Note on Schwinger mechanism and a non-Abelian instability in a non-Abelian plasma

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We point out that there is a non-Abelian instability for a non-Abelian plasma which does not allow both for a net nonzero color charge and the existence of field configurations which are coherent over a volume v whose size is determined by the chemical potential. The basic process which leads to this result is the Schwinger decay of chromoelectric fields, for the case where the field arises from commutators of constant potentials, rather than as the curl of spacetime dependent potentials. The case where instability is obtained can be expressed in terms of fields (with constant potentials) as $F^{a\mu\nu}F^a_{\mu\nu} < 0$, $A^{a\mu}A^a_\mu < 0$.

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I. INTRODUCTION

The identification of the deconfined phase of quarks and gluons at the Relativistic Heavy Ion Collider, a phase akin to a non-Abelian plasma, has led to a number of investigations on instabilities in a non-Abelian plasma [1,2]. While some of these are concerned about an upgraded version of instabilities in an Abelian plasma, such as the Weibel instability, there have been numerical studies of the evolution of instabilities in the hard thermal loop approximation and beyond. The purpose of this note is to point out that there is an instability, and a certain no-go statement, which is quite general and arises purely from non-Abelian effects. It is fairly straightforward to understand how this effect arises. For a statistical distribution of nonzero color charge, we need a chemical potential. Because the charge is non-Abelian in nature, the chemical potential is a matrix in the Lie algebra of the color group. In fact, it may be viewed as a background value for the time-component of the potential $A_0 = -it^a A_0^a$, where t^a form an orthonormal basis for the Lie algebra of the color group G. (We may actually take this matrix to be diagonal, but it is not important at this stage.) If we have a constant background A_0, then there is an electric field generated via the commutator term $[A_0, A_i]$ in the field strength tensor. For modes of A_i of wavelength λ, this gives an electric field approximately constant over this length scale. This electric field will then develop a Schwinger instability decaying via pair production. If the particles which are produced have a mass, there is an exponential suppression, but in the non-Abelian plasma, we have effectively massless modes. The end result of this argument is the following. Consider the plasma coarse-grained over a distance scale λ. Then one possibility is that the color charge density is zero when coarse-grained over this scale. The other possibility is that the plasma cannot have A_i which are coherent over length scales exceeding λ. This is the essence of our no-go statement.

The possibility of color charge density being zero has been studied in the context of color superconductivity [3]. In the limit of large baryon number density, we expect a color superconducting phase and it is important to have color neutrality. Such a requirement can be imposed on analyses of color superconductivity, but how it is achieved is really a dynamical issue. (This is not the setting for our question. We are concerned about a deconfined state, not superconducting and for us the baryon chemical potential can be zero. But there are points of connection.) Nonzero charges can lead to large electric fields which are unstable, can lead to energy being nonextensive and this is one reason why stable matter must be neutral under gauge charges [3]. Nevertheless, it is interesting to analyze some of the nuances of how neutrality is achieved. Since the chemical potential may be taken as a background value for A_0, the corresponding equation of motion (or integration of the constant mode of A_0 in the functional integral) seems to imply zero color charge. Strictly speaking this argument needs to be qualified, since it is equivalent to imposing the Gauss law integrated over functions which do not vanish at spatial infinity. The true gauge transformations of the theory go to the identity element at spatial infinity and so test functions for the Gauss law must vanish at infinity. Imposing the Gauss law with constant values for the gauge parameters is equivalent to eliminating all charged states by fiat, which we do not want to do. One can use a compact spatial manifold and then approach the limit of large volumes to preserve the zero charge condition. This provides a method for carrying out the analyses, including many of the calculations in the literature, but it is not quite an explanation. All this makes it useful to ask the question we are asking: If we have a deconfined state of gluons (and maybe quarks), and we try to have nonzero color charge, what instabilities can arise?

The density matrix for a statistical distribution in equilibrium is given by $\rho = \exp[-(H - \sum_i \mu_i Q_i)/T]$ where H is the Hamiltonian, Q_i are conserved charges, μ_i are the corresponding chemical potentials and T is the temperature. We are interested in time-dependent processes in this distribution, so we are concerned with

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real-time propagators and vertices averaged over states with the density matrix ρ. The result is equivalent to calculations at zero chemical potential, but with a Hamiltonian $H - \sum_i \mu_i Q_i$. Since the constant mode of A_0 couples to Q,it is clear that we can treat μ as a background value for A_0. Consider now the non-Abelian charge density due to quarks, say, $J_0^a = \bar{q}\gamma^0 t^a q$, or its matrix version, $(J_0)_{ij} = \bar{q}_i \gamma^0 q_j$, i, j being color labels for the quarks. Under a gauge transformation $g(x) \in G$, this matrix changes as

$$J_0 \to J_0^g = g^{-1} J_0 g. \tag{1}$$

It is thus possible to choose $g(x)$ such that J_0 is diagonal at each point. In other words, the gauge-invariant information contained in J_0 may be taken as the diagonal charge densities. Thus, to specify a charge distribution, we need only chemical potentials for the Cartan elements of the Lie algebra. There are other ways to see this as well. For example, if the charged particles form some irreducible representation R (which may be thought of as arising from the decomposition of a product of the representations of the individual particles), then we know that such a representation can be obtained by quantizing the coadjoint orbit action

$$S = i \int dt \sum_k w_k \,\mathrm{Tr}(h_k g^{-1} \dot{g}) \tag{2}$$

where w_k are the highest weights defining the representation R and h_k are the diagonal generators of the Lie algebra. We see that the diagonal charges are sufficient for our purpose. In a statistical distribution, we have to think of such a representation for the global color charge over each coarse-grained volume element, and this action can be generalized to obtain the fluid flow equations for color charge [4].

In the case of a non-Abelian plasma, there is an added complication. While it is possible to define a gauge-covariant charge density for the quarks (and other matter particles), there is no gauge-covariant charge density for the gluons. The integrated total charge has a gauge-invariant expression. The chemical potential, introduced as a background value for A_0, does couple to this global charge correctly. This also leads to terms quadratic in μ in the action, which is to be expected since the current for a charged bosonic system depends on A_μ in addition to the charged fields themselves. All these effects are included in the replacement $A_0 \to A_0 + \mu$. Since the diagonalization of the charge density happens only by choice of $g(x)$, the general ansatz for the background value of A_0 is

$$A_0 = g^{-1} \mu g + g^{-1} \partial_0 g. \tag{3}$$

The group element g can be removed by an overall gauge transformation,

$$\begin{aligned} A_0 &\to g A_0 g^{-1} - \partial_0 g g^{-1} = \mu \\ A_i &\to g A_i g^{-1} - \partial_i g g^{-1}. \end{aligned} \tag{4}$$

Designating the new spatial components of the potential as A_i again, we see that we can use μ as the background value for A_0.

II. CALCULATING THE EFFECTIVE LAGRANGIAN

We shall carry out the calculations in Euclidean space. While this is not necessary, as for many other calculations at finite temperature and density, this is slightly simpler. This means that the background value of the A_0 becomes imaginary. Thus the basic calculation to check for instability reduces to the following. Taking constant matrices for A_0 and A_i as the background values, we consider fluctuations in the fields. The integration of the action to quadratic order in the fluctuations leads to the standard determinant. This has to be evaluated as a function of the background values. The result is then analytically continued to imaginary values of the background A_0. The result can then be analyzed for instabilities. The instability of interest to us is the Schwinger decay of the chromoelectric field. This has been studied in some detail in the non-Abelian case for electric fields which are given by the curl of the gauge potentials [5], but, here, we are interested in the case when the field arises from the commutator term of the potentials. For the calculations which follow, we will consider the group $SU(2)$ since it is sufficient to capture the effect we are interested in.

The integration over the quadratic fluctuations can be phrased as an effective Lagrangian given by

$$\begin{aligned} L_{\mathrm{eff}} = \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \int_0^\infty \frac{ds}{s} \,\mathrm{Tr}[\exp(-s[-(D^2)^{ab}\eta_{\mu\nu} \\ - 2 f^{acb} F^c_{\mu\nu}])] - \int \frac{d^D p}{(2\pi)^D} \\ \times \int_0^\infty \frac{ds}{s} \,\mathrm{Tr}[\exp(-s[-D^2])] \end{aligned} \tag{5}$$

where the second term is the contribution from the ghosts. Here $D^2 = (\partial_\mu + A_\mu)(\partial^\mu + A^\mu)$ is the gauge-covariant Laplacian with the background field A^a_μ; it is a 3×3-matrix in color space, as indicated by the color indices a, b. Thus the operator $-(D^2)^{ab}\eta_{\mu\nu} - 2f^{acb}F^c_{\mu\nu}$ can be considered as a 12×12-matrix, in addition to its coordinate space properties. The evaluation of the action will follow a method which is similar to what was used many years ago by Brown and Weisberger [6]. Writing the $SU(2)$ field $A^{ab}_\mu = f^{acb} A^c_\mu = \epsilon^{acb} A^c_\mu$, we can simplify D^2 as

$$-(D^2)_{ab} = p^2 + Y - Y_{ab} - 2ip \cdot A_{ab} \tag{6}$$

where $p_\mu = -i\partial_\mu$, $Y^{ab} = A^a_\mu A^b_\mu$ and $Y = \mathrm{Tr} Y^{ab}$. The matrix Y^{ab} can be diagonalized by a suitable gauge transformation, with eigenvalues λ_a. These eigenvalues give the gauge-invariant characterization of the chromo-electric and chromomagnetic fields. The λ's are positive in the case of Euclidean signature for the contraction of spacetime indices in $A^a_\mu A^b_\mu$, but one eigenvalue can be negative with Minkowski signature. A choice of A^a_μ amounts to choosing three four-vectors, and in the Euclidean metric we are using, we can always make the choice

$$A^a_\mu = \sqrt{\lambda_a}\delta^a_\mu, \qquad a, \mu = 1, 2, 3 \qquad A^a_4 = 0. \tag{7}$$

With this choice

$$Y_{ab} + 2i(p\cdot A)_{ab} = \begin{bmatrix} \lambda_1 & -2ip_3\sqrt{\lambda_3} & 2ip_2\sqrt{\lambda_2} \\ 2ip_3\sqrt{\lambda_3} & \lambda_2 & -2ip_1\sqrt{\lambda_1} \\ -2ip_2\sqrt{\lambda_2} & 2ip_1\sqrt{\lambda_1} & \lambda_3 \end{bmatrix}. \tag{8}$$

For our purpose, it is not necessary to consider this matrix in full generality, we can take $\lambda_3 = 0$. In this case the only nontrivial component of the field strength tensor is $F^3_{12} = -F^3_{21} = \sqrt{\lambda_1\lambda_2}$. Obviously, this will not give the most general class of fields. However, we are aiming for the instability which arises from $[A_0, A_i]$ and this choice is adequate to illustrate the point. (Later, we will identify the x^1-direction with time.) For the choice of $\lambda_3 = 0$, schematically, we have

$$[Y_{ab} + 2i(p\cdot A)_{ab}]\eta_{\mu\nu} + 2F_{ab\mu\nu} = \begin{bmatrix} Y + 2ip\cdot A & 2F_{12} & 0 & 0 \\ -2F_{12} & Y + 2ip\cdot A & 0 & 0 \\ 0 & 0 & Y + 2ip\cdot A & 0 \\ 0 & 0 & 0 & Y + 2ip\cdot A \end{bmatrix} \tag{9}$$

where each block is a 3×3 matrix in color space. From this block diagonal form,

$$\mathrm{Tr}_{12\times12}\exp[s\{(Y + 2ip\cdot A)\eta_{\mu\nu} + 2F_{\mu\nu}\}] = 2\,\mathrm{Tr}_{3\times3}e^{s(Y+2ip\cdot A)} + \mathrm{Tr}_{6\times6}e^{s[(Y+2ip\cdot A)\eta_{\mu\nu}+2F_{\mu\nu}]}. \tag{10}$$

The first term on the right-hand side cancels exactly the similar contribution from ghosts. The remaining 6×6 matrix corresponds to the indices 1, 2, for spacetime and the 3×3 matrix in color space. The effective Lagrangian is thus

$$L_{\mathrm{eff}} = \frac{1}{2}\int\frac{d^D p}{(2\pi)^D}\int_0^\infty \frac{ds}{s}e^{-s(p^2+Y)}\,\mathrm{Tr}_{6\times6}e^{-s\mathbb{X}} \tag{11}$$

where $\mathbb{X}$ is the 6×6 matrix

$$(-\mathbb{X}) = \begin{bmatrix} \lambda_1 & 0 & 2ip^2\sqrt{\lambda_2} & 0 & -2\sqrt{\lambda_1\lambda_2} & 0 \\ 0 & \lambda_2 & -2ip^1\sqrt{\lambda_1} & 2\sqrt{\lambda_1\lambda_2} & 0 & 0 \\ -2ip^2\sqrt{\lambda_2} & 2ip^1\sqrt{\lambda_1} & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{\lambda_1\lambda_2} & 0 & \lambda_1 & 0 & 2ip^2\sqrt{\lambda_2} \\ -2\sqrt{\lambda_1\lambda_2} & 0 & 0 & 0 & \lambda_2 & -2ip^1\sqrt{\lambda_1} \\ 0 & 0 & 0 & -2ip^2\sqrt{\lambda_2} & 2ip^1\sqrt{\lambda_1} & 0 \end{bmatrix}. \tag{12}$$

For evaluating the remaining trace, it is convenient to use the integral representation

$$\mathrm{Tr}\,e^{-s\mathbb{X}} = \oint\frac{dz}{2\pi i}e^{-sz}\frac{\partial}{\partial z}\,\mathrm{logdet}(z - \mathbb{X}) \tag{13}$$

where the integration contour encircles all zeros of $\det(z - \mathbb{X})$.

The determinant is easy to evaluate,

$$\det(z - \mathbb{X}) = \{z^3 + z^2(\lambda_1 + \lambda_2) - [4p_1^2(z\lambda_1 + \lambda_1^2) + 4p_2^2(z\lambda_2 + \lambda_2^2) + 3z\lambda_1\lambda_2]\}^2 \tag{14a}$$

$$= \left\{z[z^2 + z(\lambda_1 + \lambda_2) - 3\lambda_1\lambda_2]\left[1 - \frac{4p_1^2(z\lambda_1 + \lambda_1^2) + 4p_2^2(z\lambda_2 + \lambda_2^2)}{z[z^2 + z(\lambda_1 + \lambda_2) - 3\lambda_1\lambda_2]}\right]\right\}^2. \tag{14b}$$

When this is used in (11) and (13), with the ∂_z carried out, we get contributions from the poles which correspond to the roots of the cubic polynomial inside the braces in (14a). It is then convenient to split the expression for L_{eff} as $L_1 + L_2$ with

$$L_1 = \int \frac{d^D p}{(2\pi)^D} \int_0^\infty \frac{ds}{s} e^{-s(p^2+\lambda_1+\lambda_2)} \oint \frac{dz}{2\pi i} e^{-sz} \times \frac{\partial}{\partial z} \log[z(z^2 + z(\lambda_1 + \lambda_2) - 3\lambda_1\lambda_2)] \quad (15a)$$

$$L_2 = \int \frac{d^D p}{(2\pi)^D} \int_0^\infty \frac{ds}{s} e^{-s(p^2+\lambda_1+\lambda_2)} \oint \frac{dz}{2\pi i} e^{-sz} \times \frac{\partial}{\partial z} \log\left[1 - \frac{4p_1^2(z\lambda_1 + \lambda_1^2) + 4p_2^2(z\lambda_2 + \lambda_2^2)}{z[z^2 + z(\lambda_1 + \lambda_2) - 3\lambda_1\lambda_2]}\right]. \quad (15b)$$

The evaluation of L_1 is simple. The zeros of the relevant cubic polynomial are $z = 0$ and $z = z_\pm$ with

$$z_\pm = \frac{1}{2}[-(\lambda_1 + \lambda_2) \pm \sqrt{(\lambda_1 + \lambda_2)^2 + 12\lambda_1\lambda_2}]. \quad (16)$$

We then find

$$L_1 = \frac{\Gamma(-D/2)}{(4\pi)^{D/2}}[(\lambda_1 + \lambda_2)^{D/2} + (\lambda_1 + \lambda_2 + z_+)^{D/2} + (\lambda_1 + \lambda_2 + z_-)^{D/2}]. \quad (17)$$

Γ is the Eulerian gamma function. Notice that there are singularities in this expression for $D = 4$. These are, of course, the standard renormalization singularities and can be isolated by expanding $(\mu)^{4-D} L_1$ in powers of ϵ with $D = 4 - \epsilon$. (The μ-factor is the usual one for ensuring the correct dimension for L_1.) This leads to the expression

$$\begin{aligned}\mu^{4-D} L_1 = {} & \frac{1}{(4\pi)^2 \epsilon}[(\lambda_1 + \lambda_2)^2 + (\lambda_1 + \lambda_2 + z_+)^2 \\ & + (\lambda_1 + \lambda_2 + z_-)^2] + \frac{(\lambda_1 + \lambda_2)^2}{(4\pi)^2} \\ & \times \left(\frac{3}{4} - \frac{1}{2}\log(\lambda_1 + \lambda_2)/\tilde{\mu}^2\right) + \frac{(\lambda_1 + \lambda_2 + z_+)^2}{(4\pi)^2} \\ & \times \left(\frac{3}{4} - \frac{1}{2}\log(\lambda_1 + \lambda_2 + z_+)/\tilde{\mu}^2\right) \\ & + \frac{(\lambda_1 + \lambda_2 + z_-)^2}{(4\pi)^2} \\ & \times \left(\frac{3}{4} - \frac{1}{2}\log(\lambda_1 + \lambda_2 + z_-)/\tilde{\mu}^2\right) + \mathcal{O}(\epsilon) \quad (18)\end{aligned}$$

where $\tilde{\mu}^2 = 4\pi e^{-\gamma}\mu^2$, γ being the Euler-Mascheroni constant.

The first term on the right-hand side of (18) is the potentially divergent contribution which is removed by renormalization. The remainder gives the finite expression we need for L_1.

The evaluation of L_2 is a little more involved and is sketched out in the appendix. The final result is

$$L_2 = -\frac{1}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^\infty dz \int_0^1 dx \frac{(1-x)^{-1+D/2}}{x} z^{-1+D/2} \times \left[1 - \frac{1}{\sqrt{1 - xA_1}\sqrt{1 - xA_2}}\right] \quad (19)$$

where

$$A_1 = \frac{4z\lambda_1(z + \lambda_2)}{(z + \lambda_1 + \lambda_2)[(z + \lambda_1)(z + \lambda_2) - 4\lambda_1\lambda_2]} \quad (20)$$

and A_2 is obtained by the exchange $\lambda_1 \leftrightarrow \lambda_2$ in the above expression. In (19) also, there is a potentially divergent contribution arising from the large z behavior of the integrand. Its removal, along with the potentially divergent term from (18) is discussed in the appendix.

A. The nature of the instability

We are now in a position to consider how instabilities can arise from these results. In continuing the expressions for L_1, L_2 to Minkowski space, one of the directions has to be identified as the time-direction. We will take this to be the 1-direction. Chromoelectric fields in Minkowski space will thus correspond to the choice $\lambda_1 < 0$, $\lambda_2 > 0$. The choice of λ_1, $\lambda_2 > 0$ will correspond to the purely chromomagnetic case, with 1-direction being interpreted as spatial direction now. We will consider various possibilities for the λ's one by one.

B. Case a

Consider first the case of $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_1 + \lambda_2 > 0$. In this case, the factor $(\lambda_1 + \lambda_2)^2 + 12\lambda_1\lambda_2$ is positive for $\lambda_2 \gg |\lambda_1|$. For this region

$$\lambda_1 + \lambda_2 + z_\pm = \frac{\lambda_1 + \lambda_2 \pm \sqrt{(\lambda_1 + \lambda_2)^2 + 12\lambda_1\lambda_2}}{2} > 0 \quad (21)$$

and hence there is no instability in L_1. As we come down in the value of λ_2, this factor changes sign at $\lambda_2 = (7 + \sqrt{48})|\lambda_1|$. For the region $|\lambda_1| < \lambda_2 < (7 + \sqrt{48})|\lambda_1|$, the quantities $\lambda_1 + \lambda_2 + z_+$ and $\lambda_1 + \lambda_2 + z_-$ are complex conjugates of each other. Writing these as $\alpha e^{\pm i\theta}$, we can easily see from (17) that there is no imaginary part in L_1 for this region as well. Thus, there is no instability resulting from L_1.

Turning to $\mathrm{Im} L_2$, notice that we can set $D = 4$ at this stage because the integration range for z for the imaginary part does not extend to infinity and so the issue of divergences do not arise. The analysis of L_2 then reduces to the analysis of the condition $A_2(z) > 1$. The polynomial factor in the denominator of the A's, namely, that $(z + \lambda_1)(z + \lambda_2) - 4\lambda_1\lambda_2 = z^2 + z(\lambda_1 + \lambda_2) + 3|\lambda_1|\lambda_2$ is easily seen to be positive. Thus $A_1(z) < 0$ and the factor $\sqrt{1 - xA_1}$ is real for the full range $(z > 0)$ of integration for z. On other hand, $A_2(z)$, whose numerator is $4z\lambda_2(z + \lambda_1)$ will show a change of sign for $z = -\lambda_1 > 0$. However, even though $A_2(z) > 0$ for $z > \lambda_1$, we have $A_2(z) \leq 1$. This is easily seen from the fact that

$$(z + \lambda_1 + \lambda_2)[(z + \lambda_1)(z + \lambda_2) - 4\lambda_1\lambda_2] \geq (z - |\lambda_1|)[(z - |\lambda_1|)(z + \lambda_2) + 4|\lambda_1|\lambda_2]. \quad (22)$$

The quantity in the square brackets on the right hand side is $\geq 4z\lambda_2$ for $z > |\lambda_1|$. Thus the factor $\sqrt{1 - xA_2}$ is also real and hence there is no instability for this case from either L_1 or L_2.

C. Case b

Now we turn to the case $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_1 + \lambda_2 < 0$. The region $(7 - \sqrt{48})|\lambda_1| < \lambda_2 < |\lambda_1|$ has complex conjugate values for $\lambda_1 + \lambda_2 + z_\pm$ and there is no imaginary part resulting from the last two terms in L_1, as in the previous case for $\lambda_2 < (7 + \sqrt{48})|\lambda_1|$. There is an imaginary part from the $\log(\lambda_1 + \lambda_2)$ term in L_1, which will give an instability for this range of λ_2. For $\lambda_2 < (7 - \sqrt{48})|\lambda_1|$ (or $|\lambda_1| > (7 + \sqrt{48})\lambda_2$) we have

$$\lambda_1 + \lambda_2 + z_\pm = \frac{\lambda_1 + \lambda_2 \pm \sqrt{(\lambda_1 + \lambda_2)^2 + 12\lambda_1\lambda_2}}{2} < 0. \quad (23)$$

There is then a nontrivial imaginary part in L_1 which leads to an instability. Thus we get instability from L_1 for all λ_1, λ_2 corresponding to this case.

Turning to L_2, we may notice that the factor $(z + \lambda_1 + \lambda_2)$ in the denominator of A_1, A_2 changes sign at $z = -(\lambda_1 + \lambda_2)$. The additional factor in the denominator, namely, $[(z + \lambda_1)(z + \lambda_2) - 4\lambda_1\lambda_2]$ has two positive roots if $|\lambda_1|/\lambda_2 > 7 + \sqrt{48} \approx 14$. Otherwise, there are no real roots and this factor is positive. The graphs of $A_1(z)$ as a function of z are as shown in Fig. 1. We see that for all values of $|\lambda_1|/\lambda_2$, there are regions of z-integration for which $A_1(z) > 1$, leading to an imaginary part for L_2. Similar statements apply for A_2, see Fig. 2.

D. Case c

Even though it is not germane to our present discussion, we may note that if we have the purely chromomagnetic case with $\lambda_1 > 0$, $\lambda_2 > 0$, then

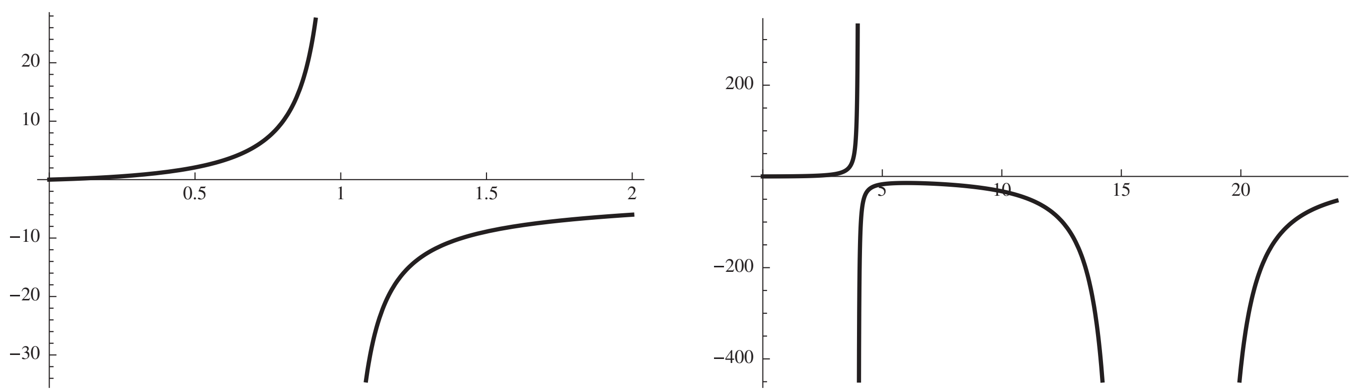

FIG. 1. Sample graphs of $A_1(z)$ for $1 < |\lambda_1|/\lambda_2 < 7 + \sqrt{48}$ (left) and for $|\lambda_1|/\lambda_2 > 7 + \sqrt{48}$ (right). The value of A_1 between 15 and 20 is large and positive and outside the frame of the graph on the right.

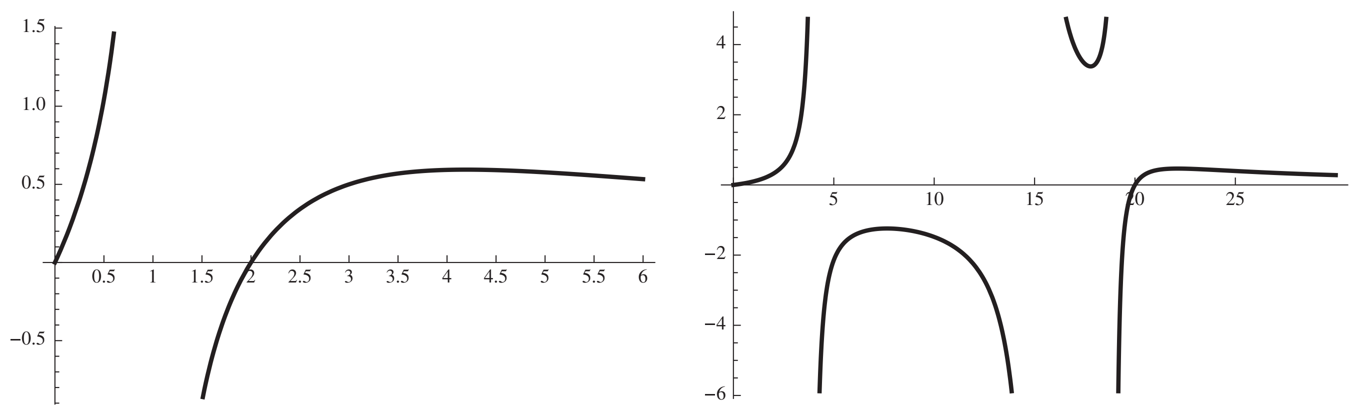

FIG. 2. Sample graphs of $A_2(z)$ for $1 < |\lambda_1|/\lambda_2 < 7 + \sqrt{48}$ (left) and for $|\lambda_1|/\lambda_2 > 7 + \sqrt{48}$ (right).

$$\lambda_1 + \lambda_2 + z_- = \frac{(\lambda_1 + \lambda_2) - \sqrt{(\lambda_1 + \lambda_2)^2 + 12\lambda_1\lambda_2}}{2} < 0 \tag{24}$$

Thus the last term on the right hand side in (18) has an imaginary component. For L_2, the polynomial $(z + \lambda_1) \times (z + \lambda_2) - 4\lambda_1\lambda_2$ in the denominators of A_1, A_2 has roots $z_\pm$. For $\lambda_1, \lambda_2 > 0$, one root is negative and the other is positive. $A_1(z)$ is positive for $z > z_+$ and goes to zero for large z, with $A_1(z) \to \infty$ for $z - z_+ \to 0_+$. Thus there is a range of z for which $\sqrt{1 - xA_1}$ has an imaginary part. Again a similar statement applies to A_2. Thus for both L_1 and L_2 we get an instability for $\lambda_1, \lambda_2 > 0$. This is the well-known vacuum instability in a chromomagnetic field.

It is interesting to characterize the instability in terms of invariants of the field. We see easily that $F^a_{\mu\nu}F^{a\mu\nu} = (\mathrm{Tr}Y)^2 - \mathrm{Tr}Y^2 = 2\lambda_1\lambda_2$, and $Y = \mathrm{Tr}(Y^{ab}) = \lambda_1 + \lambda_2$. The case where we have obtained instability may be characterized in terms of the fields as

$$F^{a\mu\nu}F^a_{\mu\nu} < 0, \qquad A^{a\mu}A^a_\mu < 0. \tag{25}$$

III. DISCUSSION

The calculation we have presented shows the Schwinger decay of chromoelectric fields for the case when the field is generated by the commutator term, rather than the curl of the potentials. For the purpose of demonstrating this result, by virtue of the Euclidean and $SU(2)$ rotational symmetries, it is sufficient to consider the case of one component of the field strength, say F^3_{12} (analytically continued to F^3_{02}), being nonzero. This is the choice we have made in Eq. (9). However, our result is general in terms of demonstrating an instability in the non-Abelian plasma. For, in order to incorporate nonzero non-Abelian charge, one needs to introduce a chemical potential. In such a situation, any A^a_i which is constant, or very slowly varying, over some length scale will lead to an effective chromoelectric field. Our calculation shows that some kind of instability is then unavoidable. The resolution of the situation may be that the plasma cannot sustain A^a_i which are slowly varying over a length scale compared to the scale of the chemical potential. Or it could be that nonzero charges are not possible over a similar scale of coarse graining.

A further remark concerning the length scale is in order as well. The key issue is that we can demonstrate the decay of the field for a constant chromoelectric field. This is basically a calculational limitation. The question then is whether this calculation has any implications for inhomogeneous fields. One would expect that, if the field is slowly varying, the qualitative result of instability should still hold. A direct demonstration of this for the non-Abelian fields would be nice, but is beyond the scope of the present paper. However, we may note that some calculations for inhomogeneous fields are available for the Abelian case. One finds that the results for a constant field are in fact obtained for inhomogeneous fields as well, provided $F_{02}L^2 \gg 1$, L being the scale on which the field varies significantly [7]. Since the effective electric field for our case is of the order of the chemical potential, one could expect that the relevant length scale is determined by the chemical potential. This is the reason for the expectation, stated in the introduction, that the instability should afflict fields coherent over scales exceeding the scale of the chemical potential.

In conclusion, we may note that the instability we are discussing hints at how statistical distributions tend to move to color neutrality or a disordered state with no coherent fields over distances long compared to the dimension given by the chemical potential.

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Note added:—After this paper was posted on the archives, Dr. Shahin Mamedov informed us that some aspects of instability of a color electric field are discussed in [8].

APPENDIX

1. Calculation of L_2

For L_2, we start with the representation

$$\log A = \int_0^\infty \frac{dt}{t}(e^{-t} - e^{-tA}). \tag{A1}$$

Using this and eliminating ∂_z by partial integration, the expression (15b) for L_2 becomes

$$\begin{aligned} L_2 &= \int \frac{d^D p}{(2\pi)^D} \int_0^\infty \frac{ds}{s} e^{-s(p^2+\lambda_1+\lambda_2)} \oint \frac{dz}{2\pi i} e^{-sz} \int_0^\infty \frac{dt}{t} e^{-st} \\ &\quad \times \left[1 - \exp\left(4st\frac{p_1^2\lambda_1(z+\lambda_1) + p_2^2\lambda_2(z+\lambda_2)}{z(z^2 + z\lambda_1 + z\lambda_2 - 3\lambda_1\lambda_2)}\right)\right] \\ &= \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} e^{-s(z+t+\lambda_1+\lambda_2)} \oint \frac{dz}{2\pi i} \int_0^\infty \frac{dt}{t} \\ &\quad \times \left[1 - \frac{1}{C_1(z)C_2(z)}\right] \end{aligned} \tag{A2}$$

where

$$C_k(z) = \sqrt{1 - \frac{4t\lambda_k(z+\lambda_k)}{z(z^2 + z\lambda_1 + z\lambda_2 - 3\lambda_1\lambda_2)}} \tag{A3}$$

for $k = 1, 2$. For the second line of Eq. (A2) we have carried out the p-integration. Note that the exponents involving p_k^2 show that we need to take the z-contour to be large enough, $|z| > 2\sqrt{t\lambda}$. Effectively, this means that we should do the z-interal before doing the t-integral. In (A2), we can further carry out the s-integration to get

$$L_2 = \frac{\Gamma(1-D/2)}{(4\pi)^{D/2}} \int_0^\infty \frac{dt}{t} \oint \frac{dz}{2\pi i}(z+t+\lambda_1+\lambda_2)^{-1+D/2} \times \left[1 - \frac{1}{C_1(z)C_2(z)}\right]. \quad \text{(A4)}$$

The factor $(z+t+\lambda_1+\lambda_2)^{-1+D/2}$ shows that, for the z-integration, we have a branch cut along the negative real axis starting at $z=-t-\lambda_1-\lambda_2$. We can deform the original contour which is a large circle around the origin, via the contour shown in Fig. 3, to the contour in Fig. 4. Notice that because of the arguments given earlier, the branch point $z=-t-\lambda_1-\lambda_2$ is always outside the original contour, while the singularities of the square root factors are always inside the contour. Integration along the cut in Fig. 4 gives

$$L_2 = \frac{\Gamma(1-D/2)}{(4\pi)^{D/2}} \int_0^\infty \frac{dt}{t} \int_{t+\lambda_1+\lambda_2}^\infty dz(z-t-\lambda_1-\lambda_2)^{-1+D/2} \times \left[\frac{e^{i\pi(D/2-1)} - e^{-i\pi(D/2-1)}}{2\pi i}\right]\left[1 - \frac{1}{C_1(-z)C_2(-z)}\right]. \quad \text{(A5)}$$

Using

$$\frac{e^{i\pi(D/2-1)} - e^{-i\pi(D/2-1)}}{2\pi i} = -\frac{1}{\Gamma(1-D/2)\Gamma(D/2)} \quad \text{(A6)}$$

and shifting the variable of integration to $z-\lambda_1-\lambda_2$, we can write (A5) as

$$L_2 = -\frac{1}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^\infty \frac{dt}{t} \int_t^\infty dz(z-t)^{-1+D/2} \times \left[1 - \frac{1}{\sqrt{1-tA_1/z}\sqrt{1-tA_2/z}}\right] \quad \text{(A7)}$$

where

$$A_1 = \frac{4z\lambda_1(z+\lambda_2)}{(z+\lambda_1+\lambda_2)[(z+\lambda_1)(z+\lambda_2) - 4\lambda_1\lambda_2]} \quad \text{(A8)}$$

and A_2 is given by the same expression with $\lambda_1 \leftrightarrow \lambda_2$. Changing the order of integration and making the substitution $t = zx$, we finally get

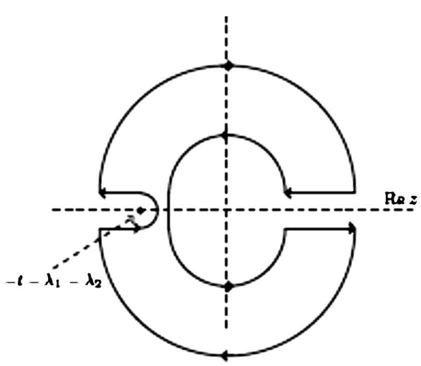

FIG. 3. Deformation of contour for branch cut at $z=-t-\lambda_1-\lambda_2$.

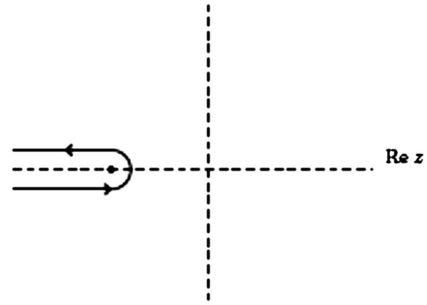

FIG. 4. Contour for evaluating L_2.

$$L_2 = -\frac{1}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^\infty dz \int_0^1 dx \times \frac{(1-x)^{-1+D/2}}{x} z^{-1+D/2}\left[1 - \frac{1}{\sqrt{1-xA_1}\sqrt{1-xA_2}}\right]. \quad \text{(A9)}$$

This is the expression quoted in the text.

2. Renormalization: A consistency check

The potentially divergent part of L_1 was obtained in Eq. (18) as

$$\mu^{4-D}L_{1\text{div}} = \frac{1}{(4\pi)^2\epsilon}[(\lambda_1+\lambda_2)^2 + (\lambda_1+\lambda_2+z_+)^2 + (\lambda_1+\lambda_2+z_-)^2]. \quad \text{(A10)}$$

Using the expressions for $z_\pm$ from (16), this simplifies to

$$\mu^{4-D}L_{1\text{div}} = \frac{1}{(4\pi)^2\epsilon}[2(\lambda_1^2+\lambda_2^2) + 10\lambda_1\lambda_2]. \quad \text{(A11)}$$

The expression for L_2 can be recast as

$$L_2 = -\frac{1}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^\infty d\tau \tau^{-1-D/2} G(\tau) \quad \text{(A12)}$$

where $\tau = 1/z$ and

$$G(\tau) = \int_0^1 \frac{dx}{x}(1-x)^{-1+D/2}\left[1 - \frac{1}{\sqrt{1-x\tilde{A}_1}\sqrt{1-x\tilde{A}_2}}\right] \quad \text{(A13)}$$

and $\tilde{A}$'s correspond to A's with $z = 1/\tau$; i.e.,

$$\tilde{A}_1 = \frac{4\tau\lambda_1(1+\tau\lambda_2)}{[1+\tau(\lambda_1+\lambda_2)][(1+\tau\lambda_1)(1+\tau\lambda_2) - 4\tau^2\lambda_1\lambda_2]} \quad \text{(A14)}$$

with $\lambda_1 \leftrightarrow \lambda_2$ to obtain $\tilde{A}_2$ from $\tilde{A}_1$. The divergence now corresponds to small values of τ. Carrying out a small τ-expansion,

$$G(\tau) = -\frac{4}{D}\tau(\lambda_1 + \lambda_2) + \frac{8\tau^2}{D(D+2)} \times [D\lambda_1\lambda_2 + (D-1)(\lambda_1^2 + \lambda_2^2)] + \mathcal{O}(\tau^3). \quad \text{(A15)}$$

We can use this expansion in (A12) and integrate; we are interested in the small τ region, so we use a cutoff $e^{-\tau}$ in the integrand. (Whether we use this or something else, such as $e^{-a\tau}$ for some a does not matter for the term of the form $\Gamma((4-D)/2)$.) The term proportional to $1/\epsilon$ is then found to be

$$L_{2\text{div}} = -\frac{1}{(4\pi)^2\epsilon}\left[2(\lambda_1^2 + \lambda_2^2) + \frac{8}{3}\lambda_1\lambda_2\right]. \quad \text{(A16)}$$

Combining this with (A11), we find

$$L_{\text{div}} = \frac{1}{(4\pi)^2\epsilon}\frac{22}{3}\lambda_1\lambda_2 = \frac{1}{(4\pi)^2\epsilon}\frac{11}{3}F^a_{\mu\nu}F^{a\mu\nu}. \quad \text{(A17)}$$

This is the expected and correct renormalization of the action, and is consistent with the β-function of

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\frac{22}{3} \quad \text{(A18)}$$

for $SU(2)$.

[1] S. Mrowczynski, Phys. Lett. B **314**, 118 (1993); P. Arnold, J. Lenaghan, and G. D. Moore, J. High Energy Phys. 08 (2003) 002; P. Arnold and J. Lenaghan, Phys. Rev. D **70**, 114007 (2004); **70**, 114007 (2004); D. Bödeker, J. High Energy Phys. 10 (2005) 092; P. Arnold, J. Lenaghan, G. D. Moore, and L. G. Yaffe, Phys. Rev. Lett. **94**, 072302 (2005); P. Arnold, G. D. Moore, and L. G. Yaffe, Phys. Rev. D **72**, 054003 (2005); A. Dumitru and Y. Nara, Phys. Lett. B **621**, 89 (2005); B. Schenke, M. Strickland, C. Greiner, and M. H. Thoma, Phys. Rev. D **73**, 125004 (2006); Y. Nara, Nucl. Phys. **A774**, 783 (2006); P. Arnold and G. D. Moore, Phys. Rev. D **73**, 025006 (2006);

[2] A. Rebhan, P. Romatschke, and M. Strickland, Phys. Rev. Lett. **94**, 102303 (2005); J. High Energy Phys. 09 (2005) 041; D. Bodeker and K. Rummukainen, J. High Energy Phys. 07 (2007) 022; P. Arnold and G. D. Moore, Phys. Rev. D **76**, 045009 (2007); A. Dumitru, Y. Nara, and M. Strickland, Phys. Rev. D **75**, 025016 (2007).

[3] See, for example, M. G. Alford, K. Rajagopal, T. Schaefer, and A. Schmitt, Rev. Mod. Phys. **80**, 1455 (2008).

[4] B. Bistrovic, R. Jackiw, H. Li, V. P. Nair, and S-Y. Pi, Phys. Rev. D **67**, 025013 (2003); R. Jackiw, V. P. Nair, S-Y. Pi, and A. P. Polychronakos, J. Phys. A **37**, R327 (2004).

[5] G. Nayak and P. van Nieuwenhuizen, Phys. Rev. D **71**, 125001 (2005); F. Cooper and G. Nayak, Phys. Rev. D **73**, 065005 (2006); G. Nayak, Eur. Phys. J. C **59**, 715 (2009); S. P. Gavrilov and D. M. Gitman, Eur. Phys. J. C **64**, 81 (2009); G. Nayak, Int. J. Mod. Phys. A **25**, 1155 (2010).

[6] L. S. Brown and W. I. Weisberger, Nucl. Phys. **B157**, 285 (1979); **B161**, 61 (1979).

[7] S. P. Kim and D. N. Page, Phys. Rev. D **73**, 065020 (2006); **75**, 045013 (2007); see also, G. V. Dunne, in *From Fields to Strings: Circumnavigating Theoretical Physics*, edited by M. Shifman, A. Vainshtein, and J. Wheater (World Scientific, Singapore, 2005), Vol. I, pp. 445–522.

[8] Sh. S. Agaev, A. S. Vshivtsev, V. Ch. Zhukovsky, and P. G. Midodashvili, Vestn. Mosk. Univ., Fiz. **26**, 12 (1985) (In Russian).