Classification of six derivative Lagrangians of gravity and static spherically symmetric solutions

Julio Oliva[1,*] and Sourya Ray[2,†]

[1]*Instituto de Física, Facultad de Ciencias, Universidad Austral de Chile, Valdivia, Chile*
[2]*Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile*
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We classify all the six-derivative Lagrangians of gravity, whose traced field equations are of second or third order, in arbitrary dimensions. In the former case, the Lagrangian in dimensions greater than six reduces to an arbitrary linear combination of the six-dimensional Euler density and the two linearly independent cubic Weyl invariants. In five dimensions, besides the independent cubic Weyl invariant, we obtain an interesting cubic combination, whose field equations for static spherically symmetric spacetimes are of second order. In the latter case, in arbitrary dimensions we obtain two combinations, which in dimension three, are equivalent to the complete contraction of two Cotton tensors. Moreover, we also recover all the conformal anomalies in six dimensions. Finally, we present the general static, spherically symmetric solution for some of these Lagrangians.

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I. INTRODUCTION

Einstein's general relativity is not only the most successful classical theory of gravity in four dimensions, it is also the simplest theory possessing some nice properties. Two of the most important characteristics are general covariance and second-order field equations. In fact, it was shown by Lovelock that, in four dimensions, general relativity is the unique generally covariant theory of gravity (up to an addition of a cosmological constant) which gives second-order equations of motion [1]. However, in higher dimensions, there exist higher curvature theories, namely, the Lovelock theories, which also give second-order field equations. These theories are generically characterized by higher curvature invariants in the action. At each order k, the combination of higher curvature invariants is unique, the integral of which on a compact manifold of dimension $2k$ gives the Euler characteristic of the manifold. There are also other interesting characteristics of the Lovelock class of theories. Primarily, exact analytic black hole solutions are known to exist [2,3] (see also [4,5], and references therein). Black holes are widely believed to exist in nature as a final state of a gravitational collapse. They are also the simplest objects to study in any gravitational theory. Therefore, exact black hole solutions are of significant importance. More recently, in the context of AdS/CFT correspondence, exact asymptotically AdS black hole solutions in a gravity theory have been proven to be useful in studying the holographic properties of a finite temperature conformal field theory on the boundary [6]. Second, the Lovelock theories also admit Birkhoff's theorem, which states that any solution which has spherical, planar, or hyperbolic symmetry must be locally isometric to the corresponding static black hole solution [7]. Generically, the admittance of Birkhoff's theorem suggests the lack of spin-0 mode excitations in the linearized field equations.

There are also other theories of gravity which admit exact analytic black hole solutions and further admit Birkhoff's theorem. These theories, being outside the Lovelock class, are generically higher-derivative theories and consequently possess ghost degrees of freedom. One well-known higher-derivative theory is the conformal theory of gravity in four dimensions, which is obtained from an action quadratic in the conformal Weyl tensor. The action is thus invariant under Weyl rescalings, and the field equations are traceless. Birkhoff's theorem in conformal gravity states that modulo a conformal factor the most general spherically symmetric solution is static [8]. Considering the same action in dimensions other than four, one loses the property of invariance under Weyl rescalings. Even then, the theory admits exact analytic black hole solutions with spherical, planar, or hyperbolic symmetry, and further admits Birkhoff's theorem [9]. Note that, in arbitrary dimensions, though the field equations are of fourth order, the trace of the field equations are of order two. This can easily be seen as follows. Varying the action gives

$$\delta I := \int \delta(\sqrt{-g}\mathcal{L}) = \int \sqrt{-g}\mathcal{E}_{\mu\nu}\delta g^{\mu\nu}. \qquad (1)$$

Now, consider infinitesimal Weyl rescalings of the metric $\delta g^{\mu\nu} = \lambda g^{\mu\nu}$. Under such variations, the Lagrangian will vary as $\lambda(2 - D/2)\sqrt{-g}\mathcal{L}$, which implies $\mathcal{E}^{\mu}{}_{\mu} = (\frac{4-D}{2})\mathcal{L}$. This shows that the trace of the field equations, being proportional to the Lagrangian density, must be of second order.

*julio.oliva@docentes.uach.cl
†ray@cecs.cl

Recently, a theory of massive gravity was constructed in three dimensions [10], where the Lagrangian is a particular combination of quadratic curvature invariants given by

$$K = R_{ab}R^{ab} - \tfrac{3}{8}R^2. \tag{2}$$

Again, the theory admits exact analytic black hole solutions [11]. The Lagrangian of this theory has a unique property in three dimensions, that the field equations have a second-order trace. So, it is natural to wonder if there could be other theories of gravity which, although nonrealistic, have some special properties which allow one to obtain exact analytic black hole solutions and can serve as toy models of gravitational theories. Specifically, it might be useful to classify higher-derivative theories of gravity whose traced field equations have a reduced order.

In this paper we construct the most general Lagrangian which is a linear combination of scalars of the form

$$R_{....}R_{....}R_{....}, \quad \text{and} \quad \nabla_{.}R_{....}\nabla_{.}R_{....}, \tag{3}$$

which are characterized by the number of derivatives of the metric (hereafter, the degree of differentiation) $n = 6$, such that the trace of the field equations is of order three or less.

We will show that when the trace is restricted to be of order two, then, in dimensions six or higher, there are only three linearly independent possible invariants which have a second-order trace. They are the six-dimensional Euler density and the two linearly independent scalars constructed by contracting all the indices of three conformal Weyl tensors. However, in five dimensions, we obtain a peculiar independent invariant which can be thought of as a cubic generalization of (2) and has been studied separately in [12–14]. We also obtain the general static spherically symmetric solution for some of these theories. Based on our analysis for six-derivative theories, we present a conjecture classifying all the scalars of arbitrary order, which give second-order traced field equations in various dimensions.

When the trace is restricted to be of order three, in arbitrary dimensions, we obtain two additional scalars. These two scalars are not independent in dimensions three and six. In six dimensions, they reduce to one of the conformal anomalies.

One future direction of study is to see which of these theories admit a Birkhoff's theorem.

In Sec. II, for completeness we review the $n = 4$ case. The case $n = 6$ is analyzed in Sec. III. In Sec. IV we focus our attention on obtaining the general static, spherically symmetric solution for some of the theories defined in Sec. III in arbitrary dimensions.

II. QUADRATIC COMBINATIONS. $n = 4$

In this section, we review how to obtain the most general quadratic Lagrangian, having second-order traced field equations [15,16].

The most general quadratic combination of curvature invariants in arbitrary dimension is given by[1]

$$\mathcal{L}_Q := aR^{abcd}R_{abcd} + bR^{ab}R_{ab} + cR^2, \tag{4}$$

where a, b, and c are arbitrary constants. The trace of the field equations coming from this Lagrangian are

$$G_a^{(2)a} = \left(4a + \frac{D}{2}b + 2(D-1)c\right)\nabla_a\nabla^a R - 4a\nabla_a\nabla_b R^{ab} - \frac{D-4}{2}\mathcal{L}_Q. \tag{5}$$

Imposing the trace $G_a^{(2)a}$ to be of second order implies that the coefficients a, b, and c must be chosen such that the first two terms in (5) vanish, i.e.,

$$\nabla_a\nabla_b\left[-4aR^{ab} + g^{ab}\left(4a + \frac{D}{2}b + 2(D-1)c\right)R\right] = 0. \tag{6}$$

The above equation is satisfied only if the term inside the bracket is proportional to the Einstein tensor, which is the most general divergenceless, symmetric, rank two tensor linear in the curvature.[2] Consequently, the coefficients in (6) must fulfill

$$-4a = \gamma \quad \text{and} \quad 4a + \frac{D}{2}b + 2(D-1)c = -\frac{\gamma}{2}. \tag{7}$$

Since there are four variables (a, b, c, γ) and two equations, there is a bi-parametric family of solutions given by

$$\mathcal{L}_Q = -\frac{\gamma}{4}R^{abcd}R_{abcd} + bR^{ab}R_{ab} + \frac{\gamma - bD}{4(D-1)}R^2. \tag{8}$$

One can further factor out the four-dimensional Euler density to write (8) in the form

$$\mathcal{L}_Q = \alpha\mathcal{N}_4 + \beta\mathcal{E}_4, \tag{9}$$

where $\alpha = b - \gamma$ and $\beta = -\frac{\gamma}{4}$ are arbitrary constants; $\mathcal{N}_4$ is defined by

$$\begin{aligned}\mathcal{N}_4 &:= 4R^{ab}R_{ab} - \frac{D}{(D-1)}R^2 \\ &= \frac{1}{2^4}\left(\frac{D-2}{D-3}\right)\delta^{a_1b_1a_2b_2}_{c_1d_1c_2d_2}(C_{a_1b_1}{}^{c_1d_1}C_{a_2b_2}{}^{c_2d_2} \\ &\quad - R_{a_1b_1}{}^{c_1d_1}R_{a_2b_2}{}^{c_2d_2}),\end{aligned} \tag{10}$$

and $\mathcal{E}_4 := R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$ is the Gauss-Bonnet combination which corresponds to the four-dimensional Euler density. Thus, we have shown that for combinations quadratic in the curvature, the most general

[1]Note that the only nonquadratic term with degree of differentiation 4 is $\Box R$, which is boundary term.

[2]Since a divergenceless vector J^a cannot be constructed locally out of the curvature, the equation $\nabla_a J^a = 0$, with $J^a := \nabla_b[-4aR^{ab} + g^{ab}(4a + \frac{D}{2}b + 2(D-1)c)R]$, does not have any nontrivial solution.

Lagrangian, which has a second-order traced field equation, can be expressed as a linear combination of the four-dimensional Euler density $\mathcal{E}_4$ and $\mathcal{N}_4$ defined in Eq. (10).

In dimensions higher than three, one can further use the following relation:

$$C^{abcd}C_{abcd} = \mathcal{E}_4 + \left(\frac{D-3}{D-2}\right)\mathcal{N}_4, \tag{11}$$

where C_{abcd} is the Weyl tensor and $C^{abcd}C_{abcd}$ is the four-dimensional conformal Weyl invariant. This means that for dimensions higher than three, $\mathcal{L}_Q$ in (8) can be equivalently expressed as a linear combination of the four-dimensional Weyl invariant and the Euler density.

III. LAGRANGIANS WITH $n = 6$

In this section, we generalize the previous discussion for Lagrangians of degree of differentiation six in arbitrary dimensions D. We start by considering a generic combination of the 12 linearly independent [17], curvature invariants of degree six

$$\mathcal{L} = \sum_{i=1}^{12} A^i L_i, \tag{12}$$

where A^i's are arbitrary coefficients and L_i's are given by

$$\begin{aligned}
&L_1 = R^{abcd}R_{cdef}R^{ef}{}_{ab}, && L_2 = R^{ab}{}_{cd}R^{ce}{}_{bf}R^{df}{}_{ae},\\
&L_3 = R^{abcd}R_{cdbe}R^{e}{}_{a}, && L_4 = RR^{abcd}R_{abcd},\\
&L_5 = R^{abcd}R_{ac}R_{bd}, && L_6 = R^{ab}R_{bc}R^{c}{}_{a},\\
&L_7 = RR^{ab}R_{ab}, && L_8 = R^3,\\
&L_9 = \nabla_a R\nabla^a R, && L_{10} = \nabla_a R_{bc}\nabla^a R^{bc},\\
&L_{11} = \nabla_p R_{abcd}\nabla^p R^{abcd}, && L_{12} = \nabla_a R_{bc}\nabla^b R^{ac}.
\end{aligned} \tag{13}$$

Note however that when we neglect a total derviative, the 12 terms are not linearly independent, as one can write two of the invariants in terms of the other ten in the following way:[3]

$$\begin{aligned}
L_{11} &= 2L_1 - 4L_2 + 2L_3 - 4L_5 + 4L_6 - L_9 + 4L_{10}\\
&\quad + \nabla_a\nabla_c[2R^{abde}R^{c}{}_{bde} - 8R^{abcd}R_{bd} + 8RR^{ac}\\
&\quad - 12R^{ab}R^{c}{}_{b} + 2g^{ac}(2R^{bd}R_{bd} - R^2)]\\
L_{12} &= L_5 - L_6 + \frac{1}{4}L_9 + \nabla_a\nabla_c\left[R^{a}{}_{b}R^{bc} - RR^{ac} + \frac{1}{4}g^{ac}R^2\right].
\end{aligned} \tag{14}$$

Using the above relations, one can rewrite the Lagrangian as a linear combination of only ten curvature invariants with coefficients $\tilde{A}^i$ $(i = 1, \cdots, 10)$. Extremizing the action constructed with the Lagrangian (12) with respect to the metric gives the field equations

$$\mathcal{G}^{(3)}_{ab} := \sum_{i=1}^{10} \tilde{A}^i G^{(3)}_{(i)ab} = 0, \tag{15}$$

where $G^{(3)}_{(i)ab}$ are defined, respectively, in Eqs. (B3)–(B10) in the Appendix B. Now, requiring the trace $\mathcal{G}^{(3)a}_{a}$ to be of order three is the same as imposing trace to be proportional to $\mathcal{L}$ (see Appendix A). This gives us a set of eight equations (B14) for the 12 variables (A^i) and a parameter u [analoguous to γ in the quadratic case (7)]. Solving these equations for arbitrary dimensions, we obtain a five-parameter family of solutions. The details of the equations and its solution are given in the Appendix B. This implies that in $D > 6$, there are five linearly independent curvature invariants (of degree of differentiation six) which gives rise to third (or lower) order traced field equations. They are as follows. First, the six-dimensional Euler density given by

$$\begin{aligned}
\mathcal{E}_6 := {}& 2L_1 + 8L_2 + 24L_3 + 3L_4 + 24L_5 + 16L_6\\
& - 12L_7 + L_8,
\end{aligned} \tag{16}$$

obviously gives second-order traced field equations. Second, there are two independent algebraic invariants constructed out of three Weyl tensors, namely $W_1 = C^{ab}{}_{cd}C^{cd}{}_{ef}C^{ef}{}_{ab}$ and $W_2 = C_{abcd}C^{ebcf}C^{a}{}_{ef}{}^{d}$, which also give second-order traced filed equations. These two Weyl invariants are given in terms of the L_i's as

$$\begin{aligned}
W_1 = {}& L_1 + \frac{12}{D-2}L_3 + \frac{6}{(D-1)(D-2)}L_4\\
& + \frac{24}{(D-2)^2}L_5 + \frac{16(D-1)}{(D-2)^3}L_6\\
& - \frac{24(2D-3)}{(D-1)(D-2)^3}L_7 + \frac{8(2D-3)}{(D-1)^2(D-2)^3}L_8,
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
W_2 = {}& -\frac{1}{4}L_1 + L_2 + \frac{3}{D-2}L_3 + \frac{3}{2(D-1)(D-2)}L_4\\
& + \frac{3D}{(D-2)^2}L_5 + \frac{2(3D-4)}{(D-2)^3}L_6\\
& - \frac{3(D^2+D-4)}{(D-1)(D-2)^3}L_7 + \frac{(D^2+D-4)}{(D-1)^2(D-2)^3}L_8.
\end{aligned} \tag{18}$$

In addition, there are two other curvature invariants Σ and Θ listed in Eqs. (B17) and (B18), respectively, which give third order traced field equations. However, in dimensions $D \leq 6$, the above curvature invariants are not all linearly independent. For example, in $D = 3$ and 6, the invariants Σ

[3] We would like to thank Nicolas Boulanger for pointing this out to us.

TABLE I. Here C_{abc} denotes the Cotton tensor. Note that in dimension five, a new combination $\mathcal{N}_6 := -24L_3 - \frac{21}{4}L_4 - 40L_5 - \frac{320}{9}L_6 + \frac{97}{3}L_7 - \frac{31}{9}L_8$ appears.

G_a^a	$D=3$	$D=4$	$D=5$	$D=6$	$D>6$
$\partial^2 g$	$\nexists$	$W_1 \sim W_2$	$W_1 \sim W_2,\ \mathcal{N}_6$	$W_1,\ W_2$	$W_1,\ W_2,\ \mathcal{E}_6$
$\partial^3 g$	$C_{abc}C^{abc}$	$\Sigma,\ \Theta$	$\Sigma,\ \Theta$	$\Sigma \sim \Theta$	$\Sigma,\ \Theta$

and Θ are proportional to each other modulo a total derivative. It is interesting to note that in six dimensions, requiring the traced field equations to be of third order (or less), we recover all four (1 type-A and 3 type-B) nontrivial conformal anomalies [18–20]. In Table I below, we list all the curvature invariants in dimensions greater than or equal to 3, which lead to third (or less) order traced field equations.

Returning to the set of invariants that gives second-order traced field equations, we find that in dimensions $D \neq 5$, they are spanned by the basis set $\{\mathcal{E}_6, W_1, W_2\}$ up to a total derivative. However, in $D = 5$ this is not the case. In particular, there exists a "special" linearly independent invariant which generalizes $\mathcal{N}_4$ to the cubic case. This is realized by noting that the following relation is analogous to Eq. (11):

$$4W_1 + 8W_2 = \mathcal{E}_6 + \left(\frac{D-5}{D-2}\right)\mathcal{N}_6, \tag{19}$$

where

$$\mathcal{N}_6 := -24L_3 - \frac{3(D+2)}{(D-1)}L_4 - \frac{24D}{D-2}L_5 - \frac{16D(D-1)}{(D-2)^2}L_6 + \frac{12(D^3-2D^2+6D-8)}{(D-2)^2(D-1)}L_7 - \frac{(D^4-3D^3+10D^2+4D-24)}{(D-2)^2(D-1)^2}L_8 \tag{20}$$

is the cubic counterpart of $\mathcal{N}_4$. Let us rewrite Eq. (19) in the form

$$\mathcal{N}_6 := \frac{D-2}{D-5}(4W_1 + 8W_2 - \mathcal{E}_6) \tag{21}$$

$$= \frac{1}{2^3}\left(\frac{D-2}{D-5}\right)\delta^{a_1b_1a_2b_2a_3b_3}_{c_1d_1c_2d_2c_3d_3}(C_{a_1b_1}{}^{c_1d_1}C_{a_2b_2}{}^{c_2d_2}C_{a_3b_3}{}^{c_3d_3} - R_{a_1b_1}{}^{c_1d_1}R_{a_2b_2}{}^{c_2d_2}R_{a_3b_3}{}^{c_3d_3}). \tag{22}$$

The term inside the parenthesis on the right-hand side vanishes identically in dimensions lower than five, since for $D \leq 5$,

$$\mathcal{E}_6 = 4W_1 + 8W_2 \equiv 0. \tag{23}$$

However, in $D = 5$ (and greater than 5) this gives a nonvanishing invariant as can be seen by expressing W_1, W_2, and $\mathcal{E}_6$ in terms of $\{L_1, \cdots, L_8\}$, thereby obtaining the expression (20). This implies that in $D = 5$, the basis is $\{W_1(\sim W_2), \mathcal{N}_6\}$ up to a total derivative. Similar results have been found for quartic invariants (see Appendix B of Ref. [12]) i.e., in dimensions $D \neq 7$, any invariant giving second-order traced field equations can be expressed as a linear combination of the eight-dimensional Euler density, and all the linearly independent Weyl invariants; however, in $D = 7$ there is an additional special invariant which completes the basis. Based on these results, we present the following conjecture.

Conjecture: (i) In dimensions $D \neq 2k - 1$, any curvature invariant of order k,[4] which gives second (or less) order traced field equations, can be expressed as a linear combination of the $2k$-dimensional Euler density, the Weyl invariants, and a total derivative.

(ii) In dimensions $D = 2k - 1$, any curvature invariant of order k, which gives second-order traced field equations can be expressed as a linear combination of the Weyl invariants,[5] a total derivative, and a special invariant which can be obtained by evaluating

$$\mathcal{N}_{2k} := \frac{1}{2^k}\left(\frac{D-2}{D-2k+1}\right)\delta^{a_1b_1\cdots a_kb_k}_{c_1d_1\cdots c_kd_k} \times (C_{a_1b_1}{}^{c_1d_1}\cdots C_{a_kb_k}{}^{c_kd_k} - R_{a_1b_1}{}^{c_1d_1}\cdots R_{a_kb_k}{}^{c_kd_k}) \tag{24}$$

in $D = 2k - 1$.

We now show that $\mathcal{N}_{2k}$ evaluated in $D = 2k - 1$ indeed gives second-order traced field equations. First, consider the following invariant of order k:

$$\frac{1}{2^k}\delta^{a_1b_1\cdots a_kb_k}_{c_1d_1\cdots c_kd_k}(C_{a_1b_1}{}^{c_1d_1}\cdots C_{a_kb_k}{}^{c_kd_k} - R_{a_1b_1}{}^{c_1d_1}\cdots R_{a_kb_k}{}^{c_kd_k}). \tag{25}$$

Obviously, the above invariant vanishes in dimensions lower than $2k$. However, if one expands the Weyl tensor in terms of the Riemann tensor, then it can be factorized by $(D - 2k + 1)$. This can be seen as follows. Consider the basis set of kth order Riemann invariants in arbitrary dimensions. In $D = 2k - 1$, not all elements of this set

[4]By a curvature invariant of order k, we mean a scalar constructed out of k curvature tensors without any derivatives acting on them.

[5]Note that the number of linearly independent Weyl invariants of order k in dimensions $D = 2k - 1$ is one less than that in dimensions $D \geq 2k$, due to the identity $C_{[a_1b_1}{}^{a_1b_1}\ldots C_{a_kb_k]}{}^{a_kb_k} = 0$.

are linearly independent. In fact, the basis set contains one less invariant than in $D \geq 2k$. This is because of the vanishing of the kth order Lovelock density. Now, after the expanding in terms of the Riemann tensors, the term (25) will not contain any $(Riemann)^k$. So, this invariant cannot vanish identically in $D = 2k - 1$ unless it is factorized by $(D - 2k + 1)$.[6] Further expanding all the Weyl tensors, one can be convinced that the dimensional dependence of the coefficient of the term with $k - 1$ Riemann tensors and one Ricci tensor must be $(D - 2k + 1)/(D - 2)$. We can now divide this factor out to get a nonvanishing invariant in $D = 2k - 1$. Thus, we write the kth order generalization of $\mathcal{N}_4$ by evaluating

$$\frac{1}{2^k}\left(\frac{D-2}{D-2k+1}\right)\delta^{a_1b_1\cdots a_kb_k}_{c_1d_1\cdots c_kd_k} \times (C_{a_1b_1}{}^{c_1d_1}\cdots C_{a_kb_k}{}^{c_kd_k} - R_{a_1b_1}{}^{c_1d_1}\cdots R_{a_kb_k}{}^{c_kd_k}) \quad (26)$$

in $D = 2k - 1$. Note that, by construction, the trace of the field equation arising from the above invariant is of second order in all dimensions.

IV. EXACT SOLUTIONS

In this section, we present exact, static solutions for the theories defined previously. For simplicity, we will first focus on the theories having fourth order field equations, defined by an arbitrary linear combination of the invariants W_1 and W_2, defined, respectively, in Eqs. (17) and (18). The theory defined by the combination $\mathcal{N}_6$ in the Table I, has further interesting properties in five dimensions, which we discuss in detail in Ref. [12]. Finally, we comment on the new three-dimensional theory shown in Table I, which possesses third order traced field equations.

The class of metrics considered is

$$ds_D^2 = -F(R)dt^2 + \frac{dR^2}{G(R)} + R^2 d\Sigma^2_{D-2,\gamma}, \quad (27)$$

where $d\Sigma_{D-2,\gamma}$ is the line element of a $(D-2)$-dimensional compact, orientable Euclidean manifold of constant curvature γ. For $\gamma = 1$, the manifold $\Sigma_{D-2}$is locally equivalent to the sphere S^{D-2}, while for $\gamma = 0$, it reduces to a locally flat manifold. Finally, for $\gamma = -1$, the geometry of Σ_{D-2} is given by the quotient H_{D-2}/Γ, where Γ is a freely acting, discrete subgroup of $O(D-2, 1)$.

After a coordinate transformation and a redefinition of the arbitrary functions, the line element (27) takes the form

$$ds_D^2 = N(r)\left[-f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma^2_{D-2,\gamma}\right]. \quad (28)$$

As shown below, this gauge choice is much more convenient for our purposes.

A. C^3 theories

Here we will consider Lagrangians of the form

$$\mathcal{L} = \alpha W_1 + \beta W_2. \quad (29)$$

It has been proven in [21] that, for the metric (28), the two invariants W_1 and W_2 defined in (17) and (18), respectively, are proportional. Consequently, for a particular choice of α/β, both the Lagrangian $\mathcal{L}$ and the field equations vanish identically. In such a situation, any metric within the family (28) is a solution of the system. Hereafter, we assume that α and β are generic.

Since, in six dimensions the gravity theories defined by combinations of W_1 and W_2 are invariant under Weyl rescalings, let us concentrate on this case first.

(i) $D = 6$: Using Weyl rescalings, one can gauge away the function $N(r)$ in (28). Then, the solution reduces to

$$ds_6^2 = -\left(ar^2 + br + K - \frac{c(1+er)^{5/2}}{r^{1/2}}\right)dt^2 + \frac{dr^2}{ar^2 + br + K - \frac{c(1+er)^{5/2}}{r^{1/2}}} + r^2 d\Sigma^2_{4,\gamma}, \quad (30)$$

where a, b, K, c, and e are constants, which are related by

$$\begin{cases} K = \gamma, & \text{and} \quad c(b - 2Ke) = 0 \\ K = -\frac{1}{2}\gamma, & \text{and} \quad c = 0. \end{cases} \quad (31)$$

The Ricci scalar of this geometry diverges at $r = r_{s1} := 0$, while at $r = r_{s2} := -e^{-1}$, the differential scalar $\nabla_\mu R \nabla^\mu R$ diverges. For negative e, the region $r > r_{s2}$ must be removed from the spacetime, since otherwise the metric is imaginary, unless c vanishes. For vanishing c and $K = \gamma$, we obtain a conformally flat solution which may possess one or two horizons surrounding the singularity at the origin. For $K = -\gamma/2$, the spacetime is not conformally flat and may also describe a black hole.

Let us note that, since in six dimensions the theory is conformally invariant, any metric conformally related to (30) will be a solution of the system.

(ii) $D \neq 6$: For dimensions other than six, the situation is different. Since the theory defined by (29) is not invariant under local Weyl rescaling, one naively expects the factor $N(r)$ in (29) to be fixed by the field equations. However, this is not the case, and for arbitrary dimensions, the most general solution within the family (28), for the theory (29) is

[6] This argument cannot be extended to dimensions $2k - 2$, since one obtains another identity involving the Riemann invariants which is obtained by contracting the Ricci tensor with the $(k-1)$-th order Lovelock equation.

$$ds_D^2 = N(r)\left[-(ar^2 + br + \gamma)dt^2 + \frac{dr^2}{ar^2 + br + \gamma} + r^2 d\Sigma_{D-2,\gamma}^2\right], \quad (32)$$

$N(r)$ being an arbitrary function.

This can be easily seen as follows: Since in dimensions other than six, the trace of the field equations for the theory (29) is proportional to the Lagrangian, the invariants $W_1 \sim W_2$ evaluated on a solution should vanish. For the spacetime under consideration (28), it has been shown in [21] that all the components of the Weyl tensor are proportional to a single function X, such that the vanishing of X implies that the metric should be conformally flat. Since the restriction $W_1 = W_2 = 0$ transforms covariantly under Weyl rescalings, it does not involve the function $N(r)$, and the mentioned restriction reduces to $X^3 = 0$, which implies $f(r) = ar^2 + br + \gamma$. Then one is left with a conformally flat space, and since the field equations explicitly contain a Weyl tensor, they are fulfilled for any arbitrary function $N(r)$.[7] For a smooth conformal factor $N(r)$, the spacetime (32) is conformally flat and it has been studied within the context of conformal gravity in four dimensions in [8] for $\gamma = 1$ and in Ref. [23] for arbitrary γ. The three-dimensional cousin of this metric, in which $d\Sigma_\gamma$ is replaced by a single compact direction $d\phi$ and γ is an integration constant, can be obtained through "conformal gluing" of Bañados-Teitelboim-Zanelli black holes, and it is a solution of three-dimensional conformal gravity [24]. In [11] this metric was obtained within the context of Bergshoeff-Hohm-Townsend new massive gravity [10,25] at the special point where the two possible maximally symmetric solutions of the theory coincide. In that case, γ is an arbitrary constant, the parameter b plays the role of a gravitational hair, while a is fixed in terms of the coupling constant.

The metric (32) may possess an event and a Cauchy horizon, depending on the zeros of g_{tt}. It generically possesses a curvature singularity located at $r = 0$, and depending on the sign of the integration constant a, it represents an asymptotically locally (A)dS or flat spacetime for $(a > 0)$ $a < 0$ or $a = 0$, respectively. The details of the different causal structures are given in [11].

B. The five-dimensional combination $\mathcal{N}_6$

As stated in Table I, in five dimensions, there are two linearly independent invariants whose traced field equations are of second order. Now, consider the following linear combination as the Lagrangian

$$\mathcal{L} = \tfrac{7}{4}W_1 - \tfrac{1}{3}\mathcal{N}_6 \quad (33)$$

evaluated on $D = 5$. This is the unique cubic invariant for which all the components of field equation, for static spherically symmetric spacetimes, are of second order.

As shown in Ref. [12], the most general, nondegenerate spherically symmetric solution is given by

$$ds^2 = -(cr^{2/3} + \gamma)dt^2 + \frac{dr^2}{cr^{2/3} + \gamma} + r^2 d\Sigma_\gamma^2, \quad (34)$$

where c is an integration constant and $\gamma = \pm 1$, and 0 is the curvature of Σ_3. Let us note that this spacetime is not conformally flat (it possesses a nonvanishing Weyl tensor) unless $c = 0$. For positive c and $\gamma = -1$, the metric (34) represents a black hole possessing an event horizon located at located at $r_+ = c^{-3/2}$. In this case the geometry of the horizon is given by H_3/Γ, where Γ is a freely acting discrete subgroup of $O(3, 1)$. The horizon hides the curvature singularity located at $r = 0$, and the asymptotic region $(r \to \infty)$ is locally flat. Further interesting features of this solution are discussed in [12]. It is also interesting to note that among the class of theories considered here, this is the only one which does not admit an (A)dS solution, in the same way as the pure K combination of BHT new massive gravity [10].

C. The three-dimensional case

As shown in Table I, within the family considered, the only nontrivial theory having third order traced field equations in three dimensions can be written as $C_{abc}C^{abc}$, where C_{abc} is the Cotton tensor. In this theory, the most general static, spherically symmetric solution is given by

$$ds^2 = N(r)\left[-(ar^2 + br - \mu)dt^2 + \frac{dr^2}{ar^2 + br - \mu} + r^2 d\phi^2\right], \quad (35)$$

where a, b, and μ are integration constants and $N(r)$ is an arbitrary function. For smooth $N(r)$, as mentioned above, this metric has an event and a Cauchy horizon, depending on the value of the parameters.

It will be interesting to study the thermodynamical properties of the black hole within the context of AdS/CFT.

V. SUMMARY

In this work, we have classified all the six-derivative Lagrangians of gravity for which the trace of the field equations have a reduced order. We have seen that, in dimensions greater or equal to six, when the trace of the field equations from a generic Lagrangian is restricted to order two, we obtain an arbitrary linear combination of three linearly independent curvature invariants, namely, the six-dimensional Euler density and the two independent Weyl invariants. These invariants are no longer

[7]Note that the same argument is valid for any theory with a Lagrangian of the form $\overbrace{C\ldots C}^{n}$ provided $D \neq 2n$. In four dimensions, for $n = 3$, this solution was found in [22], where it was mentioned that the corresponding model is the simplest one that does not admit Schwazschild horizons.

independent in lower dimensions due to the Schouten identities. However, in five dimensions, there is a special invariant $\mathcal{N}_6$, which also gives field equations with second-order trace. These invariants can be used to construct interesting cubic theories of gravity that can serve as toy models for higher-derivative theories. We have also provided a conjecture regarding all the possible invariants of arbitrary order which gives second-order traced field equations in any dimensions. In addition, we have obtained the general spherically symmetric solutions for a subclass of such theories in arbitrary dimensions. Our analysis shows that this is possible due to the reduced order of the trace of the field equations. When the order of the trace is restricted to three, we obtain two further invariants Σ and Θ in arbitrary dimensions. These two invariants are not globally independent in three and six dimensions. In six dimensions, they reduce to the third type-B anomaly;[8] whereas in three dimensions, they are equivalent to the square of the cotton tensor $\sim C_{abc}C^{abc}$. We have further obtained a general spherically symmetric solution of this theory in three dimensions.

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APPENDIX A: TRACE OF THE FIELD EQUATION

Here, we prove a general property of the trace of the field equations for any Lagrangian of the form $\mathcal{L}(g_{ab}, R_{abcd}, \nabla_e)$ with a fixed degree of differentiation n. It has been shown in [27] that the Lagrangian can always be reexpressed as

$$\mathcal{L}[g_{ab}, R_{abcd}, \nabla_{a_1}R_{bcde}, \ldots, \nabla_{(a_1,\cdots,a_p)}R_{bcde}]. \tag{A1}$$

The field equations obtained by variation of the action with respect to the metric take the form

$$-T^{ab} = \frac{\partial\mathcal{L}}{\partial g_{ab}} + E^{a}{}_{cde}R^{bcde} + 2\nabla_c\nabla_d E^{acdb} + \frac{1}{2}g^{ab}\mathcal{L}, \tag{A2}$$

$$E^{bcde} = \frac{\partial\mathcal{L}}{\partial R_{bcde}} - \nabla_{a_1}\frac{\partial\mathcal{L}}{\partial\nabla_{a_1}R_{bcde}} + \cdots + (-1)^p\nabla_{(a_1}\cdots\nabla_{a_p)}\frac{\partial\mathcal{L}}{\partial\nabla_{(a_1}\cdots\nabla_{a_p)}R_{bcde}}, \tag{A3}$$

where T^{ab} is the energy-momentum tensor of the matter fields. Taking the trace of the field equations, we obtain

$$-T^{a}{}_{a} = g_{ab}\frac{\partial\mathcal{L}}{\partial g_{ab}} + E^{abcd}R_{abcd} + \frac{D}{2}\mathcal{L} + \text{tot deriv}. \tag{A4}$$

Now, if the Lagrangian is of fixed n, then it can be expressed as a linear combination of terms of the form

$$[g^{..}]^{q_1}[R_{....}]^{q_2}[\nabla_{.}R_{....}]^{q_3}\cdots[\underbrace{\nabla_{.}\cdots\nabla_{.}}_{p\text{ times}}R_{....}]^{q_{p+2}}, \tag{A5}$$

such that

$$2q_2 + 3q_3 + \cdots + (p+2)q_{p+2} = n. \tag{A6}$$

Then, under the scaling $g^{ab}\to t^{-1}g^{ab}$, $R_{bcde}\to tR_{bcde}, \ldots, \overbrace{\nabla_{a_1}\cdots\nabla_{a_p}}^{p\text{ times}}R_{bcde}\to \overbrace{t\nabla_{a_1}\cdots\nabla_{a_p}}^{p\text{ times}}R_{bcde}$, the Lagrangian scales as $\mathcal{L}\to t^{-q_1+q_2+\cdots+q_{p+2}}\mathcal{L}$. However, q_1 can be expressed in terms of other q_p's as

$$\begin{aligned} q_1 &= \frac{1}{2}[4q_2 + 5q_3 + \cdots + (p+4)q_{p+2}] \\ &= \frac{1}{2}[2(q_2+q_3+\cdots+q_{p+2}) + (2q_2+3q_3+\cdots \\ &\quad + (p+2)q_{p+2})] = (q_2+q_3+\cdots+q_{p+2}) + \tfrac{n}{2}. \end{aligned} \tag{A7}$$

This implies that the Lagrangian scales as $t^{-(n/2)}\mathcal{L}$. Now, one can apply Euler's theorem of homogenous functions to write the following relation:

[8]In fact, the anomalies are called global conformal invariants. It was first conjectured by Deser and Schwimmer [26] that any global conformal invariant can be expressed as a linear combination of the Euler density and the local conformal invariants. Recently, the conjecture has been proved by differential geometric techniques by Alexakis [29] and cohomological techniques by Boulanger [30]. For six-derivative invariants, in arbitrary dimensions, in addition to the two independent Weyl invariants which are purely algebraic, there is a third local conformal invariant which involves derivative of the curvature. Our analysis shows that this invariant does not give field equations with reduced order trace in arbitrary dimensions. However, they coincide (equivalent up to a total derivative [26]) with our Σ and Θ in dimensions three and six.

$$\begin{aligned}
-\frac{n}{2}\mathcal{L} &= -g^{ab}\frac{\partial\mathcal{L}}{\partial g^{ab}} + R_{bcde}\frac{\partial\mathcal{L}}{\partial R_{bcde}} \\
&\quad + \nabla_{a_1}R_{bcde}\frac{\partial\mathcal{L}}{\partial\nabla_{a_1}R_{bcde}} + \cdots \\
&\quad + \nabla_{(a_1}\cdots\nabla_{a_p)}R_{bcde}\frac{\partial\mathcal{L}}{\partial\nabla_{(a_1}\cdots\nabla_{a_p)}R_{bcde}} \\
&= g_{ab}\frac{\partial\mathcal{L}}{\partial g_{ab}} + R_{bcde}\frac{\partial\mathcal{L}}{\partial R_{bcde}} + \nabla_{a_1}R_{bcde}\frac{\partial\mathcal{L}}{\partial\nabla_{a_1}R_{bcde}} \\
&\quad + \cdots + \nabla_{(a_1}\cdots\nabla_{a_p)}R_{bcde}\frac{\partial\mathcal{L}}{\partial\nabla_{(a_1}\cdots\nabla_{a_p)}R_{bcde}} \\
&= g_{ab}\frac{\partial\mathcal{L}}{\partial g_{ab}} + E^{bcde}R_{bcde} + \text{tot deriv.}
\end{aligned} \tag{A8}$$

Therefore, the trace of the field equations can be written in the form

$$T^i{}_i = \frac{n-D}{2}\mathcal{L} + \text{tot deriv.} \tag{A9}$$

Now, suppose that the trace of the field equations, from a Lagrangian of $n = 6$, is of third order. Then it must be some linear combination of the invariants $L_1, \ldots, L_{12}$. According to (A9), in dimensions $D \neq n$, the Lagrangian must be proportional to this combination up to a total derivative. In dimensions $D = n$, since the trace of the field equations is a total derivative, the only way the trace can be of at most third order is when it identically vanishes, which is the case for conformally invariant theories.

APPENDIX B: EQUATIONS OF MOTION

In this appendix, we provide the details of the analysis for the classification presented in Table I. The equations of motion for each term in the general Lagrangian are listed below [28]:

$$\begin{aligned}
G^{(3)}_{1ab} &= 3R_{aq}{}^{ef}R_b{}^{qcd}R_{cdef} + 6\nabla_p\nabla_q(R_a{}^{qcd}R_b{}^p{}_{cd}) \\
&\quad - \frac{1}{2}g_{ab}L_1,
\end{aligned} \tag{B1}$$

$$\begin{aligned}
G^{(3)}_{2ab} &= 3R_{ahd}{}^gR_b{}^{prd}R_{pgr}{}^h - 3\nabla_p\nabla_q(R^p{}_g{}^q{}_hR_a{}^g{}_b{}^h \\
&\quad - R^p{}_{hbg}R_a{}^{gqh}) - \frac{1}{2}g_{ab}L_2,
\end{aligned} \tag{B2}$$

$$\begin{aligned}
G^{(3)}_{3ab} &= R_{acbd}R^{cspq}R_{pqs}{}^d - R_a{}^{qcd}R_{cdb}{}^hR_{qh} \\
&\quad + R_b{}^{dqc}R_{adc}{}^hR_{qh} - \nabla_p\nabla_q(R_{ah}R_b{}^{qhp} + R_{bh}R_a{}^{qhp} \\
&\quad + R^q{}_hR_a{}^h{}_b{}^p + R^p{}_hR_a{}^q{}_b{}^h + \frac{1}{2}(g^{pq}R_a{}^{hcd}R_{bhcd} \\
&\quad + g_{ab}R^{prcd}R^q{}_{rcd} - g_a{}^pR_b{}^{rcd}R^q{}_{rcd} \\
&\quad - g_b{}^pR_a{}^{rcd}R^q{}_{rcd})) - \frac{1}{2}g_{ab}L_3,
\end{aligned} \tag{B3}$$

$$\begin{aligned}
G^{(3)}_{4ab} &= 2R_{apcd}R_b{}^{pcd}R + R_{ab}R^{pqcd}R_{pqcd} \\
&\quad + \nabla_p\nabla_q(4RR_a{}^q{}_b{}^p - g_a{}^pg_b{}^qR^{rscd}R_{rscd} \\
&\quad + g^{pq}g_{ab}R^{rscd}R_{rscd}) - \frac{1}{2}g_{ab}L_4,
\end{aligned} \tag{B4}$$

$$\begin{aligned}
G^{(3)}_{5ab} &= R_{ac}R_b{}^{fcd}R_{fd} + 2R_{acbd}R^{cfdg}R_{fg} + \nabla_p\nabla_q(R_{ab}R^{pq} \\
&\quad - R_a{}^pR_b{}^q + g^{pq}R_{acbd}R^{cd} + g_{ab}R^{pcqd}R_{cd} \\
&\quad - g_a{}^pR^q{}_{cbd}R^{cd} - g_b{}^pR^q{}_{cad}R^{cd}) - \frac{1}{2}g_{ab}L_5,
\end{aligned} \tag{B5}$$

$$\begin{aligned}
G^{(3)}_{6ab} &= 3R_{acbd}R^{ec}R_e{}^d + \frac{3}{2}\nabla_p\nabla_q(g^{pq}R_a{}^cR_{bc} + g_{ab}R^{ep}R_e{}^q \\
&\quad - g_b{}^pR^{qc}R_{ac} - g_a{}^pR^{qc}R_{bc}) - \frac{1}{2}g_{ab}L_6,
\end{aligned} \tag{B6}$$

$$\begin{aligned}
G^{(3)}_{7ab} &= R_{ab}R^{cd}R_{cd} + 2RR^{cd}R_{acbd} + \nabla_p\nabla_q(g_{ab}g^{pq}R^{cd}R_{cd} \\
&\quad + g^{pq}RR_{ab} - g_a{}^pg_b{}^qR^{cd}R_{cd} + g_{ab}RR^{pq} \\
&\quad - g_b{}^pRR_a{}^q - g_a{}^pRR_b{}^q) - \frac{1}{2}g_{ab}L_7,
\end{aligned} \tag{B7}$$

$$\begin{aligned}
G^{(3)}_{8ab} &= 3R^2R_{ab} + 3\nabla_p\nabla_q(g_{ab}g^{pq}R^2 - g_a{}^pg_b{}^qR^2) \\
&\quad - \frac{1}{2}g_{ab}L_8,
\end{aligned} \tag{B8}$$

$$\begin{aligned}
G^{(3)}_{9ab} &= \nabla_aR\nabla_bR - 2\Box RR_{ab} - 2(g_{ab}g_{cd} \\
&\quad - g_{ac}g_{bd})\nabla^c\nabla^d\Box R - \frac{1}{2}g_{ab}L_9,
\end{aligned} \tag{B9}$$

$$\begin{aligned}
G^{(3)}_{10ab} &= \nabla_aR^{cd}\nabla_bR_{cd} + 2\nabla_cR_a{}^d\nabla^cR_{bd} - \Box^2R_{ab} \\
&\quad - \nabla_c\nabla_d\Box R^{cd}g_{ab} + \nabla_a\nabla_c\Box R^c{}_b + \nabla_b\nabla_c\Box R^c{}_a \\
&\quad - 2R_{acbd}\Box R^{cd} + 2R_{c(a}\Box R^c{}_{b)} + 2\nabla_c[R_d{}^c\nabla_{(b}R_{a)}{}^d \\
&\quad - R_{d(a}\nabla^cR^d{}_{b)} - R_{d(b}\nabla_{a)}R^{cd}] - \frac{1}{2}g_{ab}L_{10},
\end{aligned} \tag{B10}$$

$$\begin{aligned}
G^{(3)}_{11ab} &= 2G^{(3)}_{1ab} - 4G^{(3)}_{2ab} + 2G^{(3)}_{3ab} - 4G^{(3)}_{5ab} + 4G^{(3)}_{6ab} \\
&\quad - G^{(3)}_{9ab} + 4G^{(3)}_{10ab},
\end{aligned} \tag{B11}$$

$$G^{(3)}_{12ab} = G^{(3)}_{5ab} - G^{(3)}_{6ab} + \frac{1}{4}G^{(3)}_{9ab}. \tag{B12}$$

Therefore, the trace of the full field equations can be expressed as

$$
\begin{aligned}
A^i G^{(3)a}_{ia} = {} & (3 - D/2)A^i L_i + \nabla_p \nabla_q \Big[\Big(6A^1 + 3A^2 - \frac{D-2}{2}A^3 - 4A^{11}\Big) R^{pabc} R^q{}_{abc} + (-3A^2 - 2A^3 + (D-2)A^5 \\
& + 2(D-2)A^{10} + 24A^{11} + (D-2)A^{12}) R_{ab} R^{apbq} + \Big(-2A^3 - A^5 + \frac{3(D-2)}{2}A^6 - 2A^{10} + 16A^{11} \\
& - (D+1)A^{12}\Big) R^p{}_a R^{qa} + \Big(4A^4 + A^5 + (D-2)A^7 - (D-4)A^{10} - 12A^{11} - \frac{D-8}{2}A^{12}\Big) R R^{pq} \\
& + \Big(-\frac{1}{2}A^3 + (D-1)A^4 - A^{11}\Big) g^{pq} R^{abcd} R_{abcd} + \Big(A^5 + \frac{3}{2}A^6 + (D-1)A^7 - \frac{D}{2}A^{10} - 10A^{11} \\
& - \frac{1}{2}A^{12}\Big) g^{pq} R^{ab} R_{ab} + \Big(A^7 + 3(D-1)A^8 - A^9 + \frac{D-4}{4}A^{10} + 3A^{11} + \frac{D-8}{8}A^{12}\Big) g^{pq} R^2 \\
& + \Big(-2(D-1)A^9 - \frac{D}{2}A^{10} - 2A^{11} - \frac{D-1}{2}A^{12}\Big) g^{pq} \Box R \Big]. \qquad \text{(B13)}
\end{aligned}
$$

Now we impose the trace to be proportional to the Lagrangian. This, in turn, requires the second term on the right-hand side to vanish. To realize this, one has to choose the coefficients in such a way that the symmetric tensor quadratic in curvature inside the operator $\nabla_p \nabla_q$ is proportional to the Gauss-Bonnet field equations. This gives us a set of 8 equations in 12 variables and one arbitrary parameter u. They are

$$
\begin{aligned}
6A^1 + 3A^2 - \frac{D-2}{2}A^3 - 4A^{11} &= -2u, \\
-\frac{1}{2}A^3 + (D-1)A^4 - A^{11} &= u/2, \\
-3A^2 - 2A^3 + (D-2)A^5 + 2(D-2)A^{10} + 24A^{11} + (D-2)A^{12} &= 4u, \\
-2A^3 - A^5 + \frac{3(D-2)}{2}A^6 - 2A^{10} + 16A^{11} - (D+1)A^{12} &= 4u, \\
4A^4 + A^5 + (D-2)A^7 - (D-4)A^{10} - 12A^{11} - \frac{D-8}{2}A^{12} &= -2u, \\
A^5 + \frac{3}{2}A^6 + (D-1)A^7 - \frac{D}{2}A^{10} - 10A^{11} - \frac{1}{2}A^{12} &= -2u, \\
A^7 + 3(D-1)A^8 - A^9 + \frac{D-4}{4}A^{10} + 3A^{11} + \frac{D-8}{8}A^{12} &= u/2, \\
-2(D-1)A^9 - \frac{D}{2}A^{10} - 2A^{11} - \frac{D-1}{2}A^{12} &= 0.
\end{aligned} \qquad \text{(B14)}
$$

The matrix of linear equations has rank 8, which implies that the general solution can be written in terms of 5 arbitrary parameters x, y, z, u, and v. In $D > 5$, the solution is given as

$$
\begin{aligned}
A^1 &= \frac{1}{12}[2(D^2 + 5D - 10)x + 2(D-2)^2 y - 6(3D+2)z + (D^2 - 4)v + 3(D-2)u], \\
A^2 &= -\frac{1}{6}[8(2D-3)x + 2(D-2)^2 y - 8(2D+3)z + (D^2 - 4)v + 4(D-1)u], \\
A^3 &= 2(D-1)x - 2z - u, \qquad A^4 = x, \qquad A^5 = -4x - (D-2)y + \frac{8(D-1)}{D-2}z - \frac{D}{2}v - 2u, \\
A^6 &= \frac{1}{3}\Big[8x - 2y - \frac{24}{D-2}z + v\Big], \qquad A^7 = y \\
A^8 &= -\frac{1}{24(D-1)^2}[8(D-1)y + 16Dz - (D^2 - D + 2)v - 4(D-1)u], \\
A^9 &= \frac{1}{4(D-1)(D-2)}[8z + (D-2)v], \qquad A^{10} = -\frac{4}{D-2}z - v, \qquad A^{11} = z, \qquad A^{12} = v. \qquad \text{(B15)}
\end{aligned}
$$

In dimensions $D > 5$, one can apply the following transformation:

$$\begin{aligned} x &\rightarrow \frac{3}{2(D-2)(D-1)}[2(D-2)(D-1)a + 4b + c], \\ y &\rightarrow -\frac{3}{(D-2)^3(D-1)}[4(D-2)^3(D-1)a + 8(2D-3)b + (D^2+D-4)c], \\ z &\rightarrow d, \\ u &\rightarrow 6(D-5)a, \\ v &\rightarrow 24e, \end{aligned} \tag{B16}$$

such that

$$A^i L_i = a\mathcal{E}_6 + bW_1 + cW_2 + d\Sigma + e\Theta,$$

where

$$\begin{aligned} \Sigma &= -\frac{1}{2}(3D+2)L_1 + \frac{4}{3}(2D+3)L_2 - 2L_3 + \frac{8(D-1)}{D-2}L_5 - \frac{8}{D-2}L_6 - \frac{2D}{3(D-1)^2}L_8 + \frac{2}{(D-2)(D-1)}L_9 \\ &\quad - \frac{4}{D-2}L_{10} + L_{11} \\ &= -\frac{1}{2}(3D-2)L_1 + \frac{8D}{3}L_2 + \frac{4D}{D-2}L_5 + \frac{4(D-4)}{D-2}L_6 - \frac{2D}{3(D-1)^2}L_8 - \frac{D(D-3)}{(D-2)(D-1)}L_9 + \frac{4(D-3)}{D-2}L_{10} \\ &\quad + \text{total derivative}, \end{aligned} \tag{B17}$$

$$\begin{aligned} \Theta &= 2(D^2-4)L_1 - 4(D^2-4)L_2 - 12DL_5 + 8L_6 + \frac{D^2-D+2}{(D-1)^2}L_8 + \frac{6}{D-1}L_9 - 24L_{10} + 24L_{12} \\ &= 2(D^2-4)L_1 - 4(D^2-4)L_2 - 12(D-2)L_5 - 16L_6 + \frac{D^2-D+2}{(D-1)^2}L_8 + \frac{6D}{D-1}L_9 - 24L_{10} + \text{total derivative}. \end{aligned} \tag{B18}$$

Note that the determinant of the transformation (B16) is $\frac{2592(D-5)}{(D-2)^3(D-1)}$. Now in $D \leq 5$, one needs to solve the system of Eqs. (B14) for each value of D separately. The Lagrangians obtained are tabulated in Table I.

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