Nielsen-Olesen vortices for large Ginzburg-Landau parameter

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Using analytic and numerical techniques Nielsen-Olesen vortices, which in the context of Ginzburg-Landau theory are known as Abrikosov vortices of type-II superconductors, are studied for large Ginzburg-Landau parameter λ. We show that their energy is equal to $(\pi n^2/2)\log\lambda$ to leading order, where n is the winding number of the vortex, and find that the limit of the gauge field can be expressed in terms of the modified Bessel function K_1. The leading terms of the asymptotic expansion of the solution are given, and the different contributions to the energy are analyzed.

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I. INTRODUCTION

Of all the localized finite-energy solutions of classical gauge theories, the vortices of the Abelian Higgs model in 2 space dimensions, the prototype of a gauge field theory with spontaneous symmetry breaking, should be the ones easiest to understand. However, none of these solutions is given in terms of known functions. Nielsen and Olesen [1] found the time-independent, radially symmetric, localized finite-energy solutions of the Abelian Higgs model in 2 space dimensions, the Nielsen-Olesen vortices, by reducing the equations of motion to two second-order equations for two radial functions. The mathematically rigorous proof that the resulting equations for the two radial functions have solutions with the required properties was given by Tyupkin *et al.* [2] and Berger and Chen [3].

In the context of Ginzburg-Landau theory, which is the time-independent Abelian Higgs model without an electric field, the Nielsen-Olesen vortices are known as Abrikosov vortices of type-II superconductors [4]. This means that the properties of Nielsen-Olesen vortices can be, and have been, studied in experiments. The Nielsen-Olesen vortices also provide a simple example of cosmic strings [5], which might explain some of the structures seen in the Universe today.

With the solution not available in terms of known functions, numerical computations become all the more important. For the Nielsen-Olesen vortices the numerical work started soon after the solutions were found [6,7]. Asymptotic analysis is another technique often applied when the explicit solution is not known. For the Nielsen-Olesen vortices, Berger and Chen [3] obtained some asymptotic results for a large Ginzburg-Landau parameter. The asymptotic analysis of the monopole structure was given by Kirkman and Zachos [8]. More recently, the same techniques were used for the Skyrmion [9] and a Skyrme-like monopole [10]. In this paper, we perform a similar asymptotic analysis for the Nielsen-Olesen vortices.

II. RADIALLY SYMMETRIC SOLUTIONS

The Hamiltonian density of the time-independent Abelian Higgs model in 2 space dimensions is given by

$$\mathcal{H} = \frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}(D_i\phi)(D^i\phi)^* + \frac{\lambda}{8}(|\phi|^2 - 1)^2. \quad (1)$$

Here $D_i\phi = \partial_i\phi - \imath A_i\phi$ and $F_{ij} = \partial_i A_j - \partial_j A_i$ $(i, j = 1, 2)$ are the covariant derivative and the field strength, respectively, and the metric is $g = \mathrm{diag}(1, 1)$. $\mathcal{H}$ in Eq. (1) is also the Ginzburg-Landau free energy of a superconductor. In this model, the Ginzburg-Landau parameter λ is equal to 1 at the point between type-I and type-II superconductivity. The corresponding Euler-Lagrange equations are

$$D_iD^i\phi - \frac{\lambda}{2}\phi(|\phi|^2 - 1) = 0,$$
$$\partial_i F^{ji} + \frac{\imath}{2}[\phi^* D^j\phi - \phi(D^j\phi)^*] = 0. \quad (2)$$

The Euler-Lagrange equations have radially symmetric solutions of the form

$$\phi = f(r)e^{\imath n\theta}, \qquad A_i = -\frac{a(r)}{r^2}\varepsilon_{ij}x^j, \quad (3)$$

where $n = \pm1, \pm2, \ldots$ is the winding number. The radial functions satisfy the equations

$$a'' - \frac{1}{r}a' + f^2(n - a) = 0,$$
$$f'' + \frac{1}{r}f' - \frac{(n-a)^2}{r^2}f = \frac{\lambda}{2}(f^2 - 1)f, \quad (4)$$

and the boundary conditions for regular vortex solutions to exist are

$$f(0) = a(0) = 0, \quad \lim_{r\to\infty} f(r) = 1, \quad \lim_{r\to\infty} a(r) = n. \quad (5)$$

These solutions are the Nielsen-Olesen vortices [1] of the Abelian Higgs model, or for $\lambda > 1$ the Abrikosov vortices of type-II superconductors. The existence proof for such solutions was given by Tyupkin *et al.* [2]. The proof is based on the fact that the Nielsen-Olesen solution minimizes the energy

$$E[a(r), f(r)] = \int_0^\infty \mathcal{E} dr = 2\pi \int_0^\infty \left[\frac{a'^2}{2r} + \frac{r}{2} f'^2 + \frac{1}{2r}(n-a)^2 f^2 + \frac{\lambda}{8} r(f^2-1)^2\right] dr. \tag{6}$$

The asymptotic behavior of the solutions for $r \ll 1$ (and finite λ) is

$$\begin{aligned} f &= f_n r^n - \frac{(\lambda + 4na_2)f_n}{8(n+1)} r^{n+2} + \cdots, \\ a &= a_2 r^2 - \frac{f_n^2}{4(n+1)} r^{2n+2} + \cdots. \end{aligned} \tag{7}$$

For $r \gg 1$ we have [11]

$$\begin{aligned} a &= n + \alpha\sqrt{r} e^{-r} + \cdots, \\ f &= \begin{cases} 1 + \beta \frac{e^{-\sqrt{\lambda} r}}{\sqrt{r}} + \cdots & (\lambda \le 4), \\ 1 + \frac{\alpha^2 e^{-2r}}{(4-\lambda) r} + \cdots & (\lambda > 4). \end{cases} \end{aligned} \tag{8}$$

Equations (4) with boundary conditions Eq. (5) cannot be solved analytically. By employing a collocation method for boundary-value ordinary differential equations equipped with an adaptive mesh selection procedure in a compactified grid [12], we have solved numerically the equations with high accuracy (global tolerance 10^{-9}) for a large range of values of λ. In Fig. 1 we show the energy E as a function of λ for small values of λ. We clearly see that E/n does not depend on n at $\lambda = 1$ and is increasing with n for $\lambda > 1$ and decreasing with n for $\lambda < 1$. That E/n does not depend on n means that the forces balance at $\lambda = 1$, which makes it possible for solutions corresponding to vortices at arbitrary separation to exist [13].

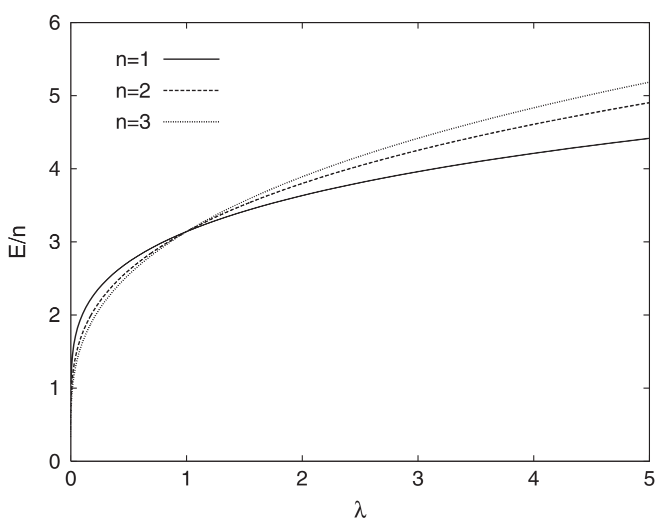

FIG. 1. Energy per vortex number E/n versus λ for Nielsen-Olesen solutions with $n = 1, 2, 3$.

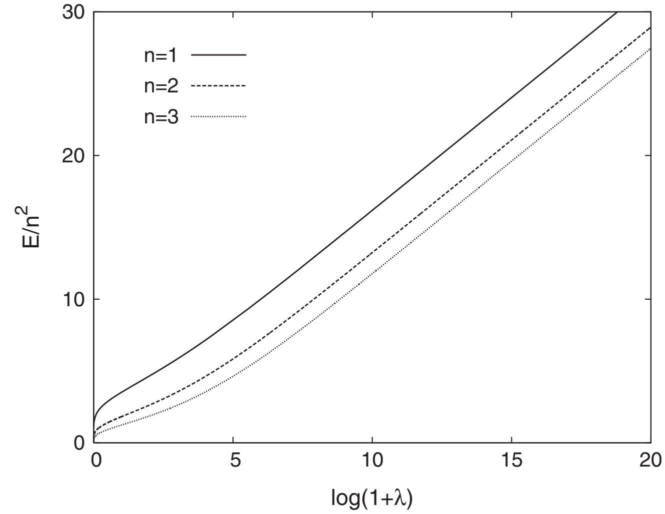

FIG. 2. Energy over n^2 versus $\log(1+\lambda)$ for Nielsen-Olesen solutions with $n = 1, 2, 3$.

Extending the computations for larger values of λ we observe numerically a logarithmic divergence of the energy. This is exhibited in Fig. 2. One can also see that, to leading order, the energy increases quadratically with the vortex number n. A detailed analysis of the numerical data reveals that the energy follows the following asymptotic formula:

$$\frac{E^{\rm num}}{n^2} = \frac{\pi}{2} \log\lambda + \Delta(n) + o(1), \tag{9}$$

where

$$\Delta(n) = \begin{cases} 0.471\,99, & n = 1, \\ -2.481\,72, & n = 2, \\ -3.953\,72, & n = 3. \end{cases} \tag{10}$$

In the next section we will prove rigorously that the energy behaves like that by performing an asymptotic analysis of Nielsen-Olesen solutions.

III. ASYMPTOTIC ANALYSIS

For our asymptotic analysis, we first split the energy Eq. (6) into four parts:

$$E_1 = 2\pi \int_0^\infty \frac{a'^2}{2r} dr, \tag{11}$$

$$E_2 = 2\pi \int_0^\infty \frac{r}{2} f'^2 dr, \tag{12}$$

$$E_3 = 2\pi \int_0^\infty \frac{1}{2r}(n-a)^2 f^2 dr, \tag{13}$$

$$E_4 = 2\pi \int_0^\infty \frac{\lambda}{8} r(f^2-1)^2 dr. \tag{14}$$

These four contributions to the total energy correspond to the gauge field contribution (E_1), the Higgs dynamical contribution (E_2 and E_3), and the contribution of the potential (E_4), respectively.

To study the dependence of the energy on λ we differentiate Eq. (6) with respect to λ and obtain

$$\frac{dE}{d\lambda} = \int_0^\infty \Big(\frac{\partial \mathcal{E}}{\partial \lambda} + \frac{\partial a}{\partial \lambda}\frac{\partial \mathcal{E}}{\partial a} + \frac{\partial a'}{\partial \lambda}\frac{\partial \mathcal{E}}{\partial a'} + \frac{\partial f}{\partial \lambda}\frac{\partial \mathcal{E}}{\partial f} + \frac{\partial f'}{\partial \lambda}\frac{\partial \mathcal{E}}{\partial f'}\Big) dr = \int_0^\infty \frac{\pi r}{4}(f^2-1)^2 dr > 0. \tag{15}$$

Here we have used integration by parts, the equations for a and f [Eq. (4)], and have assumed that $(a'/r)(\partial a/\partial \lambda)$ and $rf'(\partial f/\partial \lambda)$ vanish as $r \to 0$ and as $r \to \infty$. We see that the energy increases with λ and, if the energy is bounded, that $f = 1$ $(r > 0)$ in the limit $\lambda \to \infty$.

We will now show that the energy of the Nielsen-Olesen vortex is not bounded for $\lambda \to \infty$ but that nevertheless f will approach the singular limit $f = 1$ $(r > 0)$. We start by considering both possibilities. If we do not have $f = 1$ $(r > 0)$ in the limit, the integral in Eq. (15) does not go to zero and E_4 is at least of order λ for large λ. That is in contradiction to numerics. In Fig. 3 we exhibit E_4 as a function of λ. It clearly tends to $n^2\pi/2$ in the limit $\lambda \to \infty$, so it is bounded in that limit.

On numerical evidence, we conclude that function f tends to the singular limit $f = 1$ $(r > 0)$ when $\lambda \to \infty$. [Pursuing this possibility, we will later also conclude that f must tend to the singular limit $f = 1$ $(r > 0)$ based on a series of analytic arguments alone.] The way this limit is approached may be understood by plotting f as a function of the scaled radial coordinate $\sqrt{\lambda} r$. The shape of $f(\sqrt{\lambda} r)$ depends on λ very slightly, reaching the profile of the limiting case ($\lambda = \infty$) very quickly, above $\lambda \approx 100$. We show this fact in Fig. 4, where f is plotted as a function of $\sqrt{\lambda} r$ for Nielsen-Olesen solutions with $n = 1$ and several values of λ. The main consequence of this is that the region where f differs from 1 for large λ has a width of order $1/\sqrt{\lambda}$.

In the limit $\lambda \to \infty$, the function a satisfies the equation

$$a'' - \frac{1}{r}a' - a = -n \qquad (r > 0). \tag{16}$$

The general solution of this equation is

$$a = n + c_1 r K_1(r) + c_2 r I_1(r), \tag{17}$$

in terms of the modified Bessel functions K_1 and I_1. The condition for $r \to \infty$ implies $c_2 = 0$, the condition $a(0) = 0$ means $c_1 = -n$, and we have

$$E_3 = \pi n^2 \int_0^\infty r K_1^2(r) dr, \tag{18}$$

which is divergent, since the integrand is of order $1/r$ for small r. So the energy is definitely not bounded in the limit $\lambda \to \infty$.

That $a \to n - nrK_1(r)$ and therefore $F_{12} = a'(r)/r = nK_0(r)$ as $\lambda \to \infty$ has been shown before by Berger and Chen [3]. Berger and Chen study the equation for the magnetic field F_{12}. They show that the equation for F_{12} linearizes and is of the form

$$-\Delta F_{12}(\vec{x}) + F_{12}(\vec{x}) = 2\pi n \delta(\vec{x}), \tag{19}$$

in the limit $\lambda \to \infty$. $F_{12} = nK_0(r)$ is the solution of this equation.

Before we derive the asymptotic behavior of E_3 for large λ, we calculate the $\lambda \to \infty$ limit of E_1 and E_4. For $a = n - nrK_1(r)$ we have

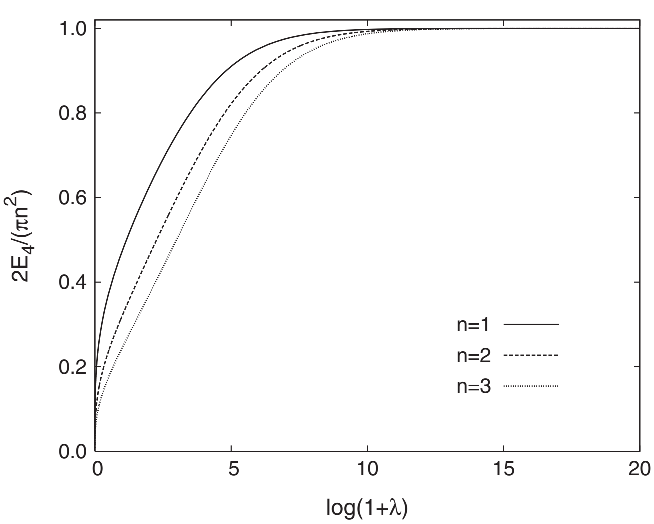

FIG. 3. E_4 versus $\log(1 + \lambda)$ for Nielsen-Olesen solutions with $n = 1, 2, 3$.

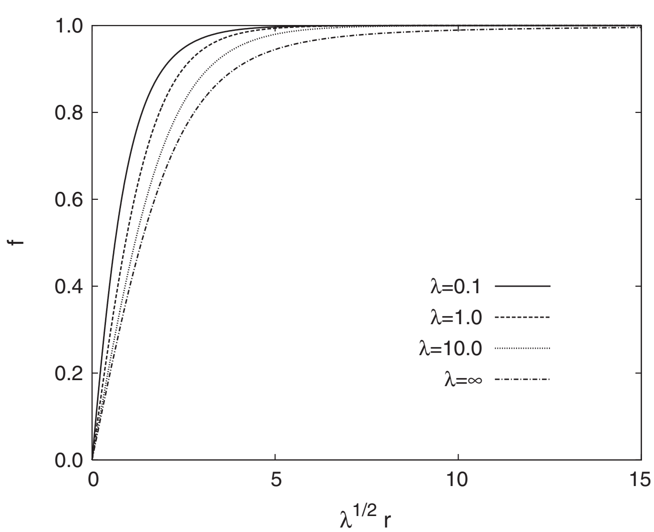

FIG. 4. Function f as a function of $\sqrt{\lambda} r$ for $\lambda = 0.1, 1.0, 10.0$ and the limiting case $\lambda = \infty$.

$$E_1 = \pi n^2 \int_0^\infty r K_0^2 dr = \frac{\pi n^2}{2}[r^2(K_0^2 - K_1^2)]_0^\infty = \frac{\pi n^2}{2}. \tag{20}$$

Because the solution $(a(r), f(r))$ minimizes the energy, we have a family of functions $(a(\gamma r), f(\gamma r))$ that satisfies

$$\begin{aligned}\frac{d}{d\gamma}E[a(\gamma r), f(\gamma r)]|_{\gamma=1} &= \frac{d}{d\gamma}(\gamma^2 E_1[a(r)] + E_2[f(r)] \\ &\quad + E_3[a(r), f(r)] \\ &\quad + \gamma^{-2}E_4[f(r)])|_{\gamma=1} \\ &= 2(E_1[a(r)] - E_4[f(r)]) = 0,\end{aligned} \tag{21}$$

which is a manifestation of Derrick's theorem. Therefore, both E_1 and E_4 approach the finite value $\pi n^2/2$ in the limit $\lambda \to \infty$, in agreement with the numerical computations (see Fig. 3). Using the asymptotic value of E_4 in Eq. (15), we get

$$\frac{dE}{d\lambda} = \frac{\pi n^2}{2\lambda} \Leftrightarrow E = \frac{\pi n^2}{2}\log\lambda, \tag{22}$$

to leading order. Since the solution minimizes the energy, the second possibility, where f tends to $f = 1$ $(r > 0)$, must be the one that is realized. We have already seen that the energy is at least of order λ for large λ, if f does not tend to $f = 1$ $(r > 0)$.

The logarithmic divergence of the energy of Nielsen-Olesen solutions in the limit of large λ comes from the contribution E_3, since E_2 remains finite. E_2 and E_3 have the following behavior for large λ:

$$E_2 = n^2\Delta_2(n) + o(1), \tag{23}$$

$$E_3 = \frac{\pi}{2}n^2\log\lambda + n^2\Delta_3(n) + o(1), \tag{24}$$

where the first three values of the functions $\Delta_2(n)$ and $\Delta_3(n)$ are

$$\Delta_2(n) = \begin{cases} 0.876\,79, & n = 1 \\ 0.325\,89, & n = 2 \\ 0.177\,08, & n = 3 \end{cases} \tag{25}$$

and

$$\Delta_3(n) = \begin{cases} -3.546\,39, & n = 1 \\ -5.949\,20, & n = 2 \\ -7.272\,39, & n = 3, \end{cases} \tag{26}$$

respectively.

Before we continue with our asymptotic analysis, we look at the variational analysis by Hill, Hodges, and Turner [7] for a large Ginzburg-Landau parameter. Hill, Hodges, and Turner use the functions

$$f = 1 - e^{-\mu r}, \qquad a = n(1 - e^{-hr})^2 \tag{27}$$

and minimize the energy with respect to μ and h. (From our previous discussion we know that μ should go to infinity and h should go to a constant as $\lambda \to \infty$, if there is any chance of approximating the correct asymptotic results.) With this ansatz the four terms of the energy are

$$\begin{aligned} E_1 &= 4\pi n^2 h^2 \log\frac{9}{8}, \\ E_2 &= \pi/4, \\ E_3 &= \pi n^2 G(s), \\ E_4 &= \frac{89\pi\lambda}{576\mu^2}, \end{aligned} \tag{28}$$

where $s = \mu/h$ and

$$G(s) = \log\frac{3^4(s+2)^7(2s+3)^4(s+4)^2}{2^{11}(s+3)^8(s+1)^4}. \tag{29}$$

Minimizing the energy with respect to μ and h leads to the equations

$$\frac{h}{\mu^3} = \frac{288n^2G'(s)}{89\lambda}, \qquad \frac{h^3}{\mu} = \frac{G'(s)}{8\log(9/8)}. \tag{30}$$

For large s, $G(s) = \log s + \log(3^4/2^7) + O(1/s^2)$,

$$\mu = \frac{\sqrt{89\lambda}}{12\sqrt{2}n} + O\left(\frac{1}{\sqrt{\lambda}}\right), \qquad h = \frac{1}{2\sqrt{2\log(9/8)}} + O\left(\frac{1}{\lambda}\right), \tag{31}$$

and

$$\begin{aligned} E_1 &= \frac{\pi n^2}{2}, \\ E_2 &= \pi/4, \\ E_3 &= \pi n^2\left(\frac{1}{2}\log\lambda + \log\frac{3^3\sqrt{89\log(9/8)}}{2^8 n}\right), \\ E_4 &= \frac{\pi n^2}{2}, \end{aligned} \tag{32}$$

up to order $o(1)$. We see that this approximation gives the correct leading terms for E_1 and E_3. Using the argument we used in Eq. (21) on the energy $E(\mu, h)$ we get $E_1 = E_4$, and therefore the leading term of E_4 must also be correct. The $O(1)$ terms in E_2 and E_3 and the total energy E are not correct. For $n = 1$, e.g., the variational method gives the upper bound $E = (\pi/2)\log\lambda + 0.551$, whereas the correct value is $E = (\pi/2)\log\lambda + 0.472$, as we saw previously [see Eq. (10)]. That we do not obtain the correct values is no surprise. For $\lambda \to \infty$ the function f in Eq. (27) goes to the step function, which is the correct asymptotic limit. The function a in Eq. (27), however, does not go to $n - nrK_1(r)$. Furthermore, the limit is not approached using the asymptotic expansions of solutions. The functions in Eq. (27) do not even have the correct asymptotic behavior [Eq. (8)] for large r.

We now give the correct asymptotic approximation for large λ. Motivated by Fig. 4 and its interpretation, we are looking for a family of approximations with the following features: In the outer region, f approaches 1, and a approaches $n - nrK_1$. In the boundary layer (for $r \lesssim r_0$), f gets steeper with increasing λ and the width of the layer goes to zero in the limit. This means that the outer approximation $a = n - nrK_1$ extends down to $r = 0$ in the limit $\lambda \to \infty$, although $n - nrK_1$ does not have the asymptotic behavior [Eq. (7)] of a, since $1 - rK_1 = -(r/2)\log r + \cdots$ for small r; i.e., the limit is singular.

Away from the boundary layer, we look for an outer solution of the form

$$f = 1 - \frac{1}{\lambda}\tilde{f} + \cdots, \quad a = n - nrK_1 + \frac{1}{\lambda}\tilde{a} + \cdots \quad (r > r_0) \tag{33}$$

and find

$$\begin{aligned} \tilde{f} &= n^2 K_1^2, \\ \tilde{a} &= k_n r K_1 + 2n^3 r K_1 \int_r^\infty s I_1(s) K_1^3(s) ds \\ &\quad - 2n^3 r I_1 \int_r^\infty s K_1^4(s) ds, \end{aligned} \tag{34}$$

where k_n is a constant. For $r \gg 1$ the solutions are of the form of Eq. (8) with $\alpha = \sqrt{\pi/2}n$. Also $\tilde{f}/\lambda \ll 1$ holds for $r \gg 1/\sqrt{\lambda}$.

In the boundary layer, a stays very small and f rises rapidly. As an approximation we can therefore use for the inner solution the equation

$$f'' + \frac{1}{r}f' - \frac{n^2}{r^2}f = \frac{\lambda}{2}(f^2 - 1)f \qquad (0 < r < r_0) \tag{35}$$

with

$$f(0) = 0, \qquad f(r_0) = 1 - \frac{1}{\lambda}\tilde{f}(r_0), \tag{36}$$

instead of using the second-order equation for f in Eq. (4). The solution of this boundary-value problem, denoted by $\hat{f}$, has to be found numerically. Given $\hat{f}$, we then have to solve the equation

$$a'' - \frac{1}{r}a' + \hat{f}^2(n - a) = 0 \tag{37}$$

with

$$a(0) = 0, \qquad a(r_0) = n - nrK_1(r_0) + \frac{1}{\lambda}\tilde{a}(r_0). \tag{38}$$

We will denote this inner function a by $\hat{a}$.

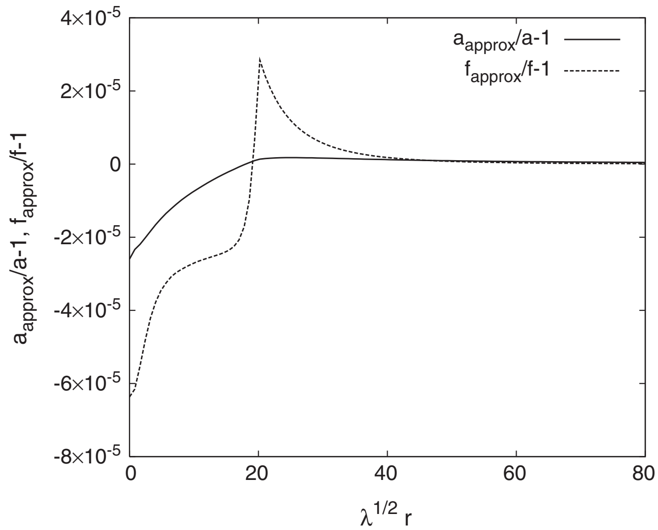

FIG. 5. Comparison of the numerical (exact) functions a and f for $n = 1$ and $\lambda = 100$ with the linear approximations a_{approx} and f_{approx} given by Eq. (33) away from the boundary layer and $\hat{a}$ and $\hat{f}$ in the boundary layer.

In order to show that a good linear approximation of the functions a and f for large λ is given by Eq. (33) away from the boundary layer and $\hat{a}$ and $\hat{f}$ in the boundary layer, we compare in Fig. 5 the numerical (exact) functions a and f with the corresponding linear approximations a_{approx} and f_{approx} for $n = 1$ and $\lambda = 100$. We observe that for a value of the location of the layer r_0 such that $\sqrt{\lambda}r_0 \approx 20$, the relative deviation of the approximation with respect to the exact values is of the order of 10^{-5}. This agreement improves as λ is increased.

IV. CONCLUSIONS

To complete the study of the four terms which contribute to the energy, we have used numerical computations. The asymptotic result for the total energy [Eq. (22)], however, follows from a simple chain of analytic arguments, as we have seen. In contrast to Hill, Hodges, and Turner [7] we make no assumptions about the class of functions to be considered. An important step in our chain of arguments is that in the $\lambda \to \infty$ limit the Higgs field takes its vacuum value for $r > 0$. In this regard, the vortex behaves like the monopole [8]. The crucial difference is that after the Higgs field has decoupled, the energy from the interaction of the Higgs field and the gauge field diverges in the case of vortices, whereas it is finite in the case of monopoles.

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