^D)_{11}}\sum_{j=2,3}\mathrm{Im}[\{(\tilde{M}_\nu^{D\dagger}\tilde{M}_\nu^D)_{1j}\}^2]\frac{M_{R1}}{M_{Rj}}, \tag{87}$$

where M_{Ri}, $i = 1, \ldots, 3$ are the masses for RH neutrinos, and $\tilde{M}_\nu^D$ is the Dirac neutrino mass matrix in the basis where the Majorana RH neutrino mass matrix is diagonal.[5] Explicitly we have

$$\begin{aligned} (\tilde{M}_\nu^{D\dagger}\tilde{M}_\nu^D)_{11} &= (2|C_D|^2 + |A_D|^2 + |B_D|^2 - A_D B_D^* - A_D^* B_D), \\ (\tilde{M}_\nu^{D\dagger}\tilde{M}_\nu^D)_{12} &= \sqrt{2}(-C_D^* A_D + C_D A_D^* + C_D^* B_D - C_D B_D^*), \\ (\tilde{M}_\nu^{D\dagger}\tilde{M}_\nu^D)_{13} &= 0, \end{aligned} \tag{88}$$

which gives a vanishing lepton asymmetry. Thus, in this seesaw-type mechanism the baryon asymmetry, generated by lepton asymmetry, is zero provided S^I is an exact symmetry. Certainly, our symmetry S^I is not exact, and the breaking term [the C part in Eq. (12), which can originate from higher dimensional operators suppressed

[5]One has to go to the basis where the RH neutrino mass matrix is diagonal because the lepton asymmetry comes from the decay of the RH neutrino mass eigenstate.

by a heavy scale] has to be added in order to lift the degeneracy among the neutrino masses. In this case, one can compute the asymmetry in terms of the S^I-symmetry breaking parameters. We shall not dwell on this, rather we shall discuss the other phenomenologically motivated possibility of leptogenesis in the type-II seesaw mechanism.

VIII. THE NEUTRINO MASS MATRIX AND TYPE-II SEESAW SCENARIO

In this scenario we introduce two SM triplet fields Σ_A, $A = 1, 2$ which are also assumed to be singlet under the flavor symmetry S^I. The Lagrangian part relevant for the neutrino mass matrix is

$$\mathcal{L} = \lambda^A_{\alpha\beta} L^T_\alpha C^{-1} \Sigma_A i\tau_2 L_\beta + \mathcal{L}(H, \Sigma_A) + \text{H.c.}, \tag{89}$$

where $A = 1, 2$ and

$$\begin{aligned}\mathcal{L}(H, \Sigma_A) = {}& \mu_H^2 H^\dagger H + \frac{\lambda_H}{2}(H^\dagger H)^2 + M_A \operatorname{Tr}(\Sigma_A^\dagger \Sigma_A) \\ &+ \frac{\lambda_{\Sigma_A}}{2}[\operatorname{Tr}(\Sigma_A^\dagger \Sigma_A)]^2 + \lambda_{H\Sigma_A}(H^\dagger H)\operatorname{Tr}(\Sigma_A^\dagger \Sigma_A) \\ &+ \mu_A H^T \Sigma_A^\dagger i\tau_2 H + \text{H.c.},\end{aligned} \tag{90}$$

where

$$H = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix},$$

and

$$\Sigma_A = \begin{pmatrix} \frac{\Sigma^+}{\sqrt{2}} & \Sigma^0 \\ \Sigma^{++} & -\frac{\Sigma^+}{\sqrt{2}} \end{pmatrix}_A. \tag{91}$$

The neutrino mass matrix due to the exchange of the two triplets, Σ_1 and Σ_2, is

$$(M_\nu)^A_{\alpha\beta} \simeq v^2\left[\lambda^1_{\alpha\beta}\frac{\mu_1}{M^2_{\Sigma_1}} + \lambda^2_{\alpha\beta}\frac{\mu_2}{M^2_{\Sigma_2}}\right], \tag{92}$$

where M_{Σ_i} is the mass of the neutral component Σ_i^0 of the triplet Σ_i, $i = 1, 2$.

Appropriately, we present some remarks here. First, the symmetry S^I implies that the symmetric matrices λ_1 and λ_2 have the structure given in Eq. (23):

$$\lambda_a = \begin{pmatrix} A_a - B_a & 0 & 0 \\ 0 & A_a & B_a \\ 0 & B_a & A_a \end{pmatrix}, \qquad a = 1, 2.$$

Second, due to the "tadpole" term (the μ_A term) in $\mathcal{L}(H, \Sigma_A)$, which forbids explicitly the "unwanted" majorons, which would have resulted from the spontaneous breaking of the lepton number, one can arrange the parameters so that minimizing the potential gives a nonzero vev for the neutral component Σ^0 of the triplet. This would generate a mass term for the neutrinos, a procedure which is equivalent to integrating out the heavy triplets leading to the same mass formula. Third, the flavor changing neutral current due to the triplet is highly suppressed as a result of the heaviness of the triplet mass scale, or equivalently the smallness of the neutrino masses.

One can discuss now the baryon asymmetry generated by leptogenesis. We show at present that even though the neutrino Yukawa couplings are real it is possible to generate a baryon to photon density consistent with the observations. In fact, since the triplet Σ_A can decay into lepton pairs $L_\alpha L_\beta$ and HH, it implies that these processes violate total lepton numbers (by two units) and may establish a lepton asymmetry. As the universe cools further, the sphaleron interaction [15] converts this asymmetry into baryon asymmetry. At temperature of the order $\max\{M_1, M_2\}$, the heaviest triplet would decay via lepton number violating interactions. Nonetheless, no asymmetry will be generated from this decay since the rapid lepton number violating interactions due to the lightest Higgs triplet will erase any previously generated lepton asymmetry. Therefore, only when the temperature becomes just below the mass of the lightest triplet Higgs the asymmetry would be generated.

With just one triplet, the lepton asymmetry will be generated at the two loop level and it is highly suppressed. We justify this in that one can always redefine the phase of the Higgs field to make the μ real resulting in the absorptive part of the self-energy diagram becoming equal to zero. The choice of having more than one Higgs triplet is necessary to generate the asymmetry [16]. In this case, the CP asymmetry in the decay of the lightest Higgs triplet (which we choose to be Σ_1) is generated at the one loop level due to the interference between the tree and the one loop self-energy diagram[6] and it is given by

$$\epsilon_{CP} \approx -\frac{1}{8\pi^2}\frac{\operatorname{Im}[\mu_1\mu_2^* \operatorname{Tr}(\lambda^1\lambda^{2\dagger})]}{M_2^2}\frac{M_1}{\Gamma_1}, \tag{93}$$

where Γ_1 is the decay rate of the lightest Higgs triplet and it is given by

$$\Gamma_1 = \frac{M_1}{8\pi}\left[\operatorname{Tr}(\lambda^{1\dagger}\lambda^1) + \frac{\mu_1^2}{M_1^2}\right]. \tag{94}$$

If we denote the phases of $A_a - B_a$, $A_a + B_a$, μ_a by α_a, β_a, ϕ_a $(a = 1, 2)$, respectively, then by redefining the fields, $\Sigma_a \to e^{-i\alpha_a}\Sigma_a$, one can remove the phases α_a in the Yukawa couplings. For $\mu_a \approx M_{\Sigma_a} \sim 10^{13}$ GeV, $a = 1, 2$ (which give a neutrino masses in the sub-eV range) we get

[6] There is no one loop vertex correction because the triplet Higgs is not self-conjugate.

$$\epsilon_{CP} \approx -\frac{1}{\pi}\frac{2|A_1 - B_1||A_2 - B_2|\sin(\phi_1 - \phi_2) + |A_1 + B_1||A_2 + B_2|\sin(\phi_1 - \phi_2 + \beta_1 - \beta_2)}{1 + 2|A_1 - B_1|^2 + |A_1 + B_1|^2}, \tag{95}$$

Note that even if $A_a = B_a$, so that to kill the first term in the numerator, the CP violation responsible of the lepton asymmetry still depends on the relative phases between μ_1, μ_2 and/or $(A + B)_1$, $(A + B)_2$.

The baryon to photon density is approximately given by

$$\eta_B \equiv \frac{n_B}{s} = \frac{1}{3}\eta_L \simeq \frac{1}{3}\frac{1}{g_*}\kappa\epsilon_{CP}, \tag{96}$$

where $g_* \sim 100$ is the number of relativistic degrees of freedom at the time when the Higgs triplet decouples from the thermal bath and κ is the efficiency factor which takes into account the fraction of out of equilibrium decays and the washout effect. In the case of strong washout, the efficiency factor can be approximated by (H is the Hubble parameter)

$$\kappa \simeq \frac{H}{\Gamma_1}(T = M_1). \tag{97}$$

With the above numerical values and with an efficiency factor of order 10^{-4}, we get, for $\beta_1 = \beta_2$, a baryon asymmetry:

$$\eta_B \approx 10^{-7}\frac{\mathrm{Tr}(\lambda^1\lambda^{2\dagger})}{\mathrm{Tr}(\lambda^{1\dagger}\lambda^1) + 1}\sin(\phi_2 - \phi_1). \tag{98}$$

Thus, one can produce the correct baryon-to-photon ratio of $\eta_B \simeq 10^{-10}$ by choosing λ's of order 0.1 and not too small relative phase between μ_1 and μ_2.

IX. SUMMARY AND CONCLUSIONS

We presented here a method to find the most general symmetry implementing the form invariance property satisfied by the neutrino mass matrix. Applying the method for the tripartite model with two degenerate masses, we found the underlying symmetry to be the Abelian group $U(1)$, which may possibly be enlarged to be $U(1) \times Z_2$, and we have given a realization of it. The symmetry can be implemented in a complete setup including charged leptons, and we presented some models to account for the lepton mass hierarchies and the possibility of baryogenesis through leptogenesis.

It is important to note that in the seesaw models we presented, the renormalization group equations (RGE) effects may become important especially for the degenerate mass case. Actually, since the seesaw mechanism (for both types I and II) occurs at very high scale, the radiative corrections of the neutrino mass matrix parameters could have important effects on the lepton mixing matrix and neutrino masses [17]. In [18], it has been shown that, in the minimal supersymmetric standard model, the effects of the RGE to the tri-bimaximal model can be sizable. However, in the nonsupersymmetric extension of the SM (which is the case in this paper) with a right-handed neutrino or Higgs triplet, the effect of the RGE is suppressed in the case of a hierarchy or inverted hierarchy, and within the currently allowed range for the case of quasidegenerate spectrum with masses of order (or smaller) than 0.1 eV [19].

The setup as a whole can be seen as a first step approximation, which can be perturbed, with a breaking scale proportional to C_ν and so to the neutrino mass splitting [Eqs. (12) and (13)], so that to lead to tripartite model without degeneracy.

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