High-energy asymptotics of D-brane decay amplitudes from Coulomb gas electrostatics

Niko Jokela,[1,2,*] Matti Järvinen,[3,†] and Esko Keski-Vakkuri[4,5,‡]

[1]*Department of Physics Technion, Haifa 32000, Israel*
[2]*Department of Mathematics and Physics University of Haifa at Oranim, Tivon 36006, Israel*
[3]*CP³-Origins, Campusvej 55, DK-5230 Odense M, Denmark*
[4]*Helsinki Institute of Physics, P.O. Box 64, FIN-00014 University of Helsinki, Finland*
[5]*Department of Physics, University of Helsinki, P.O. Box 64, FIN-00014 University of Helsinki, Finland*

We study the high-energy limit of tree-level string production amplitudes from decaying D-branes in bosonic string theory, interpreting the vertex operators as external charges interacting with a Coulomb gas corresponding to the rolling tachyon background, and performing an electrostatic analysis. In particular, we consider two open string–one closed string amplitudes and four open string amplitudes, and calculate explicit formulas for the leading exponential behavior.

I. INTRODUCTION

A famous feature of string theory is the soft behavior of scattering amplitudes in the high-energy limit. Remarkably, at each order (worldsheet genus) in the perturbation theory, the dominant saddle-point contribution has a simple electrostatic interpretation—the exponent can be identified as the electrostatic energy of point charges at the equilibrium [1–3]. Recently, the electrostatic approach was applied to string scattering from decaying D-branes [4,5]. In the original work [1–3], there was no background charge in addition to the point charges, due to momentum conservation. A new ingredient for strings scattering from a decaying D-brane is the condensing tachyon on the brane, which provides a background charge distribution that interacts with the point charges. More precisely, the background consists of a Coulomb gas of unit charges at finite temperature and (imaginary) chemical potential. This electrostatic interpretation is valid at all energies. Exact calculations are very complicated, but simplifications can be found in the high-energy limit. In a previous work, we applied this approach to study closed string pair production from a decaying D-brane at disk amplitude level [5].

In this paper, we study further examples of the electrostatic approach to decay amplitude calculations. We start with a detailed general description of the approach. For a uniform discussion, we review previous calculations before applying the method to new cases. We verify that the approach predicts the known high-energy behavior of the bulk-boundary amplitude [6] and show that the amplitude does not diverge at high energies in the kinematically allowed region. We also present an improved analysis of the ("antipodal") two-point function, related to open string pair production by the decaying brane. Then we discuss n-point functions with $n > 2$ and the associated string

production amplitudes.[1] As a completely new result, we apply our method to calculate the high-energy limit of a three-point amplitude involving a pair of open strings and a closed string. For stable D-branes, such amplitudes are, e.g., relevant for emission from or absortion by a charged black hole [7]. Finally, we extend previous work on n-point boundary functions [8] and study the high-energy asymptotic behavior of the amplitude for production of four open strings, before ending with conclusions and an outlook. Technical details are presented in two appendices.

II. AMPLITUDES FROM ELECTROSTATICS APPROACH

A. Preliminaries

We review quickly the structure of the disk amplitudes for string scattering from a decaying D-brane in bosonic string theory, with the "half S-brane" rolling tachyon background [9,10]. The correlation functions are path integrals

$$A_n(\{\omega_a, \vec{k}_a\}_{a=1}^n) = \int \prod_{a=1}^n \frac{d^2 w_a}{2\pi}$$
$$\times \left\langle \prod_{a=1}^n V(w_a, \bar{w}_a) e^{-\delta S_{\mathrm{bdry}}} \right\rangle_{\mathrm{free}}, \quad (1)$$

where the rolling tachyon boundary deformation representing the half S-brane decay mode for the D-brane is

$$\delta S_{\mathrm{bdry}} = \lambda \int dt e^{X^0(t)}. \quad (2)$$

The calculational complications arising from rolling tachyon background are similar for all open and closed string vertex operators, when they are represented in timelike gauge [11]:

*najokela@physics.technion.ac.il
†mjarvine@cp3.sdu.dk
‡esko.keski-vakkuri@helsinki.fi

---

[1]In this paper we consider only tachyon vertex operators.

$$V(k) = V_{sp}(\vec{k})e^{-i\omega X^0 + i\vec{k}\cdot\vec{X}}, \qquad (3)$$

where $V_{sp}(\vec{k})$ involves only space directions $\vec{X}$. The contributions from the contractions from the spacelike sector are similar to those in scattering from stable (or nondeformed unstable) D-branes, as they do not involve the timelike background (2). Our focus is in developing techniques for calculating the contribution from the nontrivial timelike sector. Therefore, for our purposes it is sufficient to focus on scattering amplitudes which involve only closed and open string tachyon vertex operators

$$V(w_a, \bar{w}_a) = e^{ik_a \cdot X(w_a, \bar{w}_a)}. \qquad (4)$$

Closed string vertices are placed in the interior of the unit disk $|w_a| < 1$, whereas the open string vertices lie at the boundary $w_a = e^{i\tau_a}$, $\tau_a = 0 \ldots 2\pi$. We will consider open-closed n-point amplitudes, with $n = n_c + n_o$ where n_c (n_o) is the number of closed (open) strings. We adopt a notation $\xi_a = -i\omega_a$ and break up the spatial momentum to parallel and perpendicular directions to the unstable Dp-brane: $\vec{k}_a = (\vec{k}_a^{\parallel}, \vec{k}_a^{\perp})$. On-shell conditions for the bosonic closed string tachyons are $k_a^2 = \xi_a^2 + (\vec{k}_a)^2 = -4(N_a - 1)$ and for the open string tachyons they read $k_a^2 = \xi_a^2 + (\vec{k}_a^{\parallel})^2 = -(N_a - 1)$ [6]. The overall spatial parallel momentum will be conserved: $\sum_{a=1}^n \vec{k}_a^{\parallel} = 0$.

The worldsheet correlation functions can be evaluated by first isolating the zero modes from the oscillators, $X^\mu = x^\mu + X'^\mu$, and then expanding the boundary deformation into a power series in λ. This yields

$$A_{n_c+n_o} = \int dx^0 d^p\vec{x}^{\parallel} e^{i\sum_a k_a^\mu x_\mu} \sum_{N=0}^{\infty} \frac{(-z)^N}{N!}$$
$$\times \int \prod_{a=1}^{n_c} \frac{d^2 w_a}{2\pi} \prod_{a=n_c+1}^{n} \frac{d\tau_a}{2\pi} \times \prod_{i=1}^{N} \frac{dt_i}{2\pi}$$
$$\times \left\langle \prod_{i=1}^{N} e^{X'^0(t_i)} \prod_{a=1}^{n} e^{ik_a \cdot X'(w_a, \bar{w}_a)} \right\rangle, \qquad (5)$$

where we introduced $z = 2\pi\lambda e^{x^0}$ and fixed the indexing of the vertex operators such that closed strings have smaller values of a.

After a straightforward calculation (see Appendix A), we can write (5) as

$$A_{n_c+n_o} = \int dx^0 d^p\vec{x}^{\parallel} e^{i\sum_a k_a^\mu x_\mu} \bar{A}_{n_c+n_o}(x^0), \qquad (6)$$

$$\bar{A}_{n_c+n_o}(x^0) = \sum_{N=0}^{\infty} (-z)^N I_{n_c+n_o}(\{w_a, k_a\}; N), \qquad (7)$$

$$= \sum_{N=0}^{\infty} (-z)^N \int \prod_{a=1}^{n_c} \frac{d^2 w_a}{2\pi} \prod_{a=n_c+1}^{n} \frac{d\tau_a}{2\pi}$$
$$\times \prod_{1 \leq a < b \leq n} |w_a - w_b|^{2\vec{k}_a^{\parallel} \cdot \vec{k}_b^{\parallel}}$$
$$\times \prod_{1 \leq a < b \leq n_c} |w_a - w_b|^{-k_a^{\parallel} \cdot k_b^{\parallel} + \vec{k}_a^{\perp} \cdot \vec{k}_b^{\perp}} \qquad (8)$$
$$\times \prod_{a,b=1}^{n_c} |1 - w_a\bar{w}_b|^{1/2(k_a^{\parallel} \cdot k_b^{\parallel} - \vec{k}_a^{\perp} \cdot \vec{k}_b^{\perp})}$$
$$\times Z_{n_c+n_o}(\{w_a, k_a\}; N),$$

where

$$Z_{n_c+n_o}(\{w_a, k_a\}; N) = \frac{1}{N!} \prod_{1 \leq a < b \leq n} |w_a - w_b|^{2\xi_a\xi_b}$$
$$\times \int \prod_{i=1}^{N} \frac{dt_i}{2\pi} \prod_{1 \leq i < j \leq N} |e^{it_i} - e^{it_j}|^2$$
$$\times \prod_{a=1}^{n} |1 - w_a e^{-it_i}|^{2\xi_a}. \qquad (9)$$

B. Applying the electrostatic approach

In general, (9) is too complicated to calculate, so we look for an approximation scheme in a special limit. As suggested by the notation, $Z_{n_c+n_o}$ can be interpreted as a Coulomb gas partition function[2] at the inverse temperature $\beta = 2$. We can evaluate the partition function in a saddle-point approximation corresponding to electrostatic equilibrium, at large N and ξ_a (with the ratios ξ_a/N fixed). The method is discussed in detail in a companion paper [4]. The leading term of $\log Z_{n_c+n_o}$ can be found by evaluating the electrostatic energy of the Coulomb gas, unless $n_c + n_o$ is too large. The next-to-leading term was also analyzed in [4]. In this article we shall continue the analysis of [4] by discussing in detail how the high-energy limit of string amplitudes arises from the electrostatic approach. String production at large $\xi_a = -i\omega_a$ is a physically interesting regime, since we expect the unstable D-brane to mainly decay to very massive string modes [11].

First we note that the correlator $\sim Z_{n_c+n_o}$ needs to be summed over N and integrated over x^0 to obtain the string scattering amplitude,

$$A_{n_c+n_o} \propto \int dx^0 e^{\sum_a \xi_a x^0} \sum_{N=0}^{\infty} (-z)^N Z_{n_c+n_o}(\{\xi_a\}; N), \qquad (10)$$

where $z = 2\pi\lambda e^{x^0}$ and we dropped the complicated proportionality factor for clarity. As discussed in [8], under

---

[2]We have explored many other aspects of this connection for decaying D-branes in bosonic and superstring theories in [12–16].

certain conditions the sum and the integral can be carried out exactly,

$$\int dx^0 e^{\sum_a \xi_a x^0} \sum_{N=0}^{\infty} (-z)^N Z_{n_c+n_o}(\{\xi_a\}; N)$$

$$= \frac{\pi (2\pi\lambda)^{-\sum_a \xi_a}}{\sin(\pi \sum_a \xi_a)} Z_{n_c+n_o}\left(\{\xi_a\}; N = -\sum_a \xi_a\right). \quad (11)$$

This result was originally found in [17] in the context of Liouville field theory. $Z_{n_c+n_o}$ is in general unknown but simplifies in the large N limit ($N, \xi_a \rightarrow \infty$ with N/ξ_a fixed) [4]. Since $N = -\sum_a \xi_a$ in (11), the large N limit corresponds to the high-energy ($\omega_a = i\xi_a$) limit for the string production amplitudes. In particular, the leading term gives the saddle-point approximation

$$A_{n_c+n_o} = \frac{\pi (2\pi\lambda)^{-\sum_a \xi_a}}{\sin(\pi \sum_a \xi_a)} \int (dw d\bar{w} d\tau) \gamma(w, \bar{w}, \tau)$$

$$\times \exp\left[-2\mathcal{E}|_{N=-\sum_a \xi_a}\right], \quad (12)$$

where $\mathcal{E}$ is the electrostatic energy of the corresponding Coulomb gas configuration in the continuum limit and can be computed explicitly [4]. Above $\int(dw d\bar{w} d\tau)$ denotes the w_a and τ_a integrals, and $\gamma(w, \bar{w}, \tau)$ denotes the contribution from spacelike contractions in (6). Their explicit form depends on the particular amplitude. Note that in general also $\mathcal{E}$ is w_a, τ_a dependent. If possible, we perform the integrals explicitly, but in general we follow the approach of [1] and replace w_a by their electrostatic equilibrium values (for details, see Appendix B and the examples in Secs. III and IV).

The use of (11) requires an analytic continuation of $Z_{n_c+n_o}(N)$ in the parameter N to noninteger N, which is subject to some constraints. The partition function should be analytic for $\mathrm{Re}N > 0$, and it should not grow exponentially as $N \rightarrow \infty$, when $|\arg N| < \pi/2$. The asymptotic behavior of $Z_{n_c+n_o}$ (with ξ_a fixed) may be analyzed using random matrix theory techniques. Using a random matrix theory interpretation, $Z_{n_c+n_o}$ describes an expectation value of a periodic function in the circular ensemble of $U(N)$ random matrices, CUE(N). The expectation value can in turn be converted to a Toeplitz determinant of the Fourier coefficients of periodic function (see [6,18,19] for more discussion); the advantage then is that the asymptotic behavior of the determinant at $N \rightarrow \infty$ simplifies, and is described by the Szegö formula and its generalization. Using the result of [20], we find the power-law behavior

$$Z_{n_c+n_o}(N) \underset{N\rightarrow\infty}{\sim} N^{\sum_{a=n_c+1}^{n} \xi_a^2} \quad (13)$$

(see [5] for a discussion with $n_c = 0$). This result holds also for the leading term in the $N \rightarrow \infty$ limit when N/ξ_a is

fixed, an explicit discussion for the boundary one-point partition function is given in [4].

The above analysis needs to be modified if the large N limit of the partition function is not analytic in N. As explained in [4,5], nonanalytic behavior may be associated with suddenly appearing or disappearing gaps[3] in the continuous charge distribution of the Coulomb gas picture, created in the vicinity of the external charges at some large value of N. This may happen in the presence of bulk charges: a charge near the boundary of the disk generates a gap, which disappears at sufficiently large N [5]. Similar phenomenon may occur in higher order boundary amplitudes: two nearby charges will create only one gap for small N, which separates into two gaps as N increases. In these cases, as suggested in [5], one can first integrate over the locations of the charges. The integrated partition function is analytic in N, and one can proceed with the analytic continuation.[4]

In order to calculate the electrostatic energy, we need to first assume that ξ_a are real valued, and then continue analytically to physical energies $\omega_a = i\xi_a$ in the end. We justify this by noting that $Z_{n_c+n_o}$ is analytic for $\mathrm{Re}\xi_a \geq -1/2$, as seen from (9).[5]

In summary, we have developed a prescription for calculating the high-energy approximation of the string scattering amplitudes, containing the following steps:

(1) Find the series expansion for the scattering amplitude [see Eqs. (6)–(9)] and identify the Coulomb gas partition function.

(2) Solve the electrostatic potential problem to find $\mathcal{E}$, and the leading partition function $\log Z \simeq -2\mathcal{E}$ at $\beta = 2$.

(3) If necessary, integrate the leading result over the positions of the external charges (the unfixed modular parameters of the string vertex operators).

(4) Do the summation over N and integration over time by continuing analytically N to the total external charge $N \rightarrow -\sum_a \xi_a$ in the result for $\log Z$, as shown in Eq. (11).

(5) Continue analytically to physical energies $\omega_a = i\xi_a$.

Finally, let us comment on the precision of the obtained approximation. We only used the leading term of $\log Z_{n_c+n_o}$ in the large N limit, which was shown to be $\mathcal{O}(N^2)$ in [4]. Since we fixed $N = -\sum_a \xi_a$, our result (12) is the leading

---

[3]A gap may also exist first and then disappear, or many gaps may be created or join together.

[4]Another method which typically avoids nonanalyticities is to fix the positions by the equilibrium equations, which follow from minimizing the electrostatic energy and will be discussed in Appendix B.

[5]There may be one caveat: the amplitude contains an integration over the unfixed moduli parameters, which might in principle cause problems in the case of higher-point amplitudes, since the integrals are not necessarily well defined for all $\mathrm{Re}\xi_a \geq 0$. This needs to be checked case by case.

$\mathcal{O}(\omega^2)$ term of $\log A_{n_c+n_o}$ in the limit of large $\omega = \sum \omega_a$ with the ratios ω_a/ω_b fixed. We also calculated the next-to-leading $\mathcal{O}(N)$ term of $\log Z_{n_c+n_o}$ in [4]. However, this term vanishes at $\beta = 2$. Therefore, the first nontrivial corrections to our high-energy approximation are obtained from the $\mathcal{O}(N^0)$ term of $\log Z_{n_c+n_o}$, resulting in $\mathcal{O}(\omega^0)$ (possibly logarithmic) corrections to $\log A_{n_c+n_o}$.

C. Electrostatic energies

In the remainder of the paper we study the easiest string scattering amplitudes following the above systematic prescription. In the second step, we need the electrostatic potential energy $\mathcal{E}$, as in Eq. (12). We have already studied this problem in [4,5], and summarize the results here. First, the energy with one bulk charge $\xi = \xi_1$ at $w = w_1 = r$ $(0 < r < 1)$ reads [5]

$$
\begin{aligned}
\mathcal{E}^{(\text{bulk})} = {} & \theta(r_c - r)\frac{\xi^2}{2}\log(1 - r^2) + \theta(r - r_c) \\
& \times \left[-\frac{(N + 2\xi)^2}{4}\log\frac{1 + \chi}{1 + \delta(r)} - \frac{N^2}{4} \right. \\
& \left. \times \log\frac{1 - \chi}{1 - \delta(r)} + \frac{\xi^2}{2}\log\frac{4\chi}{(1 + \delta(r))^2} \right],
\end{aligned}
\tag{14}
$$

where

$$
r_c = \frac{N}{N + 2\xi}; \qquad \delta(r) = \frac{1 - r}{1 + r}; \qquad \chi = \frac{\xi}{N + \xi}.
\tag{15}
$$

Several configurations with charges on the boundary of the disk were solved in [4]. In the simplest case there is one boundary charge ξ on the boundary, giving

$$
\begin{aligned}
\mathcal{E}^{(\text{1pt})} = {} & \tfrac{1}{2}[-2F(\xi) + F(2\xi) - F(N) - F(N + 2\xi) \\
& + 2F(N + \xi)],
\end{aligned}
\tag{16}
$$

where

$$
F(x) = \tfrac{1}{2}xf(x) = \tfrac{1}{2}x^2\log x.
\tag{17}
$$

We will also use the result for the configuration where two boundary charges ξ_1 and ξ_2 lie at exactly antipodal points on the unit circle. Then

$$
\begin{aligned}
\mathcal{E}^{(\text{2pt})} = {} & \{F(N + \xi_1 + \xi_2) - F(\xi_1 + \xi_2) - \tfrac{1}{4}[F(N + 2\xi_1 \\
& + 2\xi_2) + F(N + 2\xi_1) + F(N + 2\xi_2) + F(N)] \\
& + \tfrac{1}{4}[F(2\xi_1 + 2\xi_2) + F(2\xi_1) + F(2\xi_2)] \\
& - \xi_1\xi_2\log 2\}.
\end{aligned}
\tag{18}
$$

As an example of a more complicated situation we consider a symmetric four-point case where two particles with equal charges ξ_1 are, say, at $\tau = \arg w = 0$ and at $\tau = \pi$, and two additional particles with charges ξ_2 are at $\tau = \pi/2$ and at $\tau = 3\pi/2$. This configuration gives

$$
\begin{aligned}
\mathcal{E}^{(\text{4pt})} = {} & 2F(N/2 + \xi_1 + \xi_2) - 2F(\xi_1 + \xi_2) - \tfrac{1}{2}[F(N/2 \\
& + 2\xi_1 + 2\xi_2) + F(N/2 + 2\xi_1) + F(N/2 + 2\xi_2) \\
& + F(N/2)] + \tfrac{1}{2}[F(2\xi_1 + 2\xi_2) + F(2\xi_1) \\
& + F(2\xi_2)] - (\xi_1 + \xi_2)^2\log 2.
\end{aligned}
\tag{19}
$$

III. TWO-POINT AMPLITUDES

A. Bulk-bulk amplitude

The high-energy production amplitude of two closed strings from a decaying D-brane was derived in [5] by using the framework of Sec. II with the result of Eq. (14). For completeness, we review the results here. We fix the charge ξ_2 at $w_2 = 0$, and the charge ξ_1 at $0 < w_1 = r < 1$ by using conformal symmetry. In this case Eq. (12) becomes

$$
A_{0+2}(\xi_1, \xi_2) = \frac{\pi(2\pi\lambda)^{-\xi_1-\xi_2}}{\sin\pi(\xi_1 + \xi_2)}\left\{\int_0^1 dr k(r) \right. \\
\left. \times \exp[-2\mathcal{E}^{(\text{bulk})}]\right\}_{N=-\xi_1-\xi_2},
\tag{20}
$$

where we restored the proportionality factor

$$
k(r) = rr^{k_1\cdot k_2}(1 - r^2)^{1/2((k_1^{\|})^2 - (\vec{k}_1^{\perp})^2)}.
\tag{21}
$$

For the special case where the strings have equal energies, $\omega_1 = \omega_2 \equiv \omega$, the result simplifies to

$$
\begin{aligned}
A_{0+2}(\omega) = {} & -\frac{i\pi e^{i(\pi s/2)}(2\pi\lambda)^{2i\omega}}{2\sinh 2\pi\omega} \\
& \times \left[e^{i(\pi u/2)}\frac{\Gamma(\frac{t}{2} - 1)}{\Gamma(\frac{t+u}{2} - 2)} + \frac{\Gamma(\frac{s}{2} + 1)}{\Gamma(\frac{s+u}{2})} \right]\Gamma\left(\frac{u}{2} - 1\right),
\end{aligned}
\tag{22}
$$

where

$$
s = k_1 \cdot k_2
$$

$$
t = 2(k_1^{\|})^2 = 2(k_2^{\|})^2 = 8 - (\vec{k}_1^{\perp})^2 - (\vec{k}_2^{\perp})^2
\tag{23}
$$

$$
u = \frac{(\vec{k}_1^{\perp} - \vec{k}_2^{\perp})^2}{2}.
$$

In the definitions of the parameters we used the on-shell conditions $k_a^2 = -\omega_a^2 + \vec{k}_a^2 = -4(N_a - 1)$ for the tachyonic states $N_a = 0$. Notice that the results for the leading high-energy asymptotics (see [5] for an extensive analysis) remain valid also for higher N_a's as long as they are much smaller than the (squared) energy scale.

B. Bulk-boundary amplitude

Let us then discuss the asymptotics of the bulk-boundary scattering amplitude, and verify that our result matches the exact one [6]. We follow [6,11,18] and use conformal symmetry to place the bulk operator (charge ξ_c) to the

origin, rather than integrating over its position as suggested in (8). Then the bulk charge decouples from the Coulomb gas calculation, and we may use the results with one boundary charge (ξ_o) for the partition function. According to Eq. (12), analytic continuation of (16) to $N = -\xi_c - \xi_o$ at $\beta = 2$ gives

$$A_{1+1}(\xi_c, \xi_o) \simeq \frac{\pi(2\pi\lambda)^{-\xi_c-\xi_o}}{\sin\pi(\xi_c + \xi_o)}$$
$$\times \exp[-2\mathcal{E}^{(1pt)}]|_{\xi=\xi_o,N=-\xi_c-\xi_o}. \quad (24)$$

Setting here $\xi_a = -i\omega_a$ we find the asymptotics

$$A_{1+1}(\omega_o, \omega_c) = \frac{i\pi(2\pi\lambda)^{i(\omega_c+\omega_o)}}{\sinh\pi(\omega_c + \omega_o)} \exp\left[\omega_o^2 \log\frac{\omega_o}{\omega_c + \omega_o}\right.$$
$$+ \omega_c^2 \log\frac{\omega_c}{\omega_c + \omega_o} - \frac{1}{2}(\omega_c - \omega_o)^2$$
$$\times \log\frac{\omega_c - \omega_o}{\omega_c + \omega_o} + \omega_o^2(2\log 2 \pm i\pi)$$
$$\left. + \mathcal{O}(\log\omega)\right], \quad (25)$$

where we restored the expected size of the next-to-leading order correction. This indeed matches with the asymptotics of the exact amplitude [6] up to the branch choice of the logarithm [± in the phase factor, on the last line in (25)] which is hard to obtain from the electrostatic approach. Notice, however, that the absolute value of the amplitude is independent of the branch.

Let us make one comment about this result. After using momentum conservation parallel to the D-brane, 26-momenta of the strings become

$$k_c = (\omega_c, \vec{k}^{\parallel}, \vec{k}^{\perp}), \qquad k_o = (\omega_o, -\vec{k}^{\parallel}, 0). \quad (26)$$

At high energy, and for low-lying excitations ($N_a \ll \omega_a^2$), the mass-shell conditions $-\omega_c^2 + \vec{k}_c^2 = -4(N_c - 1)$ and $-\omega_o^2 + \vec{k}_o^2 = -(N_o - 1)$ give

$$\omega_c \simeq \sqrt{(\vec{k}^{\parallel})^2 + (\vec{k}^{\perp})^2} \qquad \omega_o \simeq |\vec{k}^{\parallel}|, \quad (27)$$

so asymptotically $\omega_c \gtrsim \omega_o$. The leading term in (25) can be written as

$$\omega_o^2 \log\frac{\omega_o}{\omega_c + \omega_o} + \omega_c^2 \log\frac{\omega_c}{\omega_c + \omega_o} - \frac{1}{2}(\omega_c - \omega_o)^2$$
$$\times \log\frac{\omega_c - \omega_o}{\omega_c + \omega_o} + 2\omega_o^2 \log 2 = (\omega_c + \omega_o)^2$$
$$\times \left[\alpha^2 \log\alpha + (1 - \alpha)^2 \log(1 - \alpha)\right.$$
$$\left. - \frac{1}{2}(1 - 2\alpha)^2 \log(1 - 2\alpha) + 2\alpha^2 \log 2\right], \quad (28)$$

where $\alpha \equiv \omega_o/(\omega_c + \omega_o)$. In the kinematically allowed region $0 < \alpha < 1/2$ the function in the square brackets in (28) is negative, and it vanishes at the endpoints $\alpha = 0$,

1/2. Thus the amplitude vanishes for high energies in the kinematically allowed region as $A \sim e^{-\pi(\omega_c+\omega_o)}$ if $\omega_c \gg \omega_o$ or $\omega_c = \omega_o$, and faster ($\sim e^{-\#(\omega_c+\omega_o)^2}$) if the energies are comparable but unequal. We observed similar behavior for the bulk two-point amplitude (22) at high energies in [5].

C. Boundary-boundary amplitude

Finally, we shall analyze the boundary two-point amplitude. Momentum conservation fixes $\vec{k}_1^{\parallel} = -\vec{k}_2^{\parallel} \equiv \vec{k}^{\parallel}$ and from the on-shell conditions for low mass excitations $-\omega_a^2 + \vec{k}_a^2 = \mathcal{O}(\omega^0)$ we get $\omega_1 \simeq \omega_2 \equiv \omega$ at high energy. The electrostatic two-point partition function at equal charges was found only numerically for general $\tau = \tau_2 - \tau_1$ in [4]. Therefore, we shall calculate the amplitude in the equilibrium configuration. Notice that due to symmetry the configuration where the charges lie at antipodal points, $\tau_2 = \tau_1 + \pi$, is always a solution to the equilibrium equations (B5): if we set $\tau_1 = 0$ and $\tau_2 = \pi$ the charge distribution is symmetric with respect to the real axis, and the imaginary parts in both of the terms of (B5) vanish. The total energy is found by setting $\xi_1 = \xi_2$ in Eq. (18) which yields

$$\mathcal{E}^{(2pt,s)} = \tfrac{1}{2}F(N + 2\xi) - \tfrac{1}{4}F(N) - \tfrac{1}{4}F(N + 4\xi) + \xi^2 \log 4\xi. \quad (29)$$

Including the spatial momentum dependence from Eq. (8), and by using Eq. (11), we find

$$A_{0+2}(\omega) \simeq \frac{i\pi(2\pi\lambda)^{2i\omega}}{\sinh 2\pi\omega} 2^{-2(\vec{k}^{\parallel})^2} \exp(-2\mathcal{E}^{(2pt,s)}|_{N=2i\omega,\xi=-i\omega}). \quad (30)$$

After using the on-shell condition $-\omega^2 + \vec{k}_{\parallel}^2 = \mathcal{O}(\omega^0)$ the amplitude becomes

$$A_{0+2}(\omega) = \frac{i\pi(2\pi\lambda)^{2i\omega}}{\sinh 2\pi\omega} e^{\pm i\pi\omega^2 + \mathcal{O}(\log\omega)}. \quad (31)$$

The result vanishes for large energies as $A \sim e^{-2\pi\omega}$. It is in accord with the one suggested in [21] and also matches with the bulk-boundary amplitude (25) asymptotically at $\omega_c = \omega_o$.

IV. HIGHER-POINT AMPLITUDES

A. Bulk-boundary-boundary amplitude

We can extend our method also to the three-point amplitude A_{1+2}. As above, we place the bulk charge at the origin, where it decouples from the Coulomb gas analysis. We then fix the boundary charges at the equilibrium configuration, where they are at antipodal points on the circle. The relevant total energy is given by Eq. (18). By using Eqs. (8) and (11), we find

$$A_{1+2}(\omega_c, \omega_1, \omega_2) \simeq \frac{i\pi(2\pi\lambda)^i \sum_a \omega_a}{\sinh(\pi \sum_a \omega_a)} 2^{2\vec{k}_1^\parallel \cdot \vec{k}_2^\parallel}$$
$$\times \exp\left[-2\mathcal{E}^{(2\mathrm{pt})}\big|_{N=i\sum_a \omega_a, \xi_a=-i\omega_a}\right], \tag{32}$$

where the subscripts 1, 2 (c) refer to the open strings (closed string), and $a = 1, 2, c$ in the sums. The 26-momenta can be written as

$$k_1 = (\omega_1, \vec{k}_1^\parallel, 0), \qquad k_2 = (\omega_2, \vec{k}_2^\parallel, 0),$$
$$k_c = (\omega_c, -\vec{k}_1^\parallel - \vec{k}_2^\parallel, \vec{k}^\perp). \tag{33}$$

At high energies mass-shell conditions give $\omega_a \simeq |\vec{k}_a^\parallel|$, $a = 1, 2$, and therefore

$$\omega_c^2 \simeq (\vec{k}_1^\parallel + \vec{k}_2^\parallel)^2 + (\vec{k}^\perp)^2 \geq \omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos\phi$$
$$\geq (\omega_1 - \omega_2)^2, \tag{34}$$

where ϕ is the angle between the vectors $\vec{k}_1^\parallel$ and $\vec{k}_2^\parallel$. The result for the amplitude may be written as

$$A_{1+2}(\omega_c, \omega_1, \omega_2) = \frac{i\pi(2\pi\lambda)^i \sum_a \omega_a}{\sinh(\pi \sum_a \omega_a)}$$
$$\times \exp\{(\omega_1 + \omega_2 + \omega_c)^2[\Xi(\alpha_1, \alpha_2, \phi)$$
$$\pm i\pi\Theta(\alpha_1, \alpha_2)] + \mathcal{O}(\log\omega)\}. \tag{35}$$

Here the functions Ξ and Θ are defined as

$$\Xi(\alpha_1, \alpha_2, \phi) = -\tfrac{1}{2}[F(|1 - 2\alpha_1 - 2\alpha_2|) + F(1 - 2\alpha_1)$$
$$+ F(1 - 2\alpha_2)] + \tfrac{1}{2}[F(2\alpha_1 + 2\alpha_2)$$
$$+ F(2\alpha_1) + F(2\alpha_2)] - 2F(\alpha_1 + \alpha_2)$$
$$+ 2F(1 - \alpha_1 - \alpha_2) - 2(1 - \cos\phi)$$
$$\times \alpha_1\alpha_2 \log 2, \tag{36}$$

$$\Theta(\alpha_1, \alpha_2) = \alpha_1^2 + \alpha_2^2 - \tfrac{1}{4}(1 - 2\alpha_1 - 2\alpha_2)^2$$
$$\times \theta(2\alpha_1 + 2\alpha_2 - 1), \tag{37}$$

where $F(x) = (x^2 \log x)/2$ as usual, θ is the step function, and $\alpha_a = \omega_a / \sum_b \omega_b$. Notice that the second inequality in Eq. (34) restricts $0 \leq \alpha_a \leq 1/2$, i.e., neither of the open strings can alone carry more than half of the total emitted energy.

It is crucial that the function Ξ is negative in the physical region for the result to make sense: the amplitude must not diverge at high energies. The first inequality in Eq. (34) may be written as

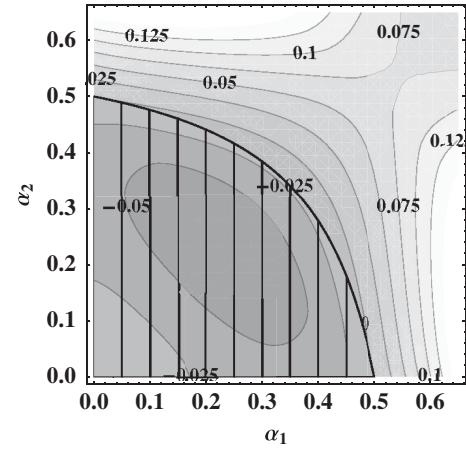$$2\alpha_1\alpha_2(1 - \cos\phi) \geq 2\alpha_1 + 2\alpha_2 - 1. \tag{38}$$

FIG. 1. The function Ξ for $\cos\phi = 0.6$. The curves lie at constant Ξ as indicated by the labels, and the striped region is the kinematically allowed one, given by Eq. (38).

We verified numerically that Ξ is negative (or zero) whenever this inequality is satisfied. Figure 1 shows the situation for $\cos\phi = 0.6$: Ξ is indeed negative in the whole kinematically allowed (striped) region. This is a most nontrivial check of our result. Notice that Ξ vanishes if $\cos\phi = \pm 1$ and Eq. (38) is saturated. In fact, Ξ vanishes if and only if the spatial momenta of all the strings are (asymptotically) parallel, with the understanding that zero is always parallel to any vector. Thus the amplitude decays exponentially

$$A_{1+2}(\omega_c, \omega_1, \omega_2) \sim e^{-\pi \sum_a \omega_a} \tag{39}$$

for large energies in these configurations, and the decay is even faster

$$A_{1+2}(\omega_c, \omega_1, \omega_2) \sim \exp\left[-\#\left(\sum_a \omega_a\right)^2\right] \tag{40}$$

in other cases.

B. Boundary amplitudes

We can also give other conjectures on the asymptotics of n-point amplitudes with $n \geq 3$. We are, however, limited to special kinematic settings, which can be accessed by solutions for the symmetric potential problems in [4]. For example, let us consider "pairwise" production of four open strings, with $\vec{k}_1^\parallel = -\vec{k}_3^\parallel$, and $\vec{k}_2^\parallel = -\vec{k}_4^\parallel$, and with possibly different energies $\omega_a \simeq |\vec{k}_a^\parallel|$, $a = 1, 2$. We fix the charges symmetrically as explained above before Eq. (19). In the case of production to orthogonal directions ($\vec{k}_1^\parallel \cdot \vec{k}_2^\parallel = 0$) this configuration also solves the equilibrium equations (B5). Proceeding as above with $\mathcal{E}^{(4\mathrm{pt})}$ from Eq. (19),

$$A_{0+4}(\omega_1, \omega_2) \simeq \frac{i\pi(2\pi\lambda)^{2i(\omega_1+\omega_2)}}{\sinh[2\pi(\omega_1 + \omega_2)]}$$
$$\times \exp[F(2\omega_1) + F(2\omega_2)$$
$$- 2F(\omega_1 + \omega_2) - 2F(|\omega_1 - \omega_2|)$$
$$- 2(\omega_1^2 + \omega_2^2)\log 2]. \tag{41}$$

One can check that the result vanishes asymptotically for any fixed ratio ω_1/ω_2 as $A \sim e^{-\#(\omega_1+\omega_2)^2}$.

V. CONCLUSIONS AND OUTLOOK

In this paper we used electrostatic techniques to investigate string scattering amplitudes of D-brane decay in bosonic string theory. In particular, we studied the high-energy limits of open and closed string emission amplitudes in the half S-brane background. We considered pair production of open strings and closed strings. We also derived a result for a mixed amplitude with one closed string and a pair of open strings and briefly discussed n-point open string amplitudes with $n \geq 3$.

Overall, our analysis revealed the expected exponential falloff behavior at high energies—the amplitudes decay with sums of the emitted energies in the exponent. However, in many cases the decay was found to be even faster, depending on the conditions for the spatial momenta.

An attractive feature of the electrostatic method is that it provides intuitive insight into the high-energy behavior of the string amplitudes. It would be worthwhile to generalize our investigations to other unstable systems. Some cases to study are (1) full S-brane, which corresponds to a collection of positive and negative unit charges [14], (2) non-BPS Dp-branes for p odd/even in Type IIA/B superstring, corresponding to paired Coulomb gases [15], and (3) inhomogeneous or lightlike decays, possibly corresponding to two sets of distinct Coulomb gases. The continuum limit with appropriate external bulk or boundary charges in each case would help one to find an approximate high-energy emission amplitude for closed or open strings.

The high-energy closed string pair production amplitude which we obtained is currently the only explicit result for this process. There are two remaining puzzles which we have so far failed to solve. First, in order to maintain symmetry in exchanging the closed strings we had to fix the energies of the closed strings to be equal [5]. This requirement is a limitation. It does not arise from the electrostatic approximation—in the exact power series expression of the amplitude, each term in the expansion is by itself asymmetric.[6] However, it would be natural for the final amplitude to depend symmetrically on the energies of the emitted closed strings.

The second puzzle is associated with open-closed duality [22]. (For discussion on the issue, see e.g. [23–26].) A striking mismatch of the duality is the failure to connect the closed string IR channel to the open string UV channel. Consider an open string annulus diagram, with rolling tachyon background on a decaying p-brane at both edges. The two natural ways of cutting the annulus, and the two corresponding different kinematical limits, give total amplitudes for open string or closed string pair production, with UV region of the open string channel corresponding to the IR region in the closed string channel [23]. However, the closed string production rate is finite for $p \leq 23$, whereas the open string pair production rate in the UV is finite for $p \leq 22$. In an ongoing work, we have tried to improve the open string analysis by including logarithmic corrections to the exponent, using the electrostatic approach, as in (31). However, the mismatch between the open string and closed string production rates seems to become even more pronounced. We have estimated the correction numerically and have found the open string pair production rate to become UV finite for all Dp-branes, with a bound $p \leq 26$.[7] So the question remains, is it valid to think of the bulk one-point and the boundary two-point amplitudes as coming from the same vacuum open string one-loop amplitude?

There are some caveats. First, there is no rigorous justification of the analytic continuation method from Euclidean to Minkowski signature, proposed in [21], to obtain the exact open string pair production amplitude. Furthermore, it seems to be very difficult to extract the IR limit of the open string amplitude in [21], in order to make contact with the closed string UV channel. As far as we know, no results in the open string IR channel are known. Second, there are no results for amplitudes in superstring theory beyond the bulk-boundary amplitude [18]. Bulk one-point amplitudes have been calculated in [27], and the closed string production rate in the IR can be easily extracted to be finite for $p \leq 7$. It would be interesting to generalize our electrostatics methods to type II superstring and find out how the open string pair production amplitude behaves at high energies.

---

[6]Notice that each term in the expansion is highly off-shell, so one does not really expect individual terms to be manifestly exchange symmetric.

[7]The emitted energy still diverges for $p = 25$.

APPENDIX A: VARIOUS CONTRACTIONS

In this appendix we will fill in some gaps between (5) and (6). For ease of reference, let us record (5)

$$A_{n_c+n_o} = \int dx^0 d^p \vec{x}_{\parallel} e^{i\sum_a k_{\mu}^{(a)} x^{\mu}} \sum_{N=0}^{\infty} \frac{(-z)^N}{N!} \int \prod_{a=1}^{n_c} \frac{d^2 w_a}{2\pi}$$

$$\times \prod_{a=n_c+1}^{n} \frac{d\tau_a}{2\pi} \times \prod_{i=1}^{N} \frac{dt_i}{2\pi}$$

$$\times \left\langle \prod_{i=1}^{N} e^{X'^0(t_i)} \prod_{a=1}^{n_c+n_o} e^{ik_a \cdot X'(w_a, \bar{w}_a)} \right\rangle. \quad (A1)$$

These yield (the singular self-contractions are dropped)

$$K_{n_c+0} = \prod_{1 \le a < b \le n_c} |1 - w_a \bar{w}_b|^{k_a^{\parallel} \cdot k_b^{\parallel} - \vec{k}_a^{\perp} \cdot \vec{k}_b^{\perp}} |w_a$$

$$- w_b|^{k_a^{\parallel} \cdot k_b^{\parallel} + \vec{k}_a^{\perp} \cdot \vec{k}_b^{\perp}} \prod_{a=1}^{n_c} |1 - w_a \bar{w}_a|^{1/2(k_a^{\parallel})^2 - (1/2)(\vec{k}_a^{\perp})^2}.$$

$$(A4)$$

Now we wish to take into account open strings, i.e., $n_o \ne 0$. They only couple to the parallel parts of the fields and have the Neumann boundary conditions. The contribution is thus easily accounted for as follows:

$$K_{n_c+n_o} = K_{n_c+0} \prod_{n_c+1 \le a < b \le n} |w_a - w_b|^{2k_a^{\parallel} \cdot k_b^{\parallel}} \prod_{a=1}^{n_c} \prod_{b=n_c+1}^{n}$$

$$\times |w_a - w_b|^{2k_a^{\parallel} \cdot k_b^{\parallel}}. \quad (A5)$$

Notice that there is a factor of 2 relative to bulk-bulk case in the exponents, since the two terms in (A3) add up.

Finally, we wish to include the contribution from the boundary deformation. Since the deformation only involves the field X^0, we get the contribution only from the temporal direction:

$$\left\langle \prod_{i=1}^{N} e^{X'^0(t_i)} \prod_{a=1}^{n_c+n_o} e^{ik_a \cdot X'(w_a, \bar{w}_a)} \right\rangle = K_{n_c+n_o} \prod_{1 \le i < j \le N}$$

$$\times |e^{it_i} - e^{it_j}|^2 \prod_{i=1}^{N} \prod_{a=1}^{n_c+n_o} |w_a - e^{it_i}|^{2\xi_a}. \quad (A6)$$

Inserting (A6) to (A1), with the expression for $K_{n_c+n_o}$ (A5) and for K_{n_c+0} in (A4), yields (6).

To calculate the full contraction, it is useful to include various contributions one by one. Denote

$$K_{n_c+n_o} \equiv \left\langle \prod_{a=1}^{n_c+n_o} e^{ik_a \cdot X'(w_a, \bar{w}_a)} \right\rangle. \quad (A2)$$

Let us first focus just on closed strings, $n_o = 0$. It is important to recall that the closed strings have mixed boundary conditions, Neumann for parallel ones ($\mu = 0, \ldots, p$) and Dirichlet for perpendicular directions ($\mu = p + 1, \ldots, 25$). This is encoded in the Green's functions [6]

$$\langle X'^{\mu}(w_a, \bar{w}_a) X'^{\nu}(w_b, \bar{w}_b) \rangle = \begin{cases} -\frac{1}{2}\eta^{\mu\nu}(\log|w_a - w_b|^2 + \log|1 - w_a \bar{w}_b|^2), & \text{Neumann} \\ -\frac{1}{2}\eta^{\mu\nu}(\log|w_a - w_b|^2 - \log|1 - w_a \bar{w}_b|^2), & \text{Dirichlet}. \end{cases} \quad (A3)$$

APPENDIX B: EQUILIBRIUM CONDITIONS

We shall look for the (global) equilibrium configuration, which is the electrostatic configuration in the Coulomb gas picture. Let us start with the boundary n-point amplitude A_{0+n} and set all spatial momenta at zero, $\vec{k}_a = 0$. Since the external charges lie on the unit circle, $\bar{w}_a = w_a^{-1}$. Therefore, we may extend the Coulomb gas partition function to an analytic function of w_a:

$$Z(N) = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{dt_i}{2\pi} \prod_{1 \le i < j \le N} |e^{it_i} - e^{it_j}|^2$$

$$\times \exp\left\{ \sum_{i,a} \xi_a [\log(e^{it_i} - w_a) + \log(e^{-it_i} - w_a^{-1})] \right.$$

$$\left. + \sum_{a<b} \xi_a \xi_b [\log(w_a - w_b) + \log(w_a^{-1} - w_b^{-1})] \right\}. \quad (B1)$$

We find that the saddle-point equations can be written as

$$0 = \frac{1}{2} \partial_{w_c} \log Z(N)$$

$$= \xi_c \left[\sum_{a \ne c} \frac{\xi_a}{w_a - w_c} + N \left\langle \frac{1}{e^{it_1} - w_c} \right\rangle + \frac{N + \sum_{a \ne c} \xi_a}{2w_c} \right], \quad (B2)$$

where the expectation value is defined by $Z(N)$. Notice that (B2) is basically (the expectation value of the conjugate of) the electric force felt by the particle at $w = w_c$. The first term is due to the self-interactions of the external charges at $w = w_a$, and the second term is the expectation value of force due to the unit charges created by the tachyon profile. This suggests that the solution of the equations for all w_c is the equilibrium configuration.

The last term in (B2) is an electric force due to a special charge at $w = 0$. The origin of this term is understood as

follows. Note that (B1) is real (for real ξ_a) when all w_a are on the unit circle. Hence the complex derivative with respect to any w_c must return a tangential force, i.e., $\propto iw_c^{-1}$ where the proportionality constant is real. The radial force equation is automatically satisfied when all $|w_a| = 1$. In Eq. (B2), this is explicitly realized by the additional charge at the origin, which cancels the radial pressure due to the interactions of the charged particles.

We may verify these arguments explicitly by splitting (B2) into radial and tangential components. We write

$$
\frac{w_c}{2} \partial_{w_c} \log Z(N) + \frac{\bar{w}_c}{2} \overline{\partial_{w_c} \log Z(N)}
$$
$$
= \frac{1}{N} \sum_{a \neq c} \left[\frac{\xi_a w_c}{w_a - w_c} + \frac{\xi_a \bar{w}_c}{\bar{w}_a - \bar{w}_c} \right]
$$
$$
+ \left\langle \frac{w_c}{e^{it_1} - w_c} + \frac{\bar{w}_c}{e^{-it_1} - \bar{w}_c} \right\rangle + \frac{N + \sum\limits_{a \neq c} \xi_a}{N};
$$

$$
\frac{iw_c}{2} \partial_{w_c} \log Z(N) - \frac{i\bar{w}_c}{2} \overline{\partial_{w_c} \log Z(N)}
$$
$$
= \frac{i}{N} \sum_{a \neq c} \left[\frac{\xi_a w_c}{w_a - w_c} - \frac{\xi_a \bar{w}_c}{\bar{w}_a - \bar{w}_c} \right]
$$
$$
+ i \left\langle \frac{w_c}{e^{it_1} - w_c} - \frac{\bar{w}_c}{e^{-it_1} - \bar{w}_c} \right\rangle,
$$

(B3)

where the former expression is the radial force and the latter one is tangential. Since both w_a and w_c lie on the unit circle,

$$
\frac{w_c}{w_a - w_c} + \frac{\bar{w}_c}{\bar{w}_a - \bar{w}_c} = -1, \qquad |w_a| = |w_c| = 1,
$$

(B4)

i.e., the radial electric field at w_c due to a particle at w_a is independent of both w_a and w_c. Hence one sees immediately that the radial component in (B3) vanishes identically, so the equilibrium configuration is fixed by the tangential equation. If one uses rotational symmetry to fix $w_c = 1$ the tangential equation becomes

$$
\frac{1}{N} \sum_{a \neq c} \xi_a \, \mathrm{Im} \, \frac{1}{w_a - 1} + \mathrm{Im} \left\langle \frac{1}{e^{it_1} - 1} \right\rangle = 0.
$$

(B5)

The above derivation was done for fixed N. When the partition function is summed over N we expect that the final equilibrium equations are found by continuing analytically to $N = -\sum_a \xi_a$, in analogue with the partition function. In the text we shall apply the equations only to such cases where the saddle-point configuration is independent of N.

The case of nonzero $\vec{k}_a$ is also interesting. As is easy to see from (6), for the boundary amplitude this means replacing ξ_a by an effective charge $\xi_a^{(c)} = \xi_a + \vec{k}_a^{\parallel} \cdot \vec{k}_c^{\parallel}/\xi_c$ in the above formulas. Extension to bulk charges is simple as well. In this case w_a and $\bar{w}_a$ can be taken to be independent, and the electric force obtained by differentiation has two nontrivial components.

[1] D. J. Gross and P. F. Mende, Phys. Lett. B **197**, 129 (1987).
[2] D. J. Gross and P. F. Mende, Nucl. Phys. **B303**, 407 (1988).
[3] D. J. Gross and J. L. Manes, Nucl. Phys. **B326**, 73 (1989).
[4] N. Jokela, M. Järvinen, and E. Keski-Vakkuri, J. Phys. A **43**, 425006 (2010).
[5] N. Jokela, M. Järvinen, and E. Keski-Vakkuri, Phys. Rev. D **80**, 126010 (2009).
[6] V. Balasubramanian, E. Keski-Vakkuri, P. Kraus, and A. Naqvi, Commun. Math. Phys. **257**, 363 (2005).
[7] S. R. Das and S. D. Mathur, Nucl. Phys. **B482**, 153 (1996).
[8] N. Jokela, M. Järvinen, and E. Keski-Vakkuri, Phys. Rev. D **79**, 086013 (2009).
[9] A. Sen, J. High Energy Phys. 04 (2002) 048.
[10] F. Larsen, A. Naqvi, and S. Terashima, J. High Energy Phys. 02 (2003) 039.
[11] N. D. Lambert, H. Liu, and J. M. Maldacena, J. High Energy Phys. 03 (2007) 014.
[12] V. Balasubramanian, N. Jokela, E. Keski-Vakkuri, and J. Majumder, Phys. Rev. D **75**, 063515 (2007).
[13] N. Jokela, M. Järvinen, E. Keski-Vakkuri, and J. Majumder, J. Phys. A **41**, 015402 (2008).
[14] N. Jokela, E. Keski-Vakkuri, and J. Majumder, Phys. Rev. D **77**, 023523 (2008).
[15] J. A. Hutasoit and N. Jokela, Phys. Rev. D **77**, 023521 (2008).
[16] N. Jokela, M. Järvinen, and E. Keski-Vakkuri, Phys. Rev. D **79**, 106005 (2009).
[17] A. B. Zamolodchikov and A. B. Zamolodchikov, Nucl. Phys. **B477**, 577 (1996).
[18] N. Jokela, E. Keski-Vakkuri, and J. Majumder, Phys. Rev. D **73**, 046007 (2006).
[19] N. Jokela, M. Järvinen, and E. Keski-Vakkuri, J. Phys. A **41**, 145003 (2008).
[20] M. E. Fisher and R. E. Hartwig, Adv. Chem. Phys. **15**, 333 (1969); H. Widom, Am. J. Math. **95**, 333 (1973); E. Basor, Trans. Am. Math. Soc. **239**, 33 (1978).
[21] M. Gutperle and A. Strominger, Phys. Rev. D **67**, 126002 (2003).
[22] A. Sen, Phys. Rev. Lett. **91**, 181601 (2003).
[23] J. L. Karczmarek, H. Liu, J. M. Maldacena, and A. Strominger, J. High Energy Phys. 11 (2003) 042.
[24] A. Sen, Int. J. Mod. Phys. A **20**, 5513 (2005).
[25] Y. Nakayama, S. J. Rey, and Y. Sugawara, J. High Energy Phys. 08 (2006) 014.
[26] Y. Song, J. High Energy Phys. 07 (2010) 030.
[27] J. Shelton, J. High Energy Phys. 01 (2005) 037.