PHYSICAL REVIEW D **82,** 125021 (2010)

Resolution of infrared divergences in gluon-gluon scattering regulated on a lightcone worldsheet lattice

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(Received 9 November 2010; published 14 December 2010)

We improve and update the discussion, given some years ago by my collaborators and me, of infrared divergences and bremsstrahlung in one-loop gluon scattering probabilities in lightcone gauge. In that work, we showed that adding soft and collinear gluon radiation, satisfying simple Lorentz invariant constraints, not only canceled all IR divergences, but resulted in compact expressions for the consequent scattering probabilities. Here we impose less restrictive (albeit noncovariant) constraints on the unobserved radiation, which increases the high energy (s) fixed momentum transfer (t) behavior of the total probabilities from $-\ln^2 s$ to $\ln s \ln t$, a behavior shared by the (IR divergent) elastic probabilities. Using this new treatment we also make a much more detailed comparison of the lightcone results to covariant calculations using dimensional regularization, finding complete agreement between the two styles of calculation.

DOI: 10.1103/PhysRevD.82.125021 PACS numbers: 11.15.Bt

I. INTRODUCTION

In this paper we seek to clarify some aspects of the one-loop QCD corrections to the scattering of glue by glue in lightcone gauge as calculated in [1,2]. Those calculations employed an infrared regulator motivated by the lightcone worldsheet lattice [3–5], which discretizes both $p^+ = (p^0 + p^3)/\sqrt{2}$ and $ix^+ = i(x^0 + x^3)/\sqrt{2}$. In the field theory context, the discretization of x^+ was immediately removed after adoption of a worldsheet friendly ultraviolet cutoff δ on the transverse momenta. But the discretization of $p^+ = Mm$, $M = 1, 2, \ldots$ was retained as an infrared cutoff.

We first address some issues stemming from the novel manner in which soft and collinear gluon emission was included to resolve the infrared divergences. Given the lightcone gauge setup, it was natural to define a jet of momentum $P_i = k + p_i$, containing two gluons, by the restriction on their momenta k, p_i:

$$\frac{(p_i^+ \boldsymbol{k} - k^+ \boldsymbol{p}_i)^2}{k^+ p_i^+} < \Delta^2, \quad \text{or} \quad \left(\boldsymbol{k} - \frac{k^+}{p_i^+}\boldsymbol{p}_i\right)^2 < \frac{k^+}{p_i^+}\Delta^2. \tag{1}$$

In fact the left side of the first inequality is just $-(k + p_i)^2 = -P_i^2$, the invariant mass squared of the jet. The restriction simply limits the "virtuality" of the jet compared to an on mass shell gluon: this jet definition is Lorentz invariant. The interpretation of gluons as jets is of course part of what is needed to define an infrared-safe scattering probability. But one needs to include soft gluon bremsstrahlung as well. Usually this additional radiation is defined by a condition such as $|k^0| < \epsilon$ on the energy of the extra gluon. Of course at the same time one has to exclude such soft gluons from the jet definition, to avoid double counting. But in [2], we pursued a Lorentz covariant alternative to this, namely: include in the bremsstrahlung part of the calculation any extra gluon with momentum k^μ satisfying the jet condition (1) for at least one of the external legs of the core process. Indeed, we showed that adding just this real radiation to the one-loop contributions to the elastic scattering probabilities canceled all the infrared divergences, in accord with the Lee-Nauenberg theorem [6]. Moreover, this treatment produces nice compact Lorentz invariant expressions for the total scattering probabilities [see (36) and (37) in Sec. III].

The only problem is that these formulas have an awkward high energy (Regge) limit $s \to \infty$ with t fixed. Inspection of the formulas shows that the dominant behavior in this limit goes as $-\ln^2 s$, with a *negative* coefficient. This clashes with the absence of a $\ln^2 s$ behavior in the known covariant dimensionally regulated elastic scattering amplitude [7]: the double log terms in these elastic amplitudes are of the form $\ln s \ln t$. The absence of $\ln^2 s$ behavior at one loop is compatible with the hypothesis that the higher order corrections put the gluon on a Regge trajectory of order g^2: a behavior $s^{1+g^2 f(t)} \to s(1 + g^2 f(t) \ln s)$. Indeed, recent interest in this possibility [8] in connection with the AdS/CFT correspondence was one motivation for the present update. Since the $-\ln^2 s$ behavior found in [2] includes only part of the inelastic bremsstrahlung processes, it is possible that including more bremsstrahlung will cancel this negative term. Had the $\ln^2 s$ term been positive, it would have been bad news for Regge behavior.

We resolve this puzzle of too little bremsstrahlung in Sec. III by employing a more traditional, and less restrictive, definition of soft gluon radiation: simply limit the k^+ of the extra gluon $k^+ < \kappa$. [To ensure that all components of k are soft, we also need to limit $k^- < O(\kappa)$. But, as we shall see, the form of the soft gluon emission amplitude

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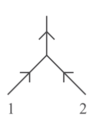
suppresses $|\boldsymbol{k}| > k^+$ sufficiently to automatically satisfy this second restriction.] The consequent probabilities will not be Lorentz invariant, but we show that in the center of mass frame the $\ln^2 s$ behavior cancels as $s \to \infty$ with t fixed. The calculations in this section differ from those in [2] in that they are done in arbitrary transverse dimension. But as long as d is set to 2 *before* integration over k^+, the two calculations are completely equivalent.

The second issue that we address involves the comparison of the results of [2] to previously known covariant calculations using dimensional regularization. In [2] we noted that the ratio of amplitudes describing different gluon polarizations, from which IR divergences cancel, agreed with the known results [7]. However, the comparison of amplitudes with a given polarization was obscured by the vagaries of IR divergences. We must compare the results for total probabilities, since it is not meaningful to compare the (gauge dependent) elastic amplitudes. So in Sec. IV we redo the bremsstrahlung calculations of Sec. III using dimensional regularization throughout: all momentum integrals are done *before* taking $d \to 2$. We then combine these with the previously known elastic scattering probabilities, obtained covariantly using dimensional regularization, to obtain expressions for the total probabilities. These agree in every detail with the results of Sec. III. In this way we provide a definitive confirmation that the discrete p^+ regularization, motivated by the lightcone worldsheet lattice, provides a reliable treatment of infrared divergences.

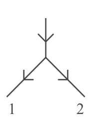
As in [1,2], we organize the Feynman diagrams of the $SU(N_c)$ Yang-Mills theory according to 't Hooft's large N_c expansion [9], and we calculate the one-loop planar diagrams surviving the $N_c \to \infty$ limit. The 't Hooft limit suppresses diagrams with quark loops, so they are not included here. We begin our discussion with a short Sec. II, which summarizes the Feynman rules for planar Yang-Mills theory and sets our notation and conventions.

II. LIGHTCONE FEYNMAN RULES FOR $N_c \to \infty$ GAUGE THEORY

Here, we use the notation and conventions in Ref. [10], according to which the values of the nonvanishing three transverse gluon vertices are

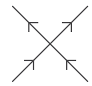


$$= \frac{2gp_3^+}{p_1^+p_2^+}\left(p_1^+p_2^\wedge - p_2^+p_1^\wedge\right) = \frac{2gp_3^+}{p_1^+p_2^+}K_{12}^\wedge, \qquad (2)$$

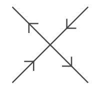


$$= \frac{2gp_3^+}{p_1^+p_2^+}\left(p_1^+p_2^\vee - p_2^+p_1^\vee\right) = \frac{2gp_3^+}{p_1^+p_2^+}K_{12}^\vee. \qquad (3)$$

The quartic vertices in this helicity basis are given by



$$= -2g^2\frac{p_1^+p_3^+ + p_2^+p_4^+}{(p_1^+ + p_4^+)^2}, \qquad (4)$$



$$= +2g^2\left(\frac{p_1^+p_2^+ + p_3^+p_4^+}{(p_1^+ + p_4^+)^2} + \frac{p_1^+p_4^+ + p_2^+p_3^+}{(p_1^+ + p_2^+)^2}\right). \qquad (5)$$

In these expressions, $\wedge$ and $\vee$ label the $\pm$ helicity of the gluon, and $p_k^\wedge = (p_k^x + ip_k^y)/\sqrt{2}$, $p_k^\vee = (p_k^x - ip_k^y)/\sqrt{2}$, and $p_k^+ = (p_k^0 + p_k^z)/\sqrt{2}$ are momenta *entering* the diagram on leg k. The coupling g is proportional to the conventional QCD coupling g_s. Note that these are lightcone gauge ($A_- = 0$) expressions and include the contributions that arise when the longitudinal field A_+ is eliminated from the formalism. These rules are given in the context of 't Hooft's $1/N_c$ expansion at fixed $N_c g_s^2$. Then the *planar* diagrams of the $SU(N_c)$ theory are correctly given if we take $g \equiv g_s\sqrt{N_c/2}$. Nonplanar diagrams with this definition of g must be accompanied by appropriate powers of $1/N_c^2$, depending on the number of "handles" in the diagram. Here we restrict attention to planar diagrams, so our results should be compared to the limit $N_c \to \infty$, fixed $g_s^2 N_c$ of those in the literature. In making such comparisons, note that our definition of g multiplies conventionally defined n-gluon tree amplitudes by a factor $N_c^{n/2-1} \to N_c$ for $n = 4$, so for each gluon scattering process we remove this factor before comparing to the literature.

III. RESOLUTION OF IR DIVERGENCES USING A DISCRETE p^+ REGULATOR

It is well understood that a consistent resolution of infrared divergences in loop corrections to scattering amplitudes involves a cancellation in the rates against corresponding infrared divergences in the rates for the emission (or absorption) of an extra gluon, whose momentum is either collinear with one of the gluons in the core process or "soft."

In the context of the large N_c limit one needs to combine coherently only bremsstrahlung diagrams with the same cyclic ordering. For example, in the diagrams shown in Fig. 1 at $N_c = \infty$ it is only necessary to square the sum of the two diagrams on each line and combine the results on different lines incoherently. Because $N_c = \infty$ suppresses nonplanar diagrams, it is convenient to take an extra gluon line attached between two outgoing gluons (as with the diagrams on the first line of Fig. 1) to be outgoing. Similarly a gluon line attached between two incoming gluons is taken to be incoming. On the other hand, both outgoing and incoming extra gluons must be considered when attached between an incoming and an outgoing gluon (as with the diagrams on the second line of Fig. 1).

Infrared and collinear divergences occur only when the bremsstrahlung gluon attaches to external legs. For example

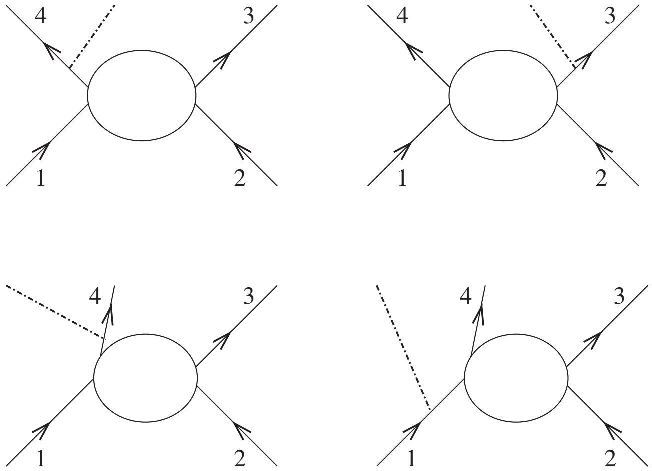


FIG. 1. The bremsstrahlung diagrams associated with glue-glue scattering involving leg 4. At $N_c = \infty$ the sum of the diagrams on each line may be independently squared to give the leading contribution to the cross section. Similar pairs of diagrams involving each of the other legs must also be included.

if the extra gluon is collinear with p_4, there is a collinear divergence in the phase-space integral of the square of the diagrams where the gluon is emitted from or absorbed by leg 4. Calling the extra gluon's four-momentum k, for fixed k^+ the collinear point is $\boldsymbol{k} = k^+ \boldsymbol{p}_4/p_4^+$, and it is convenient to write

$$\boldsymbol{k} = k^+ \frac{\boldsymbol{p}_4}{p_4^+} + \hat{\boldsymbol{k}} \tag{6}$$

and examine the phase-space integral for $|\hat{\boldsymbol{k}}|$ in a neighborhood of zero. Here we assume $k^+ = O(1)$, so the extra gluon is not soft. This is the customary jet interpretation of the scattered gluon [11]. We define the jet resolution Δ by the condition (1). This translates to $\hat{\boldsymbol{k}}^2 < |k^+|\Delta^2/|p_4^+|$.

The amplitudes for the emission of a hard collinear gluon from the right of leg 4 (as in the first diagram on the first line of Fig. 1) are given, for the two polarizations, by

$$A^{\vee}_{\text{brem}} = -2g \frac{k^+ + p_4^+}{k^+ p_4^+} \frac{K^{\vee}_{k,4} A_{\text{Core}}(p_1, p_2, p_3, k + p_4)}{(k + p_4)^2}, \quad \text{outgoing helicity,} \tag{7}$$

$$A^{\wedge}_{\text{brem}} = -2g \frac{p_4^+}{k^+(k^+ + p_4^+)} \frac{K^{\wedge}_{k,4} A_{\text{Core}}(p_1, p_2, p_3, k + p_4)}{(k + p_4)^2}, \quad \text{incoming helicity.} \tag{8}$$

When the extra gluon (with momentum k) is emitted from the left of leg 4, the amplitudes are the same except that $\boldsymbol{K}_{4,k}$ appears instead of $\boldsymbol{K}_{k,4}$. Thus the amplitudes for emission from left and right have opposite signs. The amplitudes do not cancel, however, because they have different gauge group structure. At $N_c = \infty$ the two terms enter the cross section incoherently. When the extra gluon has the same helicity as leg 4 and is collinear with p_4, it and gluon 4 are distinguished only by their p^+ values. Then we arbitrarily call the one with smaller $|p^+|$ the extra gluon.

Now it is easy to see that

$$\begin{aligned} \boldsymbol{K}_{k,4} &= -p_4^+ \hat{\boldsymbol{k}}, \\ (k + p_4)^2 &= -p_4^+ \hat{\boldsymbol{k}}^2/k^+ = -2p_4^+ \hat{k}^{\wedge} \hat{k}^{\vee}/k^+. \end{aligned} \tag{9}$$

Then we have, in d transverse dimensions,

$$\begin{aligned} &\frac{d\boldsymbol{p}_4}{2|p_4^+|} \frac{d\boldsymbol{k}}{2|k^+|(2\pi)^{d+1}} (|A^{\vee}|^2 + |A^{\wedge}|^2) \\ &\quad = \frac{d\boldsymbol{P}}{2|P^+|} \frac{d\hat{\boldsymbol{k}}}{|k^+|(2\pi)^{d+1}} \left(\frac{P^+ - k^+}{P^+}\right)^{d-1} \left(\frac{(P^+)^2}{(P^+ - k^+)^2} \right. \\ &\qquad \left. + \frac{(P^+ - k^+)^2}{(P^+)^2}\right) \frac{g^2}{\hat{\boldsymbol{k}}^2} |A_{\text{Core}}|^2, \end{aligned} \tag{10}$$

where $P^\mu = k^\mu + p_4^\mu$. The collinear divergence is now transparent in the integration over $\hat{\boldsymbol{k}}$ near zero. The coefficient of the phase-space factor $d\boldsymbol{P}/2|P^+|$ combines nicely with the square of the tree amplitudes with self-energy corrections on external lines. The collinear divergence is present at finite k^+ and is not regulated by our k^+ discretization. However, in lightcone gauge it cancels when combined with the self-energy correction on the corresponding external line. To properly arrange this cancellation on-shell we need an additional regulator. In [2] we introduced a temporary gluon mass. Here we regulate by sending $d \to 2$ from above only after the combination. For $d > 2$, the required integral is simply

$$\int_{0<\hat{k}^2|p_4^+|<|k^+|\Delta^2} \frac{d\hat{\boldsymbol{k}}}{\hat{\boldsymbol{k}}^2} = \frac{1}{(d-2)} \frac{2\pi^{d/2}}{\Gamma(d/2)} \left(\frac{|k^+|}{|p_4^+|}\Delta^2\right)^{d/2-1}. \tag{11}$$

Then the coefficient of the jet phase-space factor is

$$\begin{aligned} &\int_\Delta \frac{d\boldsymbol{k}}{2|k^+|(2\pi)^{d+1}} (|A^{\vee}|^2 + |A^{\wedge}|^2) \\ &\quad = \frac{g^2 |A_{\text{Core}}|^2}{|P^+| 4\pi^2 \Gamma(d/2)(d-2)} \left(\frac{|k^+||P^+ - k^+|}{|P^+|^2}\right)^{d/2-2} \\ &\qquad \times \left(\frac{\Delta^2}{4\pi}\right)^{d/2-1} \left(1 + \frac{|P^+ - k^+|^4}{|P^+|^4}\right). \end{aligned} \tag{12}$$

The blowup as $d \to 2$ is the collinear divergence we are seeking to resolve. According to the Lee-Nauenberg theorem, to get an infrared-safe quantity we must sum over all k^+ in the range $0 < |k^+| < |P^+|$. And we must also include collinear emission from the left of leg 4. The first term represents the emission of a gluon with identical helicity to leg 4, so when we sum that term over the whole range of k^+ we have included emission from both the left and the right of leg 4. However, the second term represents the emission of a gluon with opposite helicity, and when summed over the whole range gives only gluon emission from the right of leg 4. The emission of an opposite helicity gluon (with momentum k) from the left has the same squared amplitude, but it is convenient to switch the roles of k and p_4, so k always refers to the right gluon. Then the total emission rate is given by

$$\sum_{0<|k^+|<|P^+|}\int_\Delta \frac{d\boldsymbol{k}}{2|k^+|(2\pi)^{d+1}}(|A^\vee|^2+|A_R^\wedge|^2+|A_L^\wedge|^2) = \frac{g^2|A_{\rm Core}|^2}{|P^+|4\pi^2\Gamma(d/2)(d-2)}\sum_{k^+}\left(\frac{|k^+||P^+-k^+|}{|P^+|^2}\right)^{d/2-2} \times\left(\frac{\Delta^2}{4\pi}\right)^{d/2-1}\left(1+\frac{|P^+-k^+|^4}{|P^+|^4}+\frac{|k^+|^4}{|P^+|^4}\right). \tag{13}$$

Calling $x=|k^+|/|P^+|$, $1/x(1-x)$ times the quantity in parentheses can be rearranged

$$\frac{1}{x(1-x)}+\frac{(1-x)^3}{x}+\frac{x^3}{1-x}=2\left(x(1-x)+\frac{x}{1-x}+\frac{1-x}{x}\right). \tag{14}$$

So with this notation the squared amplitude for jet production along gluon 4 is

$$\sum_{0<|k^+|<|P^+|}\int_\Delta \frac{d\boldsymbol{k}}{2|k^+|(2\pi)^3}(|A^\vee|^2+|A_R^\wedge|^2+|A_L^\wedge|^2) = \frac{g^2|A_{\rm Core}|^2}{|P^+|4\pi^2\Gamma(d/2)(d-2)}\sum_{k^+}\left(\frac{1}{x(1-x)}+\frac{x^3}{1-x}+\frac{(1-x)^3}{x}\right)\left(\frac{x(1-x)\Delta^2}{4\pi}\right)^{d/2-1} \tag{15}$$

$$=\frac{g^2|A_{\rm Core}|^2}{|P^+|2\pi^2\Gamma(d/2)(d-2)}\sum_{k^+}\left(x(1-x)+\frac{x}{1-x}+\frac{1-x}{x}\right)\left(\frac{x(1-x)\Delta^2}{4\pi}\right)^{d/2-1}. \tag{16}$$

In Ref. [2] we obtained an IR finite on-shell wave function renormalization by introducing the same gluon mass used in the collinear emission calculation. Instead, here we simply redo the self-energy calculation with $d>2$:

$$\Pi^{\wedge\vee}=-\frac{g^2}{4\pi^2}\frac{p^2}{|P^+|}\frac{1}{(4\pi)^{d/2-1}}\sum_{k^+}\left(x(1-x)+\frac{x}{1-x}+\frac{1-x}{x}\right)\int_0^\infty\frac{dTe^{-x(1-x)p^2T}}{(T+\delta)^{d/2}} \tag{17}$$

$$\to-\frac{g^2}{4\pi^2}\frac{p^2}{|P^+|}\frac{1}{(4\pi)^{d/2-1}}\sum_{k^+}\left(x(1-x)+\frac{x}{1-x}+\frac{1-x}{x}\right)\times\int_0^\infty\frac{dT}{(T+\delta)^{d/2}} \tag{18}$$

$$Z-1\to-\frac{g^2}{4\pi^2}\frac{1}{|P^+|}\sum_{k^+}\left(x(1-x)+\frac{x}{1-x}+\frac{1-x}{x}\right)\times\frac{(4\pi\delta)^{1-d/2}}{d/2-1}. \tag{19}$$

Combining this wave function renormalization with the collinear emission rate in $d>2$ transverse dimensions gives

$$\langle|\mathcal{M}|^2\rangle_{\rm jet}=\frac{g^2}{4\pi^2}\frac{|A_{\rm Core}|^2}{|p^+|}\sum_{k^+}\left(x(1-x)+\frac{x}{1-x}+\frac{1-x}{x}\right)\times\frac{(x(1-x)\Delta^2/4\pi)^{d/2-1}-\Gamma(d/2)(4\pi\delta)^{1-d/2}}{(d/2-1)\Gamma(d/2)}$$
$$\to\frac{g^2}{4\pi^2}\frac{|A_{\rm Core}|^2}{|p^+|}\sum_{k^+}\left(x(1-x)+\frac{x}{1-x}+\frac{1-x}{x}\right)\times\ln(x(1-x)\Delta^2\delta e^\gamma) \tag{20}$$

for $d\to 2$, in complete agreement with the $\mu\neq 0$ regulation method of [2].

The collinear radiation discussed so far does not necessarily have low momentum: only when k^+ or P^+-k^+ is small will this radiation be soft. When it is soft, it is no longer valid to neglect the interference between diagrams where the soft gluon is emitted from different external lines. In [2] my collaborators and I took this interference into account for just that soft radiation that satisfied the collinear constraints for at least one of the external lines. This was sufficient to cancel all infrared divergences in the rates calculated to one-loop order. Moreover, since all the added radiation, both collinear and soft, was constrained by Lorentz invariant constraints the final results were Lorentz covariant.

Here instead we include all soft bremsstrahlung radiation satisfying a single energy constraint in addition to the collinear radiation. In the lightcone description it is natural to specify the energy constraint in terms of the + component of the extra gluon momentum: $k^+<\kappa$. We shall see that the rate for this radiation is quite simple to calculate. To avoid double counting we must at the same time exclude these soft gluons from the collinear calculation:

$$\text{soft radiation}: k^+<\kappa,\qquad \text{collinear radiation}: \kappa<k^+<P^+-\kappa. \tag{21}$$

These prescriptions guarantee that there is no double counting. One must bear in mind though that these constraints break Lorentz invariance.

In calculating the soft part of bremsstrahlung, we must be sure to combine coherently the two diagrams where the soft gluon attaches to two neighboring lines in the same cyclic ordering. For definiteness, take the coherent emission of a gluon between legs 3 and 4, both of which we assume to have outgoing helicity. Then the emission amplitudes are

$$A^{\vee} = -2gA_{\text{Core}}\left[\frac{k^+ + p_4^+}{k^+ p_4^+}\frac{K^{\vee}_{k,4}}{(k+p_4)^2} + \frac{k^+ + p_3^+}{k^+ p_3^+}\frac{K^{\vee}_{3,k}}{(k+p_3)^2}\right] \tag{22}$$

$$\sim -\frac{2gA_{\text{Core}}}{k^+}\left[\frac{K^{\vee}_{k,4}}{(k+p_4)^2} - \frac{K^{\vee}_{k,3}}{(k+p_3)^2}\right], \tag{23}$$

$$A^{\wedge} = -2gA_{\text{Core}}\left[\frac{p_4^+}{k^+(k^+ + p_4^+)}\frac{K^{\wedge}_{k,4}}{(k+p_4)^2} + \frac{p_3^+}{k^+(k^+ + p_3^+)} \times \frac{K^{\wedge}_{3,k}}{(k+p_3)^2}\right] \tag{24}$$

$$\sim -\frac{2gA_{\text{Core}}}{k^+}\left[\frac{K^{\wedge}_{k,4}}{(k+p_4)^2} - \frac{K^{\wedge}_{k,3}}{(k+p_3)^2}\right]. \tag{25}$$

In these formulas we have assumed that A_{Core} is the same in both terms, which is approximately true since all components of k are small. The squared amplitudes for small k^+ are

$$|A^{\vee}|^2 \approx |A^{\wedge}|^2 \sim \frac{2g^2|A_{\text{Core}}|^2}{k^{+2}}\left\{\frac{\boldsymbol{K}^2_{k,4}}{(k+p_4)^4} + \frac{\boldsymbol{K}^2_{k,3}}{(k+p_3)^4} - \frac{2\boldsymbol{K}_{k,3}\cdot\boldsymbol{K}_{k,4}}{(k+p_3)^2(k+p_4)^2}\right\}. \tag{26}$$

We next use

$$(k+p_4)^2 = 2\boldsymbol{k}\cdot\boldsymbol{p}_4 - k^+\boldsymbol{p}_4^2/p_4^+ - p_4^+\boldsymbol{p}^2/k^+ = -\boldsymbol{K}^2_{k,4}/k^+p_4^+,$$
$$(k+p_3)^2 = -\boldsymbol{K}^2_{k,3}/k^+p_3^+$$

to write

$$|A^{\vee}|^2 + |A^{\wedge}|^2 \sim 4g^2|A_{\text{Core}}|^2\left\{\frac{p_4^{+2}\boldsymbol{K}^2_{k,3} + p_3^{+2}\boldsymbol{K}^2_{k,4} - 2p_3^+p_4^+\boldsymbol{K}_{k,3}\cdot\boldsymbol{K}_{k,4}}{\boldsymbol{K}^2_{k,3}\boldsymbol{K}^2_{k,4}}\right\}$$
$$\sim 4g^2|A_{\text{Core}}|^2\frac{(p_4^+\boldsymbol{K}_{k,3} - p_3^+\boldsymbol{K}_{k,4})^2}{\boldsymbol{K}^2_{k,3}\boldsymbol{K}^2_{k,4}}$$
$$= 4g^2|A_{\text{Core}}|^2\frac{k^{+2}\boldsymbol{K}^2_{3,4}}{\boldsymbol{K}^2_{k,3}\boldsymbol{K}^2_{k,4}}. \tag{27}$$

The fact that this expression for soft gluon emission behaves as $1/\boldsymbol{k}^4$ for large transverse momentum is the reason that a soft constraint on k^+ imposes a soft constraint on all components of k^μ. Then the probability for emission of a soft gluon between legs 3 and 4 is in d transverse dimensions (note that the integral over $\boldsymbol{k}$ converges in both the IR and UV for $2 < d < 4$),

$$|\mathcal{M}^{\text{Soft}}_{34}|^2 = 4g^2|A_{\text{Core}}|^2\sum_{k^+<\kappa}\int\frac{d^d\boldsymbol{k}}{2k^+(2\pi)^{d+1}}\frac{k^{+2}\boldsymbol{K}^2_{3,4}}{p_3^{+2}p_4^{+2}(\boldsymbol{k}-k^+\boldsymbol{v}_3)^2(\boldsymbol{k}-k^+\boldsymbol{v}_4)^2}$$
$$= 4g^2|A_{\text{Core}}|^2\sum_{k^+<\kappa}\frac{k^+\boldsymbol{K}^2_{3,4}}{4\pi p_3^{+2}p_4^{+2}}\int\frac{d^d\boldsymbol{k}}{(2\pi)^d}\int_0^1 dt\frac{1}{[\boldsymbol{k}^2 + k^{+2}t(1-t)(\boldsymbol{v}_3-\boldsymbol{v}_4)^2]^2}$$
$$= 4g^2|A_{\text{Core}}|^2\sum_{k^+<\kappa}k^{+d-3}\frac{\boldsymbol{v}^2_{34}}{4\pi}\int_0^1 dt[t(1-t)\boldsymbol{v}^2_{34}]^{d/2-2}\int\frac{d^d\boldsymbol{k}}{(2\pi)^d}\frac{1}{[\boldsymbol{k}^2+1]^2}$$
$$= 4g^2|A_{\text{Core}}|^2\frac{[\boldsymbol{v}^2_{34}]^{d/2-1}}{4\pi}\sum_{k^+<\kappa}k^{+d-3}\frac{\Gamma(d/2-1)^2}{\Gamma(d-2)}\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}$$
$$= c_d\frac{g^2|A_{\text{Core}}|^2}{4\pi^2}[\boldsymbol{v}^2_{34}]^{d/2-1}\sum_{k^+<\kappa}k^{+d-3}\frac{2}{d/2-1},$$
$$c_d \equiv \frac{\Gamma(d/2)^2\Gamma(2-d/2)}{\Gamma(d-1)(4\pi)^{d/2-1}}, \tag{28}$$

where $\boldsymbol{v}_k \equiv \boldsymbol{p}_k/p_k^+$, $\boldsymbol{v}_{kl} \equiv \boldsymbol{v}_k - \boldsymbol{v}_l$, and we have defined c_d, which goes to 1 for $d = 2$, for simplicity of writing. In this section keeping k^+ discrete serves as our IR regulator. It is convenient to include the part of this soft radiation that also satisfies the collinear constraint for one of the external legs in the collinear calculation, so that the cancellation of the collinear divergence with the self-energy occurs for the full range of k^+. The part of the leg 4 and leg 3 collinear emission contributing to the 34 soft radiation is

$$\frac{g^2|A_{\text{Core}}|^2}{2\pi^2\Gamma(d/2)(d-2)}\sum_{k^+<\kappa}\frac{1}{k^+}\left\{\left(\frac{k^+\Delta^2}{4\pi|p_4^+|}\right)^{d/2-1} + \left(\frac{k^+\Delta^2}{4\pi|p_3^+|}\right)^{d/2-1}\right\}. \tag{29}$$

We check that for $d \sim 2$

$$\frac{1}{c_d\Gamma(d/2)(4\pi)^{d/2-1}} \sim \frac{1+(d-2)\Gamma'(1)}{(1+3(d/2-1)\Gamma'(1))(1-(d/2-1)\Gamma'(1))} = 1+O((d-2)^2). \quad (30)$$

Subtracting the soft collinear rate from the soft rate we then find

$$|\mathcal{M}_{34}^{\rm Soft}|^2 - |\mathcal{M}_{34}^{\rm Soft\&Col}|^2 \sim c_d \frac{g^2|A_{\rm Core}|^2}{4\pi^2}\sum_{k^+<\kappa}\frac{1}{k^+}\frac{1}{d/2-1} \times\left(2[k^{+2}\boldsymbol{v}_{34}^2]^{d/2-1} - \left(\frac{k^+\Delta^2}{|p_4^+|}\right)^{d/2-1} - \left(\frac{k^+\Delta^2}{|p_3^+|}\right)^{d/2-1}\right) \to \frac{g^2|A_{\rm Core}|^2}{4\pi^2}\sum_{k^+<\kappa}\frac{1}{k^+}\ln\frac{k^{+2}\boldsymbol{v}_{34}^4 p_3^+p_4^+}{\Delta^4}, \qquad d\to 2. \quad (31)$$

As already mentioned, the bremsstrahlung included in the calculations of [2] satisfied different constraints. These were simply the union of the four regions $R_1 \cup R_2 \cup R_3 \cup R_4$:

$$R_i\colon\ \frac{(p_i^+\boldsymbol{k} - k^+\boldsymbol{p}_i)^2}{|k^+P_i^+|} < \Delta^2. \quad (32)$$

Avoiding double counting was a tedious headache, but eventually the result for soft minus collinear radiation assumed a reasonably compact form:

$$|\mathcal{M}_{34}^{\rm Soft,CQT}|^2 - |\mathcal{M}_{34}^{\rm Soft\&Col,CQT}|^2 = +\frac{g^2|A_{\rm Core}|^2}{4\pi^2}\sum_{|k^+|<\Delta^2/|P_4^+|v_{34}^2}\frac{1}{|k^+|}\ln\frac{k^{+2}\boldsymbol{v}_{34}^4|p_3^+p_4^+|}{\Delta^4} \quad (33)$$

$$\approx +\frac{g^2|A_{\rm Core}|^2}{4\pi^2}\left[\sum_{|k^+|<A}\frac{1}{|k^+|}\ln\frac{k^{+2}\boldsymbol{v}_{34}^4|P_3^+P_4^+|}{\Delta^4} - \ln\frac{\Delta^2}{A|P_4^+|v_{34}^2}\ln\frac{\Delta^2}{A|P_3^+|v_{34}^2}\right]. \quad (34)$$

Here A is chosen much larger than the k^+ discretization unit. This formula is of course insensitive to the choice of A. But by choosing $A=\kappa$ we find a very simple relation between the bremsstrahlung radiation calculated in the present article and that calculated in [2].

$$|\mathcal{M}_{34}^{\rm brem}|^2 = |\mathcal{M}_{34}^{\rm brem,CQT}|^2 + \frac{g^2|A_{\rm Core}|^2}{4\pi^2}\ln\frac{\Delta^2}{\kappa|P_4^+|v_{34}^2} \times\ln\frac{\Delta^2}{\kappa|P_3^+|v_{34}^2}. \quad (35)$$

Thus we can immediately write down the new probabilities for glue-glue scattering by making the appropriate adjustment to the results of [2]:

$$P_{\wedge\wedge\vee\vee}^{\rm CQT} = |A_{\wedge\wedge\vee\vee}|^2\left[1+\frac{g^2}{4\pi^2}\left[-2\log^2\frac{\Delta^2}{s} - 2\log^2\frac{\Delta^2}{|t|} - \frac{\pi^2}{3}+\frac{67}{9}-\frac{11}{3}\left[\log(\Delta^2\delta e^\gamma)+\log\frac{\Delta^2}{|t|}\right] + \log^2\frac{s}{|t|}\right]\right], \quad (36)$$

$$P_{\wedge\vee\wedge\vee}^{\rm CQT} = |A_{\wedge\vee\wedge\vee}|^2\left[1+\frac{g^2}{4\pi^2}\left[-2\log^2\frac{\Delta^2}{s} - 2\log^2\frac{\Delta^2}{|t|} - \frac{\pi^2}{3}+\frac{67}{9}-\frac{11}{3}\left[\log(\Delta^2\delta e^\gamma) + \frac{1}{2}\log\frac{\Delta^4}{s|t|}\right] + \frac{(s^2+st+t^2)^2}{(t+s)^4}\log^2\frac{s}{|t|} + \frac{(5st^2-5s^2t+11t^3-11s^3)}{6(t+s)^3}\cdot\log\frac{s}{|t|} - \frac{ts}{(t+s)^2}\right]\right]. \quad (37)$$

We simply have to add the four terms

$$S \equiv \ln\frac{\Delta^2}{\kappa|p_4^+|v_{34}^2}\ln\frac{\Delta^2}{\kappa|p_3^+|v_{34}^2} + \ln\frac{\Delta^2}{\kappa|p_1^+|v_{12}^2}\ln\frac{\Delta^2}{\kappa|p_2^+|v_{12}^2} + \ln\frac{\Delta^2}{\kappa|p_4^+|v_{14}^2}\ln\frac{\Delta^2}{\kappa|p_1^+|v_{14}^2} + \ln\frac{\Delta^2}{\kappa|p_2^+|v_{23}^2}\ln\frac{\Delta^2}{\kappa|p_3^+|v_{23}^2} \quad (38)$$

inside the square brackets. We use

$$v_{ij}^2 = \frac{(p_i+p_j)^2}{p_i^+p_j^+} = \frac{|(p_i+p_j)^2|}{|p_i^+p_j^+|}$$

and $|(p_1+p_2)^2| = |(p_3+p_4)^2| = |s|$, $|(p_1+p_4)^2| = |(p_2+p_3)^2| = |t|$, to combine these terms with the first two terms in square brackets

$$S - 2\ln^2\frac{\Delta^2}{s} - 2\ln^2\frac{\Delta^2}{|t|} = \ln\frac{|p_3^+|}{\kappa}\ln\frac{|p_4^+|}{\kappa} + \ln\frac{|p_2^+|}{\kappa}\ln\frac{|p_1^+|}{\kappa} + \ln\frac{\Delta^2}{s}\ln\frac{\prod_i|p_i^+|}{\kappa^4} + \ln\frac{|p_3^+|}{\kappa}\ln\frac{|p_2^+|}{\kappa} + \ln\frac{|p_4^+|}{\kappa}\ln\frac{|p_1^+|}{\kappa} + \ln\frac{\Delta^2}{|t|}\ln\frac{\prod_i|p_i^+|}{\kappa^4} = \ln\frac{|p_3^+p_1^+|}{\kappa^2}\ln\frac{|p_2^+p_4^+|}{\kappa^2} - \ln\frac{s|t|}{\Delta^4}\ln\frac{\prod_i|p_i^+|}{\kappa^4}. \quad (39)$$

Then our new results are

$$P_{\wedge\wedge\vee\vee} = |A_{\wedge\wedge\vee\vee}|^2\Big[1 + \frac{g^2}{4\pi^2}\Big[\ln\frac{|p_3^+ p_1^+|}{\kappa^2}\ln\frac{|p_2^+ p_4^+|}{\kappa^2} - \ln\frac{s|t|}{\Delta^4}\ln\frac{|p_1^+ p_2^+ p_3^+ p_4^+|}{\kappa^4} - \frac{\pi^2}{3} + \frac{67}{9} - \frac{11}{3}\Big[\log(\Delta^2\delta e^\gamma) + \log\frac{\Delta^2}{|t|}\Big] + \log^2\frac{s}{|t|}\Big]\Big], \tag{40}$$

$$P_{\wedge\vee\wedge\vee} = |A_{\wedge\vee\wedge\vee}|^2\Big[1 + \frac{g^2}{4\pi^2}\Big[\ln\frac{|p_3^+ p_1^+|}{\kappa^2}\ln\frac{|p_2^+ p_4^+|}{\kappa^2} - \ln\frac{s|t|}{\Delta^4}\ln\frac{|p_1^+ p_2^+ p_3^+ p_4^+|}{\kappa^4} - \frac{\pi^2}{3} + \frac{67}{9} - \frac{11}{3}\Big[\log(\Delta^2\delta e^\gamma) + \frac{1}{2}\log\frac{\Delta^4}{s|t|}\Big] + \frac{(s^2+st+t^2)^2}{(t+s)^4}\log^2\frac{s}{|t|} + \frac{(5st^2 - 5s^2t + 11t^3 - 11s^3)}{6(t+s)^3}\cdot\log\frac{s}{|t|} - \frac{ts}{(t+s)^2}\Big]\Big]. \tag{41}$$

It is evident that this second definition of the bremsstrahlung to be included in describing gluon scattering depends on the Lorentz frame. This is in contrast to the first Lorentz invariant definition. However it is more physically meaningful, because it includes all radiation satisfying a single energy constraint $k^+ < \kappa$. This is particularly significant in high energy scattering $s \to \infty$ at fixed t, the Regge limit. In the center of mass frame in the case that the scattering plane is in the transverse direction, all the $|p_i^+| = \sqrt{s/8}$. Then it is simple to see that the terms quadratic in $\ln s$ cancel. In contrast the first definition included so little radiation at large s and fixed t that the coefficient of the $\ln^2 s$ term was negative.

IV. COMPARISON TO COVARIANT CALCULATIONS

In order to make a definitive comparison of the noncovariant lightcone gauge calculations of [2] to covariant calculations, it is necessary to compare physical quantities. The elastic amplitudes contain infrared divergences and calculations in different gauges depend on the infrared cutoff: they need not agree, and indeed they do not.

We need to compare infrared-safe quantities, such as the probabilities for jet scattering plus unobserved soft radiation. The dimensionally regulated elastic one-loop amplitudes have long been available, e.g. in [7]. For comparison to our results we need to redo our bremsstrahlung calculations in dimensional regularization. We have presented our intermediate results for general transverse dimension $d > 2$. We simply need to take k^+ continuous in the results and explicitly evaluate the k^+ integrals at fixed $d > 2$. The collinear jet production probability on leg 4 becomes at continuous $\kappa < k^+ < |p_4^+| - \kappa$

$$\begin{aligned}
&\sum_{\kappa<|k^+|<|P^+|-\kappa}\int_\Delta \frac{d\boldsymbol{k}}{2|k^+|(2\pi)^3}(|A^\vee|^2 + |A_R^\wedge|^2 + |A_L|^2)\\
&\to \frac{g^2|A_{\rm Core}|^2}{2\pi^2\Gamma(d/2)(d-2)}\int_{\kappa/|p_4^+|}^{1-\kappa/|p_4^+|} dx\Big(x(1-x) + \frac{x}{1-x} + \frac{1-x}{x}\Big)\Big(\frac{x(1-x)\Delta^2}{4\pi}\Big)^{d/2-1}\\
&\sim c_d\frac{g^2|A_{\rm Core}|^2}{4\pi^2}\int_{\kappa/|p_4^+|}^{1-\kappa/|p_4^+|} dx\Big(x(1-x) - 2 + \frac{2}{x}\Big)\Big(\frac{1}{d/2-1} + \ln x(1-x)\Delta^2\Big)\\
&\sim c_d\frac{g^2|A_{\rm Core}|^2}{4\pi^2}\Big[\Big(\frac{1}{d/2-1} + \ln\Delta^2\Big)\Big(2\ln\frac{|p_4^+|}{\kappa} - \frac{11}{6}\Big) - \ln^2\frac{|p_4^+|}{\kappa} - \frac{\pi^2}{3} + 2\Big(-\frac{1}{4} + \frac{1}{9} + 2\Big)\Big]\\
&\sim c_d\frac{g^2|A_{\rm Core}|^2}{4\pi^2}\Big[\Big(\frac{1}{d/2-1} + \ln\Delta^2\Big)\Big(2\ln\frac{|p_4^+|}{\kappa} - \frac{11}{6}\Big) - \ln^2\frac{|p_4^+|}{\kappa} - \frac{\pi^2}{3} + \frac{67}{18}\Big] + O\Big(\frac{\kappa}{|p_4^+|}, d-2\Big).
\end{aligned} \tag{42}$$

There are of course similar expressions for the collinear radiation from the other external legs.

Next we turn to the soft radiation between legs 3 and 4.

$$\begin{aligned}
|\mathcal{M}_{34}^{\rm Soft}|^2 &\to c_d\frac{g^2|A_{\rm Core}|^2}{4\pi^2}[\boldsymbol{v}_{34}^2]^{d/2-1}\frac{2}{d/2-1}\int_0^\kappa dk^+ k^{+d-3} = c_d\frac{g^2|A_{\rm Core}|^2}{4\pi^2}[\kappa^2\boldsymbol{v}_{34}^2]^{d/2-1}\frac{1}{(d/2-1)^2},\\
\mathcal{M}_{34}^{\rm Soft}|^2 &\sim c_d\frac{g^2|A_{\rm Core}|^2}{4\pi^2}\Big[\frac{1}{(d/2-1)^2} + \frac{1}{(d/2-1)}\ln\kappa^2\boldsymbol{v}_{34}^2 + \frac{1}{2}\ln^2\kappa^2\boldsymbol{v}_{34}^2\Big],\\
|\mathcal{M}_{34}^{\rm Soft}|^2 &\sim c_d\frac{g^2|A_{\rm Core}|^2}{4\pi^2}\Big[\frac{1}{(d/2-1)^2} + \frac{1}{(d/2-1)}\ln\frac{\kappa^2 s}{|p_3^+ p_4^+|} + \frac{1}{2}\ln^2\frac{\kappa^2 s}{|p_3^+ p_4^+|}\Big].
\end{aligned} \tag{43}$$

And there are three more such contributions associated with radiation between legs 1, 2; 1, 4; and 2, 3.

Collecting all the contributions to bremsstrahlung radiation gives

$$|\mathcal{M}^{\mathrm{brem,total}}|^2 \sim c_d \frac{g^2|A_{\mathrm{Core}}|^2}{4\pi^2}\bigg[\bigg(\frac{1}{d/2-1}+\ln\Delta^2\bigg)\bigg(2\sum_i \ln\frac{|p_i^+|}{\kappa}-\frac{44}{6}\bigg)-\sum_i \ln^2\frac{|p_i^+|}{\kappa}-4\frac{\pi^2}{3}+4\frac{67}{18}+\frac{4}{(d/2-1)^2} + \frac{1}{(d/2-1)}\ln\frac{\kappa^8 s^2|t|^2}{|p_1^+p_2^+p_3^+p_4^+|^2}+\frac{1}{2}\ln^2\frac{\kappa^2 s}{|p_1^+p_2^+|}+\frac{1}{2}\ln^2\frac{\kappa^2 s}{|p_3^+p_4^+|}+\frac{1}{2}\ln^2\frac{\kappa^2|t|}{|p_1^+p_4^+|}+\frac{1}{2}\ln^2\frac{\kappa^2|t|}{|p_2^+p_3^+|}\bigg] \tag{44}$$

$$\sim c_d \frac{g^2|A_{\mathrm{Core}}|^2}{4\pi^2}\bigg[\frac{4}{(d/2-1)^2}+\frac{1}{d/2-1}\bigg(\ln s^2|t|^2-\frac{22}{3}\bigg)+(\ln\Delta^2)\bigg(2\sum_i\ln\frac{|p_i^+|}{\kappa}-\frac{22}{3}\bigg)-\sum_i\ln^2\frac{|p_i^+|}{\kappa}-4\frac{\pi^2}{3} + 4\frac{67}{18}+\frac{1}{2}\ln^2\frac{\kappa^2 s}{|p_1^+p_2^+|}+\frac{1}{2}\ln^2\frac{\kappa^2 s}{|p_3^+p_4^+|}+\frac{1}{2}\ln^2\frac{\kappa^2|t|}{|p_1^+p_4^+|}+\frac{1}{2}\ln^2\frac{\kappa^2|t|}{|p_2^+p_3^+|}\bigg]. \tag{45}$$

This must be added to the contribution of the one-loop corrections to the elastic scattering probability, which we take from [7]. (We set their mass scale $\mu = 1$.)

$$P^{\mathrm{1Loop}}_{\wedge\wedge\vee\vee} = c_d\frac{g^2|A_{\mathrm{Core}}|^2}{4\pi^2}\bigg[|t|^{d/2-1}\bigg(-\frac{4}{(d/2-1)^2}+\frac{1}{d/2-1}\bigg(\frac{11}{3}+2\ln\frac{|t|}{s}\bigg)+\pi^2-\frac{67}{9}\bigg)+\frac{11}{3}\frac{1}{d/2-1}\bigg]$$

$$\sim c_d\frac{g^2|A_{\mathrm{Core}}|^2}{4\pi^2}\bigg[-\frac{4}{(d/2-1)^2}-\frac{4}{d/2-1}\ln|t|-2\ln^2|t|+\frac{1}{d/2-1}\bigg(\frac{11}{3}+2\ln\frac{|t|}{s}\bigg)+\ln|t|\bigg(\frac{11}{3}+2\ln\frac{|t|}{s}\bigg) + \pi^2-\frac{67}{9}+\frac{11}{3}\frac{1}{d/2-1}\bigg]$$

$$\sim c_d\frac{g^2|A_{\mathrm{Core}}|^2}{4\pi^2}\bigg[-\frac{4}{(d/2-1)^2}+\frac{1}{d/2-1}\bigg(\frac{22}{3}-2\ln|t|s\bigg)+\ln|t|\bigg(\frac{11}{3}-2\ln s\bigg)+\pi^2-\frac{67}{9}\bigg]. \tag{46}$$

Combining elastic plus bremsstrahlung, the divergences as $d\to 2$ cancel:

$$P_{\wedge\wedge\vee\vee}\sim c_d\frac{g^2|A_{\mathrm{Core}}|^2}{4\pi^2}\bigg[\ln|t|\bigg(\frac{11}{3}-2\ln s\bigg)-\frac{\pi^2}{3}+\frac{67}{9}+(\ln\Delta^2)\bigg(2\sum_i\ln\frac{|p_i^+|}{\kappa}-\frac{22}{3}\bigg)-\sum_i\ln^2\frac{|p_i^+|}{\kappa}+\frac{1}{2}\ln^2\frac{\kappa^2 s}{|p_1^+p_2^+|} + \frac{1}{2}\ln^2\frac{\kappa^2 s}{|p_3^+p_4^+|}+\frac{1}{2}\ln^2\frac{\kappa^2|t|}{|p_1^+p_4^+|}+\frac{1}{2}\ln^2\frac{\kappa^2|t|}{|p_2^+p_3^+|}\bigg] \tag{47}$$

$$\sim c_d\frac{g^2|A_{\mathrm{Core}}|^2}{4\pi^2}\bigg[\frac{11}{3}\ln\frac{|t|}{\Delta^2}+\ln^2\frac{s}{|t|}-\frac{\pi^2}{3}+\frac{67}{9}-\frac{11}{3}\ln\Delta^2-\sum_i\ln^2\frac{|p_i^+|}{\kappa}+\frac{1}{2}\ln^2\frac{\kappa^2}{|p_1^+p_2^+|}+\frac{1}{2}\ln^2\frac{\kappa^2}{|p_3^+p_4^+|} + \frac{1}{2}\ln^2\frac{\kappa^2}{|p_1^+p_4^+|}+\frac{1}{2}\ln^2\frac{\kappa^2}{|p_2^+p_3^+|}+\ln\frac{s|t|}{\Delta^4}\ln\frac{\kappa^4}{|p_1^+p_2^+p_3^+p_4^+|}\bigg]$$

$$\sim c_d\frac{g^2|A_{\mathrm{Core}}|^2}{4\pi^2}\bigg[\ln^2\frac{s}{|t|}-\frac{\pi^2}{3}+\frac{67}{9}-\frac{11}{3}\bigg(\ln\Delta^2-\ln\frac{|t|}{\Delta^2}\bigg)+\ln\frac{\kappa^2}{|p_1^+p_3^+|}\ln\frac{\kappa^2}{|p_2^+p_4^+|}+\ln\frac{s|t|}{\Delta^4}\ln\frac{\kappa^4}{|p_1^+p_2^+p_3^+p_4^+|}\bigg]. \tag{48}$$

This result agrees in all respects with that obtained from the discrete k^+ regularization (40), apart from the dependence on the ultraviolet cutoff δ, which had been explicitly subtracted in the results presented in [7]. It was already noted in [2] that the infrared insensitive ratio $P_{\wedge\wedge\vee\vee}/P_{\wedge\vee\wedge\vee}$ was in complete agreement with that presented in [7]. So we now have a definitive confirmation that the IR regulation supplied by the worldsheet lattice is completely equivalent to that provided by dimensional regularization.

V. CONCLUSION

In this paper we have clarified two aspects of the glue-glue scattering calculations of [2]. First, we have placed less restrictive constraints on the soft bremsstrahlung radiation which we combine with the one-loop probabilities to cancel infrared divergences. Since more soft radiation is allowed by the new constraints, the scattering probabilities are increased over those obtained in [2]. In particular, the large s limit at fixed t now behaves as $\ln s$ instead of $-\ln^2 s$. This behavior is compatible with Regge behavior.

Secondly, we extended our new calculations of bremsstrahlung to general continuous transverse dimensions. When combined with previous dimensionally regulated covariant calculations of one-loop corrections, we obtained results in complete accord with those regulated using the worldsheet lattice. This is a much more detailed comparison than that made in [2], which was limited by the lack of a common treatment of bremsstrahlung radiation. Thus, apart from collinear divergences, which only occur for self-energy insertions on on-shell external lines, this discretization provides a viable infrared regulator for lightcone calculations.

ACKNOWLEDGMENTS

This research was supported in part by the Department of Energy under Grant No. DE-FG02-97ER-41029.

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