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# Classical stability of the BTZ black hole in topologically massive gravity

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We demonstrate the classical stability of the BTZ black hole within the context of topologically massive gravity. The linearized perturbation equations can be solved exactly in this case. By choosing standard boundary conditions appropriate to the stability problem, we demonstrate the absence of modes which grow in time, for all values of the Chern-Simons coupling.

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## I. INTRODUCTION

Topologically massive gravity (TMG) [1], in three-dimensional anti-de Sitter space has been the subject of considerable interest recently (see, for example, [2–7] for a partial list of references). As in any theory, a first step is to search for solutions to the classical equations of motion. Fortunately, the structure of the equations for TMG guarantee that the known solutions without a Chern-Simons coupling $\mu = \infty$ are also solutions when $\mu$ is finite. In particular, then, we already have a constant curvature black hole solution at our disposal, namely, the BTZ black hole [8]. One of the most remarkable features of the BTZ black hole is the role that it has played in understanding many aspects of the AdS/CFT correspondence, and its role in the near-horizon geometry of higher-dimensional black holes.

Given a black hole solution, a first order of business is to examine its classical stability properties. Typically, this is accomplished by resorting to the linearized approximation, and exploring solutions to the associated boundary value problem. Our purpose here is to examine the stability of the BTZ black hole within the context of topologically massive gravity, for all values of the Chern-Simons coupling $\mu$.

The key to our analysis lies in the fact that the perturbation equations can be solved exactly. We can thus search explicitly for modes which grow in time; the presence of such modes would indicate a classical instability. Of course, a crucial ingredient in this analysis is the choice of boundary conditions. The original asymptotic boundary conditions for three-dimensional anti-de Sitter gravity were determined by Brown and Henneaux [9]. Recently, it has been shown that one may relax these conditions slightly in topologically massive gravity, for certain ranges of the coupling $\mu$ [10]. We adopt these generalized boundary conditions at asymptotic infinity as the appropriate conditions to impose on the linear perturbations. In the presence of a black hole background, one also needs to impose boundary conditions at the horizon. We establish the fact that by simply demanding boundedness of the perturbation (necessary for the linearized approximation to be valid), the absence of unstable modes is guaranteed. Stability of black holes in warped anti-de Sitter has recently been discussed in [11].

The plan of this paper is as follows. In Sec. II, we present the basic equations of topologically massive gravity and solve directly the first order equations of motion in the BTZ background. In Sec. III, we identify potential unstable modes and confront them with the appropriate boundary conditions. In Sec. IV, we present an alternative approach to the problem, which is based on the second order analysis of [4]. The absence of unstable modes in also confirmed from this viewpoint. In Sec. V, we conclude with some brief remarks.

## II. METRIC PERTURBATIONS

The action for topologically massive gravity is taken in the form

$$S = -\frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R + \frac{2}{l^2}\right) - \frac{1}{32\mu\pi G} \times \int d^3x\sqrt{-g}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}_{\nu\rho} + \frac{2}{3}\Gamma^{\sigma}_{\mu\tau}\Gamma^{\tau}_{\nu\rho}\right), \quad (1)$$

where $\mu$ is the Chern-Simons coupling, and the parameter $l$ sets the scale of the cosmological constant of anti-de Sitter space, $\Lambda = -1/l^2$.

In the following, we will be interested in the linear approximation around a background spacetime, and therefore decompose the metric tensor as $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$. Working in the transverse traceless gauge, with $\nabla^\mu h_{\mu\nu} = g^{\mu\nu}h_{\mu\nu} = 0$, the equation of motion for a background

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which is locally isometric to anti-de Sitter space takes the form

$$(\nabla^2+2)\left[h_{\mu\nu}+\frac{1}{\mu}\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha h_{\beta\nu}\right]=0. \tag{2}$$

It is convenient to define the operators

$$(\mathcal{D}^M)_\mu{}^\beta=\delta_\mu{}^\beta+\frac{1}{\mu}\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha,$$
$$(\mathcal{D}^{L/R})_\mu{}^\beta=\delta_\mu{}^\beta\pm l\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha. \tag{3}$$

The third order equation of motion can then be written in the form [3]

$$(\mathcal{D}^L\mathcal{D}^R\mathcal{D}^M h)_{\mu\nu}=0. \tag{4}$$

The first order equation for a massive graviton is given by $(\mathcal{D}^M h)_{\mu\nu}=0$, namely

$$\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha h_{\beta\nu}+\mu h_{\mu\nu}=0. \tag{5}$$

The BTZ black hole metric can be written in the form

$$ds^2=-\sinh^2\rho dt^2+\cosh^2\rho d\phi^2+d\rho^2, \tag{6}$$

where we have introduced the radial coordinate $r=\cosh\rho$; the horizon and infinity then correspond to $\rho=0,\ \infty$, respectively. We choose units such that the mass of the black hole is $M=1$ and set $l=1$. In the following, we will also use coordinates $u=t+\phi$ and $v=t-\phi$.

To solve the equation of motion (5), we make an ansatz for the perturbation in the form

$$h_{\mu\nu}=e^{-i\omega t-ik\phi}\begin{pmatrix}F_{uu} & F_{uv} & F_{u\rho}\\ F_{vu} & F_{vv} & F_{v\rho}\\ F_{\rho u} & F_{\rho v} & F_{\rho\rho}\end{pmatrix}. \tag{7}$$

Using $\epsilon^{\rho uv}=\frac{1}{\sqrt{-g}}=\frac{4}{\sinh 2\rho}$, the equations of motion can be written in the form [7]

$$\bar{h}F_{uu}-hF_{uv}=\left(\frac{-\mu-1}{4i}\right)\sinh(2\rho)F_{u\rho}, \tag{8}$$

$$\bar{h}F_{uv}-hF_{vv}=\left(\frac{-\mu+1}{4i}\right)\sinh(2\rho)F_{v\rho}, \tag{9}$$

$$\bar{h}F_{u\rho}-hF_{v\rho}=\frac{i}{\sinh(2\rho)}[-F_{vv}(-\mu+1)-F_{uu}(-\mu-1)+2\mu\cosh(2\rho)F_{uv}], \tag{10}$$

$$F_{\rho\rho}=\frac{4}{\sinh^2(2\rho)}[2\cosh(2\rho)F_{uv}+F_{uu}+F_{vv}], \tag{11}$$

together with the differential equations

$$\frac{dF_{uv}}{d\rho}=\left(\frac{-\mu+1}{\sinh(2\rho)}\right)\left[F_{uv}\left(\frac{4h\bar{h}}{(-\mu+1)^2}-\cosh(2\rho)\right)-F_{vv}\left(1+\frac{4h^2}{(-\mu+1)^2}\right)\right], \tag{12}$$

$$\frac{dF_{vv}}{d\rho}=\left(\frac{-\mu+1}{\sinh(2\rho)}\right)\left[-F_{vv}\left(\frac{4h\bar{h}}{(-\mu+1)^2}-\cosh(2\rho)\right)+F_{uv}\left(1+\frac{4\bar{h}^2}{(-\mu+1)^2}\right)\right], \tag{13}$$

where we have defined $h=(\omega+k)/2$, $\bar{h}=(\omega-k)/2$. The case of chiral gravity $\mu=\pm1$ will be treated separately below.

The Eqs. (12) and (13) are readily transformed into a second order hypergeometric equation (see [7] for details). The asymptotic $\rho$ dependence of the two solutions of the second order equation is of the form $F_{vv}\propto e^{(1-\mu)\rho}$ and $F_{vv}\propto e^{(3-\mu)\rho}$, respectively. The solutions with the former asymptotic behavior are obtained as descendants of a "highest weight" solution [6] of (8)–(13), which is obtained by setting $F_{uu}=F_{uv}=F_{u\rho}=0$. We then have

$$F_{v\rho}=\frac{i}{\sinh(2\rho)}\left(\frac{-\mu+1}{h}\right)F_{vv}, \tag{14}$$

$$F_{\rho\rho}=\frac{4}{\sinh^2(2\rho)}F_{vv}, \tag{15}$$

$$h=\pm\frac{i}{2}(-\mu+1). \tag{16}$$

Choosing the branch $h=\frac{i}{2}(-\mu+1)$, we find a right-moving solution

$$h^R_{\mu\nu}=e^{(1-\mu)t+ik(t-\phi)}(\sinh\rho)^{1-\mu}(\tanh\rho)^{ik}\times\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & \frac{2}{\sinh(2\rho)}\\ 0 & \frac{2}{\sinh(2\rho)} & \frac{4}{\sinh^2(2\rho)}\end{pmatrix}, \tag{17}$$

while the branch $h=-\frac{i}{2}(-\mu+1)$ also leads to a right-moving solution

$$H^R_{\mu\nu}=e^{(\mu-1)t+ik(t-\phi)}(\sinh\rho)^{1-\mu}(\tanh\rho)^{-ik}\times\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & -\frac{2}{\sinh(2\rho)}\\ 0 & -\frac{2}{\sinh(2\rho)} & \frac{4}{\sinh^2(2\rho)}\end{pmatrix}. \tag{18}$$

From the above expressions we see explicitly that (17) and (18) are ingoing and outgoing at the horizon, respectively. The outgoing modes are relevant for white holes while the ingoing modes are relevant for black holes.[1] We will thus focus on the mode (17).

[1]In fact, it turns out that even if we were to keep the outgoing modes, they would be eliminated by the boundary conditions given below.

The solutions of the second order equation with asymptotic behavior $F_{vv} \propto e^{(\mu-3)\rho}$ are descendants of an ingoing highest weight solutions obtained from (17) upon substitution $u \leftrightarrow v$, $h \leftrightarrow \bar{h}$, $\mu \to -\mu$, leading to

$$h^L_{\mu\nu} = e^{(1+\mu)t-ik(t+\phi)}(\sinh\rho)^{1+\mu}(\tanh\rho)^{-ik} \times \begin{pmatrix} 1 & 0 & \frac{2}{\sinh(2\rho)} \\ 0 & 0 & 0 \\ \frac{2}{\sinh(2\rho)} & 0 & \frac{4}{\sinh^2(2\rho)} \end{pmatrix}. \tag{19}$$

## III. STABILITY

The task now is to identify those solutions which obey the generalized boundary conditions [10] at infinity, and which grow in time. In terms of the coordinates $(\rho, u, v)$, an admissible metric perturbation has to satisfy either the boundary conditions

$$\begin{aligned} h_{\rho\rho} &= e^{-2\rho} f_{\rho\rho}, \\ h_{\rho u} &= e^{-2\rho} f_{\rho u}, \\ h_{\rho v} &= k_{\rho v} e^{-(1+\mu)\rho} + e^{-2\rho} f_{\rho v}, \\ h_{uu} &= f_{uu}, \\ h_{uv} &= f_{uv}, \\ h_{vv} &= k_{vv} e^{(1-\mu)\rho} + f_{vv}, \end{aligned} \tag{20}$$

or the boundary conditions

$$\begin{aligned} h_{\rho\rho} &= e^{-2\rho} f_{\rho\rho}, \\ h_{\rho u} &= k_{\rho u} e^{(-1+\mu)\rho} + e^{-2\rho} f_{\rho u}, \\ h_{\rho v} &= e^{-2\rho} f_{\rho v}, \\ h_{uu} &= k_{uu} e^{(1+\mu)\rho} + f_{uu}, \\ h_{uv} &= f_{uv}, \\ h_{vv} &= f_{vv}. \end{aligned} \tag{21}$$

Here, the functions $f$ and $k$ may only depend on $(u, v)$ but are otherwise unrestricted. The additional $\mu$-dependent terms are absent for $|\mu| > 1$. Upon examination of the above solutions, we observe that the ingoing solutions grow exponentially in time, and also satisfy these boundary conditions, provided $|\mu| < 1$. These solutions thus represent potentially unstable modes. The right-moving perturbation $h^R_{\mu\nu}$ obeys the boundary conditions (20), while the left-moving solution $h^L_{\mu\nu}$ obeys the boundary conditions (21). We wish to determine if such solutions can be eliminated by imposing a physically acceptable boundary condition at the horizon. To see this, it suffices to invoke boundedness of the solution, which is required in order for the linear approximation to be valid [12–14]. Boundedness of the solution near the horizon is most easily seen by transforming to Kruskal coordinates [15]

$$R = \tanh\frac{\rho}{2}\cosh t, \qquad T = \tanh\frac{\rho}{2}\sinh t. \tag{22}$$

Since the Kruskal coordinates are well defined at the horizon, we must also require the perturbation to be well behaved there. However, one can check that the Kruskal components of $h^L_{\mu\nu}$ and $h^R_{\mu\nu}$ diverge at the horizon, thus excluding them as physically acceptable. For example, if we examine the relation between the components of $h^R_{\mu\nu}$ in the $(\rho, u, v)$ and $(R, T, \phi)$ coordinate systems near the horizon at $t = 0$, we find that $h_{\rm RR} \sim h_{\rho\rho}$, $h_{\rm TR} \sim \rho^{-1} h_{v\rho}$, $h_{\rm TT} \sim \rho^{-2} h_{vv}$. From (17), it is then clear that boundedness of the perturbation in the Kruskal coordinates requires $\mu < -1$, which is therefore incompatible with the condition $|\mu| < 1$. Alternatively, one may state that boundedness of the perturbation requires the existence of a nonsingular linearized diffeomorphism, $\epsilon_\mu$ with $h_{\mu\nu} \to \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\nu$ such that $g^{\mu\alpha} h_{\alpha\nu} \ll 1$, and this leads to boundary conditions both at the horizon and infinity. Given the explicit expression

$$(h^L)^\mu{}_\nu = e^{(1+\mu)t-ik(t+\phi)}(\sinh\rho)^{1+\mu}(\tanh\rho)^{-ik} \times \begin{pmatrix} -\frac{4}{\sinh^2(2\rho)} & 0 & -\frac{8}{\sinh^3(2\rho)} \\ -\frac{4\cosh(2\rho)}{\sinh^2(2\rho)} & 0 & -\frac{8\cosh(2\rho)}{\sinh^3(2\rho)} \\ \frac{2}{\sinh(2\rho)} & 0 & \frac{4}{\sinh^2(2\rho)} \end{pmatrix}, \tag{23}$$

which satisfies the asymptotic boundary conditions at infinity for $|\mu| < 1$, it is not difficult to show that a nonsingular diffeomorphism that renders the perturbation bounded at the horizon cannot exist for this range of $\mu$. For example, in order to be able to write $(h^L)^\rho_\rho$ in the form $\nabla_\rho \epsilon_\rho$ requires a diffeomorphism $\epsilon_\rho \sim \rho^\mu$ near the horizon. However, this is nonsingular only when $\mu > 1$, which is thus incompatible with the requirement $|\mu| < 1$. A similar argument applies to the right-moving perturbation $(h^R)^\mu{}_\nu$. Thus, the solutions are excluded as physically unacceptable. It should also be noted that for $|\mu| > 1$, the relevant boundary conditions are given by (20) and (21) without the $\mu$-dependent terms. As a result, there are no potentially unstable modes which grow exponentially in time and obey these boundary conditions.

The generic metric perturbation with the same asymptotic behavior as (19) has a time dependence given by replacing $\mu \to \mu - 2n$, $n \in \mathbf{N}$ in the exponent [6]. However, for $n \neq 0$, there are no growing modes which satisfy the asymptotic boundary conditions. Similarly, generic metric perturbation with the same asymptotic behavior as (17) have a time dependence given by replacing $\mu \to \mu + 2n$, $n \in \mathbf{N}$. Again, such modes are excluded by the asymptotic boundary conditions.

The remaining case to deal with is when $\mu = \pm 1$. The modes (17) and (19) then become pure gauge transformations and thus do not represent a physical perturbation of the black hole. However, at the chiral point $\mu = 1$, a new class of logarithmic modes arises [5]. It is obtained by differentiating (17) before setting $\mu = 1$ [16], leading to a solution of the form

$$h^{\log}_{\mu\nu} = -y(\tau, \rho) h^R_{\mu\nu}, \tag{24}$$

where $y(\tau, \rho) = \tau + \ln[\sinh(\rho)]$. The generalized boundary conditions for $\mu = \pm 1$ can be obtained by making the replacement $e^{(1-\mu)\rho} \to \rho$ in (20), and $e^{(1+\mu)\rho} \to \rho$ in (21). The asymptotic $\rho$ dependence of (24) is then consistent with these generalized boundary conditions. However, as is clear from the form of $y(\tau, \rho)$, the nonboundedness of $h^R_{\mu\nu}$ at the horizon implies the nonboundedness of $h^{\log}_{\mu\nu}$, and consequently all of its descendants. The antichiral point $\mu = -1$ is treated in a similar fashion.

In conclusion, we have show that all potentially unstable solutions, growing in time and obeying the generalized boundary conditions at asymptotic infinity are excluded by the requirement of boundedness of the solution at the horizon. According to these criteria, the BTZ black hole is thus a stable solution of topologically massive gravity for all values of the Chern-Simons coupling $\mu$.

## IV. SCALAR FORMULATION OF METRIC PERTURBATIONS

In [4], it was shown that the action for topologically massive gravity can be recast in terms of a single massive scalar field, with the mass related to the Chern-Simons coupling parameter. Consequently, it was established that the perturbation equations for all gauge invariant modes can be formulated as second order massive scalar field equations. It is well know that the equation for a massive scalar field in the background of the BTZ metric can be solved exactly in terms of hypergeometric functions, and this will allow us to explicitly study the stability of the BTZ black hole. As well as confirming the analysis of the previous section, it will also highlight an alternative viewpoint on the boundary conditions relevant for a stability analysis.

It is well known that there exists a class of topological black holes in anti-de Sitter space, with line element [17]

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 h_{ij}(x)dx^i dx^j, \tag{25}$$

where

$$f(r) = \left(k - \frac{2M}{r^{d-3}} + \frac{r^2}{l^2}\right). \tag{26}$$

The parameter $k$ can take the values $k = 1, 0, -1$, and the cosmological constant is $\Lambda = -(d-1)(d-2)/2l^2$. The novel feature of these topological black holes is the fact that there exists a massless black hole when $k = -1$. The crucial point to note here is that the metric ansatz for the BTZ black hole (with mass parameter equal to one) is of the same form as the massless topological black hole with $M = 0$, $k = -1$, $d = 3$, and we set $l = 1$. Thus, the stability analysis of the massless topological black hole performed in [18] can be used to analyze the stability of the BTZ black hole within the context of topologically massive gravity. In [18], the massive scalar field equation was solved exactly. However, in order to apply those results to the case at hand, we need to specify the scalar field in terms of metric components, and re-analyze the stability question within that context.

To proceed, we consider a scalar field $\phi$ of mass $m$ in the BTZ background,

$$(\nabla^2 - m^2)\phi = 0. \tag{27}$$

As shown in [4], the essential dynamics of topologically massive gravity is encoded in a scalar field of mass $m^2 = (-\mu + 2)^2 - 1$. Furthermore, all independent gauge invariant perturbations can be recast as scalar field equations for various masses. Choosing the ansatz

$$\phi = \phi(r)e^{\omega t - ik\phi}, \tag{28}$$

brings the radial equation to the form

$$\left[-\left(f\frac{d}{dr}\right)^2 + V\right]\Phi(r) = -\omega^2\Phi(r), \tag{29}$$

where $\Phi = r^{1/2}\phi$, and

$$V = \frac{f}{r^2}\left[k^2 + \frac{1}{4} + \left(\frac{3}{4} + m^2\right)r^2\right]. \tag{30}$$

In order to investigate the stability properties of the black hole, it is useful to recast Eq. (29) as a Sturm-Liouville equation

$$A\Phi = \lambda\Phi, \tag{31}$$

where the Schrödinger operator is given by

$$A = -\frac{d^2}{dr_*^2} + V(r), \tag{32}$$

with eigenvalue $\lambda = -\omega^2$, and the tortoise coordinate $r_*$ is defined by $dr_* = \frac{dr}{f}$. Our task is to solve this equation subject to appropriate boundary conditions. Given the Sturm-Liouville form, this involves searching for eigenvalue solutions which are normalizable with respect to the standard measure [12,19],

$$1 = \int dr_* \Phi^* \Phi. \tag{33}$$

In particular, unstable modes will correspond to normalizable ($\omega > 0$) states of the Schrödinger operator $A$.

Near the horizon, this condition of normalizability demands that the solution behave as $\Phi \sim (r-1)^\epsilon$, and thus we impose a Dirichlet boundary condition $\Phi \to 0$ on the perturbation [12,19,20]. For large $r$, normalizability requires $\Phi \sim r^{(1/2)-\epsilon}$. However, the scalar field for TMG is related to a metric component by [4]

$$\Phi = z^{3/2} h_{zz}, \tag{34}$$

where the upper half-space coordinate $z \sim \frac{1}{r}$ for large $r$. In terms of this coordinate, the generalized asymptotic boundary condition is $h_{zz} = O(1)$, and thus we require $\Phi \sim \frac{1}{r^{3/2}}$, for large $r$.

To proceed towards the solution of (29), we change variables to a new radial coordinate defined by

$$z = 1 - \frac{1}{r^2}. \tag{35}$$

Thus, $z = 0$ now corresponds to the location of the horizon $r = 1$, while $z = 1$ corresponds to $r = \infty$. The master equation can then be written as

$$z(1-z)\frac{d^2\Phi}{dz^2} + \left(1 - \frac{3z}{2}\right)\frac{d\Phi}{dz} + \left[\frac{A}{z} + B + \frac{C}{1-z}\right]\Phi = 0, \tag{36}$$

where

$$A = -\frac{\omega^2}{4}, \quad B = -\frac{1}{4}\left(\frac{1}{4} + k^2\right), \quad C = -\frac{1}{4}\left(m^2 + \frac{3}{4}\right). \tag{37}$$

Defining

$$\Phi(z) = z^\alpha (1-z)^\beta F(z), \tag{38}$$

allows the master equation to be reduced to hypergeometric form

$$z(1-z)\frac{d^2F}{dz^2} + [c - (a+b+1)z]\frac{dF}{dz} - abF = 0, \tag{39}$$

provided that

$$\alpha = \pm\frac{\omega}{2}, \qquad \beta = \frac{1}{4} \pm \frac{1}{2}\sqrt{1+m^2}, \tag{40}$$

with the coefficients determined as follows:

$$a = \frac{1}{4} + \alpha + \beta + \frac{1}{2}\sqrt{-k^2}, \quad b = \frac{1}{4} + \alpha + \beta - \frac{1}{2}\sqrt{-k^2}, \quad c = 2\alpha + 1. \tag{41}$$

Without loss of generality, we can take

$$\alpha = \frac{\omega}{2}, \qquad \beta = \frac{1}{4} - \frac{1}{2}\sqrt{1+m^2}. \tag{42}$$

In the neighborhood of the horizon, the two linearly independent solutions of (39) are $F(a,b,c,z)$ and $z^{1-c}F(a-c+1, b-c+1, 2-c, z)$. With the choice (42), the solution which is regular (satisfying Dirichlet boundary conditions) at the horizon is then given by

$$\Phi(z) = z^\alpha(1-z)^\beta F(a,b,c,z). \tag{43}$$

Having imposed the Dirichlet boundary condition at the horizon, we can now analytically continue this solution to infinity. In general, the form of the solution near $z = 1$ is given by [21]

$$\begin{aligned}\Phi = {}& z^\alpha(1-z)^\beta \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \times F(a,b,a+b-c+1, 1-z) \\ &+ z^\alpha(1-z)^{\beta+c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \times F(c-a, c-b, c-a-b+1, 1-z),\end{aligned} \tag{44}$$

where $c - a - b = \frac{1}{2} - 2\beta$. The generalized asymptotic boundary condition requires that $\Phi \sim (1-z)^{3/4}$ near $z = 1$. First, we consider the case when $m^2 > 0$. Then $\beta < -\frac{1}{4}$ and the second term in (44) clearly vanishes at infinity. In order to guarantee the vanishing of the divergent first term, we must demand that

$$c - a = -n, \quad \text{or} \quad c - b = -n, \tag{45}$$

where $(n = 0, 1, 2, 3, \ldots)$. In particular, the condition $c - a = -n$ becomes

$$\omega = -1 - \sqrt{1+m^2} + \sqrt{-k^2} - 2n. \tag{46}$$

It is then clear that unstable modes with $\omega > 0$ do not exist since $k^2 \geq 0$.

For $m^2 = 0$, corresponding to $\mu = 1$, we have $\beta = -1/4$, and $c - a - b = 1$. As a result, the analytic continuation to $z = 1$ contains logarithmically divergent terms, and is given by

$$\begin{aligned}\Phi = {}& z^\alpha(1-z)^{-1/4}\frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} + \frac{\Gamma(a+b+1)}{\Gamma(a)\Gamma(b)} z^\alpha (1-z)^{3/4}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{n!(n+1)!} \\ &\times (1-z)^n[\ln(1-z) - \psi(n+1) - \psi(n+2) + \psi(a+n+1) + \psi(b+n+1)],\end{aligned} \tag{47}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$, and $\psi(z) = \Gamma'(z)/\Gamma(z)$. To guarantee the absence of the divergent first term, we now require $a + 1 = -n$ or $b + 1 = -n$. Note that these conditions also ensure the vanishing of the logarithmic terms in (47). Since $c - a - b = 1$, we can write these conditions as (45), which we have already shown have no solutions. The appearance of logarithmic terms in the solution here is equivalent to their appearance in the formalism of Sec. III.

For $-1 < m^2 < 0$, the range of $\beta$ is $-\frac{1}{4} < \beta < \frac{1}{4}$, and the solution is given by (44). The absence of unstable solutions is again guaranteed by (45).

Finally, for $m^2 = -1$, we have $\beta = \frac{1}{4}$, and $c - a - b = 0$. The solution then takes the form

$$\Phi = z^{\alpha}(1-z)^{1/4}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(n!)^2} \times [2\psi(n+1) - \psi(a+n) - \psi(b+n) - \ln(1-z)](1-z)^n. \quad (48)$$

Consistency with the generalized asymptotic boundary conditions requires $a = -n$ or $b = -n$. However, since $c - a - b = 0$, these conditions again reduce to (45).

In conclusion, we have established the absence of unstable modes for the BTZ black hole within the scalar field formulation of topologically gravity. This result confirms the first order analysis of the previous section. In the previous section, we used boundedness of the perturbation at the horizon to eliminate the potentially unstable modes. It is worth mentioning that these modes can also be eliminated by requiring the perturbation to be normalizable at the horizon. Normalizability at the horizon requires that $\Phi \sim (r-1)^{\epsilon}$, and hence $h_{\rho\rho} \sim \rho^{2+\epsilon}$. By examining the solutions $h^L$ we see that normalizability at the horizon requires $\mu > 3$. For $h^R$ normalizability at the horizon requires $\mu < -3$. Both conditions are incompatible with the generalized asymptotic boundary conditions, thereby excluding such modes.

## V. DISCUSSION

We have discussed the classical stability of the BTZ black hole as a solution of topologically massive gravity. The linearized perturbation equations are exactly solvable, and this allowed us to explicitly examine the behavior of the solutions subject to certain boundary conditions at the horizon and infinity. Using the Brown-Henneaux boundary conditions at infinity, extended to incorporate the Chern-Simons term, and boundedness of the perturbation at the horizon, allows us to establish the stability of the BTZ black hole. This result was confirmed by studying the perturbation within the scalar field formulation of [4], and using normalizability of the perturbation.

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