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Coincidence problem in $f(R)$ gravity models

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The $f(R)$ gravity models formulated in the Einstein conformal frame are equivalent to Einstein gravity together with a minimally coupled scalar field. The scalar field couples with the matter sector and the coupling term is given by the conformal factor. We use this interacting model to derive a necessary condition for alleviating the coincidence problem.

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I. INTRODUCTION

There are strong observational evidences that the expansion of the Universe is accelerating (see e.g. [1]). However, the origin of this cosmic acceleration is not well understood and remains as one of the main challenges of modern cosmology. The standard explanation invokes an unknown component, usually referred to as dark energy. It contributes to energy density of the Universe with $\Omega_d = 0.7$, where Ω_d is the corresponding density parameter [2]. A candidate for dark energy which seems to be both natural and consistent with observations is the cosmological constant [2–4]. However, in order to avoid theoretical problems [3], other scenarios have been investigated. In one of these scenarios the matter sector remains unchanged and the gravitational part suffers from some modifications. A family of these modified gravity models is obtained by replacing the Ricci scalar R in the usual Einstein-Hilbert Lagrangian density for some function $f(R)$ [5].

There are two important problems that are related to the cosmological constant. The first problem, usually known as the fine-tuning problem, is the large discrepancy between observations and theoretical predictions on its value. There have been many attempts trying to resolve this problem [3]. Most of them are based on the belief that the cosmological constant may not have such an extremely small value at all times and there should exist a dynamical mechanism working during evolution of the Universe which provides a cancellation of the vacuum energy density at late times [6]. The second problem concerns the coincidence between the observed vacuum energy density and the current matter density. While these two energy components evolve differently as the Universe expands, their contributions to total energy density of the Universe in the present epoch are the same order of magnitude. Besides the possibility that the present epoch may be a stationary regime at which the ratio of the two energy densities are constant, it is also quite possible that we live in a very special epoch, a transient epoch at which the ratio varies slowly with respect to the expansion of the Universe. A possible solution to the coincidence problem is to consider an interaction between dark energy and dark matter. If such an interaction exists the two corresponding energy densities do not scale independently. It is shown that this can lead to a constant ratio of energy densities when an appropriate coupling term is applied [7,8].

In the present paper, we will consider the coincidence problem in Einstein frame representation of $f(R)$ gravity models. In these models the dynamical variable of the vacuum sector is the metric tensor and the corresponding field equations are fourth order. This dynamical variable can be replaced by a new pair which consists of a conformally rescaled metric and a scalar partner. Moreover, in terms of the new set of variables the field equations are those of general relativity. The original set of variables is commonly called the Jordan conformal frame and the transformed set whose dynamics is described by Einstein field equations is called the Einstein conformal frame. The dynamical equivalence of Jordan and Einstein conformal frames does not generally imply that they are also physically equivalent. In fact it is shown that some physical systems can be differently interpreted in different conformal frames [9,10]. The physical status of the two conformal frames is an open question which we are not going to address here. Our motivation to work in the Einstein conformal frame is that in this frame there is a coupling between the scalar degree of freedom and matter sector induced by the conformal transformation. As previously stated, there is a large amount of interest to realize the coincidence problem as a consequence of an interaction between matter systems and the dark sector. Although the whole idea seems to be promising, the suggested interaction terms are usually phenomenological and are not generated by a fundamental theory. In our case the interaction term is given by the conformal factor. We investigate the consequences of this interaction term and derive an expression which constrains the form of the $f(R)$ function. We will show that this constraint selects those $f(R)$ models that allow for possible alleviation of the coincidence problem.

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II. FRAMEWORK

The action for an $f(R)$ gravity theory in the Jordan frame is given by

$$S_{\rm JF} = \frac{1}{2k}\int d^4x\sqrt{-g}f(R) + S_m(g_{\mu\nu}, \psi), \tag{1}$$

where $k \equiv 8\pi G$, G is the gravitational constant, g is the determinant of $g_{\mu\nu}$, and S_m is the action of (dark) matter which depends on the metric $g_{\mu\nu}$ and some (dark) matter field ψ. Stability in the matter sector (the Dolgov-Kawasaki instability [11]) imposes some conditions on the functional form of $f(R)$ models. These conditions require that the first and the second derivatives of $f(R)$ function with respect to the Ricci scalar R should be positive definite. The positivity of the first derivative ensures that the scalar degree of freedom is not tachyonic and positivity of the second derivative tells us that the graviton is not a ghost.

It is well known that $f(R)$ models are equivalent to a scalar field minimally coupled to gravity with an appropriate potential function. In fact, we may use a new set of variables

$$\bar{g}_{\mu\nu} = \Omega g_{\mu\nu}, \tag{2}$$

$$\phi = \frac{1}{2\beta\sqrt{k}}\ln\Omega, \tag{3}$$

where $\Omega \equiv \frac{df}{dR} = f'(R)$ and $\beta = \sqrt{\frac{1}{6}}$. This is indeed a conformal transformation which transforms the above action in the Jordan frame to the following action in the Einstein frame [9,12]:

$$S_{\rm EF} = \frac{1}{2}\int d^4x\sqrt{-\bar{g}}\left\{\frac{1}{k}\bar{R} - \bar{g}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - 2V(\phi)\right\} + S_m(\bar{g}_{\mu\nu}e^{2\beta\sqrt{k}\phi}, \psi). \tag{4}$$

All indices are raised and lowered by $\bar{g}_{\mu\nu}$ unless stated otherwise. In the Einstein frame, ϕ is a minimally coupled scalar field with a self-interacting potential which is given by

$$V(\phi(R)) = \frac{Rf'(R) - f(R)}{2kf'^2(R)}. \tag{5}$$

Note that the conformal transformation induces the coupling of the scalar field ϕ with the matter sector. The strength of this coupling β is fixed to be $\sqrt{\frac{1}{6}}$ and is the same for all types of matter fields. In the action (4), we take $\bar{g}^{\mu\nu}$ and ϕ as two independent field variables and variations of the action yield the corresponding dynamical field equations. Variation with respect to the metric tensor $\bar{g}^{\mu\nu}$ leads to

$$\bar{G}_{\mu\nu} = k(\bar{T}^{\phi}_{\mu\nu} + \bar{T}^{m}_{\mu\nu}), \tag{6}$$

where

$$\bar{T}^{\phi}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \tfrac{1}{2}\bar{g}_{\mu\nu}\nabla^\gamma\phi\nabla_\gamma\phi - V(\phi)\bar{g}_{\mu\nu}, \tag{7}$$

$$\bar{T}^{m}_{\mu\nu} = \frac{-2}{\sqrt{-\bar{g}}}\frac{\delta S_m(\bar{g}_{\mu\nu}, \psi)}{\delta\bar{g}^{\mu\nu}} \tag{8}$$

are stress tensors of the scalar field and the matter field system. The trace of (6) is

$$\nabla^\gamma\phi\nabla_\gamma\phi + 4V(\phi) - \bar{R}/k = \bar{T}^m, \tag{9}$$

which differentially relates the trace of the matter stress tensor $\bar{T}^m = \bar{g}^{\mu\nu}\bar{T}^m_{\mu\nu}$ to $\bar{R}$. Variation of the action (4) with respect to the scalar field ϕ gives

$$\Box\phi - \frac{dV(\phi)}{d\phi} = -\beta\sqrt{k}\bar{T}^m. \tag{10}$$

It is important to note that the two stress tensors $\bar{T}^m_{\mu\nu}$ and $\bar{T}^\phi_{\mu\nu}$ are not separately conserved. Instead they satisfy the following equation:

$$\bar{\nabla}^\mu\bar{T}^m_{\mu\nu} = -\bar{\nabla}^\mu\bar{T}^\phi_{\mu\nu} = \beta\sqrt{k}\nabla_\nu\phi\bar{T}^m. \tag{11}$$

We apply the field equations in a spatially flat homogeneous and isotropic cosmology described by Friedmann-Robertson-Walker spacetime

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{12}$$

where $a(t)$ is the scale factor. To do this, we take $\bar{T}^m_{\mu\nu}$ and $\bar{T}^\phi_{\mu\nu}$ as the stress tensors of a pressureless perfect fluid with energy density $\bar{\rho}_m$, and a perfect fluid with energy density $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and pressure $p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$, respectively. In this case, (6) and (10) take the form[1]

$$3H^2 = k(\rho_\phi + \rho_m), \tag{13}$$

$$2\dot{H} + 3H^2 = -k\omega_\phi\rho_\phi, \tag{14}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = -\beta\sqrt{k}\rho_m, \tag{15}$$

where $\omega_\phi = \frac{p_\phi}{\rho_\phi}$ is the equation of state parameter of the scalar field ϕ, and the overdot indicates differentiation with respect to cosmic time t. The trace equation (9) and the conservation equation (11) give, respectively,

$$\dot{\phi}^2 + R/k - 4V(\phi) = \rho_m, \tag{16}$$

$$\dot{\rho}_m + 3H\rho_m = Q, \tag{17}$$

$$\dot{\rho}_\phi + 3H(\omega_\phi + 1)\rho_\phi = -Q, \tag{18}$$

where

[1] Hereafter we will use unbarred characters in the Einstein frame.

$$Q = \beta\sqrt{k}\dot{\phi}\rho_m \tag{19}$$

is the interaction term. This term vanishes only for $\phi = \text{const}$, which due to (3) happens when $f(R)$ linearly depends on R. The direction of energy transfer depends on the sign of Q or $\dot{\phi}$. For $\dot{\phi} > 0$, the energy transfer is from dark energy to dark matter and for $\dot{\phi} < 0$ the reverse is true.

We emphasize that the coupling term (19) is very similar to some phenomenological coupling terms suggested in the literature. In fact, there are different kinds of interacting models which have been investigated [7,8]. A particular class of these models considers $Q = \alpha\dot{\varphi}\rho$ in which α is a coupling constant, φ is usually a quintessence field, and ρ is the energy density of dark matter [8]. Apart from the similarity of the latter with (19), there are also some important differences. First, the scalar field ϕ is not a kind of matter field and is actually given in terms of the function $f(R)$. Second, β is a universal coupling constant implying that ϕ couples with the same strength to all types of matter fields. On the contrary, it is possible to consider α as a nonuniversal coupling constant so that it may couple to dark matter and baryons with different strengths [13]. Moreover, the value of β is fixed to be $1/\sqrt{6}$, while α is constrained by observations [8]. We will return to this last point later.

III. THE COINCIDENCE PROBLEM

One of the important features of the cosmological constant problem is the present coincidence between dark energy and dark matter energy densities [14]. There is a class of models in which this observation is related to some kind of interaction between the two components [7,8]. In these models the two components are not separately conserved and there is a flow of energy from dark energy to dark matter or vice versa. In this sense, dark energy and dark matter energy densities may have the same scaling at late times due to the interaction, although they decrease with the expansion of the Universe at different rates. The important task in this context is to find a constant ratio of dark energy to dark matter energy densities for an appropriate interaction term. Despite the fact that this approach seems to be promising, there is still not a compelling form of interaction which is introduced by a fundamental theory. Therefore one usually uses different interaction terms and tries to adapt them with recent observations.

In $f(R)$ gravity models presented in the Einstein frame, there is a fixed interaction between the scalar field and matter sector. Since the form of the interaction is fixed by the conformal transformation one can therefore search for some appropriate forms of the function $f(R)$ for which the energy densities ratio of the two components takes a stationary value. This is the strategy that we are going to pursue in this section, namely, to find some conditions on the functional form of $f(R)$ that may lead to a constant $r \equiv \rho_m/\rho_\phi$.

To do this, we consider the time evolution of the ratio r,

$$\dot{r} = \frac{\dot{\rho}_m}{\rho_\phi} - r\frac{\dot{\rho}_\phi}{\rho_\phi}. \tag{20}$$

From Eqs. (17)–(19) we obtain

$$\dot{r} = 3Hr\omega_\phi + \beta\sqrt{k}\dot{\phi}r(r+1). \tag{21}$$

In this relation, we can write $\dot{r}$ in terms of the parameters r and q. We first use (13) and (14) to replace the equation of state parameter ω_ϕ with the deceleration parameter q. Applying

$$\dot{H} = -(q+1)H^2. \tag{22}$$

to Eq. (14) gives

$$\omega_\phi = \frac{(2q-1)H^2}{\sqrt{k}\rho_\phi}. \tag{23}$$

We then use (13) in the latter and substitute the result in (21), which leads to

$$\dot{r} = Hr(2q-1)(r+1) + \beta\sqrt{k}\dot{\phi}r(r+1). \tag{24}$$

On the other hand, we can combine the trace equation (16) with Eqs. (5) and (13) to obtain

$$\dot{\phi}^2 = \frac{1}{k}\left\{\frac{3H^2r}{r+1} + 3H^2(2q-3)\left(1-\frac{2}{f'}\right) - 2\frac{f}{f'^2}\right\}. \tag{25}$$

When we put this expression into (24), the result is an equation that relates $\dot{r}$ to the parameters r, q, and H. The requirement that the Universe approach a stationary stage in which r either becomes a constant or varies more slowly than the scale factor leads to the following relation:

$$g(f'; H, r_s, q) = 0, \tag{26}$$

where

$$g(f'; H, r_s, q) \equiv r_s(2q-1)(r_s+1) + \beta r_s(r_s+1)\left\{\frac{3r_s}{r_s+1} + 3(2q-3)\left(1-\frac{2}{f'}\right) - \frac{2f}{H^2f'^2}\right\}^{1/2}, \tag{27}$$

and r_s is the value of r when it takes a stationary value. It is now possible to use (26) to check that whether a particular $f(R)$ model is consistent with a late-time stationary ratio of energy densities. In general, to find such $f(R)$ gravity models one may start with a particular $f(R)$ function in the action (1) and solve the corresponding field equations for finding the form of $q(z)$ or $H(z)$. However, this approach is not efficient in view of the complexity of the field equations. An alternative approach is to start from the best fit parametrization $q(z)$ obtained directly from data and use this $q(z)$ for a particular $f(R)$ function in (26). Here we will follow the latter approach.

For a given redshift z_0 and the parameters $r_s(z_0)$, $q(z_0)$, and $H(z_0)$, the relation (26) acts as a constraint on the function $f(R)$. As an illustration, we apply this constraint to some $f(R)$ functions. Before doing this, there are some remarks to do with respect to (26). This condition is a consequence of $\dot{r} = 0$ when $r = r_s$ becomes stationary at late times. At sufficiently late times characterized by $z = z_0$, we take $r_s = r_0$ and rewrite (26) as

$$g(f'_0; H_0, r_0, q_0) = 0, \tag{28}$$

where

$$g(f'_0; H_0, r_0, q_0) \equiv r_0(2q_0 - 1)(r_0 + 1) + \beta r_0(r_0 + 1)\left\{\frac{3r_0}{r_0 + 1} + 3(2q_0 - 3)\left(1 - \frac{2}{f'_0}\right) - 2\frac{f_0}{H_0 f'^2_0}\right\}^{1/2}. \tag{29}$$

Here the functions f_0, f'_0, and f''_0 are the late-time configurations of $f(R)$, $f'(R)$, and $f''(R)$ which are obtained by replacing R with

$$R = 6(1 - q)H^2 \tag{30}$$

at the redshift z_0. Note that an $f(R)$ gravity model is usually given in terms of some parametrizations. In this sense, the condition (26) acts actually as a constraint relating the corresponding parameters of a particular $f(R)$ gravity model to the constants q_0, r_0, and H_0. We use a two-parametric reconstruction function for characterizing $q(z)$ [15,16],

$$q(z) = \frac{1}{2} + \frac{q_1 z + q_2}{(1 + z)^2}. \tag{31}$$

Fitting this model to the gold data set gives $q_1 = 1.47^{+1.89}_{-1.82}$ and $q_2 = -1.46 \pm 0.43$ [16]. We also take $z_0 = 0.25$ which, with use of (31), corresponds to $q_0 \approx -0.2$. Moreover, recent observations imply that $r_0 \equiv \frac{\rho_m(z_0)}{\rho_\phi(z_0)} \approx \frac{3}{7}$ [17].

Now let us first consider the model [18,19]

$$f(R) = R + \lambda R_0\left(\frac{R}{R_0}\right)^n. \tag{32}$$

Here R_0 is taken to be of the order of H_0^2 and λ, n are constant parameters. In terms of the values attributed to these parameters, the model (32) is divided by three cases [19]. First, when $n > 1$ there is a stable matter-dominated era which does not follow by an asymptotically accelerated regime. In this case, $n = 2$ corresponds to Starobinsky's inflation and the accelerated phase exists in the asymptotic past rather than in the future. Second, when $0 < n < 1$ there is a stable matter-dominated era followed by an accelerated phase only for $\lambda < 0$. Finally, in the case that $n < 0$ there is no accelerated and matter-dominated phases for $\lambda > 0$ and $\lambda < 0$, respectively. Thus the model (32) is cosmologically viable in the regions of the parameters space which is given by $\lambda < 0$ and $0 < n < 1$.

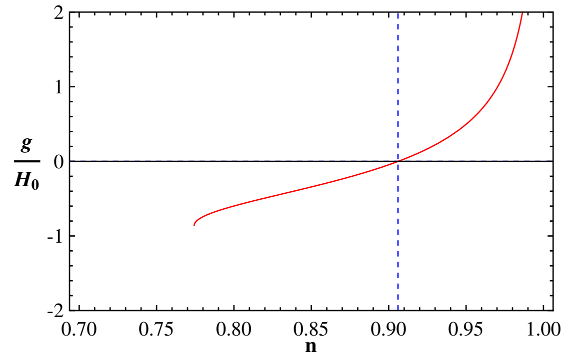

FIG. 1 (color online). The plot of $g(n, \lambda; H_0, r_0, q_0)$ for the model (32) when $\lambda = -1$, $q_0 = -0.2$, and $r_0 = 3/7$. The vertical dashed line corresponds to $n = 0.906$.

When we use (30) in the function $g(f'_0; H_0, r_0, q_0)$, it takes the form of an expression which relates the parameters n and λ to q_0, r_0, and H_0. In Fig. 1 we have plotted $g(n, \lambda; H_0, r_0, q_0)$ for $\lambda = -1$. This figure indicates that the constraint (28) is satisfied only for $n \approx 0.9$ which implies that for this value of the parameter n, the model (32) admits a late-time stationary ratio of the energy densities. Note that $n \approx 0.9$ lies in the range in which the model is cosmologically viable.

Now we consider the model presented by Starobinsky [20,21]

$$f(R) = R - \gamma R_0\left\{1 - \left[1 + \left(\frac{R}{R_0}\right)^2\right]^{-m}\right\}, \tag{33}$$

where γ, m are positive constants and R_0 is again of the order of the presently observed effective cosmological constant. Using the same procedure, we have plotted the function $g(m, \gamma; H_0, r_0, q_0)$ in Fig. 2. The figure shows that there are some regions in the parameters space for which the condition (28) is satisfied. The condition is satisfied on the upper boundary of the surface plotted in Fig. 2 where $g(m, \gamma; H_0, r_0, q_0) = 0$. Thus for the corresponding values of the parameters, the coincidence problem can be

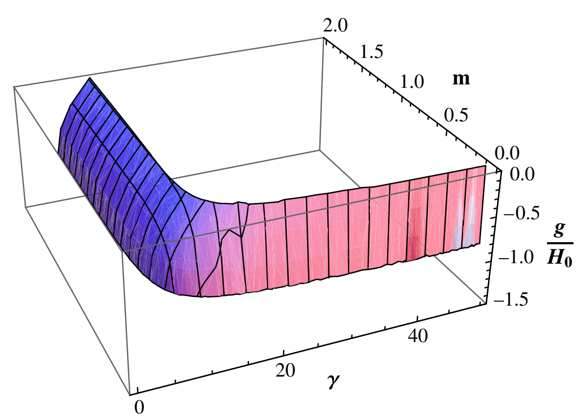

FIG. 2 (color online). The plot of $g(m, \gamma; H_0, r_0, q_0)$ for the model (33) when $q_0 = -0.2$ and $r_0 = 3/7$.

addressed in the context of the model (33). For instance, as the figure indicates the parameters space is bounded by $\gamma \geq 10.5$ and $m \geq 0.04$ so that for $m > 0.04$ the parameter γ should remain near the value 10.5.

IV. CONCLUSION

In the Einstein frame representation of $f(R)$ gravity models, the scalar partner of the metric tensor interacts with (dark) matter in such a way that the interaction term is fixed by the conformal transformation. This means that contributions of the scalar field and the (dark) matter system to total energy density do not scale independently. As a consequence, even though the two components may start with different scalings at early times, they may have the same scaling at sufficiently late times.

We have considered this feature as a possibility for addressing the coincidence problem. In fact, the interaction of dark energy and dark matter has been recently taken as a natural guidance for alleviating the coincidence problem by some authors. In absence of an interaction or coupling term based on a fundamental theory, most of the current investigations have been limited to a phenomenological level. In our case, the interaction term, Q, is given by the conformal transformation and can be written in terms of $\dot{R}$, $f'(R)$, and $f''(R)$. Because of stability considerations, any viable $f(R)$ model should satisfy $f'(R) > 0$ and $f''(R) > 0$ [22]. Thus the direction of the energy transfer is determined by the sign of $\dot{R}$ in a particular epoch. For instance, in an epoch for which $\dot{R} > 0$, the energy transfer is from dark energy (or the scalar field ϕ) to dark matter while for $\dot{R} < 0$ the reverse is true.

We have derived a relation giving the evolution of the parameter r. We have found that there is a class of $f(R)$ gravity models satisfying the condition (28) for which a late-time stationary state for r exists. As illustrations, we have shown that the model (32) lies in this class only for $n \approx 0.9$. The condition is also used for Starobinsky's model. We have shown that there is a region in the parameters space for which the coincidence problem can be addressed in this model. The region is characterized by the upper border of the surface plot of Fig. 2 for which $g(m, \gamma; H_0, r_0, q_0) = 0$.

Finally, we point out that there is not a free parameter in the interaction term (19) since β is fixed by conformal transformation. In general, the interaction of the scalar field ϕ and the matter sector may lead to a fifth force and violation of the equivalence principle. In fact, the real challenge for alleviating the coincidence problem comes from the combination of restrictions from local gravity experiments and dynamical considerations. Thus the question is how can a coupling term without a free parameter be consistent with local gravity experiments. The point is that, in our case, these experiments constrain the corresponding parameters of a particular $f(R)$ gravity model[2] rather than the coupling constant of the interaction term. For the model (33), it is shown [24] that the most stringent bound is $m > 0.9$ which comes from violation of the equivalence principle. Combining the latter with the bounds indicated in Fig. 2, one infers that alleviation of the coincidence problem requires that $\gamma \approx 10.5$.

[2]These constraints are imposed by the chameleon mechanism; see, e.g., [23] and references therein.

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