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Equivalence of covariant and light front QED: Generating instantaneous diagrams

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One loop expressions for fermion self-energy, vacuum polarization, and vertex correction in light-front time ordered perturbation theory can be obtained from respective covariant expressions by performing k^- integration. In an earlier work, we have shown that the third term in the doubly transverse gauge propagator is necessary to generate the diagrams involving instantaneous photon exchange both in the case of fermion self-energy as well as vertex correction. In this work, using the two-term photon propagator, we show that the instantaneous photon exchange diagrams in fermion self-energy as well as the IR singular terms in the propagating diagrams can be generated by taking the asymptotic limit of the covariant expression. It is further shown that this method reproduces the IR singular terms in propagating diagrams of vacuum polarization also.

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I. INTRODUCTION

The issue of equivalence of covariant perturbation theory and light-front Hamiltonian perturbation theory has attracted a lot of attention in recent years [1–4]. It is important to establish equivalence between the two approaches as light-front field theory has *spurious* divergences not present in covariant perturbation theory and it is necessary to understand how these divergences are generated in order to establish a correspondence between the light-front expressions and the covariant expressions. One of the approaches consists of establishing equivalence at the Feynman diagram level wherein the covariant expression for a Feynman diagram is integrated over the light-cone energy k^- to generate all the diagrams of light-front perturbation theory [1]. Bakker *et al.* [1] have given a general algorithm for proving equivalence in theories involving scalars as well spin-$\frac{1}{2}$ particles. Equivalence at Feynman diagram level in Yukawa theory has been discussed in detail [2]. Correspondence between the light-front Hamiltonian approach and the Lorentz-covariant approach has been discussed for QED $1 + 1$ and also for QCD by bosonization of the model [3].

As far as $3 + 1$-dimensional theories are concerned, equivalence of light-front QED (LFQED) and covariant QED in Coulomb gauge has been proven within the framework of Feynman-Dyson-Schwinger theory [5]. However, not much work has been done on proving equivalence for QED at the Feynman diagram level. In a previous work [6], we had addressed the issue of equivalence of light-front QED [7] and covariant QED at the Feynman diagram level. In Ref. [6], we have shown, at the one-loop level, how one can obtain all the propagating as well as instantaneous diagrams by performing the k^--integration carefully. The feature that sets QED apart from other cases considered in literature is the presence of diagrams involving instantaneous photon exchange. Our previous study was aimed at generating these expressions in the diagram based approach. It was shown that the equivalence cannot be established by performing k^- integration if one uses the commonly used two-term photon propagator in light-cone gauge [7,8]:

$$d_{\mu\nu} = \frac{1}{k^2 + i\epsilon}\left[-g_{\mu\nu} + \frac{\delta_{\mu+}k_\nu + \delta_{\nu+}k_\mu}{k^+}\right]. \tag{1}$$

However, if one uses the three-term photon propagator [5,8–12] given by

$$d_{\mu\nu} = \frac{1}{k^2 + i\epsilon}\left[-g_{\mu\nu} + \frac{\delta_{\mu+}k_\nu + \delta_{\nu+}k_\mu}{k^+} - \frac{k^2\delta_{\mu+}\delta_{\nu+}}{(k^+)^2}\right], \tag{2}$$

then one can generate the diagrams involving instantaneous photon exchange also which completes the proof of equivalence. In the present work, we give an alternative method to generate the instantaneous photon exchange diagrams using the two-term photon propagator only. We show how one can use the asymptotic method proposed by Bakker *et al.* [4] to generate the instantaneous photon exchange diagrams for one-loop self-energy correction. This method does not require the third term in the photon propagator. There has been some debate in literature over the relevance of the third term in the gauge boson propagator. It is usually dropped on the grounds that it does not propagate any information. In our previous work, we emphasized the importance of this term in proving equivalence at one-loop level. In the present work, we use the asymptotic method to generate the instantaneous photon exchange diagrams without the need of the third term. However, it should not be considered as undermining the importance of this term. On the contrary, the present method, being an alternative to the three-term propagator method, may be able to throw some light on the physical significance of this term.

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The plan of the paper is as follows: In Sec. II, we summarize the one-loop renormalization of LFQED [7] and briefly review the work of Ref. [6] for completeness. Here, we present only those results of Refs. [6,7] which are needed for our discussion. In Sec. III, we consider the self-energy correction and use the asymptotic method to generate the graphs involving instantaneous photon exchange. We show that in a certain asymptotic limit, the covariant expression for fermion self-energy reduces to a sum of expressions for the instantaneous photon exchange graphs and the IR singular terms of the propagating graph. We have further carried out a similar analysis for vacuum polarization. Since vacuum polarization does not have any contribution from instantaneous photon exchange vertex at the one-loop level, in this case the above mentioned limit reproduces only the IR singular terms in the propagating part. In Sec. IV, we summarize and discuss our results. Appendix A contains the notations and basics. Appendix B contains some useful formulas.

II. PROOF OF EQUIVALENCE OF COVARIANT AND LIGHT-FRONT QED USING THE THREE-TERM PHOTON PROPAGATOR

In this section, we summarize the results of Ref. [7] on one-loop renormalization of light-front QED in Hamiltonian formalism and recall how these results were obtained by performing k^- integration in Ref. [6]. The results presented here are needed for our discussion of the asymptotic method in Sec. III.

A. Fermion self-energy correction

In light-cone time ordered perturbation theory, fermion self-energy at $O(e^2)$ has three contributions given by

$$\bar{u}(p, s')\Sigma_1(p)u(p, s) = \left\langle p, s'|V_1 \frac{1}{p^- - H_0} V_1|p, s\right\rangle, \quad (3)$$

corresponding to the diagram in Fig. 1(a),

$$\bar{u}(p, s')\Sigma_2(p)u(p, s) = \langle p, s'|V_2|p, s\rangle, \quad (4)$$

corresponding to diagram in Fig. 1(b), and

$$\bar{u}(p, s')\Sigma_3(p)u(p, s) = \langle p, s'|V_3|p, s\rangle, \quad (5)$$

corresponding to the sum of diagrams in Figs. 1(c) and 1(d). V_1 is the standard three-point QED vertex and V_2 and V_3 are $O(e^2)$ nonlocal four-point vertices corresponding to an exchange of instantaneous fermion and photon, respectively. Expressions for V_1, V_2, and V_3 are given in Appendix A.

The contribution of Fig. 1(a) to δm is given by Eq. (3) and leads to the light-cone expression for the propagating part given by

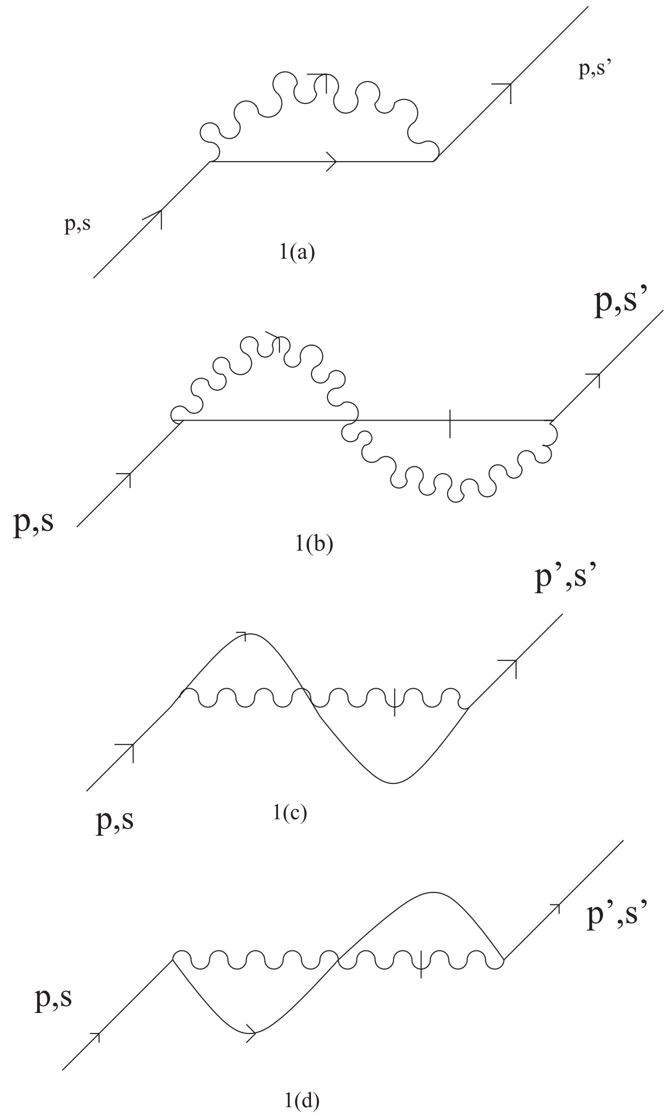

FIG. 1. Diagrams for electron mass shift in LFQED.

$$\delta m_a \delta_{s\sigma} = \frac{e^2}{m} \int \frac{d^2k_\perp}{(4\pi)^3} \int_0^{p^+} \frac{dk^+}{k^+(p^+ - k^+)} \times \frac{\bar{u}(p, \sigma)\gamma^\mu(\not{k}' + m)\gamma^\nu u(p, s)d_{\mu\nu}(k)}{p^- - k^- - k'^-}, \quad (6)$$

where all the momenta are on shell:

$$p = \left(p^+, \frac{p_\perp^2 + m^2}{2p^+}, p_\perp\right), \quad (7)$$

$$k = \left(k^+, \frac{k_\perp^2}{2k^+}, k_\perp\right), \quad (8)$$

and

$$k' = \left(p^+ - k^+, \frac{(p_\perp - k_\perp)^2 + m^2}{2(p^+ - k^+)}, p_\perp - k_\perp\right). \quad (9)$$

The contribution of Fig. 1(b) is

$$\delta m_b \delta_{ss'} = \frac{e^2 p^+ \delta_{ss'}}{2m} \int \frac{d^2 k_\perp}{(2\pi)^3} \int_0^{+\infty} \frac{dk^+}{k^+(p^+ - k^+)}, \quad (10)$$

and the sum of contributions of Figs. 1(c) and 1(d) is

$$\delta m_c \delta_{ss'} = \frac{e^2 p^+ \delta_{ss'}}{2m} \int \frac{d^2 k_\perp}{(2\pi)^3} \bigg[\int_0^{+\infty} \frac{dk^+}{(p^+ - k^+)^2} - \int_0^{+\infty} \frac{dk^+}{(p^+ + k^+)^2} \bigg]. \quad (11)$$

These integrals have potential singularities at $k^+ = 0$ and $k^+ = p^+$. To regularize them one introduces small cutoffs α and β

$$\alpha \leq k^+ \leq p^+ - \beta, \quad (12)$$

and handles the pole at $k^+ = p^+$ in δm_b and δm_c by principal value prescription as shown in Appendix A. Using this procedure, one obtains [7]

$$\delta m_a = \frac{e^2}{2m} \int \frac{d^2 k_\perp}{(2\pi)^3} \bigg[\int_0^{p^+} \frac{dk^+}{k^+} \frac{m^2}{p \cdot k} - 2\bigg[\frac{p^+}{\alpha} - 1\bigg] - \ln\bigg(\frac{p^+}{\beta}\bigg) \bigg], \quad (13)$$

$$\delta m_b = \frac{e^2}{2m} \int \frac{d^2 k_\perp}{(2\pi)^3} \ln\bigg(\frac{p^+}{\alpha}\bigg), \quad (14)$$

and

$$\delta m_c = \frac{e^2}{m} \int \frac{d^2 k_\perp}{(2\pi)^3} \bigg[\frac{p^+}{\alpha} - 1\bigg]. \quad (15)$$

To establish equivalence, one starts with the covariant expression for electron self-energy in the light-front gauge,

$$\Sigma(p) = \frac{(ie)^2}{2mi} \int \frac{d^4 k}{(2\pi)^4} \times \frac{\gamma^\mu (\not p - \not k + m) \gamma^\nu d'_{\mu\nu}(k)}{[(p-k)^2 - m^2 + i\epsilon][k^2 - \mu^2 + i\epsilon]}, \quad (16)$$

where $\frac{d'_{\mu\nu}}{k^2}$ is the photon propagator in the light-cone gauge in covariant perturbation theory with $d'_{\mu\nu}(k)$ given by Eq. (2). Substituting

$$\not p - \not k + m = \gamma^+ \bigg[\bigg(\frac{(p_\perp - k_\perp)^2 + m^2}{2(p^+ - k^+)}\bigg)\bigg] + \gamma^-(p^+ - k^+) - \gamma_\perp(p_\perp - k_\perp) + \gamma^+\bigg[p^- - k^- - \frac{(p_\perp - k_\perp)^2 + m^2}{2(p^+ - k^+)}\bigg], \quad (17)$$

and integrating over light-cone energy k^-, one obtains [6]

$$\Sigma(p) = \Sigma_1^{(a)}(p) + \Sigma_1^{(b)}(p) + \Sigma_2(p), \quad (18)$$

where

$$\Sigma_1^{(a)}(p) = \frac{e^2}{m} \int \frac{d^2 k_\perp}{(4\pi)^3} \int_0^{p^+} \frac{dk^+}{k^+(p^+ - k^+)} \times \frac{\gamma^\mu (\not k' + m) \gamma^\nu d_{\mu\nu}(k)}{p^- - k^- - k'^-} \quad (19)$$

is the propagating part leading to δm_a. $\Sigma_2(p)$ is given by

$$\Sigma_2(p) = \frac{e^2}{2m} \int_0^\infty \frac{dk^+}{2k^+} \int \frac{d^2 k_\perp}{(2\pi)^3} \frac{\gamma^\mu \gamma^+ \gamma^\nu d_{\mu\nu}(k)}{2(p^+ - k^+)}, \quad (20)$$

and leads to δm_b, whereas $\Sigma_1^{(b)}$ arises from the third term in the photon propagator and yields δm_c.

$\Sigma_1^{(a)}(p)$ differs from the covariant expression in that the fermion momentum in the loop is on shell in the light-front expression, i.e.

$$k' = \bigg(p^+ - k^+, \frac{(p_\perp - k_\perp)^2 + m^2}{2(p^+ - k^+)}, \bar p_\perp - \bar k_\perp\bigg), \quad (21)$$

whereas in the covariant expression it is off shell.

One should recall that δm_b arises when off-shell momentum in the covariant expression is replaced by on-shell momentum. In fact, in light-front perturbation theory all diagrams involving instantaneous fermion exchange are obtained by the replacement

$$k^- \to k^-_{\rm on} + (k^- - k^-_{\rm on}). \quad (22)$$

The first term here generates the LF propagating diagram and the second term generates the instantaneous fermion exchange diagram. Note that the resulting expression for δm_a still has IR singular terms. We show in Sec. III that these IR singular terms and δm_c can be obtained by taking the limit $k^+ \to p^+$, $k^- \to \infty$ in the covariant expression.

B. Vacuum polarization

In exactly the same manner as for electron self-energy, the covariant expression for photon self-energy can also be shown to be equivalent to the sum of the propagating and instantaneous diagrams of light-front field theory by changing the off-shell momenta to on-shell momenta.

One defines a tensor $\Pi^{\mu\nu}(p)$ through

$$\delta\mu^2 \delta_{\lambda\lambda'} = \epsilon^\lambda_\mu(p) \Pi^{\mu\nu}(p) \epsilon^{\lambda'}_\nu(p). \quad (23)$$

The corresponding diagrams are displayed in Fig. 2. $\delta\mu_a^2$ is given by

$$\delta\mu_a^2 \delta_{\lambda\lambda'} = \bigg\langle p, \lambda' | V_1 \frac{1}{p^- - H_0} V_1 | p, \lambda \bigg\rangle, \quad (24)$$

whereas the seagulls are given by

$$\delta\mu^2_{b+c} = \langle p, \lambda | V_2 | p, \lambda \rangle. \quad (25)$$

Inserting appropriate sets of intermediate states and following the standard procedure, one obtains

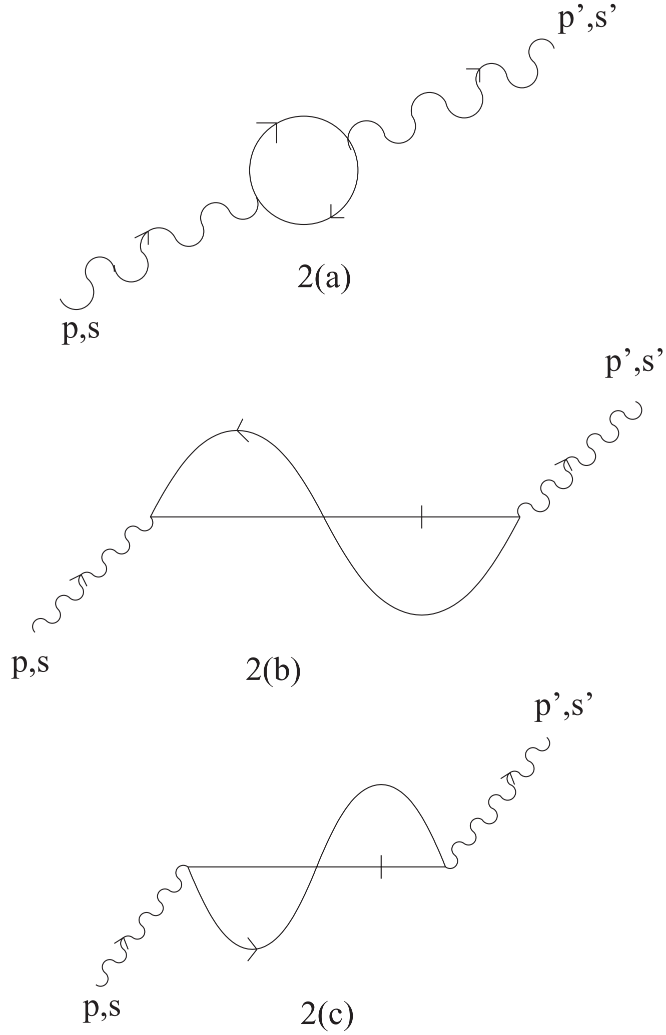

FIG. 2. Diagrams for vacuum polarization in LFQED.

$$\delta\mu_a^2 = 2e^2 \int \frac{d^2k_\perp}{(4\pi)^3} \int_\alpha^{p^+-\beta} \frac{dk^+}{k^+(p^+-k^+)} \times \frac{\mathrm{tr}[\not\epsilon^\lambda(p)(\not k + m)\not\epsilon^{\lambda'}(p)(\not k' - m)]}{p^- - k^- - k'^-}, \tag{26}$$

where

$$k = \left(k^+, \frac{k_\perp^2 + m^2}{2k^+}, k_\perp\right), \tag{27}$$

and

$$k' = \left(p^+ - k^+, \frac{(p_\perp - k_\perp)^2 + m^2}{2(p^+ - k^+)}, p_\perp - k_\perp\right). \tag{28}$$

It has been shown [7] that $\delta\mu^2$ is the sum of $\delta\mu_a^2$ and $\delta\mu_{b+c}^2$ where

$$\delta\mu_a^2 = e^2 \int \frac{d^2k_\perp}{(2\pi)^3}\left[\ln\left[\frac{\alpha\beta}{(p^{+2})}\right] + \frac{2k_\perp^2}{k_\perp^2 + m^2}\right] \tag{29}$$

corresponds to the propagating diagram and

$$\delta\mu_{b+c}^2 = e^2 \int \frac{d^2k_\perp}{(2\pi)^3} \int_0^\infty dk^+\left[\frac{1}{p^+ - k^+} - \frac{1}{p^+ + k^+}\right] \tag{30}$$

corresponds to the instantaneous fermion exchange.

One can obtain this result from the covariant expression also by performing the k^- integration in a manner similar to the one sketched above for the fermion self-energy diagram [6].

III. ASYMPTOTIC METHOD AND LIGHT-FRONT QED

In this section, we show that the diagrams involving instantaneous photon exchange in fermion self-energy can be generated by the asymptotic method discussed by Bakker *et al.* in the context of 1 + 1-dimensional theories [4]. We use the method discussed in this reference to isolate the divergent graphs and evaluate the integral by using the u-integration method of Ligterink *et al.* [1]. In QED, we find that the asymptotic method also generates the instantaneous photon exchange diagrams if we do not include the third term in the doubly transverse photon propagator. In fact, the asymptotic limit of the third term is $-\delta m_c$ which actually cancels the contribution of the third term and therefore, including the third term in the asymptotic method does not make any difference.

In general, the number of light-cone energy denominators is one less than the number of denominators in the covariant expression. This may give the impression that one can obtain the light-cone expression from the covariant expression by integrating over light-cone energy k^- using the method of residues. However, this apparently straightforward manner of proving equivalence does not reproduce all the instantaneous diagrams unless one takes into account the contribution of arc at infinity and end point contributions [4,6]. The diagrams involving instantaneous fermion exchange arise in a straightforward manner when the fermion momenta in covariant expression are replaced by on-shell momenta, as discussed in the previous section. We will not discuss this contribution here. In Ref. [6], we have shown that the diagrams involving instantaneous photon exchange arise from the third term in the photon propagator of Eq. (2). We now show that these instantaneous diagrams can also be generated by taking the asymptotic limit of the leading k^- term in the one-loop covariant expression with the conventional two-term photon pro pagator of Eq. (1). In addition, the IR divergent term in propagating part can also be generated by this method.

The asymptotic method was introduced by Bakker *et al.* in Ref. [4] in the context of (1 + 1)-dimensional theories. The proof of equivalence of light-front (LF) theory and covariant perturbation theory involves integration over LF energy using the method of residues. It was pointed out by Bakker *et al.* that if one performs the k^- integration naively by simply picking up the residues and ignoring the contribution along the arc used to close the contour at infinity, spurious divergences appear. The reason for this is that in light-front formulation there are cases in which the integrand does not vanish sufficiently fast as k^- goes to infinity. Moreover, there are point singularities at $k^+ = 0$ and $k^+ = p^+$. The asymptotic method of Bakker *et al.* consists of isolating the divergent parts by identifying the behavior of the integrand at asymptotic values of k^- and then regularizing these divergent parts in an appropriate manner. The rationale behind the method is that since the spurious divergences in integrals appear due to arc contributions at infinity and the end point contributions, one can isolate these by evaluating that part of the integrand which is dominant in the limits $k^- \to \infty$, $k^+ \to 0$ or $k^- \to \infty$, $k^+ \to p^+$ or both depending on the diagram. In Ref. [4], Bakker *et al.* regularize the divergent part by shifting the integration variables to light-front cylindrical coordinates $k^+ = R\cos\phi$ and $k^- = R\sin\phi$. The regularized integrals are then evaluated over a finite region first (keeping R finite) and finally the limit $R \to \infty$ is taken.

In this section, we use the asymptotic method to isolate the divergent parts of one-loop expressions for self-energy and vacuum polarization in QED. We then use the u-coordinate regularization [1] to evaluate these integrals. We show that this method reproduces the instantaneous photon exchange diagram as well as the divergent part of the propagating diagram even if one uses the two-term photon propagator. We consider the covariant expression in the limit when $k^- \to \infty$ and the light-cone momentum of internal fermion line approaches zero since we are interested in generating diagrams involving instantaneous photon exchange i.e. Figs. 1(c) and 1(d).

A. Fermion self-energy correction using asymptotic method

The covariant expression for electron self-energy in the light-front gauge with the two- term photon propagator is given by

$$\Sigma(p) = \frac{(ie)^2}{2mi}\int \frac{d^4k}{(2\pi)^4}\frac{N}{D_1 D_2}, \tag{31}$$

where

$$N = \gamma^\mu(\not{p} - \not{k} + m)\gamma^\nu d_{\mu\nu}(k), \tag{32}$$

$$D_1 = k^2 - \mu^2 + i\epsilon, \tag{33}$$

$$D_2 = (p-k)^2 - m^2 + i\epsilon, \tag{34}$$

and $\frac{d_{\mu\nu}(k)}{k^2}$ is the photon propagator in the light-cone gauge commonly used in light-front QED [7] given by

$$d_{\mu\nu} = -g_{\mu\nu}(k) + \frac{\delta_{\mu+}k_\nu + \delta_{\nu+}k_\mu}{k^+}. \tag{35}$$

Using Eqs. (A8)–(A12), the numerator reduces to

$$N = (p^+ - k^+)\left[2\gamma^- + \frac{4\gamma^+ k^-}{k^+} - \frac{2\gamma_\perp \cdot k_\perp}{k^+}\right] + (p^- - k^-)2\gamma^+ - \frac{2\gamma^+}{k^+}k^i(p^i - k^i) - 2m. \tag{36}$$

The numerator in $\bar{u}\Sigma(p)u$ is obtained from this by using Eqs. (A14) and (A15). In the limit $k^- \to \infty$, the numerator in $\bar{u}\Sigma(p)u$ is reduced to

$$N' = \frac{8p^+}{k^+}(p^+ - k^+)k^- - 4p^+k^- + \frac{4p^+}{k^+}k_\perp^2. \tag{37}$$

In the limit $k^- \to \infty$, $k^+ \to p^+$ D_1 reduces to $2k^+k^-$ [4] and δm reduces to

$$\delta m_{\rm asy} = \delta m_1 + \delta m_2 + \delta m_3, \tag{38}$$

where

$$\delta m_1 = \frac{ie^2p^+}{m}\int\frac{d^2k_\perp}{(2\pi)^4}\int\frac{dk^+}{k^{+2}} \times \int \frac{dk^-}{p^- - k^- - \frac{(p_\perp - k_\perp)^2 - m^2 - i\epsilon}{2(p^+ - k^+)}}, \tag{39}$$

$$\delta m_2 = -\frac{ie^2p^+}{m}\int\frac{d^2k_\perp}{(2\pi)^4}\int\frac{dk^+}{2k^+(p^+ - k^+)} \times \int \frac{dk^-}{p^- - k^- - \frac{(p_\perp - k_\perp)^2 - m^2 - i\epsilon}{2(p^+ - k^+)}}, \tag{40}$$

$$\delta m_3 = \frac{ie^2p^+}{2m}\int\frac{d^2k_\perp}{(2\pi)^4}k_\perp^2\int\frac{dk^+}{k^{+2}(p^+ - k^+)} \times \int \frac{dk^-}{k^-[p^- - k^- - \frac{(p_\perp - k_\perp)^2 + m^2 - i\epsilon}{2(p^+ - k^+)}]}. \tag{41}$$

Using Eqs. (B5)–(B9), δm_1 reduces to

$$\delta m_1 = -\frac{e^2p^+}{2m}\int\frac{d^2k_\perp}{(2\pi)^3} \times \int\frac{dk^+}{k^{+2}}[\theta(k^+ - p^+) - \theta(p^+ - k^+)], \tag{42}$$

which is the same as δm_c,

$$\delta m_2 = \frac{e^2p^+}{2m}\int\frac{d^2k_\perp}{(2\pi)^3}\left[\int_{p^+}^{\infty}\frac{dk^+}{2k^+(p^+ - k^+)} - \int_{-\infty}^{p^+}\frac{dk^+}{2k^+(p^+ - k^+)}\right], \tag{43}$$

and

$$\delta m_3 = \frac{e^2p^+}{2m}\int\frac{d^2k_\perp}{(2\pi)^3}\left[\int_{p^+}^{\infty}\frac{dk^+}{k^{+2}} - \int_{-\infty}^{p^+}\frac{dk^+}{k^{+2}}\right]. \tag{44}$$

The sum of δm_2 and δm_3, on performing k^+ integration, reduces to

$$\delta m_2 + \delta m_3 = -\frac{e^2}{2m}\int \frac{d^2k_\perp}{(2\pi)^3}\left[2\left(\frac{p^+}{\alpha} - 1\right) + \ln\left(\frac{p^+}{\beta}\right)\right], \tag{45}$$

which is the same as the IR divergent part of δm_a. Thus, the covariant expression, in the limit $k^- \to \infty$, $k^+ \to p^+$ reproduces the sum of the instantaneous photon exchange graph and the IR singular terms in the propagating graph.

B. Vacuum polarization using asymptotic method

Photon self-energy is given by

$$\delta\mu^2\delta_{\lambda\lambda'} = \epsilon_\mu^\lambda(p)\Pi^{\mu\nu}(p)\epsilon_\nu^{\lambda'}(p), \tag{46}$$

where

$$i\Pi^{\mu\nu} = -e^2\int \frac{d^2k_\perp}{(2\pi)}\int dk^+ \times \int dk^- \frac{\mathrm{Tr}[\gamma^\mu(\not{k} + m)\gamma^\nu(\not{p} - \not{k} - m)]}{D_1 D_2}. \tag{47}$$

One can rewrite Eq. (47) as

$$i\Pi^{\mu\nu}(p) = -e^2\int \frac{d^3k}{(2\pi)^3}\int \frac{dk^-}{2\pi} \frac{\mathrm{Tr}[\gamma^\mu(\not{k} + m)\gamma^\nu(\not{p} - \not{k} - m)]}{2k^+ 2(p^+ - k^+)[k^- - \frac{k_\perp^2 + m^2 - i\epsilon}{2k^+}][p^- - k^- \frac{(p_\perp - k_\perp)^2 + m^2 - i\epsilon}{2(p^+ - k^+)}]}. \tag{48}$$

Taking $k^- \to \infty$ limit in the numerator, we obtain

$$\delta\mu^2_{\rm asy} = ie^2\int \frac{d^2k_\perp}{(2\pi)^4} \times \int dk^+ dk^- \frac{[-4k^+k^- + 4(p^+ - k^+)k^- + 4k_\perp^2]}{D_1 D_2}. \tag{49}$$

In the limit $k^+ \to 0$ and $k^- \to \infty$ this reduces to

$$\delta\mu^2_{\rm asy1} = ie^2\int \frac{d^2k_\perp}{(2\pi)^4} \times \int dk^+ dk^- \frac{[-4k^+k^- + 4(p^+ - k^+)k^- + 4k_\perp^2]}{2k^+(k^- - \frac{k_\perp^2 + m^2 - i\epsilon}{2k^+})(-2)(p^+ - k^+)k^-}, \tag{50}$$

which, on using the Eqs. (B5)–(B9), reduces to

$$\delta\mu^2_{\rm asy1} = \frac{e^2}{2}\int \frac{d^2k_\perp}{(2\pi)^3}\left[\int_0^\infty \frac{dk^+}{(p^+ - k^+)} - \int_{-\infty}^0 \frac{dk^+}{(p^+ - k^+)}\right] - \frac{e^2}{2}\int \frac{d^2k_\perp}{(2\pi)^3}\left[\int_0^\infty \frac{dk^+}{k^+} - \int_{-\infty}^0 \frac{dk^+}{k^+}\right] - e^2\int \frac{d^2k_\perp}{(2\pi)^3}\left[\int_0^\infty \frac{dk^+}{p^+ - k^+} - \int_{-\infty}^0 \frac{dk^+}{p^+ - k^+}\right]. \tag{51}$$

Similarly, in the limit $k^+ \to p^+$, $k^- \to \infty$, one obtains

$$\delta\mu^2_{\rm asy2} = ie^2\int \frac{d^2k_\perp}{(2\pi)^4}\int dk^+ dk^- \frac{[-4k^+k^- + 4(p^+ - k^+)k^- + 4k_\perp^2]}{2k^+k^- 2(p^+ - k^+)[p^- - k^- - \frac{(p_\perp - k_\perp)^2 + m^2 + i\epsilon}{2(p^+ - k^+)}]}. \tag{52}$$

Thus $\delta\mu^2_{\rm asy2}$ becomes

$$\delta\mu^2_{\rm asy2} = \frac{e^2}{2}\int \frac{d^2k_\perp}{(2\pi)^3}\left[\int_{p^+}^\infty \frac{dk^+}{p^+ - k^+} - \int_{-\infty}^{p^+} \frac{dk^+}{p^+ - k^+}\right] - \frac{e^2}{2}\int \frac{d^2k_\perp}{(2\pi)^3}\left[\int_{p^+}^\infty \frac{dk^+}{k^+} - \int_{-\infty}^{p^+} \frac{dk^+}{k^+}\right] + e^2\int \frac{d^2k_\perp}{(2\pi)^3}\left[\int_{p^+}^\infty \frac{dk^+}{k^+} - \int_{-\infty}^{p^+} \frac{dk^+}{k^+}\right]. \tag{53}$$

Adding $\delta\mu^2_{\rm asy1}$ and $\delta\mu^2_{\rm asy2}$, we finally obtain

$$\delta\mu^2_{\rm asy} = e^2\int \frac{d^2k_\perp}{(2\pi)^3}\left[\ln\left[\frac{\alpha\beta}{(p^+)^2}\right]\right], \tag{54}$$

which is the IR singular part of the propagating diagram of one-loop vacuum polarization in Eq. (29).

IV. SUMMARY AND CONCLUSION

We have shown that the instantaneous photon exchange diagrams, present in the one-loop fermion self-energy calculation within light-front time ordered perturbation theory, can be generated by taking the asymptotic limit $k^+ \to p^+$, $k^- \to \infty$ of the covariant expression. In our earlier work [6], we had shown that the third term in the doubly transverse photon propagator is necessary to generate these diagrams. Here, in this alternative method of generating these diagrams, we have used the two-term photon propagator only. Thus the asymptotic method provides an alternative way to generate photon exchange diagrams. In addition, this limit also reproduces the IR divergent terms in propagating diagrams. This method does not generate instantaneous fermion exchange diagrams. It is well established that these diagrams arise when one takes the limit $k^- \to k^-_{\rm on}$ of covariant expression to obtain the propagating diagram of the light-front perturbation theory. Thus, subtracting the two limits $k^- \to (k^- - k^-_{\rm on})$ and $k^- \to \infty$, $k^+ \to p^+$ will render the covariant expression completely free of IR singularities.

In case of vacuum polarization, there are no instantaneous photon exchange diagrams, but the propagating diagram does have an IR divergent contribution. In this case, both the internal lines are fermions and, therefore, we consider both the limits $k^- \to \infty$, $k^+ \to 0$ as well as $k^- \to \infty$, $k^+ \to p^+$ to obtain the IR divergent contribution. We verify that the IR singular part of the propagating term can indeed be generated by this method. Similar to the self-energy case, one can use this method to subtract the IR singular part from the propagating diagrams. It is worth mentioning that the IR divergences we have discussed here are not the "true" IR divergences of light front field theories [13,14] but are the "spurious" IR divergences arising due to the form of LF energy momentum relation. True IR divergences shall remain after the above procedure has been applied and have to be dealt with separately.

The question about the relevance of the third term in the doubly transverse propagator has been raised by several authors. Suzuki and Sales have shown at the classical level that to get the three-term propagator, one needs to incorporate not only the usual $A^+ = 0$ condition in the gauge fixing part, but also couple it to the Lorentz condition $\partial \cdot A = 0$ [10]. Mustaki *et al.* [7] have used both of these conditions to eliminate the dependent degrees of freedom, leading to an instantaneous photon exchange vertex in the Hamiltonian. This was the reason why in our earlier work, we were able to establish equivalence between the covariant perturbation theory and the light-front perturbation theory only after including the third term in the propagator. In the present work, we do not use the third term in the propagator, but still generate the instantaneous photon exchange diagram using the asymptotic method. This indicates a connection between the Lorentz condition and the end point singularities in the light-front formulation. We are presently looking at this issue and will address it in a future work.

The procedure sketched here can also be applied to vertex correction graphs. We shall address this issue in a future communication.

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APPENDIX: A

1. Basics

We define the light-front coordinates by

$$x^+ = \frac{x^0 + x^3}{\sqrt{2}}, \tag{A1}$$

$$x^- = \frac{x^0 - x^3}{\sqrt{2}}, \tag{A2}$$

$$x_\perp = (x^1, x^2). \tag{A3}$$

The metric tensor is given by

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Dirac matrices satisfy the following properties:

$$(\gamma^+)^2 = (\gamma^-)^2 = 0, \tag{A4}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{A5}$$

$$(\gamma^0)^+ = \gamma^0, \tag{A6}$$

$$(\gamma^k)^\dagger = -\gamma^k (k = 1, 2, 3), \tag{A7}$$

$$\gamma^+\gamma^-\gamma^+ = 2\gamma^+, \tag{A8}$$

$$\gamma^-\gamma^+\gamma^- = 2\gamma^-, \tag{A9}$$

$$d_{\mu\nu}(p) = -g_{\mu\nu} + \frac{\delta_{\mu+}p_\nu + \delta_{\nu+}p_\mu}{p^+}, \tag{A10}$$

also

$$\gamma^\alpha\gamma^\beta d_{\alpha\beta}(p) = -2, \tag{A11}$$

$$\gamma^\alpha\gamma^\nu\gamma^\beta d_{\alpha\beta}(p) = \frac{2}{p^+}(\gamma^+\gamma^\nu + g^{+\nu}\not{p}), \tag{A12}$$

$$\begin{aligned} \gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\beta d_{\alpha\beta}(p) &= -4g^{\mu\nu} + 2\frac{p_\alpha}{p^+}(g^{\mu\alpha}\gamma^\nu\gamma^+ - g^{\alpha\nu}\gamma^\mu\gamma^+ \\ &\quad + g^{\alpha+}\gamma^\mu\gamma^\nu - g^{+\nu}\gamma^\mu\gamma^\alpha + g^{+\mu}\gamma^\nu\gamma^\alpha). \end{aligned} \tag{A13}$$

Dirac spinors satisfy

$$\bar{u}(p,s)u(p,s') = -\bar{v}(p,s)v(p,s) = 2m\delta_{s,s'}, \qquad (A14)$$

$$\bar{u}(p,s)\gamma^\mu u(p,s') = \bar{v}(p,s)\gamma^\mu v(p,s) = 2p^\mu\delta_{s,s'}. \qquad (A15)$$

2. Light-front Hamiltonian

P^-, the Light-front Hamiltonian, is the operator conjugate to the "time" evolution variable x^+ and is given by

$$P^- = H_0 + V_1 + V_2 + V_3, \qquad (A16)$$

where H_0 is the free Hamiltonian, and V_1 is the standard, order-e three-point interaction,

$$V_1 = e\int d^2x_\perp dx^- \bar{\xi}\gamma^\mu\xi a_\mu. \qquad (A17)$$

V_2 is an order-e^2 nonlocal effective four-point vertex corresponding to an instantaneous fermion exchange,

$$V_2 = -\frac{i}{4}e^2\int d^2x_\perp dx^- dy^- \epsilon(x^- - y^-) \times (\bar{\xi}a_k\gamma^k)(x)\gamma^+(a_j\gamma^j\xi)(y), \qquad (A18)$$

and V_3 is an order-e^2 nonlocal effective four-point vertex corresponding to an instantaneous photon exchange,

$$V_3 = -\frac{e^2}{4}\int d^2x_\perp dx^- dy^- (\bar{\xi}\gamma^+\xi)(x)|x^- - y^-|(\bar{\xi}\gamma^+\xi)(y). \qquad (A19)$$

3. Instantaneous diagrams in self-energy correction

Here, we briefly review the calculation of δm_b and δm_c in Eqs. (14) and (15). The details can be found in Appendix B of Ref. [7]. To prove the expression for δm_b in Eq. (14) starting from Eq. (10), one writes

$$\begin{aligned}\int_0^\infty dk^+\frac{p^+}{k^+(p^+-k^+)} &= \int_0^\infty dk^+\left[\frac{1}{k^+}+\frac{1}{p^+-k^+}\right] \\ &= \int_\alpha^\infty \frac{dk^+}{k^+} + \int_{p^++\eta}^\infty \frac{dk^+}{p^+-k^+} + \int_0^{p^+-\eta}\frac{dk^+}{p^+-k^+} \\ &= \ln\left[\frac{p^+}{\alpha}\right], \end{aligned} \qquad (A20)$$

where we have identified α with η.

To prove Eq. (15), we start with Eq. (11) and write

$$\begin{aligned}\int_0^\infty \frac{dk^+}{(p^+-k^+)^2} - \int_0^\infty\frac{dk^+}{(p^++k^+)^2} &= \int_0^{p^+-\eta}\frac{dk^+}{(p^+-k^+)^2} + \int_{p^++\eta}^\infty\frac{dk^+}{(p^+-k^+)^2} - \int_{p^+}^\infty\frac{dk^+}{(k^+)^2} \\ &= \left[\int_\eta^\infty + \int_\eta^{p^+} - \int_{p^+}^\infty\right]\frac{dk^+}{(k^+)^2} \\ &= 2\int_\eta^{p^+}\frac{dk^+}{(k^+)^2} = \frac{2}{p^+}\left[\frac{p^+}{\alpha}-1\right], \end{aligned} \qquad (A21)$$

where again we have identified η with α.

APPENDIX: B

In this appendix, we will give expressions for the integrals used in Sec. III. Consider the integral

$$I_1 = \int_{-\infty}^\infty \frac{dk^-}{k^- - \frac{k_\perp^2+\mu^2-i\epsilon}{2k^+}}. \qquad (B1)$$

The integrand in Eq. (B1) has a pole at $k^- = \frac{k_\perp^2+\mu^2-i\epsilon}{2k^+}$ that tends to infinity in the limit $k^+\to 0$. To evaluate the integral, we change the variable to $u=\frac{1}{k^-}$ and obtain

$$I_1 = \int_{-\infty}^\infty \frac{du}{u[1-\frac{k_\perp^2+\mu^2-i\epsilon}{2k^+}u]}. \qquad (B2)$$

Regularizing the integral by the replacement

$$\frac{1}{u} = \frac{1}{2}\left(\frac{1}{u+i\delta}+\frac{1}{u-i\delta}\right), \qquad (B3)$$

we obtain

$$I_1 = \frac{1}{2}\int\frac{du}{(u+i\delta)[1-\frac{k_\perp^2+\mu^2-i\epsilon}{2k^+}u]} + \frac{1}{2}\int\frac{du}{(u-i\delta)[1-\frac{k_\perp^2+\mu^2-i\epsilon}{2k^+}u]}. \qquad (B4)$$

Closing the contour in the lower half plane for the first integral and in the upper half plane for the second integral, we finally obtain

$$I_1 = -\pi i[\theta(k^+)-\theta(-k^+)]. \qquad (B5)$$

Similarly the integral

$$I_2 = \int\frac{dk^-}{p^- - k^- - \frac{(p_\perp-k_\perp)^2+m^2-i\epsilon}{2(p^+-k^+)}} \qquad (B6)$$

has a pole at $k^- = p^- - \frac{(p_\perp-k_\perp)^2+m^2-i\epsilon}{2(p^+-k^+)}$ which tends to infinity as $k^+\to p^+$. Again changing the variable to $u=\frac{1}{k^-}$ and using the same procedure as above we finally obtain

$$\int \frac{dk^-}{p^- - k^- - \frac{(p_\perp - k_\perp)^2 + m^2 - i\epsilon}{2(p^+ - k^+)}} = \pi i[\theta(k^+ - p^+) - \theta(p^+ - k^+)]. \tag{B7}$$

Also the integral

$$\int \frac{dk^-}{k^-[k^- - \frac{k_\perp^2 + \mu^2 - i\epsilon}{2k^+}]} = -\pi i \frac{2k^+[\theta(k^+) - \theta(-k^+)]}{k_\perp^2 + \mu^2 - i\epsilon}, \tag{B8}$$

and

$$\int \frac{dk^-}{k^-[p^- - k^- - \frac{(p_\perp - k_\perp)^2 + m^2 - i\epsilon}{2(p^+ - k^+)}]} = \pi i \frac{2(p^+ - k^+)[\theta(k^+ - p^+) - \theta(p^+ - k^+)]}{2(p^+ - k^+)p^- - (p_\perp - k_\perp)^2 - m^2 + i\epsilon}. \tag{B9}$$

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