PHYSICAL REVIEW D **82,** 124048 (2010)

Lifshitz scalar, brick wall method, and generalized uncertainty principle in Hořava-Lifshitz gravity

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(Received 2 August 2010; published 22 December 2010)

Using the brick-wall method, we study statistical entropy for spherically symmetric black holes in Hořava-Lifshitz gravity. In particular, a Lifshitz scalar field is considered in order to incorporate foliation-preserving diffeomorphism, which eventually gives a modified dispersion relation. Finally, we obtain the area law without the UV cutoff for $z > 3$ and discuss some of the consequences in connection with the generalized uncertainty principle.

DOI: 10.1103/PhysRevD.82.124048 PACS numbers: 04.70.Dy, 04.50.Kd

I. INTRODUCTION

Recently, Hořava has put forward a renormalizable theory of gravity when the scaling dimensions of space and time are different, which is called Hořava-Lifshitz (HL) gravity [1,2]. It is power-counting renormalizable for $z = d$ and superrenormalizable for $z > d$, where z and d are scaling dimension and number of spatial dimensions, respectively. Subsequently, there have been extensive studies of the HL gravity [3–17], such that various black holes [5–12] and cosmological solutions [13–17] have been intensively studied. Moreover, it has been claimed that the nonisotropic scaling of spacetime related to the foliation-preserving diffeomorphism (FPD) gives a modified dispersion relation [18,19]. As for the nonisotropic scaling, even in a flat spacetime, it generically leads to an intriguing dispersion relation of the form $E^2 - c^2[p^2 + \cdots + \Lambda_z(p^2)^z] = m^2c^4$, where c, m, and Λ_z are the speed of light, the mass of a particle, and a parameter, respectively [19]. Of course, it can be generalized in nonflat spacetimes. There have been some studies for modified dispersion relations in black hole physics similar to this dispersion relation [20,21].

On the other hand, it has been known that the entropy of a black hole is proportional to the area of its event horizon. For calculating the statistical entropy, the brick-wall method suggested by 't Hooft can be used [22], where the cutoff parameter should be introduced to handle the divergence near the event horizon. Since degrees of freedom of a field are dominant near the horizon, the brick wall can be replaced by a thin layer or a thin spherical box [23]. By the way, the cutoff parameter located just outside the horizon can be avoided if we consider the generalized uncertainty principle (GUP) [24–27]. Actually, the mode counting can be done from the horizon to the minimal length, and it gives finite density because of the modification of phase-space volume and the dispersion relation.

In this paper, we would like to study the statistical entropy of spherically symmetric black holes in the HL gravity using the (thin-layered) brick-wall method. For this purpose, we introduce a Lifshitz scalar field rather than the usual scalar field to incorporate the nonisotropic symmetry of the matter sector. In this semiclassical calculation, the resulting entropy shows that for $z > 3$, corresponding to the superrenormalizable case of HL gravity, the ultraviolet (UV) cutoff parameter can be avoided, so that a thin layer can be located just outside the horizon, similarly to the case of the GUP. Assuming the Bekenstein-Hawking entropy, which is proportional to the area of the horizon, we can naturally fix the size of the thin layer depending only on the scaling. In Sec. II, we recapitulate the Hořava-Lifshitz gravity and black hole solutions. In Sec. III, WKB approximations with the modified dispersion relation for the Lifshitz scalar field will be considered. In Sec. IV, the statistical entropy will be given by counting the number of quantum states and we will find the condition to give the area law of entropy. In Sec. V, some issues related to the modified dispersion relation will be presented. Finally, some discussions will be given in Sec. VI.

II. BLACK HOLES IN HL GRAVITY

We briefly review HL gravity in a self-contained manner and introduce black hole solutions for (3 + 1)-dimensional HL gravity. On general grounds, like the Arnowitt-Deser-Misner (ADM) decomposition of the metric in Einstein gravity, the (3 + 1)-dimensional metric can be decomposed into

$$ds^2 = -N^2c^2dt^2 + g_{ij}(dx^i + N^i dt) \times (dx^j + N^j dt), \qquad i, j = 1, 2, 3, \tag{1}$$

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where N and N^i are the usual lapse and shift functions. An anisotropic scaling transformation of time t and space $\vec{x}$ is given by

$$t \to b^z t, \qquad x^i \to b x^i, \tag{2}$$

under which g_{ij} and N are invariant, while $N^i \to b^{1-z} N^i$. We have scaling dimensions given by $[t] = z, [x^i] = 1$, and $[c] = [N^i] = 1 - z$ in units of spatial length.

The kinetic action in the Hořava-Lifshitz gravity is given by [2]

$$I_K = \frac{2}{\kappa^2} \int dt d^3x \sqrt{g} N [K_{ij} K^{ij} - \lambda K^2], \tag{3}$$

where κ^2 and λ are a coupling related to the Newton constant G_N and an additional dimensionless coupling constant, respectively. The extrinsic curvature is given by $K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)$, where the overdot denotes the derivative with respect to time t, and ∇_i is the covariant derivative with respect to the spatial metric g_{ij}. Note that the original kinetic part of the Einstein-Hilbert action can be recovered when $\lambda = 1$ and $\kappa^2 = 32\pi G_N / c^2$. Moreover, the power-counting renormalizability requires $z \geq 3$. Now, the potential term of action is determined by the "detailed balance condition" as follows [2]:

$$I_V = \frac{\kappa^2}{8} \int dt d^3x \sqrt{g} N E^{ij} \mathcal{G}_{ijkl} E^{kl}, \tag{4}$$

where E^{ij} comes from three-dimensional relativistic action in the form of

$$E^{ij} = \frac{1}{\sqrt{g}} \frac{\delta W[g_{ij}]}{\delta g_{ij}}, \tag{5}$$

and the generalized DeWitt metric $\mathcal{G}^{ijkl}$ and its inverse metric $\mathcal{G}_{ijkl}$ for $\lambda \neq 1/3$ are given by

$$\mathcal{G}^{ijkl} = \frac{1}{2}(g^{ik} g^{jl} + g^{il} g^{jk}) - \lambda g^{ij} g^{kl}, \tag{6}$$

$$\mathcal{G}_{ijkl} = \frac{1}{2}(g_{ik} g_{jl} + g_{il} g_{jk}) - \frac{\lambda}{3\lambda - 1} g^{ij} g^{kl}, \quad \text{for } \lambda \neq \frac{1}{3}, \tag{7}$$

with the normalization condition of

$$\mathcal{G}^{ijkl} \mathcal{G}_{klmn} = \frac{1}{2}(\delta^i_m \delta^j_n + \delta^i_n \delta^j_m). \tag{8}$$

In particular, the relativistic action W is expressed as $W = W_1 + W_2$ for $z = 3$ [2,13], where W_1 and W_2 are given by

$$W_1 = \mu \int d^3x \sqrt{g}(R - 2\Lambda_W), \tag{9}$$

$$W_2 = \frac{1}{w^2} \int d^3x \sqrt{g} \varepsilon^{ijk} \Gamma^m_{il} \left(\partial_j \Gamma^l_{km} + \frac{2}{3} \Gamma^l_{jn} \Gamma^n_{km} \right), \tag{10}$$

with $\varepsilon_{ijk} = \sqrt{g} \epsilon_{ijk}$ and $\epsilon_{123} = 1$. Here, μ and w^2 are coupling constants, and Λ_W is a cosmological constant.

Let us assume that the line element of a spherically symmetric black hole can be written in the form of

$$ds^2 = -f \tilde{N}^2 c^2 dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{11}$$

where $f = f(r)$ and $\tilde{N} = \tilde{N}(r)$. For an arbitrary λ for $z = 3$, there are three solutions [5]. The first one is given by

$$f = 1 - \Lambda_W r^2, \tag{12}$$

with an arbitrary function $\tilde{N}$. The others for $\lambda > 1/3$ are given by

$$f = 1 - \Lambda_W r^2 - \alpha(\sqrt{-\Lambda_W} r)^{2\lambda \pm \sqrt{6\lambda - 2}/(\lambda - 1)}, \tag{13}$$

$$\tilde{N} = (\sqrt{-\Lambda_W} r)^{-(1 + 3\lambda \pm 2\sqrt{6\lambda - 2})/(\lambda - 1)}, \tag{14}$$

where α is an integration constant. On the other hand, $\lambda = 1$ for $z = 3$, the asymptotically flat solution, along with a vanishing cosmological constant, is given by [9]

$$f = 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}, \tag{15}$$

$$\tilde{N} = 1, \tag{16}$$

where $\omega = 16\mu^2/\kappa^2$, and M is an integration constant. Moreover, in the modified Hořava-Lifshitz gravity proposed in Ref. [3], the other types of spherically symmetric solutions have been also studied [15]. In fact, we need not consider specific forms of black hole solutions as long as the spherical symmetric ansatz holds, since we shall calculate the statistical entropy near the horizon without loss of generality.

III. MODIFIED AND REDUCED DISPERSION RELATIONS

We consider a complex scalar field φ, obeying the modified Klein-Gordon equation implemented by FPD, which is assumed to be

$$-\frac{1}{Nc\sqrt{g}} \partial_t \left(\frac{\sqrt{g}}{Nc} D_t \varphi \right) + \frac{1}{Nc} \nabla_i \left(\frac{N^i}{Nc} D_t \varphi \right) - [\Lambda_0 + \Lambda_1(-\nabla^2) + \cdots + \Lambda_z(-\nabla^2)^z]\varphi = 0, \tag{17}$$

where the derivative D_t is defined by $D_t = \partial_t - N^i \partial_i$, the Laplacian is given by $\nabla^2 \equiv g^{ij} \nabla_i \nabla_j$, and the constants Λ_n will be fixed later. Note that Eq. (17) may be induced from a certain action, especially for a constant lapse function; there appears such a consideration in Ref. [28]. Now, applying a WKB approximation to Eq. (17) with $\varphi = \exp[iS(t, x^i)]$, we obtain a modified dispersion relation,

$$\frac{1}{N^2 c^2}(p_t - N^i p_i)^2 - [\Lambda_0 + \Lambda_1 p^2 + \cdots + \Lambda_z (p^2)^z] = 0, \tag{18}$$

where momenta are defined by $p_t = \partial_t S$, $p_i = \partial_i S$, and $p^2 = p_i p^i = g^{ij} p_i p_j$. In order to recover the dispersion relation in general relativity, we can take $\Lambda_0 = m^2c^2$ and $\Lambda_1 = 1$ and assume that Λ_n's are very small for $n \geq 2$. For convenience, let us define $x^\mu = (ct, x^i)$, and set the Boltzman constant to $k_B = 1$. For the spherically symmetric background of $N^i = 0$, the dispersion relation (18) can be written as

$$p_0p^0 + \sum_{n=1}^{z} \Lambda_n (p_i p^i)^n = -m^2c^2, \tag{19}$$

where p_μ is the conjugate momentum to x^μ. The constants Λ_n have a scaling dimension of $2n - 2$. Note that Eq. (19) can be reduced to $p_0p^0 + p_ip^i = -m^2c^2$ for the relativistic limit of $z = 1$.

Now, we choose the constants Λ_n as $\Lambda_n = l_{\rm P}^{2(z-1)}\delta_{nz}$ for $n \geq 1$ to obtain a reasonable entropy where $l_{\rm P}$ is the Plank length given by $l_{\rm P} = \sqrt{\hbar G_N/c^3}$. As a result, the dispersion relation (19) for the scalar field with mass m can be simply reduced to

$$p_0p^0 + l_{\rm P}^{2(z-1)}(p_ip^i)^z = -m^2c^2. \tag{20}$$

We will consider this reduced relation of the highest-momentum case for a scalar field on a spherically symmetric black hole background.

IV. ENTROPY IN A REDUCED DISPERSION RELATION

We now consider a spherically symmetric black hole whose line element can be written as

$$ds^2 = -f\tilde{N}^2c^2dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{21}$$

where $f = f(r)$ and $\tilde{N} = \tilde{N}(r)$, and the horizon r_H of the black hole is defined by $g^{rr}|_{r_H} = f(r_H) = 0$. With the help of conjugate pairs of $x^\mu = (ct, r, \theta, \phi)$ and $p_\mu = (-\omega/c, p_r, p_\theta, p_\phi)$, the dispersion relation (20) becomes

$$l_{\rm P}^{2(z-1)}\left(fp_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right)^z = \frac{\omega^2}{f\tilde{N}^2c^2} - m^2c^2. \tag{22}$$

Let us consider a spherical box specified by $r_H + \epsilon$ to $r_H + \epsilon + \delta$, where ϵ plays the role of UV cutoff in the conventional brick-wall method. It will be shown that it is unnecessary, because the UV-divergent behavior of free energy can be improved. Next, the number of quantum states with energy less than ω is calculated as

$$\begin{aligned} n(\omega) &= \frac{1}{(2\pi)^3}\int drd\theta d\phi dp_r dp_\theta dp_\phi \\ &= \frac{1}{(2\pi)^3}\int drd\theta d\phi \times V_p, \end{aligned} \tag{23}$$

where V_p is the z-dimensional volume of momentum space satisfying Eq. (22), which is explicitly

$$n(\omega) = \frac{2}{3\pi l_{\rm P}^{(3/z)(z-1)}}\int dr\frac{r^2}{\sqrt{f}}\left(\frac{\omega^2}{f\tilde{N}^2c^2} - m^2c^2\right)^{(3/2z)}, \tag{24}$$

near the horizon where ω is the energy of a scalar field with the range of $\omega \geq mc^2\tilde{N}\sqrt{f}$. For $z = 1$, it recovers the standard form of number of the quantum states,

$$n(\omega) = \frac{2}{3\pi}\int dr\frac{r^2}{\sqrt{f}}\left(\frac{\omega^2}{f\tilde{N}^2c^2} - m^2c^2\right)^{3/2}. \tag{25}$$

Then, the free energy is given by

$$F_{(z)} = -\int_{\omega_0}^{\infty} d\omega \frac{n(\omega)}{e^{\beta\omega} - 1}, \tag{26}$$

where $\omega_0 = mc^2\tilde{N}\sqrt{f}$, β^{-1} is a inverse temperature defined by $\beta^{-1} = \kappa_H/(2\pi c)$, and $\kappa_H = \frac{1}{2}c^2\tilde{N}f'|_{r_H}$ is a surface gravity. So, the entropy can be written as

$$S_{(z)} = \beta^2\frac{\partial F_{(z)}}{\partial\beta} = \beta^2\int_{\omega_0}^{\infty} d\omega\frac{\omega n(\omega)}{4\sinh^2\frac{1}{2}\beta\omega}. \tag{27}$$

For the sake of convenience, ω is replaced by $x = \frac{1}{2}\beta\omega$. Then, it can be written as

$$S_{(z)} = \int_{x_0}^{\infty} dx\frac{xn(2x/\beta)}{\sinh^2 x}, \tag{28}$$

where $x_0 = \frac{1}{2}\beta mc^2\tilde{N}\sqrt{f}$, which goes to zero near the horizon. Plugging Eq. (24) into Eq. (28), the entropy becomes

$$S_{(z)} = \frac{2}{3\pi}\int dr\frac{r^2}{\sqrt{f}}\int_{x_0}^{\infty} dx\frac{xJ(x)}{\sinh^2 x}, \tag{29}$$

where

$$J(x) \equiv \begin{cases} l_{\rm P}^{(3/z)(1-z)}\left(\frac{4x^2}{\beta^2 f\tilde{N}^2c^2} - m^2c^2\right)^{(3/2z)}, & \text{near the horizon} \\ \left(\frac{4x^2}{\beta^2 f\tilde{N}^2c^2} - m^2c^2\right)^{3/2}, & \text{for } r \gg r_H \end{cases}. \tag{30}$$

Note that for $r \gg r_H$, as expected, the integral is proportional to the volume of space as long as the metric function f approaches the nonzero constant at infinity.

On the other hand, we are concerned about the entropy near the horizon, which is given by

$$S_{(z)} = \frac{2}{3\pi l_{\rm P}^{(3/z)(z-1)}}\int dr\frac{r^2}{f^{(1+3/z)/2}}\left(\frac{2}{\beta c\tilde{N}}\right)^{3/z}\alpha(z), \tag{31}$$

where $\alpha(z) \equiv \int_0^\infty dx\frac{x^{1+3/z}}{\sinh^2 x}$. For some z's, we can find $\alpha(1) = \pi^4/30$, $\alpha(2) \approx 1.5762$, $\alpha(3) = \pi^2/6$, and $\alpha(4) \approx 1.8766$. Now, in the near horizon limit, the function $f(x)$ can be expanded as $f(r) = \frac{2\kappa_H}{\tilde{N}_H c^2}(r - r_H) + O(r - r_H)^2$, with $\tilde{N}_H \equiv \tilde{N}(r_H)$, so that one can take the first-order approximation of $\kappa_H \neq 0$ for nonextremal black holes. Therefore, the entropy (31) is explicitly calculated as

$$S_{(3)} = \frac{A}{4} \cdot \frac{c}{9\beta l_{\rm P}^2 \kappa_H} \cdot \ln\left(1 + \frac{\delta}{\epsilon}\right), \quad \text{for } z = 3, \tag{32}$$

$$S_{(z)} = \frac{A}{4} \cdot \frac{4z\alpha(z)}{3\pi^2(z-3) l_{\rm P}^{(3/z)(z-1)}} \left(\frac{2}{\beta c \tilde{N}_H}\right)^{3/z} \times \left(\frac{\tilde{N}_H c^2}{2\kappa_H}\right)^{[(1/2)+(3/2z)]} [(\epsilon + \delta)^{[(1/2)-(3/2z)]} - \epsilon^{[(1/2)-(3/2z)]}], \quad \text{for } z \neq 3. \tag{33}$$

Note that there is no continuous limit at $z = 3$. Now, let us define proper lengths for ϵ and δ, respectively,

$$\bar{\epsilon} \equiv \int_{r_H}^{r_H+\epsilon} dr \sqrt{g_{rr}} = \sqrt{\frac{2c^2 \tilde{N}_H \epsilon}{\kappa_H}}, \tag{34}$$

$$\bar{\delta} \equiv \int_{r_H+\epsilon}^{r_H+\epsilon+\delta} dr \sqrt{g_{rr}} = \sqrt{\frac{2c^2 \tilde{N}_H}{\kappa_H}} (\sqrt{\epsilon + \delta} - \sqrt{\epsilon}), \tag{35}$$

where $\epsilon = \frac{1}{2}\kappa_H \bar{\epsilon}^2/(\tilde{N}_H c^2)$ and $\epsilon + \delta = \frac{1}{2}\kappa_H(\bar{\epsilon} + \bar{\delta})^2/(\tilde{N}_H c^2)$. Then, the entropy can be written as

$$S_{(3)} = \frac{A}{4} \frac{1}{9\pi l_{\rm P}^2} \ln\left(1 + \frac{\bar{\delta}}{\bar{\epsilon}}\right) \quad \text{for } z = 3, \tag{36}$$

$$S_{(z)} = \frac{A}{4} \cdot \frac{2z\alpha(z)}{3\pi^{[2+(3/z)]}(z-3) l_{\rm P}^{(3/z)(z-1)}} \times [(\bar{\epsilon} + \bar{\delta})^{[1-(3/z)]} - \bar{\epsilon}^{[1-(3/z)]}] \quad \text{for } z \neq 3. \tag{37}$$

It is interesting to note that for the case of $z > 3$, the entropy (37) is finite

$$S_{(z)} = \frac{A}{4} \cdot \frac{2z\alpha(z)}{3\pi^{[2+(3/z)]}(z-3) l_{\rm P}^{(3/z)(z-1)}} \bar{\delta}^{[1-(3/z)]}, \tag{38}$$

even in spite of the absence of the UV cutoff, i.e., $\bar{\epsilon} \to 0$. In other words, for the case of $z \leq 3$, the UV cutoff is necessary to get some finite results. Recovering the dimension, except the Boltzman constant k_B, the entropy (38) is written as

$$S_{(z)} = \frac{c^3 A}{4\hbar G_N} \cdot \frac{2z\alpha(z)\bar{\delta}^{[1-(3/z)]}}{3\pi^{[2+(3/z)]}(z-3) l_{\rm P}^{[(z-3)/3]}}. \tag{39}$$

Then, Eq. (39) is compatible with the Bekenstein-Hawking entropy given by

$$S_{\rm BH} = \frac{c^3 A}{4\hbar G_N}, \tag{40}$$

as long as we identify the size of the box as

$$\bar{\delta} = l_{\rm P}\left[\frac{3(z-3)\pi^{[2+(3/z)]}}{2z\alpha(z)}\right]^{z/(z-3)}, \tag{41}$$

for $z > 3$. It depends on the scale parameter z. However, it is independent of the black hole hairs.

There are some special limits to be mentioned. As for the marginal case of $z = 3$, recovering dimensions, the entropy becomes

$$S_{(3)} = \frac{c^3 A}{4\hbar G_N} \cdot \frac{1}{9\pi} \ln\left(1 + \frac{\bar{\delta}}{\bar{\epsilon}}\right). \tag{42}$$

Note that Eq. (42) also agrees with the Bekenstein-Hawking entropy, assuming $\bar{\delta}/\bar{\epsilon} = e^{9\pi} - 1$. In this case, the UV cutoff is needed. On the other hand, for the limit of $z = 1$, which corresponds to the (thin-layered) brick-wall method, the well-known cutoff parameter can be obtained, $S_{(1)} = c^3 A/(4\hbar G_N) \cdot \delta l_{\rm P}^2/[90\beta\epsilon(\epsilon + \delta)]$ [29]. In these respects, excitations of the Lifshitz scalar field coupled to the gravity contribute to the finite entropy near the horizon limit without the UV cutoff for certain scaling parameters.

V. ENTROPY FROM (PARTIALLY) MODIFIED DISPERSION RELATION

We are going to devote this section to clarifying some issues related to the modified dispersion relation (19). In the course of calculations, we have considered just the highest power of nonisotropic exponent in the dispersion relation for simplicity, just as Λ_n's are chosen as $\Lambda_n = l_{\rm P}^{2(n-1)}$ for $n \geq 1$. The justification for this is needed from a general point of view, because all terms in the modified dispersion relation may contribute to the final entropy. For instance, the large infrared cutoff $\bar{\delta}$ may require all contributions of terms. To answer this question, instead of using analytic results, we want to present some numerical simulations in order to show how much the previous results change for $z = 2$ and $z = 4$ when we take a partially modified dispersion relation, which is a sort of reduced relation.

Now, for the solvability, we take a partially modified dispersion relation as follows:

$$p_0 p^0 + \Lambda_{z/2}(p_i p^i)^{z/2} + \Lambda_z (p_i p^i)^z = -m^2 c^2, \tag{43}$$

where z is even. The full dispersion relation, for instance, for the case of $z = 2$ recovers as $p_0 p^0 + \Lambda_1 p_i p^i + \Lambda_2 (p_i p^i)^2 = -m^2 c^2$. In spherically symmetric black holes described by the line element (21), the dispersion relation (43) becomes

$$p_i p^i = f p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta} = \frac{1}{(2\Lambda_z)^{2/z}} \times \left[-\Lambda_{z/2} + \sqrt{\Lambda_{z/2} + 4\Lambda_z\left(\frac{\omega^2}{f\tilde{N}^2 c^2} - m^2 c^2\right)}\right]^{2/z}. \tag{44}$$

Of course, for $\Lambda_z = 0$ and $z = 2$, the number of quantum states can be reduced to the relativistic limit

$$p_i p^i = \frac{1}{\Lambda_1}\left(\frac{\omega^2}{f\tilde{N}^2c^2} - m^2c^2\right). \qquad (45)$$

Next, from Eq. (44), the number of quantum states with energy less than ω can be written as

$$n(\omega) = \frac{2}{3\pi(2\Lambda_z)^{3/z}}\int dr \frac{r^2}{\sqrt{f}} \times \left[-\Lambda_{z/2} + \sqrt{\Lambda_{z/2}^2 + 4\Lambda_z\left(\frac{\omega^2}{f\tilde{N}^2c^2} - m^2c^2\right)}\right]^{3/z}, \qquad (46)$$

where the energy should satisfy $\omega \geq \omega_0 \equiv mc^2\tilde{N}\sqrt{f}$. Substituting Eq. (46) into Eq. (28), the entropy is written as

$$S = \frac{2}{3\pi(2\Lambda_z)^{3/z}}\int_{x_0}^{\infty} dx \frac{x}{\sinh^2 x}\int dr \frac{r^2}{\sqrt{f}} \times \left[-\Lambda_{z/2} + \sqrt{\Lambda_{z/2}^2 + \frac{16\Lambda_4}{\beta^2c^2\tilde{N}^2 f}(x^2 - x_0^2)}\right]^{3/z}, \qquad (47)$$

where x and x_0 are defined by $x \equiv \frac{1}{2}\beta\omega$ and $x_0 = \frac{1}{2}\beta mc^2\tilde{N}\sqrt{f}$. For the thin layer with the range from $r + \epsilon$ to $r + \epsilon + \delta$ near the horizon, it can be written as

$$S = \frac{2c}{3\pi(2\Lambda_4)^{3/z}}\sqrt{\frac{\tilde{N}_H}{\kappa_H}}\int_{r_H+\epsilon}^{r_H+\epsilon+\delta} dr \frac{r_H^2}{\sqrt{r - r_H}}\int_{x_0}^{\infty} dx \frac{x}{\sinh^2 x} \times \left[-\Lambda_{z/2} + \sqrt{\Lambda_{z/2}^2 + \frac{8\Lambda_z}{\beta^2\tilde{N}_H\kappa_H}\frac{x^2 - x_0^2}{r - r_H}}\right]^{3/z}. \qquad (48)$$

Actually, it is not easy to get analytic results, so we plot entropies with respect to the proper lengths $\bar{\epsilon}$ for the case of $z = 2$ and $\bar{\delta}$ for the case of $z = 4$, which are shown in Figs. 1 and 2, respectively. The coefficients of momenta in the dispersion relation (43) have been chosen as $\Lambda_n = l^{2(n-1)}$ for $n = z$ and $z/2$ for the sake of comparison with the results obtained in Sec. IV.

For $z = 2$, there are largely three cases: a relativistic limit, $R_2 \equiv (\Lambda_1, \Lambda_2) = (1, 0)$; a highest-momentum consideration, $H_2 \equiv (\Lambda_1, \Lambda_2) = (0, l^2)$; and a full consideration, $F_2 \equiv (\Lambda_1, \Lambda_2) = (1, l^2)$. As shown in Fig. 1(a), much smaller cutoffs are required compared to those of the relativistic case for the same value of the entropy, so that the curve for the R_2 case lies far above the two curves of H_2 and F_2. This fact has been discussed using a modified dispersion, which is similar to the full dispersion relation (F_2) of $z = 2$ [20,21]; in particular, it is interesting to see that for the black hole entropy, the same brick wall lies at a much smaller proper distance in the free fall time slice compared to the conventional brick-wall cutoff. While the conventional brick wall cuts off all modes at the same location, they cut off all modes at the same momentum. By the way, from Fig. 1(b), we can see that the cutoff in the F_2 case is slightly smaller than that of the H_2 case. As a result, the full consideration of the dispersion relation is very close to the highest-momentum consideration in comparison with the relativistic case, even though the former case gives a slightly smaller cutoff.

On the other hand, in order to investigate the role of the lower-momentum contribution for $z > 3$, we study the dispersion relation (43) for the specific case of $z = 4$. The entropy curves with respect to the proper infrared cutoff $\bar{\delta}$ are plotted in Fig. 2 for the highest power of momentum consideration, $H_4 \equiv (\Lambda_2, \Lambda_4) = (0, l^6)$, and the full consideration, $F_4 \equiv (\Lambda_2, \Lambda_4) = (l^2, l^6)$, respectively. It can be shown that in a very short distance, compared to the given length scale of l, the entropy profiles are almost coincident,

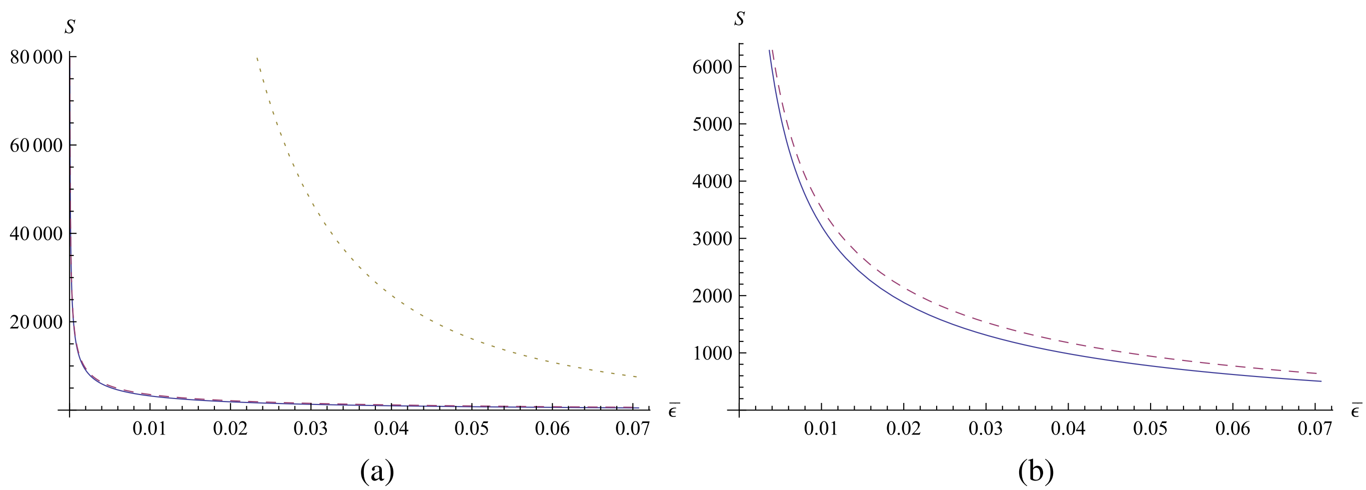

FIG. 1 (color online). The entropies can be shown as a function of the cutoff parameter $\bar{\epsilon}$ for the case of $z = 2$. The dotted, dashed, and solid lines correspond to the cases of $R_2 = (\Lambda_1, \Lambda_2) = (1, 0)$, $H_2 = (\Lambda_1, \Lambda_2) = (0, l^2)$, and $F_2 = (\Lambda_1, \Lambda_2) = (1, l^2)$, respectively, where $l = 0.01$. The variables in Eq. (48) have been simply chosen as $\delta = 0.05$, $r_H = \kappa_H = 2$, and $\tilde{N}_H = m = c = 1$. (b) The vertical axis for the entropy in (a) has been rescaled.

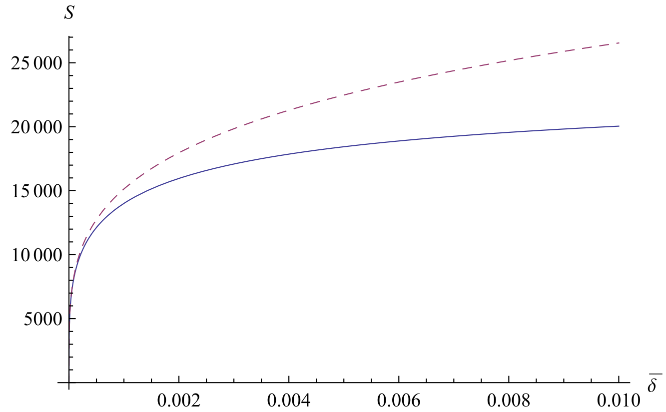

FIG. 2 (color online). The entropies can be shown as the function of the cutoff parameter $\bar{\delta}$ for the case of $z = 4$. The dashed and solid lines correspond to the cases of $H_4 = (\Lambda_2, \Lambda_4) = (0, l^6)$ and $F_4 = (\Lambda_2, \Lambda_4) = (l^2, l^6)$, respectively, where $l = 0.01$. The variables in Eq. (48) have been chosen as $\epsilon = 0$, $r_H = \kappa_H = 2$, and $\tilde{N}_H = m = c = 1$.

while they generate a little difference as the cutoff is getting large. So, if $\bar{\delta}$ is much smaller than l, then low-momentum contributions can be neglected, whereas their contributions cannot be ignored when $\bar{\delta}$ and l are at the same order of scale. It means that we have to discard the area law of the entropy at $\bar{\delta} \approx l$. Therefore, the full modified dispersion relation will give the smaller entropy, which is not compatible with the area law. Of course, this conclusion is more or less restrictive, so we hope this issue will be discussed more generally elsewhere.

VI. DISCUSSIONS

We have studied the statistical entropy of spherical symmetric black holes using the brick-wall method in HL gravity. The crucial difference from the conventional brick-wall method is that the scalar field satisfying FPD called Lifshitz scalar gives the area law of the finite entropy without the UV cutoff for $z > 3$ corresponding to the superrenormalizable sector of HL gravity, as long as the length of thin wall is identified with a certain value, depending on the scale parameter.

This result is reminiscent of the entropy calculation in the brick-wall method using the GUP, $\Delta x \Delta p \geq \hbar + \frac{\lambda}{\hbar} \times (\Delta p)^2$, and there exists a minimal length, $\Delta x_{\rm min} = 2\sqrt{\lambda}$ [30–32]. Similar to the present modes counting between the horizon and $\bar{\delta}$, it happens between just outside the horizon and the minimal length $\Delta x_{\rm min}$ without any UV cutoff in the GUP regime. For instance, in a spherical symmetric black hole based on the GUP, the number of quantum states in this minimal length is obtained as $n(\omega) = \frac{2}{3\pi} \int dr \frac{r^2(\omega^2/f - \mu^2)^{3/2}}{\sqrt{f}[1+\lambda(\omega^2/f-\mu^2)]^3}$. Following the same procedure as in the previous Sec. IV, the entropy is calculated as $S = \frac{c^3 A}{4\hbar G_N} \cdot \frac{\xi l_{\rm P}^2}{\lambda}$, where $\xi \equiv \frac{1}{3}[\frac{4}{\pi}\zeta(3) - \frac{25}{8\pi} - \frac{\pi}{6}]$. To satisfy the area law of entropy for this black hole, λ is required to be the same as $\xi l_{\rm P}^2$. In comparison with HL gravity, the thickness of the thin wall of $\bar{\delta}$ can be identified with the minimal length in the GUP $\bar{\delta} = 2\sqrt{\lambda}$. Then, using Eq. (41), it amounts to the highly superrenormalizable case of HL gravity $z \approx 342$. It implies that the Lifshitz scalar field may play a role of the usual scalar field along with the GUP, at least in the brick-wall regime. Unfortunately, it is unclear why the scaling parameter should be so large if we try to match the minimal length in the GUP and $\bar{\delta}(z)$.

ACKNOWLEDGMENTS

We would like to thank E. Son for exciting discussions. W. Kim was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0008359) and the Special Research Grant of Sogang University (Grant No. 200911044). M. Eune was supported by the National Research Foundation of Korea Grant funded by the Korean Government (Grant No. NRF-2009-351-C00109).

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