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Into the bulk: Reconstructing spacetime from the c = 1 matrix model

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We write down exact solutions in the collective field theory of the c = 1 matrix model and in dilaton-gravity coupled to a massless scalar. Using a known correspondence between these two theories at the null boundaries of spacetime, we make a connection between scalar fields in these two theories in the bulk of spacetime. In the process, we gain insight into how a theory containing gravity can be equivalent to one without gravity. We analyze a simple time-dependent background as an example.

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I. INTRODUCTION

One of the hallmarks of gauge/gravity correspondence is the emergent nature of a noncompact spacetime dimension, with the gauge theory dual living on the boundary of the gravitational—or string theory—spacetime. The situation with c = 1 string theory is somewhat similar. On the string side, we have Liouville string theory, which is a subcritical string theory with a two dimensional target spacetime. The corresponding effective spacetime action is dilaton-gravity coupled to a massless scalar known as the "tachyon". On the gauge side, we have a gauged Matrix Quantum Mechanics in a double scaling limit, which is a large N system living in one dimension: time. Using collective field formalism, this MQM can be rewritten as a 1 + 1 dimensional field theory of a single scalar, describing the density of matrix eigenvalues. The spacial dimension in which Liouville strings propagate is emergent, as it is generated from collective behavior of matrix eigenvalues. It is less clear how gravity emerges in this picture.

The purpose of this paper is to explore in some detail how gravity arises from the c = 1 matrix model when the so called "leg pole" factors are taken into account, extending the results of [1]. We can think about this construction as a toy model for holography, as follows. The spacetime of Liouville string theory [see Fig. 1(a)] is flat, and can be parametrized by two coordinates x and t, or $x^{\pm} = t \pm x$. String coupling varies with spacelike $g_s \sim \exp(2x)$ and the strong coupling region at large x is shielded by the presence of a tachyon background, $T_0 \sim \exp(2x)$, which repels strings away from this region. In addition, the same quantum improvement which leads to the inhomogeneous string coupling (and which is necessary in a noncritical string theory) also makes the tachyon massless. Finally, in two target space dimensions, there are no transverse oscillators in the quantization of the string world sheet, so the tachyon is the only propagating degree of freedom. From the infinite ladder of string states, only some special discrete states at discrete Euclidean momenta remain. These lead to short distance bulk interactions between the tachyon quanta, described at the lowest order in α' by dilaton-gravity. Tachyon pulses enter from $\mathcal{I}^-$ to be scattered by the tachyon wall and return to $\mathcal{I}^+$. In the bulk, these pulses can interact with each other via either tachyon three-point (and higher) vertices, or by exchanging gravitons and dilatons (and more massive string fields).

The matrix model and the corresponding collective field theory can be thought of as providing boundary data for tachyon scattering. In particular, together with the leg pole transform, the matrix model allows us to calculate the exact shape of the outgoing tachyon pulse given the incoming pulse. It is in this sense that the matrix model provides us with a holographic description of dilaton-gravity. We have a gravity background which two null boundaries, $\mathcal{I}^{\pm}$, and a one dimensional "gauge theory" which supplies the scattering matrix between them.

At the same time, we have an equivalence between two different theories involving a scalar field in 1 + 1 dimensions: one with gravity (dilaton-gravity interacting with the tachyon field) and one without (collective field theory for MQM). We will see in detail how it is possible for these two theories to be equivalent, shedding perhaps some light on how gravity can be an emergent theory.

Our strategy for making connections between the bulk fields in these two theories is as follows. Given a solution to the collective field theory for MQM, we reconstruct (perturbatively in the size of the fluctuation) the boundary data

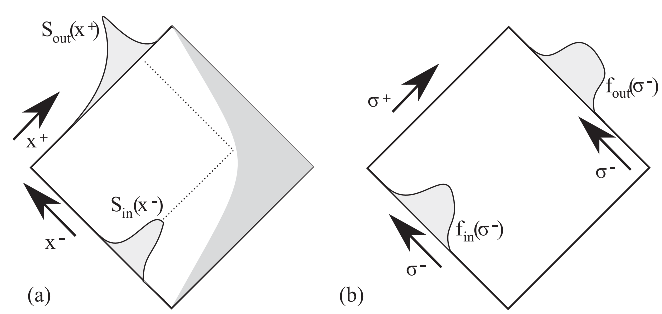

FIG. 1. (a) Penrose diagram of Liouville string theory. The shaded region is inaccessible to string degrees of freedom due to the tachyon wall. Massless tachyon pulses scatter from $\mathcal{I}^-$ to $\mathcal{I}^+$. Shape of the outgoing pulse encodes information about the interior metric and dilaton fields. (b) Penrose diagram of the collective field theory for $\mu > 0$.

in that model. We then use the leg pole transform to compute the boundary tachyon field. Finally, we use this boundary field to integrate inwards in the dilaton-gravity theory to third order in the tachyon field, reconstructing the metric, the dilaton and higher-order tachyon field corrections. As a result, we can explicitly write the metric, dilaton and tachyon field as a (nonlocal) function of the bulk field in the collective field theory (Eqs. (60) and (62)).

Naively, the spacetime on which the collective field theory lives is fixed. As was first pointed out in [2], this is not the case. Sufficiently large fluctuations of the Fermi sea of eigenvalues can in fact make the spacetime on which the collective field lives time-dependent, changing its structure, for example, by introducing spacelike infinities [3]. In such scenarios, it would be most interesting to be able to calculate the metric and dilaton of the equivalent string theory target space. We will see that our methods make this partially possible, and will calculate the metric and the dilaton for a particular time-dependent scenario.

We work in the convention where $\alpha' = 1$. In order to avoid complications arising from the tachyon background, we focus on bulk processes which do not involve it. To separate these bulk processes from the interactions with the background, we make the background parametrically small by taking the string coupling at the tachyon wall to be large.

The reminder of the paper is organized as follows: In Sec. II we discuss exact solutions to the collective field equations of motion, in the chiral, or light cone, formalism. In Sec. III we solve the equations of motion of dilaton-gravity coupled to a scalar, perturbatively to third order in the scalar field. In Sec. IV we tie the results of two previous sections together, and extract information about the relationship between the tachyon and the collective field of matrix eigenvalues. Finally, in Sec. V, we use our techniques to study a particular time-dependent background.

II. SOLUTIONS IN CHIRAL FORMALISM

In this section, we will obtain explicit formulas linking the profile of outgoing fluctuations in the collective field to the profile of incoming fluctuations. We will start with a brief review of the salient facts and definitions in the matrix model. For more details, please see [4–6] (chapter 5).

The $c = 1$ MQM has as its fundamental degrees of freedom noninteracting fermions in the upside down harmonic oscillator potential, with the Hamiltonian

$$H = \frac{1}{2}p^2 - \frac{1}{2}x^2. \tag{1}$$

Curvature of the potential is fixed by taking $\alpha' = 1$. The effective (or bosonized) picture for this system is that of a Fermi fluid moving in phase space (x, p). Because of the incompressibility of this fluid, in the classical limit it is sufficient to give the position of the Fermi surface, often presented as $p_\pm(x, t)$, the upper and lower branches of the Fermi surface as a function of x. Local density of fermions is then given by the distance between the two branches of p:

$$\varphi(x, t) \equiv \frac{1}{2}(p_+(x, t) - p_-(x, t)). \tag{2}$$

Static Fermi surfaces are constant energy hyperbolas given by $E = \frac{1}{2}p^2 - \frac{1}{2}x^2 = \mu$. For $\mu < 0$ the left and the right branches of the hyperbola do not interact: any small fluctuation around this static background evolves by moving along one arm from $x = \infty$ towards finite x and back out to $x = \infty$ along the other arm of the same branch.

Unfortunately, this description is singular at the place where the upper and the lower edge of the Fermi sea join. To avoid this singularity, we will use an equivalent description with $\mu > 0$ and allow the fluctuation to propagate from left to right along the upper branch of the hyperbola $p^2 - x^2 = 2\mu$.

To calculate the relationship between the incoming and the outgoing pulse, we will use the chiral (or light cone) formalism [7]. Our implementation of the chiral formalism is based on a simple observation that if the Fermi surface is given at $t = 0$ by $G(x, p) = 0$ for some function G, then the time evolution of this Fermi surface is given by

$$G\left(\frac{1}{2}(p + x)e^{-t} - \frac{1}{2}(p - x)e^{t}, \frac{1}{2}(p + x)e^{-t} + \frac{1}{2}(p - x)e^{t}\right) = 0. \tag{3}$$

Consider now the fluctuations of the upper branch of the hyperbola given by $p^2 - x^2 = 2\mu$, with $\mu > 0$. We define the fluctuation field η with

$$p(x, t) = \sqrt{2\mu + x^2} + 2\sqrt{\pi}\partial_x \eta. \tag{4}$$

It will turn out that η is best thought of as a function of σ such that $x = \sqrt{2\mu}\sinh\sigma$, so that $\partial_x \eta = (2\mu + x^2)^{-1/2}\partial_\sigma \eta(\sigma) \approx |1/x|\partial_\sigma \eta(\sigma)$ for large $|x|$. Fluctuations come in from $x \to -\infty$ and exit at $x \to \infty$. We would like to write down an expression connecting $\eta(x, t)$ at either infinity to the corresponding profile at finite x and t. To achieve this, consider that any pertubation around the static hyperbola can be written in the form $p^2 - x^2 = 2\mu +$ fluctuations. If the fluctuations are expressed solely in terms of either $(p - x)e^t$ or $(p + x)e^{-t}$, Eq. (3) tells us that we have a solution to the equations of motion. In this parametrization it will turn out to be easy to study the $x \to \pm\infty$ limits.

Consider then the following general exact time-dependent profile of type (3) for some function $f_{\rm in}(\sigma) << \mu$ (whose meaning will become clear in a moment):

$$p^2 - x^2 = 2\mu - 2f'_{\rm in}\left(\ln\left(\sqrt{\frac{1}{2\mu}}(p - x)e^t\right)\right). \tag{5}$$

This can be rewritten as

$$p + x = \frac{2\mu}{p-x} - \frac{2}{p-x} f'_{\rm in}\left(\ln\left(\sqrt{\frac{1}{2\mu}}(p-x)\right) + t\right). \quad (6)$$

Assuming that $f_{\rm in}(y)$ has finite support on some interval near $y = 0$, for $t \to -\infty$, $f_{\rm in}$ is nonzero only if $p - x$ is large. Then, for $t \to -\infty$, the right hand side of our equation is small, and we must have $p \approx -x$. Therefore, x is large and negative, and $p - x \approx -2x$. Substituting this in, we get

$$p = -x - \frac{\mu}{x} + \frac{1}{x} f'_{\rm in}\left(\ln\left(-\sqrt{\frac{2}{\mu}}x\right) + t\right), \quad (7)$$

which for large negative x can be rewritten as

$$p = \sqrt{x^2 + 2\mu} - \frac{1}{|x|} f'_{\rm in}(t - \sigma), \quad (8)$$

with $x = \sqrt{2\mu}\sinh\sigma \approx -\sqrt{\mu/2}\exp(-\sigma)$. We can now identify $f_{\rm in}$ with the early time η in Eq. (4), which is right-moving as expected. To be precise, for $t \to -\infty$, $\eta(\sigma, t) = (2\sqrt{\pi})^{-1} f_{\rm in}(t - \sigma)$.

The same analysis applies to late time fluctuations at large positive x. Starting with a time-dependent profile given by

$$p^2 - x^2 = 2\mu - 2f'_{\rm out}\left(-\ln\left(\sqrt{\frac{1}{2\mu}}(p+x)e^{-t}\right)\right), \quad (9)$$

we can identify $\eta(\sigma, t) = (2\sqrt{\pi})^{-1} f_{\rm out}(t - \sigma)$ at late times, $t \to \infty$, with $x \approx \sqrt{\mu/2}\exp(\sigma)$.

The crucial observation is that the profiles in Eqs. (5) and (9) are exact solutions and valid at all times. Assuming they describe the same surface, if the incoming profile is $f_{\rm in}$, the outgoing profile can be obtained from setting the right-hand sides of Eqs. (5) and (9) equal:

$$f'_{\rm in}\left(\ln\left(\sqrt{\frac{1}{2\mu}}(p-x)e^{t}\right)\right) = f'_{\rm out}\left(-\ln\left(\sqrt{\frac{1}{2\mu}}(p+x)e^{-t}\right)\right). \quad (10)$$

We now substitute the expression for $x + p$ from Eq. (6), and define $y = \ln((p - x)e^t/\sqrt{2\mu})$ to get

$$f'_{\rm in}(y) = f'_{\rm out}(y - \ln(1 - \mu^{-1} f'_{\rm in}(y))), \quad (11)$$

or, defining $y(z)$ by $z = y - \ln(1 - \mu^{-1} f'_{\rm in}(y))$, $f'_{\rm out}(z) = f'_{\rm in}(y(z))$, which is nothing more but the time delay equation in [1,8]. More interestingly, we can find the profile at any time. Given the incoming profile $f_{\rm in}$, Eq. (5) can be solved for p as a function of x treating $f_{\rm in}$ as a small perturbation. η as a function of x (or σ) and t can then be read off.

To second order in $f_{\rm in}$, we get

$$\begin{aligned} p = {}& \sqrt{x^2 + 2\mu} - \frac{1}{\sqrt{x^2+2\mu}} f'_{\rm in}(t-\sigma) \\ & + \frac{e^\sigma}{\sqrt{2\mu}(2\mu + x^2)} f'_{\rm in}(t-\sigma) f''_{\rm in}(t-\sigma) \\ & - \frac{1}{2(2\mu + x^2)^{3/2}} (f'_{\rm in}(t-\sigma))^2 + o((f'_{\rm in})^3), \end{aligned} \quad (12)$$

or

$$\begin{aligned} 2\sqrt{\pi}\partial_\sigma \eta(\sigma, t) = {}& -f'_{\rm in}(t-\sigma) + \frac{e^\sigma}{2\mu\cosh\sigma} f'_{\rm in}(t-\sigma) \\ & \times f''_{\rm in}(t-\sigma) - \frac{(f'_{\rm in}(t-\sigma))^2}{4\mu\cosh^2\sigma} \\ & + o((f'_{\rm in})^3). \end{aligned} \quad (13)$$

This can be integrated with respect to σ

$$\begin{aligned} 2\sqrt{\pi}\eta(\sigma, t) = {}& f_{\rm in}(t-\sigma) - \frac{e^\sigma}{4\mu\cosh\sigma}(f'_{\rm in}(t-\sigma))^2 \\ & + o((f'_{\rm in})^3). \end{aligned} \quad (14)$$

As a consistency check, we notice that in the large negative σ regime, $\eta = \eta_{\rm in}$, as expected, and that in the large positive σ regime, there are no left-moving terms (everything is a function of $t - \sigma$). In particular, if we take $\sigma \to \infty$ in the above equation, then

$$\begin{aligned} 2\sqrt{\pi}\eta(\sigma \to \infty) = {}& f_{\rm in}(t-\sigma) - \frac{1}{2\mu}(f'_{\rm in}(t-\sigma))^2 \\ & + o((f'_{\rm in})^3). \end{aligned} \quad (15)$$

To third order, the calculation is a bit more messy. The result for η' is again a total derivative, and can be integrated to give

$$\begin{aligned} 2\sqrt{\pi}\eta(\sigma, t) = {}& f_{\rm in}(t-\sigma) - \frac{e^\sigma}{4\mu\cosh\sigma}(f'_{\rm in}(t-\sigma))^2 \\ & - \frac{(e^{2\sigma}+3)e^\sigma}{48\mu^2\cosh^3(\sigma)}(f'_{\rm in}(t-\sigma))^3 \\ & + \frac{e^{2\sigma}}{8\mu^2\cosh^2(\sigma)}(f'_{\rm in}(t-\sigma))^2 f''_{\rm in}(t-\sigma) \\ & + o((f'_{\rm in})^4). \end{aligned} \quad (16)$$

Our procedure can be extended to any order, and gives both the collective field profile at any time, and the outgoing profile for $t \to \infty$ in terms of the incoming field, as illustrated in Fig. 1(b) and further elaborated on in Appendix A.

III. DILATON-GRAVITY COUPLED TO A MASSLESS SCALAR

Having studied the behavior of the collective field at finite x and t (i.e., interior behavior, away from the past and future null boundaries) in the collective field theory, we now turn our attention to interior behavior of the dilaton-gravity theory.

As was described above, effective field theory for Liouville string theory is dilaton-gravity coupled to the (massless) tachyon scalar. Since the tachyon is a massless field and not actually tachyonic, the action for these 3 degrees of freedom is perfectly well defined. Denoting the dilaton field with Φ and the tachyon with T, we have [1]

$$S = \frac{1}{2}\int dtdx\sqrt{-G}e^{-2\Phi}[a_1[R + 4(\nabla\Phi)^2 + 16] - (\nabla T) + 4T^2 - 2V(T)], \tag{17}$$

where we will take the tachyon potential to be

$$V(T) = \frac{a_2 T^3}{3}. \tag{18}$$

Here a_1 and a_2 are constants which were determined in [1] to be $a_1 = \frac{1}{2}$ and $a_2 = -2\sqrt{2}$.

In conformal gauge, where the metric is $ds^2 = -e^{2\rho}dx^+dx^-$, the equations of motion are [9]

$$2\partial_+^2\Phi - 4\partial_+\rho\partial_+\Phi = a_1^{-1}\partial_+T\partial_+T \tag{19}$$

$$2\partial_-^2\Phi - 4\partial_-\rho\partial_-\Phi = a_1^{-1}\partial_-T\partial_-T \tag{20}$$

$$2\partial_+\partial_-\Phi - 4\partial_+\Phi\partial_-\Phi - 4e^{2\rho} = a_1^{-1}e^{2\rho}\left(T^2 - \frac{a_2}{6}T^3\right) \tag{21}$$

$$4\partial_+\partial_-\Phi - 4\partial_+\Phi\partial_-\Phi - 2\partial_+\partial_-\rho - 4e^{2\rho} = a_1^{-1}\partial_+T\partial_-T + a_1^{-1}e^{2\rho}\left(T^2 - \frac{a_2}{6}T^3\right) \tag{22}$$

$$e^{-2\rho}(\partial_+\partial_-T - \partial_+\Phi\partial_-T - \partial_-\Phi\partial_+T) - T = -\frac{a_2}{4}T^2. \tag{23}$$

The first three equations are for the metric, the fourth is for the dilaton and the last is for the tachyon field. The last two equations can be combined to give a particularly simple relationship,

$$2\partial_+\partial_-(\rho - \Phi) + a_1^{-1}\partial_-T\partial_+T = 0. \tag{24}$$

In the absence of a tachyon field, above equation becomes $\partial_+\partial_-(\rho - \Phi) = 0$. Using up the leftover coordinate freedom $x_\pm \to \tilde{x}_\pm(x_\pm)$, we could set $\Phi = \rho$, the Kruskal gauge. However, since we are dealing with a linear dilaton background, a more natural gauge choice is the modified Kruskal gauge $\Phi = x^+ - x^- + \rho$. Either gauge choice is only possible in regions where the tachyon field is zero.

We will expand in powers of the tachyon field. To zeroth order, we have the linear dilaton background,

$$\Phi_0 = 2x = x^+ - x^-, \qquad \rho_0 = 0. \tag{25}$$

The tachyon background is a solution to the linearized version of Eq. (23) in this background,

$$\partial_+\partial_-T - \partial_-T + \partial_+T - T = 0. \tag{26}$$

The most general static solution to this equation is

$$T_0 = (b_1 x + b_2)e^{2x}. \tag{27}$$

We are working in the limit where the tachyon background can be neglected, $b_1, b_2 \to 0$, and will be expanding in powers of the incoming tachyon field: $T = T^{(1)} + T^{(2)} + T^{(3)} + \ldots$, ignoring T_0.

It will be convenient to absorb a factor of the dilaton background into T by defining a new field $S = e^{-\Phi_0}T = e^{-2x}T = S^{(1)} + S^{(2)} + S^{(3)} + \ldots$. To lowest order the equation of motion is simply

$$\partial_-\partial_+S^{(1)} = 0. \tag{28}$$

The rescaled tachyon field S is a massless scalar; above equation has solutions of the form $S^{(1)} = S_-^{(1)}(x_-) + S_+^{(1)}(x_+)$. Since the region $x \to +\infty$ is the strong coupling region, protected by the tachyon condensate, S cannot have left-moving incoming excitations, and we are left with $S^{(1)} = S_-^{(1)}(x_-)$. We are ignoring here any reflections from the tachyon wall itself.

To second order, we can linearize Eqs. (19)–(22) in gravity and dilaton fluctuations about the background, $\Phi = \Phi_0 + \delta$, to obtain

$$\partial_+^2\delta - 2\partial_+\rho = \frac{1}{2a_1}(\partial_+T^{(1)})^2 \tag{29}$$

$$\partial_-^2\delta + 2\partial_-\rho = \frac{1}{2a_1}(\partial_-T^{(1)})^2 \tag{30}$$

$$\partial_+\partial_-\delta - 4\rho + 2\partial_+\delta - 2\partial_-\delta = \frac{1}{2a_1}(T^{(1)})^2 \tag{31}$$

$$2\partial_+\partial_-\delta + 2\partial_+\delta - 2\partial_-\delta - \partial_-\partial_+\rho - 4\rho = \frac{1}{2a_1}(\partial_+T^{(1)}\partial_-T^{(1)} + (T^{(1)})^2). \tag{32}$$

The tachyon equation of motion at this level is

$$\partial_+\partial_-T^{(2)} - \partial_-T^{(2)} + \partial_+T^{(2)} - T^{(2)} = -\frac{a_2}{4}(T^{(1)})^2 \tag{33}$$

or

$$\partial_-\partial_+S^{(2)} = -\frac{a_2}{4}e^{2x}(S^{(1)})^2. \tag{34}$$

This last equation is easy to solve for $S^{(2)}(x_-)$,

$$S^{(2)} = -\frac{a_2}{4}e^{x^+}\int^{x^-} dx^- e^{-x^-}(S^{(1)}(x^-))^2. \tag{35}$$

Defining $\Omega = 2(\partial_- - \partial_+)\delta + 4\rho$, we can combine Eqs. (29)–(31) into

$$(\partial_+ - 2)\Omega = \frac{1}{a_1}((T^{(1)})^2 - (\partial_+ T^{(1)})^2) \tag{36}$$

$$(\partial_- + 2)\Omega = \frac{1}{a_1}(-(T^{(1)})^2 + (\partial_- T^{(1)})^2) \tag{37}$$

$$\partial_+\partial_-\delta = \Omega + \frac{1}{2a_1}(T^{(1)})^2 \tag{38}$$

while Eqs. (31) and (32) give

$$\partial_+\partial_-(\delta - \rho) = \frac{1}{2a_1}\partial_+ T^{(1)}\partial_- T^{(1)}. \tag{39}$$

These four equations can be integrated explicitly to give ρ and δ. Note that there are more equations (four) than unknown functions (two), and consistency between them requires that T satisfy the 1st order Eq. (26), whose most general solution is

$$T^{(1)} = \sqrt{a_1}e^{x^+ - x^-}(f_+(x^+) + f_-(x^-)). \tag{40}$$

It is easy to show that in that case, the first two equations give [10]

$$\Omega = -e^{2x^+ - 2x^-}\Big(f_+^2 + 2f_-f_+ + f_-^2 - \int_{x^+} dx^+(f_+')^2 - \int^{x^-} dx^-(f_-')^2 + 4A\Big), \tag{41}$$

and the third gives

$$\begin{aligned}\delta = {}& \frac{1}{4}\int^{x^-} dx^- e^{2x^+ - 2x^-}[(f_-')^2 - f_-^2] + \frac{1}{4}\int_{x^+} dx^+ e^{2x^+ - 2x^-}[(f_+')^2 - f_+^2] \\ & - \frac{1}{4}e^{2x^+ - 2x^-}\int_{x^+} dx^+(f_+')^2 - \frac{1}{4}e^{2x^+ - 2x^-} \times \int^{x^-} dx^-(f_-')^2 + \int^{x_-} dx^- e^{-2x^-} f_- \\ & \times \int_{x_+} dx^+ e^{2x^+} f_+ - Ae^{2x^+ - 2x^-} + \alpha_+(x^+) - \alpha_-(x^-).\end{aligned} \tag{42}$$

From the definition of Ω, ρ is then

$$\begin{aligned}\rho = {}& \frac{1}{4}\int^{x^-} dx^- e^{2x^+ - 2x^-}[(f_-')^2 - f_-^2] + \frac{1}{4}\int_{x^+} dx^+ e^{2x^+ - 2x^-}[(f_+')^2 - f_+^2] \\ & - \frac{1}{8}e^{2x^+ - 2x^-}\Big(f_+^2 + 4f_-f_+ + f_-^2 + 2\int_{x^+} dx^+(f_+')^2 + 2\int^{x^-} dx^-(f_-')^2\Big) - \frac{1}{2}e^{2x^+}f_+\int^{x_-} dx^- e^{-2x^-}f_- \\ & - \frac{1}{2}e^{-2x^-}f_-\int_{x_+} dx^+ e^{2x^+}f_+ - Ae^{2x^+ - 2x^-} + \frac{1}{2}\partial_+\alpha_+(x^+) + \frac{1}{2}\partial_-\alpha_-(x^-).\end{aligned} \tag{43}$$

The fourth equation is satisfied automatically (it is in fact implied by the other three combined with (26)).

In the above solution, A is an arbitrary integration constant, and $\alpha_\pm$ are arbitrary integration functions. $\alpha_\pm$ can be removed from the solution by a coordinate transformation which respects conformal gauge, namely, (to linear order) $x^\pm \to x^\pm + \alpha_\pm(x^\pm)$. In the interest of simplicity, we will adopt a coordinate system where $\alpha_\pm = 0$, and return to the issue of coordinate ambiguity later.

In contrast with $\alpha_\pm$, the constant A cannot be set to zero by a coordinate change. Its presence, however, is contrary to our implicit boundary conditions, since $Ae^{2x^+ - 2x^-}$ is large for $x^- \to -\infty$. If we imagine that the incoming tachyon pulse is localized (as shown in Fig. 1(a)], the metric before the pulse arrives should be flat. As we will see in a moment, inclusion of a nonzero A corresponds to a black hole background. We will therefore set $A = 0$ as well. Similar arguments apply to the region of spacetime where $x^+ \to \infty$.

The general solution in Eqs. (41)–(43) can be simplified for a certain class of problems. Because our theory has only one asymptotic weakly coupled region, and because we have ignored the presence of the background which can reflect back a scalar pulse, $S^{(1)}$ has only one component, and not two: $S^{(1)} = S^{(1)}(x_-) = \sqrt{a_1}f_-(x^-)$. Therefore (dropping the (1) subscript on S for brevity),

$$\Omega = -\frac{1}{a_1}e^{2x^+ - 2x^-}\Big(S_-^2 - \int^{x^-} dx^-(S_-')^2\Big), \tag{44}$$

$$\delta = \frac{1}{4a_1}\int^{x^-} dx^- e^{2x^+ - 2x^-}[(S_-')^2 - S_-^2] - \frac{1}{4a_1}e^{2x^+ - 2x^-}\int^{x^-} dx^-(S_-')^2 \tag{45}$$

and

$$\rho = \frac{1}{4a_1}\int^{x^-} dx^- e^{2x^+ - 2x^-}[(S_-')^2 - S_-^2] - \frac{1}{8a_1}e^{2x^+ - 2x^-}\Big(S_-^2 + 2\int^{x^-} dx^-(S_-')^2\Big). \tag{46}$$

Let us now return to the issue of integration constant A. If the incoming pulse is localized around some x^-, and we look at larger values of x^-, the metric and the dilaton outside the pulse simplify to

$$\delta = \rho = \frac{1}{4a_1} e^{2x^+} \int dx^- e^{-2x^-}[(S'_-)^2 - S_-^2] - \frac{1}{4a_1} e^{2x^+ - 2x^-} \int dx^- (S'_-)^2, \qquad (47)$$

which imply

$$\Omega = \frac{1}{a_1} e^{2x^+ - 2x^-} \int dx^- (S'_-)^2. \qquad (48)$$

This is nothing else but the 2D black hole, which in Kruskal gauge $\rho = \Phi$ is given by

$$e^{-2\rho} = e^{-2\Phi} = \frac{m}{2} - 4(y^+ - y_0^+)(y^- - y_0^-). \qquad (49)$$

Changing variables $y^\pm - y_0^\pm = \mp e^{\mp 2\tilde{y}^\pm}$ and linearizing, we get

$$e^{2\rho} = 1 - \frac{m}{8} e^{2\tilde{y}^+ - 2\tilde{y}^-} \quad \text{and} \quad e^{2\Phi} = \frac{1}{4} e^{2\tilde{y}^+ - 2\tilde{y}^-}\left(1 - \frac{m}{8} e^{2\tilde{y}^+ - 2\tilde{y}^-}\right) \qquad (50)$$

or

$$\rho = -\frac{m}{16} e^{2\tilde{y}^+ - 2\tilde{y}^-} \quad \text{and} \quad \Phi = \text{const.} + (\tilde{y}^+ - \tilde{y}^-) - \frac{m}{16} e^{2\tilde{y}^+ - 2\tilde{y}^-}. \qquad (51)$$

To compare with Eq. (47), let $\tilde{y}^+ = x^+ + Ce^{2x^+}$ and $\tilde{y}^- = x^-$. Then, for large negative x^+, the metric and the dilaton in (47) and (51) agree, with

$$m = \frac{4}{a_1} \int dx^- (S'_-)^2 \quad \text{and} \quad C = \frac{1}{4a_1} \int dx^- e^{-2x^-}[(S'_-)^2 - S_-^2]. \qquad (52)$$

Notice that the mass is simply the integral over the stress energy of the incoming pulse, as expected, and that, had we included the integration constant A, it would have contributed to the mass, signaling the presence of an undesirable black hole background unrelated to the tachyon pulse.

To compute the tachyon field to third order, we need the tachyon equation, which now includes interactions with the metric and the dilaton,

$$\partial_- \partial_+ S^{(3)} = -\frac{a_2}{2} e^{2x}(S^{(1)})(S^{(2)}) + \frac{1}{2}\Omega S^{(1)} + \partial_+ \delta \partial_- S^{(1)} + \partial_- \delta \partial_+ S^{(1)}. \qquad (53)$$

With explicit forms of Ω and δ above, this equation can be integrated as well. Since the general answer is long and not illuminating, we will not include it. It is easy to integrate the above equation given a specific tachyon profile.

IV. RELATIONSHIP BETWEEN SPACETIME AND MATRIX MODEL

In this section, we will confirm that the results of the two preceding sections are related by the leg pole transform on the boundary, and discuss a strategy towards extending the correspondence into the bulk.

The leg pole transform connects incoming tachyon field profile to incoming collective field profile via [1]

$$S_{\text{in}}(x^-) = -\int dv K(v - x^-)\eta_{\text{in}}(v) \qquad (54)$$

$$\eta_{\text{in}}(\sigma^-) = -\int dv K(\sigma^- - v) S_{\text{in}}(v), \qquad (55)$$

and the outgoing profiles via

$$S_{\text{out}}(x^+) = \int dv\, K(x^+ - v + \ln(\mu/2))\eta_{\text{out}}(v) \qquad (56)$$

$$\eta_{\text{out}}(\sigma^-) = \int dv\, K(v - \sigma^- + \ln(\mu/2)) S_{\text{out}}(v). \qquad (57)$$

$x^\pm = t \pm x$ and $\sigma^\pm = t \pm \sigma$ are light cone coordinates in spacetime and in collective field theory, as shown in Fig. 1. Kernel K of the leg pole transform is given by

$$K(v) = -\frac{w}{2} J_1(w), \quad \text{where } w = 2\left(\frac{2}{\pi}\right)^{1/8} e^{v/2}. \qquad (58)$$

It derives its name from the poles in its Fourier transform,

$$K(\omega) = \int dv e^{-i\omega v} K(v) = \left(\frac{2}{\pi}\right)^{i\omega/4} \frac{\Gamma(-i\omega)}{\Gamma(i\omega)}. \qquad (59)$$

This frequency space expression was originally derived by comparing the S-matrix of the matrix model with world sheet results in Liouville string theory [11,12].

Shifts of $\ln(\mu/2)$ in the outgoing formulas (56) and (57) are related to the position of the tachyon wall. We will be taking $\mu \to 0$, which takes the tachyon wall deeply into the strong coupling region and allows us to neglect, for the most part, scattering from the tachyon background.

Minus signs in Eqs. (54) and (55) arise from using the positive energy hyperbola to calculate the collective field scattering in Sec. II.

In Appendix A we check that our boundary results are consistent with the boundary fields being related by the leg pole transform both in the past and the future (on both $\mathcal{I}^+$ and $\mathcal{I}^-$). In particular, we reproduce the results of [1], showing that the leg pole transform of the outgoing collective field η agrees to third order with the outgoing tachyon field T if the corresponding incoming fields were related by the leg pole transform on $\mathcal{I}^-$. To this effect, Appendix B collects some useful formulas about the leg pole kernel, which are easily derived in frequency space.

Once we know the boundary profiles match up, we can explore the relationship between bulk fields. In Sec. III we computed the bulk tachyon field S from the boundary field $S_{\rm in}$ to be

$$S(x^-, x^+) = S_{\rm in}(x^-) + \frac{1}{\sqrt{2}} e^{x^+} \times \int^{x^-} dx^- e^{-x^-} (S_{\rm in}(x^-))^2 + S^{(3)}. \qquad (60)$$

To write the above, we used Eq. (35). $S^{(3)}$ is written explicitly in Eq. (53). We can now obtain our desired relationship between S and η at the same time t up to third order by solving for $S_{\rm in}$ in terms of $\eta(\sigma, t)$ at a fixed time t. Equation (16) implies that, to third order in η,

$$\eta_{\rm in}(y) = \eta(t-y, t) + \frac{\sqrt{\pi} e^{t-y}}{2\mu \cosh(t-y)} (\partial_y \eta(t-y, t))^2 + \frac{\pi(e^{2(t-y)} + 3) e^{t-y}}{12\mu^2 \cosh^3(t-y)} (\partial_y \eta(t-y, t))^3 + o(\eta^4). \qquad (61)$$

We will now apply the equal-time leg pole transform to obtain $S_{\rm in}$

$$\begin{aligned} S_{\rm in}(t-x) &= \int dv\, K(-v+x) \eta_{\rm in}(t-v) \\ &= \int dv\, K(-v+x) \bigg(\eta(v, t) + \frac{\sqrt{\pi} e^{v}}{2\mu \cosh v} \\ &\quad \times (\partial_v \eta(v, t))^2 - \frac{\pi(e^{2v} + 3) e^{v}}{12\mu^2 \cosh^3 v} (\partial_v \eta(v, t))^3 \bigg) \end{aligned} \qquad (62)$$

which we can then plug into Eq. (60), to obtain S to third order in η. Moreover, we can substitute (62) into (46) and (45) to obtain the metric and the dilaton [13]. Since the third order expression is quite cumbersome, here we just state the result to second order:

$$\begin{aligned} S(x^-, x^+) &= \int dv\, K(-v+x) \bigg(\eta(v, t) \\ &\quad + \frac{\sqrt{\pi} e^{v}}{2\mu \cosh v} (\partial_v \eta(v, t))^2 \bigg) + \frac{1}{\sqrt{2}} e^{x^+} \\ &\quad \times \int^{x^-} dx^- e^{-x^-} \bigg(\int dv\, K(-v+x) \eta(v, t) \bigg)^2. \end{aligned} \qquad (63)$$

The above formula gives the bulk tachyon field S as a function of the bulk collective field η. The procedure outlined above can be used to obtain a similar result to third order, but little is to be gained from writing it down explicitly. It follows from simply combining Eqs. (46), (45), (53), (60), and (62).

Our procedure could be extended to even higher orders in perturbation theory. Therefore, at least in principle, it is possible write explicit field redefinitions linking dilaton-gravity coupled to a massless scalar to a theory with only a single scalar field. The resulting map is nonlocal, which should come as no surprise, since it is a result of integrating out dilaton-gravity. Notice that going beyond the third order would require the inclusion of effects of heavy string states into the gravity action.

Because of diffeomorphism invariance in the dilaton-gravity theory, this field redefinition cannot be unique. We have implicitly fixed the coordinate invariance by asking that the fields to be related at equal times, hence picking a particular coordinate system in the gravity theory. For localized pulses, the coordinate ambiguity results in at most exponentially small corrections at large x.

The argument for this last fact rests on form of the detailed agreement between the collective field and the tachyon on the boundary. Let us assume a well localized incoming tachyon pulse. Under the transform (55), for σ^- large and negative, the incoming collective field has a form $\eta_{\rm in} = A_1 e^{\sigma^-} + A_2 e^{2\sigma^-} + \ldots$, while for σ^- large and positive, the falloff is much more rapid. The outgoing collective field has the same form. Therefore, the outgoing tachyon field for large negative x^+ must also be sum of terms of the form e^{kx^-} with k a positive integer. Now, consider the effect of a change of coordinates $x^+ \to \tilde{x}^+$ on Eqs. (35) and (A8). For these equations to only contribute terms in the form $e^{k\sigma^-}$, the change of coordinates much be limited to $x^+ \to x^+ + \sum_k B_k e^{kx^-}$. Therefore, on the boundary the coordinates can be fixed up to exponentially small ambiguity. A similar argument holds for the incoming boundary, and the coordinate x^-.

In the bulk, the coordinate changes are limited to those which maintain conformal gauge. This is because at the lowest order, both the collective field and the rescaled tachyon field S are massless scalars, with equations of motion $\partial_+ \partial_- S = \partial_+ \partial_- \eta = 0$. To maintain conformal gauge, the bulk coordinate changes must be of the form $x^\pm \to X^\pm(x^\pm)$, where the functions $X^\pm$ must be of the form discussed in the previous paragraph, and are fixed up to exponentially small corrections.

While explicit, our procedure for connecting bulk fields in the two theories described in this section is not straightforward. In practice, it is probably easier to match incoming field shapes and solve equations of motion in the two theories independently. In the next section, we will do just that to analyze an interesting example which goes beyond localized pulses.

V. EXAMPLE: TIME-DEPENDENT BACKGROUND

In this section we will employ our results to make a connection between the matrix model and spacetime physics in a time-dependent scenario. For convenience, especially when comparing our results with previous work on this background [2,16–19], in this section we take our matrix model background to be the left branch of $x^2 - p^2 = 2\mu$, and define the fluctuation field η in the standard way [20], $(p_+ - p_-)/2 = \sqrt{x^2 - 2\mu} + \sqrt{\pi}\partial_x \eta$, which is compatible with our definition in Eq. (4). On the left branch of the hyperbola, we have $x = -\sqrt{2\mu}\cosh\sigma$ and we will take σ to be negative, so that for large x, $x \approx -\sqrt{\mu/2}\exp(-\sigma)$.

We will focus on the following exact time-dependent profile in eigenvalue phase space [2]

$$(x + p + \lambda e^t)(x - p) = 2\mu, \tag{64}$$

which at large x and large negative t corresponds to $\eta \approx -\frac{\lambda}{2\sqrt{\pi}} e^t x \approx \frac{\lambda}{2}\sqrt{\mu/2\pi} e^{t-\sigma}$. This is the incoming η profile.

As has been shown in [16,19], exact effective action for the fluctuation η in the background with $\lambda \neq 0$ is the same as the effective action in the static background ($\lambda = 0$) under a change of coordinates from σ to $\tilde{\sigma}$ given by $\sqrt{2\mu}\cosh\sigma = \sqrt{2\mu}\cosh\tilde{\sigma} + (\lambda/2)e^t$. For σ and $\tilde{\sigma}$ large and negative, the change of coordinates is

$$e^{-\sigma^+} = e^{-\tilde{\sigma}^+} + \tilde{\lambda} \tag{65}$$

with $\tilde{\lambda} = \lambda/\sqrt{2\mu}$, $\sigma^+ = t + \sigma$ and $\tilde{\sigma}^+ = t + \tilde{\sigma}$.

To analyze spacetime behavior in this background, we first notice that under the leg pole transform, the incoming profile $\eta_{\rm in} \sim e^{t-\sigma}$ changes only by an infinite normalization constant. We have, therefore, an incoming field given by $S_{\rm in} \sim e^{x^-}$. Consider, therefore, a tachyon background $T_\lambda = \tilde{\mu}(e^{x^+ - x^-} + \hat{\lambda} e^{x^+})$ where $\hat{\lambda}$ is a renormalized constant, and we have added back the standard stationary background term to regularize our problem. In this background, small tachyon fluctuations must satisfy a linear version of Eq. (23),

$$\partial_+ \partial_- S = -\frac{a_2}{2} T_\lambda S. \tag{66}$$

This equation can be transformed into one where the background is simply $T_0 = \tilde{\mu} e^{\tilde{x}^+ - \tilde{x}^-}$ by the following change of variables

$$e^{-\tilde{x}^-} = e^{-x^-} - \hat{\lambda} x^-, \qquad \tilde{x}^+ = x^+. \tag{67}$$

In the matrix model the time-dependent background is equivalent to one which is static, and the same appears to be true in dilaton-gravity. The combined change of coordinates (65) and (67) relate these static backgrounds to each other, at least at large x (or σ). Behavior near the potential barrier is more complicated, and hard to study in the spacetime picture since the exact form of the tachyon potential is not well defined.

The physical picture is illustrated in Fig. 2. Figure 2(a) shows the time-dependent spacetime generated by the decaying Fermi sea [16] whose $\mathcal{I}^+$ is incomplete and which ends on a "moving tachyon wall". Figure 2(b) shows how this incomplete spacetime is related to the static spacetime obtained from the collective theory in the new coordinates. Figure 2(c) shows this relationship for dilaton-gravity. The metric and the dilaton are trivial in the $\tilde{x}^\pm$ coordinates to this order.

To next order, we can calculate the second order tachyon field, as well as the metric and the dilaton:

$$T^{(2)} = \frac{1}{\sqrt{2}}(\tilde{\mu}\,\hat{\lambda})^2 e^{2x^+} \tag{68}$$

$$\delta = \frac{1}{4}(\tilde{\mu}\,\hat{\lambda})^2 e^{2x^+} \tag{69}$$

$$\rho = \frac{1}{2}(\tilde{\mu}\,\hat{\lambda})^2 e^{2x^+}. \tag{70}$$

For sake of completeness, let us also find the tachyon field to third order:

$$T^{(3)} = (\tilde{\mu}\,\hat{\lambda})^3 e^{3x^+}. \tag{71}$$

While the general conformal structure and the presence of the moving tachyon wall has been studied extensively in [2,16–18], the metric has not been determined before. Our result that the metric and the dilaton are both nontrivial in the vicinity of the wall complements the analysis in the above works: not only does spacetime "dissolve" [16] but length scales are distorted as well. Away from the wall, the metric, the dilaton and higher corrections to the tachyon field are all small, and do not have an effect on the spacetime.

VI. CONCLUSION AND FURTHER DIRECTIONS

The results of Sec. III can be used to rewrite the theory of a scalar coupled to dilaton-gravity without the need for the dilaton and gravity fields, at least to lowest order in those fields. Simply take the expressions (42) and (43) and substitute them back to the tachyon equation of motion, (23). The resulting equation of motion is of course nonlocal, as is expected when trying to integrate out gravitational interaction. Diffeomorphism invariance of the original theory manifest itself in the presence of the integration functions $\alpha_\pm(x^\pm)$.

In Sec. IV we outlined a procedure for relating the solution of this nonlocal action to the simpler solutions

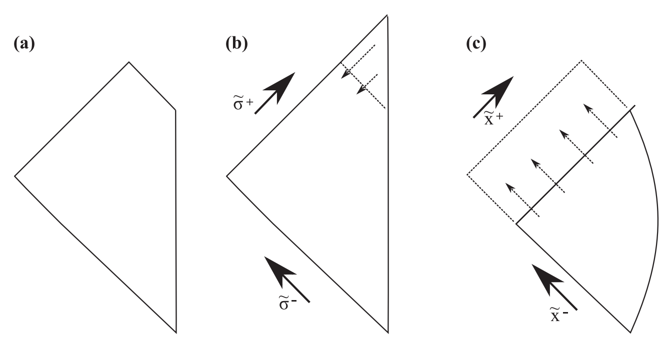

FIG. 2. Spacetime of the time-dependent solution in different coordinate systems. (a) $\sigma^\pm$ or $x^\pm$ (b) $\tilde{\sigma}^\pm$ (c) $\tilde{x}^\pm$. Dashed arrows indicate coordinate change relating the regions in (b) and (c) to the region in (a).

of the collective field theory, using the known boundary correspondence. This is a toy model for the much more complicated problem of reconstructing spacetime dynamics in AdS/CFT. Our simple example in Sec. V demonstrates how our results can be used to study time-dependent scenarios in Liouville string theory. It would be very interesting to see how these results can be used in more complicated scenarios, such as those involving spacelike future boundaries [3,21].

ACKNOWLEDGMENTS

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APPENDIX A: COMPARISON WITH THE KNOWN BOUNDARY RESULTS

In this Appendix, we will compare our bulk expressions with known results for the scattering matrix in flat spacetime, following the work of [1].

First, let us compute the large σ limit of expression (16). We could do it directly, but as a useful check, let us compute it instead from the time delay Eq. (10) $y = z + \ln(1 - \mu^{-1} f'_{\rm in}(y))$. Expanding, we get that

$$y(z) = z - \mu^{-1} f'_{\rm in}(z) + \mu^{-2} f'_{\rm in}(z) f''_{\rm in}(z) - \frac{1}{2}\mu^{-2}(f'_{\rm in}(z))^2 + o((f'_{\rm in})^3), \quad \text{(A1)}$$

and therefore

$$f'_{\rm out}(z) = f'_{\rm in}(y(z)) = f'_{\rm in}(z) - \mu^{-1} f'_{\rm in}(z) f''_{\rm in}(z) + \mu^{-2}(f'_{\rm in}(z)(f''_{\rm in}(z))^2 - \frac{1}{2}\mu^{-2}(f'_{\rm in}(z))^2 f''_{\rm in}(z)) + \frac{1}{2}\mu^{-2}(f'_{\rm in}(z))^2 f'''_{\rm in}(z) + o((f'_{\rm in})^4), \quad \text{(A2)}$$

which can be integrated to give

$$f_{\rm out} = f_{\rm in} - \frac{1}{2\mu}(f'_{\rm in})^2 - \frac{1}{6\mu^2}(f'_{\rm in})^3 + \frac{1}{2\mu^2} f''_{in}(f'_{\rm in})^2 + o((f_{\rm in})^4), \quad \text{(A3)}$$

implying that

$$\eta_{\rm out} = \eta_{\rm in} - \frac{\sqrt{\pi}}{\mu}(\eta'_{\rm in})^2 - \frac{2\pi}{3\mu^2}(\eta'_{\rm in})^3 + \frac{2\pi}{\mu^2}\eta''_{\rm in}(\eta'_{\rm in})^2 + o((\eta_{\rm in})^4). \quad \text{(A4)}$$

This agrees with Eq. (16) when $\sigma \to \infty$.

Next, let us treat a special case, of Eq. (53) where we will imagine that the incoming field is made up of two well separated pulses with finite support, $S_-^{(1)}(x^-) = S_-^{(1A)}(x^-) + S_-^{(1B)}(x^-)$, with the A pulse centered around x_A^- and the B pulse centered around $x_B^- = x_A^- + T$, with T large. We will think of $S_-^{(1A)}(x^-)$ as a source for the second order fields (tachyon, dilaton and metric) and examine scattering of the second pulse, B, from this background. For $x_- \to +\infty$, the outgoing third order tachyon field is

$$S^{(3)} = -\frac{a_2}{2}\int^{x^+} dx^+ \int dx^- e^{x^+ - x^-}(S_-^{(1B)})(S_-^{(2)A}) \quad \text{(A5)}$$

$$+ \int^{x^+} dx^+ \int dx^- \left(\frac{1}{2}\Omega - \partial_+\partial_-\delta\right) S_-^{(1B)}. \quad \text{(A6)}$$

Combining all our previous results,

$$-\frac{a_2}{2} e^{x^+ - x^-}(S^{(2)A}) + \frac{1}{2}\Omega - \partial_+\partial_-\delta = \frac{a_2^2}{8} e^{2x^+ - x^-}\int dx^- e^{-x^-}(S^{(1A)}(x^-))^2 - \frac{1}{2a_1} e^{2x^+ - 2x^-}\int dx^- (\partial_{x^-} S_-^{(1A)})^2, \quad \text{(A7)}$$

and therefore

$$S^{(3)} = \frac{a_2^2}{16} e^{2x^+}\int dx^- e^{-x^-} S^{(1B)}(x^-)\int dx^- e^{-x^-}(S^{(1A)}(x^-))^2 - \frac{1}{4a_1} e^{2x^+}\int dx^- e^{-2x^-} S^{(1B)}(x^-) \times \int dx^- (\partial_{x^-} S_-^{(1A)}(x_-))^2. \quad \text{(A8)}$$

The first term is due to a Feynman diagram shown in Fig. 3(a) and the second due to that in Fig. 3(b). In the latter case, it is the total stress energy of the pulse which determines the result, in other words, pulse B scatters from the dilaton-gravity background created by the first pulse.

We will now combine the scattering formula (A4) with the leg pole transforms to reproduce this result from the matrix model. We will assume that the incoming pulse is well localized (with Gaussian falloff, for example) around $x^- = 0$. Our calculation will reproduce the results of [1].

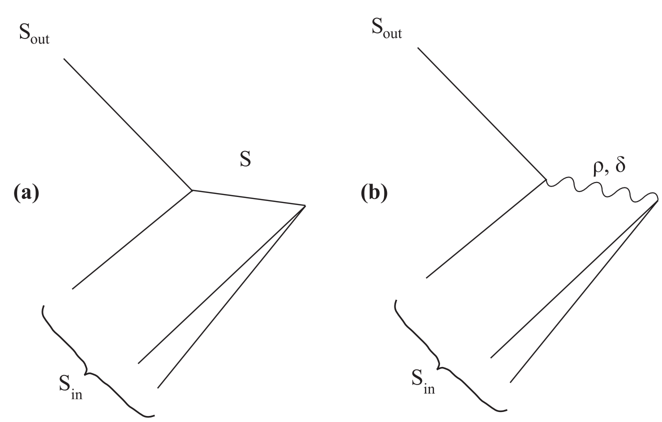

FIG. 3. Feynman diagram for the third order scattering of tachyons: (a) with a tachyon as an intermediate state (b) with a dilaton or a graviton as an intermediate state.

To first order in $S_{\rm in}$ (and $\eta_{\rm in}$) we have

$$S^{(1)}_{\rm out}(x^+) = -\int du\Big(\int dv K(x^+ + \ln(\mu/2) - v)K(v-u)\Big)S_{\rm in}(u). \tag{A9}$$

The kernel in the bracket can be, at first approximation, thought to be local, and centered around $u = x^+ + \ln(\mu/2)$. Therefore, for an incoming pulse centered around x^-, the bulk of the outgoing pulse is centered around $x^+ = x^- - \ln(\mu/2)$, indicating that the scattering takes place at string coupling $e^{2x} = e^{x^+-x^-} = 1/\mu = g_{\rm st}$, as expected (up to a coupling independent shift). The shape of the tachyon background can be deduced from the detailed shape of the scattered pulse, but we are not interested in it.

To second order, we have

$$\begin{aligned}S^{(2)}_{\rm out}(x^+) &= \int dv\, K(x^+ + \ln(\mu/2) - v)\Big(-\frac{\sqrt{\pi}}{\mu}\eta'_{\rm in}(v)^2\Big)\\ &= -\frac{1}{2\mu}\int du_1du_2\Big(\int dv\, K(x^+ + \ln(\mu/2) - v)K'(v-u_1)K'(v-u_2)\Big)S_{\rm in}(u_1)S_{\rm in}(u_2).\end{aligned} \tag{A10}$$

We are interested in the region where $x^+ + \ln\mu$ is large and negative, so we can use formula (B3) in the Appendix B to obtain

$$\begin{aligned}S^{(2)}_{\rm out}(x^+) &= \frac{\sqrt{2}}{\mu}\int du\, e^{x^+ + \ln(\mu/2) - u}(S_{\rm in}(u))^2\\ &= \frac{1}{\sqrt{2}}\int du\, e^{x^+-u}(S_{\rm in}(u))^2.\end{aligned} \tag{A11}$$

Notice that the answer is independent of μ; this is bulk scattering, and does not depend on the position of the tachyon wall. The answer is in agreement with Eq. (35), with $a_2 = -2\sqrt{2}$.

To third order, we will assume that the incoming tachyon profile is made up of two pulses, just like we did in Sec. II. Then,

$$\begin{aligned}S^{(3)}_{\rm out}(x^+) &= \int dv\, K(x^+ + \ln(\mu/2) - v)\times\Big(\frac{2\pi}{3\mu^2}(\partial_v - 1)(\eta'_{\rm in}(v))^3\Big)\\ &= \frac{2\pi}{3\mu^2}\int du_1du_2du_3\,(3S^B_{\rm in}(u_1)\,S^A_{\rm in}(u_2)\times S^A_{\rm in}(u_3))\Big(\int dv(1-\partial)K(x^+ + \ln(\mu/2) - v)\\ &\quad\times K'(v-u_1)K'(v-u_2)K'(v-u_3)\Big).\end{aligned} \tag{A12}$$

Using Eq. (B4) in the Appendix B, this becomes

$$\begin{aligned}S^{(3)}_{\rm out}(x^+) &= \frac{1}{2}\int du_1du_2du_3(e^{2x^+-u_1-u_2}\delta(u_2-u_3) + e^{2x^+-2u_1}\delta''(u_2-u_3))S^B_{\rm in}(u_1)S^A_{\rm in}(u_2)S^A_{\rm in}(u_3)\\ &= \frac{1}{2}\int du_1 e^{2x^+-u_1}S^B_{\rm in}(u_1)\int du_2 e^{-u_2}(S^A_{\rm in}(u_2))^2 - \frac{1}{2}\int du_1 e^{2x^+-2u_1}S^B_{\rm in}(u_1)\int du_2(\partial S^A_{\rm in}(u_2))^2,\end{aligned} \tag{A13}$$

which agrees with Eq. (A8) for $a_1 = \frac{1}{2}$, and $a_2 = -2\sqrt{2}$ as before.

APPENDIX B: INTEGRALS INVOLVING THE LEG POLE KERNEL K

Using the Fourier transform form of K, it is easy to show that the following integrals are true:

$$\int dy K(y-x_1)K(y-x_2) = \delta(x_1-x_2), \tag{B1}$$

for any x_1 and x_2;

$$\int dy K(x-y)K(y-x_1) = \sqrt{\frac{2}{\pi}}\Big((x-x_1) + 4\gamma - 2 + \ln\sqrt{\frac{2}{\pi}}\Big)e^{x-x_1}, \tag{B2}$$

for $x - x_1$ large and negative;

$$\int dy K(x-y)\partial K(y-x_1)\partial K(y-x_2) = -\sqrt{\frac{2}{\pi}}e^{x-x_1}\delta(x_1-x_2), \tag{B3}$$

for $x - x_i$, $i = 1, 2$ large and negative; and finally, for $x - x_i$ large and negative, and with $x_1 - x_2 \gg |x_2 - x_3|$,

$$\int dy(1-\partial)K(x-y)\partial K(y-x_1)\partial K(y-x_2)\partial K(y-x_3) = \frac{1}{\pi}e^{2x-x_1-x_2}\delta(x_2-x_3) + \frac{1}{\pi}e^{2x-2x_1}\delta''(x_2-x_3). \tag{B4}$$

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