

Remarks on asymptotically safe inflation

S.-H. Henry Tye and Jiajun Xu

Laboratory for Elementary Particle Physics, Cornell University, Ithaca, New York 14853, USA

(Received 18 August 2010; published 28 December 2010)

We comment on Weinberg's interesting analysis of asymptotically safe inflation [S. Weinberg, *Phys. Rev. D* **81**, 083535 (2010)]. We find that even if the gravity theory exhibits an ultraviolet fixed point, the energy scale during inflation is way too low to drive the theory close to the fixed point value. We choose the specific renormalization group flow away from the fixed point towards the infrared region that reproduces the Newton's constant and today's cosmological constant. We follow this renormalization group flow path to scales below the Planck scale to study the stability of the inflationary scenario. Again, we find that some fine-tuning is necessary to get enough e folds of inflation in the asymptotically safe inflationary scenario.

DOI: 10.1103/PhysRevD.82.127302

PACS numbers: 04.60.-m, 04.50.Kd, 11.10.Hi, 98.80.Cq

I. INTRODUCTION: Among all the approaches to realize quantum gravity, the asymptotic safety of gravity [1,2] is a brilliant approach, in which the ultraviolet (UV) behavior of the theory is controlled by a non-Gaussian fixed point (NGFP) of the renormalization group (RG) flow, with a finite dimensional critical surface of trajectories attracted to the UV fixed point. Recently, this scenario has received increasing attention due to the mounting evidence that such a fixed point exists in the UV, and the critical surface turns out to be three-dimensional [3–6].

A natural area to apply asymptotically safe gravity is the early universe [7]. Recently, the interesting idea of asymptotically safe inflation was investigated by Weinberg in Ref. [8]. Starting with a general covariant theory of gravitation, Weinberg showed that such a theory allows de Sitter space as a solution to its classical gravitational field equations. Including time dependence in the Hubble parameter naturally introduces instabilities in the de Sitter solution that can terminate inflation. Assuming the theory is at the fixed point during inflation, and using the known numerical values for the couplings at the fixed point, Weinberg concludes that, in the absence of some fine-tuning, in some known examples with asymptotic safety, inflation ends prematurely without enough number of e folds achieved.

In this paper, we assume the gravity theory not only has a fixed point in the UV, but also reproduces today's Newton's constant and the cosmological constant (as the dark energy) in the IR. That is, the renormalization group trajectory that fits the IR data flow to a fixed point in the UV. If this assumption does not hold, the whole idea of asymptotic safety does not describe nature and so is irrelevant for the cosmology of our Universe. Under this assumption, we reexamine Weinberg's analysis. We find that even if the theory admits a non-Gaussian fixed point in the UV, the energy scale generically has to be above the reduced Planck scale m_{pl} in order for the coupling constants to approach the fixed point. However, inflation proceeds at a much lower energy scale, where the Hubble parameter $H \sim 10^{-5} m_{\text{pl}}$ and the values of the coupling parameters can be quite different from the fixed point values; this may make a

difference in terms of how many e folds of inflation one gets. By solving the RG equations, we determine how the couplings flow away from the fixed point, as the energy is lowered below the Planck scale. In the infrared (IR), we use the known experimental values, namely, the Newton's constant G_N and the cosmological constant Λ as well as constraints on higher order terms, to determine the particular RG trajectory in the critical surface that flows away from the UV fixed point towards the IR. Along this trajectory at $H/m_{\text{pl}} \sim 10^{-5}$, we find that fine-tuning is still needed to achieve enough e folds before instability sets in. Compared with Ref. [8], the fine-tuning we find depends on the ratio H/m_{pl} , but is insensitive to the values of couplings at the fixed point. This is the main point of this comment.

II. SETUP: We start with a general covariant action of gravitation with higher-derivative terms as in Ref. [8],

$$S = - \int dx^4 \sqrt{|g|} [g_0 \mu^4 + g_1 \mu^2 R + g_{2a} R^2 + g_{2b} R_{\mu\nu} R^{\mu\nu} + \dots], \quad (1)$$

where μ is the cutoff scale of the theory. We have extracted powers of μ explicitly to make the coupling constants g_i dimensionless. The dependence of g_i on μ is suppressed in the notation. In principle, matter fields and all higher-derivative terms are there from an effective field theory point of view. The running of the couplings g_i satisfy the RG equations of the form

$$\mu \frac{dg_i}{d\mu} = \beta_i(g_j). \quad (2)$$

In order for $g_i(\mu)$ to approach a fixed point g_i^* as $\mu \rightarrow \infty$, it is necessary that $\beta_i(g_j^*) = 0$.

III. ASYMPTOTICALLY SAFE INFLATION: With rotational and translational symmetries, we start with the flat space Friedmann-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t) dx_i^2.$$

The classical gravitational field equation can be solved through the single equation

$$\mathcal{N}(\mu, t) = 0, \quad (3)$$

$$\begin{aligned} \mathcal{N}(\mu, t) \equiv & -g_0 - 6\mu^{-2}g_1H^2 - \mu^{-4}g_{2a}(216H^2\dot{H} \\ & - 36\dot{H}^2 + 72H\ddot{H}) + \mu^{-4}g_{2b}(72H^2\dot{H} \\ & - 12\dot{H}^2 + 24H\ddot{H}) + \dots \end{aligned} \quad (4)$$

To search for de Sitter solutions, we specify the scale factor

$$a(t) = \exp(\bar{H}t) \quad (5)$$

with \bar{H} the constant Hubble parameter. We see that

$$\frac{\bar{H}}{\mu} = \sqrt{-\frac{g_0}{6g_1}} \quad (6)$$

is a solution of the classical field equation $\mathcal{N}(\mu, t) = 0$. The Hubble scale is determined by the value of the coupling constants g_0 and g_1 .

We now discuss how inflation can end in the asymptotically safe scenario. So far we have only considered the Hubble parameter to be a time independent constant. Including the time dependence, we can write

$$H(t) = \bar{H} + \delta H(t). \quad (7)$$

If there exists an unstable mode, then $\delta H(t)$ will grow, thus ending inflation.

With a time dependent $H(t)$, $\mathcal{N}(\mu, t) = 0$ now requires that

$$c_0(\mu, \bar{H})\frac{\delta H}{\bar{H}} + c_1(\mu, \bar{H})\frac{\delta \dot{H}}{\bar{H}^2} + c_2(\mu, \bar{H})\frac{\delta \ddot{H}}{\bar{H}^3} + \dots = 0$$

with c_i 's Taylor expansion coefficients of $\mathcal{N}(\mu, t)$ around \bar{H} , whose forms are worked out in Ref. [8].

If we schematically write $\delta H(t) \sim \exp(\xi \bar{H}t)$, we get

$$c_0 + c_1\xi + c_2\xi^2 + \dots = 0. \quad (8)$$

Roots with $\text{Re}(\xi) > 0$ will represent instabilities of the solution, and de Sitter expansion will only last for $1/\text{Re}(\xi)$ e folds. If all the roots have $\text{Re}(\xi) < 0$, then de Sitter expansion will be an attractor solution and, ignoring the effect of the inflaton potential, inflation will last forever.

IV. A SPECIFIC EXAMPLE: Consider the action

$$- \int d^4x \sqrt{|g|} \left[\frac{\mu^2}{g_N} (2\lambda\mu^2 - \mathcal{R}) + \frac{1}{2s} \mathcal{C}^2 - \frac{\omega}{3s} \mathcal{R}^2 \right]. \quad (9)$$

Here μ is the energy scale, g_N , λ , ω , s are dimensionless parameters. \mathcal{R} is the Ricci scalar, and \mathcal{C} is the Weyl tensor. We require $s > 0$ so that the Euclidean functional integral is damping.

The β functions for s and ω are given by one-loop perturbation theory [2],

$$\mu \frac{ds}{d\mu} = -\frac{1}{(4\pi)^2} \frac{133}{10} s^2, \quad (10)$$

$$\mu \frac{d\omega}{d\mu} = -\frac{s}{(4\pi)^2} \left(\frac{5}{12} + \frac{183}{10} \omega + \frac{10}{3} \omega^2 \right), \quad (11)$$

from which we notice that ω has a stable fixed point at

$$\omega^* = -0.0228 \quad (12)$$

and s is asymptotically free.

As we approach the inflation energy scale, higher order terms become important, and as we approach the Planck scale, presumably all higher order terms can be relevant in determining the existence and location of the UV fixed point. Here we are interested only in gravity models with asymptotic safety. Such models may consist of particular sets of higher order terms as well as appropriate matter fields. The flow of Newton's constant and the cosmological constant have been studied in special cases. For example, it has been argued that there are singularities in the β function of the quadratically truncated theory terminating the flow to the IR [2], which suggests that such truncated models may not be sufficient to describe our Universe. Here we start with today's Universe, which is well described by Einstein gravity, and assume asymptotic safety. (Otherwise, asymptotic safety is irrelevant to our Universe.) Our main concern is to illustrate how to flow down from the Planck scale to the inflation energy scale, and further to today's Universe at 10^{-3} eV. We implicitly assume that a truncated theory will give, within orders of magnitude, approximately the same flow as the full theory.

Given the assumptions above, We simplify the problem by fixing $\omega = \omega^*$, $s = 0$ and consider the two-dimensional flow of g_N and λ first. We write [9]

$$\mu \frac{dg_N}{d\mu} = 2g_N - \gamma_1 g_N^2 + \mathcal{O}(g_N^3), \quad (13)$$

$$\mu \frac{d\lambda}{d\mu} = -2\lambda + a_1 g_N + a_2 g_N \lambda + a_3 g_N^2 + \mathcal{O}(g_N^2 \lambda), \quad (14)$$

where a_1, a_2, a_3 are known coefficients at the fixed point, which are functions of ω and s . Setting $\omega = \omega^*$ and $s = 0$, and using the analysis in Ref. [10], we have

$$\gamma_1 = -a_2 = \frac{2u_2^*}{(4\pi)^2}, \quad a_1 = \frac{2u_1^*}{(4\pi)^2} \quad (15)$$

with $u_1^* = 1.38$, $u_2^* = 0.73$. One should note that the relation $\gamma_1 = -a_2$ may not hold if one introduces matter fields into the theory.

The RG flow can now be easily solved

$$\lambda = \frac{(\mu/\mu_\lambda)^{-4} + u_1^*}{2(\mu/\mu_0)^{-2} + 2u_2^*}, \quad (16)$$

$$g_N = (4\pi)^2 \frac{(\mu/\mu_0)^2}{1 + u_2^*(\mu/\mu_0)^2} \quad (17)$$

with μ_0 and μ_λ free parameters.

So, in the UV limit $\mu \rightarrow \infty$, g_N and λ flow to the fixed point

$$\frac{g_N^*}{(4\pi)^2} = \frac{1}{u_2^*} = 1.37, \quad \lambda^* = \frac{u_1^*}{2u_2^*} = 0.95. \quad (18)$$

In the IR limit $\mu \rightarrow 0$, the Einstein-Hilbert term in the action (9) reads

$$\frac{\mu^2}{g_N} (2\lambda\mu^2 - \mathcal{R}) = \frac{\mu_\lambda^4 - \mu_0^2 \mathcal{R}}{(4\pi)^2} \equiv m_{\text{pl}}^2 \left(\Lambda - \frac{\mathcal{R}}{2} \right),$$

where we have demanded that

$$\mu_0^2 = 8\pi^2 m_{\text{pl}}^2 = \frac{\pi}{G_N}, \quad \Lambda = \frac{\mu_\lambda^4}{(4\pi)^2 m_{\text{pl}}^2}. \quad (19)$$

The cosmological constant Λ contributes energy density $\Lambda m_{\text{pl}}^2 \sim 10^{-120} m_{\text{pl}}^4$, which means for positive and small Λ , the scale μ_λ and μ_0 has to obey $\frac{\mu_\lambda}{\mu_0} \sim 10^{-30}$.

V. THE HUBBLE SCALE DURING INFLATION:

Using Eq. (6), during inflation, we have $H = \sqrt{\lambda/3}\mu$. At the fixed point $\lambda^* \sim 1$, $H \sim \mu$. However, from observational data, we are pretty confident that

$$\frac{H}{m_{\text{pl}}} \ll 1 \Rightarrow \mu \ll \mu_0.$$

Imposing $H/m_{\text{pl}} \ll 1$ forces the theory to flow away from the UV fixed point.

Since the theory cannot be at the fixed point during inflation, we want to know how the coupling constants depend on the ratio H/m_{pl} . Away from fixed point, on scales $H_0 \ll \mu \lesssim m_{\text{pl}}$ (H_0 being the Hubble constant today), Eq. (16) gives

$$\lambda \approx \frac{u_1^*}{2} \left(\frac{\mu}{\mu_0} \right)^2, \quad (20)$$

so we have

$$H \approx \sqrt{\frac{u_1^*}{6}} \frac{\mu^2}{\mu_0}, \quad \frac{H}{m_{\text{pl}}} \approx \pi \sqrt{\frac{4u_1^*}{3}} \frac{\mu^2}{\mu_0^2} \sim \frac{\mu^2}{\mu_0^2}. \quad (21)$$

Using Eqs. (16) and (17), the coupling constants g_N and λ are

$$\lambda \approx \frac{u_1^*}{2} \frac{H}{m_{\text{pl}}}, \quad g_N \approx (4\pi)^2 \frac{H}{m_{\text{pl}}}. \quad (22)$$

In terms of the ratio H/μ , we find that

$$\frac{H}{\mu} \sim \sqrt{\lambda} \sim \sqrt{\frac{H}{m_{\text{pl}}}} \ll 1, \quad (23)$$

so we are in a regime that higher derivatives terms beyond those included in the action (9) are negligible.

VI. INSTABILITY OF DE SITTER SOLUTION: In the absence of the four or higher order derivative terms in the action (9), the de Sitter solution is stable. The presence of appropriate matter fields will introduce a positive u_2^* into Eq. (13) so that g_N has an UV fixed point. Introducing four and higher order derivative terms into the action (9)

introduces instability to the de Sitter solution. A perturbation to the de Sitter solution will grow, thus destabilizing the inflationary phase.

To find the unstable mode to H , Eq. (8) reduces to

$$\xi^2 + 3\xi - A = 0, \quad A = -\frac{3s}{2\omega\lambda g_N}. \quad (24)$$

If $A \ll 1$, the two roots are $\xi \approx -3$ and $\xi \approx A/3$. The negative ξ does not lead to any instability. If we pick $\xi = A/3$, the de Sitter phase can last for $3/A$ e folds. For inflation, we need $A \sim 1/20$.

Using Eqs. (12) and (22) for λ , g_N and ω , we get

$$A \sim s \left(\frac{H}{m_{\text{pl}}} \right)^{-2}. \quad (25)$$

In order to get $A \sim 1/20$, we need, at the inflation energy scale,

$$s(\mu = H) \lesssim 10^{-1} \left(\frac{H}{m_{\text{pl}}} \right)^2.$$

For $H \sim \mu_0 \sim m_{\text{pl}}$, s can be of order unity. However, for the realistic inflationary scenario, we require $H/m_{\text{pl}} \sim 10^{-5}$; so $s \lesssim 10^{-11}$, which is close to the bound obtained by Weinberg [8].

The parameter s is asymptotically free,

$$s = s_0 \left/ \left[1 + s_0 \frac{133}{160\pi^2} \ln(\mu/\mu_s) \right] \right. \quad (26)$$

The \mathcal{C}^2 term in the action (9) introduces a massive tensor mode while the \mathcal{R}^2 term introduces a massive scalar mode. They will modify the gravitational force which is well checked up to distance as small as sub-millimeters. This implies that the extra modes must be massive enough ($m > 10^{-3}$ eV) so these extra forces are Yukawa damped, which goes like e^{-mr}/r . Since $m^2 \sim s\mu^2/g_N$, this implies that

$$s > 10^{-60}, \quad (27)$$

where coincidentally, $m \geq u_\lambda$. So $s \lesssim 10^{-11}$ is consistent with the bound and may not be hard to arrange in the early universe. It is not clear whether the exponentially small s is a fine-tuning or not, since other than the bound (27), there is no guiding principle for a natural value of s .

For $s \sim 10^{-11}$ during inflation, the parameter s will remain small for the whole energy range from μ_λ to m_{pl} , due to its asymptotically free property. This justifies our approximation to set $s \rightarrow 0$ in the flow equation. At the same time, a tiny s parameter will ensure that the running of ω is also small according to Eq. (11), which justifies setting ω at its fixed point value ω^* in our analysis.

VII. THE ISSUE WITH THE GHOST POLE:

Because of the fourth derivative terms in the action (9), the graviton propagator contains, in addition to the usual massless graviton, a massive spin two particle of negative residue, i.e. a ghost. The presence of the ghost leads to

violation of unitarity and makes the theory inconsistent at quantum level.

Let us first look at the scale of the ghost pole. Schematically, the ghost propagator takes the form

$$\frac{-1}{p^2 - m_2^2}, \quad m_2^2 = \frac{s}{g_N} \mu^2.$$

We now compare the ghost mass m_2 with the Hubble scale. Since away from the fixed point, $H/\mu \sim \sqrt{\lambda}$, we have

$$\frac{m_2}{H} \sim \sqrt{\frac{s}{g_N \lambda}}. \quad (28)$$

Note that the total number of e folds is

$$N_e = \frac{3}{A} = \frac{-2\omega^* \lambda g_N}{s}, \quad (29)$$

and we therefore have

$$\frac{m_2}{H} = \sqrt{\frac{-2\omega^*}{N_e}} \lesssim \frac{1}{10}. \quad (30)$$

We see that during inflation, the energy scale is above the ghost mass, if the ghost is present at its naive value.

However, the presence of the ghost may be an artifact of the truncation of the theory. If the theory is not truncated, the propagator denominator could have nontrivial forms which removes the ghost pole. The idea is to absorb quadratic counterterms (those which contribute to the propagator) by field redefinitions and include all (infinitely many) counterterms generated in the bare action [11]. Since the counterterms are all powers of H/μ , the classical solution we find above should still hold as long as $H/\mu \ll 1$, which is imposed by observation.

Even if the theory is truncated, with running couplings, it is not clear whether the tree level ghost pole will be hit or not [5].

VIII. MATTER FIELDS: The ghost problem is totally absent if the 4- and higher-derivative terms are not

introduced. In the Einstein gravity theory, g_N has a UV fixed point if there is an appropriate set of matter fields so that $\gamma_1 \propto 4n_V + 2n_D - n_S > 0$ in Eq. (13), where n_V (n_D, n_S) is the number of vector (Dirac, scalar) fields [12]. In this simple example, the instability discussed above is also absent so the ending of inflation in this scenario follows from the properties of the inflaton potential. This is the standard scenario.

IX. POWER-LAW INFLATION: Instead of searching for de Sitter solution, one can also consider power-law expansion with $a(t) \sim t^p$ ($p > 1$) [13]. For a power-law background, we expect that $p \gtrsim 100$, so that the tensor mode $r = 16/p$ satisfies the observational bound.

We use the action (9) for illustration. To analyze the instability of the power-law solution, we need to expand around $\bar{H} = p/t$. The only difference is that $\bar{H} = p/t$ is time dependent, so $\dot{\bar{H}}$ and $\ddot{\bar{H}}$ cannot be ignored. The resulting instability Eq. (8) has a positive root (leading order in the large p limit)

$$\xi = -\sqrt{0.51 \frac{H}{m_{\text{pl}}}} + \sqrt{\frac{1.37s}{H/m_{\text{pl}}} + 0.51 \frac{H}{m_{\text{pl}}}}. \quad (31)$$

In reaching the above result, we have used Eq. (22) for the couplings away from the fixed points.

If we take $H/m_{\text{pl}} \sim 10^{-5}$, we have

$$\xi \sim -0.002 + 0.002\sqrt{1 + 2.7 \times 10^9 s}. \quad (32)$$

We see that if during inflation $s \sim 1$, $N_e \ll 1$. To get enough e folds, we need at least $s(\mu = H) \lesssim 10^{-7}$. This requirement of an exponentially small s during inflation is again tied to the ratio $H/m_{\text{pl}} \ll 1$, and is insensitive to the values of g_N and λ at the fixed points.

We thank Eanna Flanagan and Gary Shiu for discussion. This work is supported by the National Science Foundation under Grant No. PHY-0355005.

-
- [1] S. Weinberg, in *General Relativity, An Einstein Centenary Survey*, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, United Kingdom, 1979).
- [2] M. Niedermaier and M. Reuter, *Living Rev. Relativity* **9**, 5 (2006).
- [3] A. Codello, R. Percacci, and C. Rahmede, *Int. J. Mod. Phys. A* **23**, 143 (2008).
- [4] A. Codello, R. Percacci, and C. Rahmede, *Ann. Phys. (N.Y.)* **324**, 414 (2009).
- [5] D. Benedetti, P.F. Machado, and F. Saueressig, *Mod. Phys. Lett. A* **24**, 2233 (2009).
- [6] D. Benedetti, P.F. Machado, and F. Saueressig, *Nucl. Phys.* **B824**, 168 (2010).
- [7] A. Bonanno and M. Reuter, *Phys. Rev. D* **65**, 043508 (2002); M. Reuter and F. Saueressig, *J. Cosmol. Astropart. Phys.* **09** (2005) 012; M. Reuter and F. Saueressig, A. Bonanno, and M. Reuter, *J. Phys. Conf. Ser.* **140**, 012008 (2008).
- [8] S. Weinberg, *Phys. Rev. D* **81**, 083535 (2010).
- [9] M. Reuter and F. Saueressig, arXiv:0708.1317.
- [10] M.R. Niedermaier, *Phys. Rev. Lett.* **103**, 101303 (2009).
- [11] J. Gomis and S. Weinberg, *Nucl. Phys.* **B469**, 473 (1996).
- [12] R. Percacci, *Phys. Rev. D* **73**, 041501 (2006).
- [13] A. Bonanno, A. Contillo, and R. Percacci, arXiv:1006.0192.