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Linear theory of gravitational wave propagation in a magnetized, relativistic Vlasov plasma

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We consider propagation of gravitational waves in a magnetized plasma, using the linearized Maxwell-Vlasov equations coupled to Einstein's equations. A set of coupled electromagnetic-gravitational wave equations are derived that can be straightforwardly reduced to a single dispersion relation. We demonstrate that there is a number of different resonance effects that can enhance the influence of the plasma on the gravitational waves.

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I. INTRODUCTION

The propagation of small amplitude gravitational waves (GWs) in the presence of matter and/or electromagnetic fields has been studied by many authors. A large variety of different approximation schemes and basic assumptions of the matter content are possible. In case the influence of the GW on matter and fields is in focus, a test matter approach can be sufficient [1–4]. When the backreaction on the GW is of interest, a self-consistent approach is necessary [1]. The physics close to the GW source may require a nonlinear approach [4–17], whereas a linear treatment [3,18–21] is adequate for larger distances from the source. Depending on the basic assumptions, the calculations may be of relevance for cosmology [22–26], astrophysics [4,5,13,20,27,28], or gravitational wave detectors [2,29,30]. For many applications in cosmology and astrophysics the relevant state of matter is the plasma state. Here a large variety of models can be relevant, e.g. magnetohydrodynamic models [8,20,21,31], multifluid models [4,7], or kinetic models [6,18,19].

In the present work we will perform a self-consistent, linearized treatment of GW propagation in a magnetized plasma, modeled by the Vlasov equation, as is relevant for a collisionless system. We will here focus on the direct contribution from matter and fields to the dispersion relation, rather than the indirect contribution from the background curvature, and thus we will consider a flat Minkowski background. Within this context a general dispersion relation is derived describing the linear coupling of GWs to electromagnetic waves (EMWs). Note, however, that the division into direct and indirect contributions contains certain complications. For a detailed discussion of this see e.g. the Appendix in Ref. [19]. A similar effort to ours has previously been undertaken by Ref. [18]. Here we extend that work in a number of directions. First, we continue the analysis a step further before going to special cases, by solving the integrals over the azimuthal angles in momentum space in general. Second, we keep the relativistic regime in the calculation until the end stage in the general case. Typically the contribution to the GW dispersion relation from matter is weak, for relevant energy densities. However, it is also shown that numerous resonant mechanisms exist that can increase the coupling to matter. A list of special cases exhibiting various types of resonances enhancing the influence on the GW dispersion relation is presented. The case of resonant coupling to high-frequency waves for GWs propagating at an arbitrary angle has not been studied before, to the best of our knowledge. Finally our results are summarized and discussed.

II. BASIC EQUATIONS

We consider the linear coupling between weak gravitational waves and electromagnetic perturbations in a collisionless plasma in an external magnetic field, assuming that the wavelength of the GW is much shorter than the characteristic radius of the background curvature (i.e. employing the high-frequency approximation). As noted by e.g. Refs. [1,19], the deviation from the vacuum dispersion relation of the GW has two types of contribution: direct effects of matter, and indirect effects of matter, where the latter comes from the background curvature. Here we will focus only on the direct effects of matter, and hence we will simply take the background metric as the flat Minkowski metric. This is motivated by the fact that we are mainly interested in enhanced GW couplings due to various resonances, which all follow from the direct effects of matter. For a detailed discussion of the separation into direct and indirect effects of matter, see the Appendix in Ref. [19]. In the present approach it is thus the *perturbations* of the electromagnetic and material fields, contributing to the perturbed energy-momentum tensor that are of interest.

For simplicity, the unperturbed plasma is assumed to be static, isotropic, and homogeneous. Linearized, the Einstein field equations take the form

$$\Box h_{ab} = -2\kappa[\delta T_{ab} - \tfrac{1}{2}\delta T \eta_{ab}], \tag{1}$$

provided the gauge condition $h^{ab}{}_{,b} = 0$ is satisfied, which is fulfilled only if tensorial perturbations are present. Note that for driven gravitational perturbations, this gauge condition is not necessarily fulfilled, but this is not the case of interest here. For a detailed discussion of the (approximate) fulfillment of the gauge condition, see e.g. Sec. IV in

Ref. [15]. Here $\Box \equiv [c^{-2}\partial_t^2 - \partial_z^2]$, h_{ab} is the small deviation from the Minkowski background metric, i.e. $g_{ab} = \eta_{ab} + h_{ab}$, $\kappa \equiv 8\pi G/c^4$, δT_{ab} is the part of the energy-momentum tensor containing small electromagnetic and material field perturbations associated with the gravitational waves and $\delta T = \delta T_a^a$. In our notation $a, b, c, \ldots = 0, 1, 2, 3$ and $i, j, k, \ldots = 1, 2, 3$ and the metric has the signature $(- + + +)$.

In vacuum, a linearized gravitational wave can be transformed into the transverse and traceless gauge. Then we have the following line element and corresponding orthonormal frame basis:

$$ds^2 = -c^2dt^2 + [1 + h_+(\chi)]dx^2 + [1 - h_+(\chi)]dy^2 + 2h_\times(\chi)dxdy + dz^2, \tag{2}$$

$$\mathbf{e}_0 \equiv c^{-1}\partial_t, \qquad \mathbf{e}_1 \equiv (1 - \tfrac{1}{2}h_+)\partial_x - \tfrac{1}{2}h_\times\partial_y, \tag{3}$$

$$\mathbf{e}_2 \equiv (1 + \tfrac{1}{2}h_+)\partial_y - \tfrac{1}{2}h_\times\partial_x, \qquad \mathbf{e}_3 \equiv \partial_z, \tag{4}$$

where $\chi \equiv z - ct$ and $h_+, h_\times \ll 1$. Moreover, the gravitational waves take this form also in the particular case (propagation in a magnetized plasma) that we are considering. The difference to the vacuum case will be that $\chi = z - v_{\rm ph}t$, where $v_{\rm ph}$ is the phase velocity of the gravitational wave. Note, however, that the theory will be limited to the case of small deviation from the vacuum dispersion relation (i.e. we will have $v_{\rm ph} \approx c$), and to the high-frequency approximation. From now on we will refer to tetrad components rather than coordinate components.

We follow the approach presented in [32] for handling gravitational effects on the electromagnetic and material fields. Suppose an observer moves with 4-velocity u^a. This observer will measure the electric and magnetic fields $E_a \equiv F_{ab}u^b$ and $B_a \equiv \frac{1}{2}\epsilon_{abc}F^{bc}$, respectively, where F_{ab} is the electromagnetic field tensor and ϵ_{abc} is the volume element on hypersurfaces orthogonal to u^a. It is convenient to introduce a 3-vector notation $\mathbf{E} \equiv (E^i) = (E^1, E^2, E^3)$ etc. and $\nabla \equiv \mathbf{e}_i$. Note that for the spatial components we can raise and lower indices with the Kronecker delta. From now on we will assume that $u^0 = c$ is the only nonzero component of u^a. As has been presented in e.g. Ref. [19] the Maxwell equations contain terms coupling the electromagnetic field to the gravitational radiation field. Including terms that are linear in h_+ and $h_\times$ [33], Maxwell's equations are written as

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\partial_t\mathbf{E} + \mu_0(\mathbf{j} + \mathbf{j}_E), \tag{5}$$

$$\nabla \times \mathbf{E} = -\partial_t\mathbf{B} - c\mu_0\mathbf{j}_B, \tag{6}$$

$$\nabla \cdot \mathbf{B} = \frac{\rho_B}{c\varepsilon_0}, \tag{7}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\varepsilon_0} + \frac{\rho_E}{\varepsilon_0}, \tag{8}$$

where $\mathbf{j}_E$, $\mathbf{j}_B$, ρ_E, and ρ_B are effective current and charge densities due to the GWs; see e.g. Ref. [7]. For the particular case of linearized theory, and a background magnetic field directed as

$$\mathbf{B}_0 = B_0(\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_3), \tag{9}$$

the effective charge densities vanish and the effective currents reduce to

$$\mathbf{j}_E = \frac{B_0\sin\theta}{2\mu_0}(\dot{h}_\times\mathbf{e}_1 - \dot{h}_+\mathbf{e}_2), \tag{10}$$

$$\mathbf{j}_B = \frac{B_0\sin\theta}{2\mu_0}\frac{v_{\rm ph}}{c}(\dot{h}_+\mathbf{e}_1 + \dot{h}_\times\mathbf{e}_2), \tag{11}$$

where the dot denotes derivatives with respect to χ. The physical current and charge density are denoted $\mathbf{j}$ and ρ_c, respectively.

Next we need an evolution equation for particles with mass m and charge q. We then apply kinetic plasma theory, representing each particle species by a distribution function f governed by the Vlasov equation. In tetrad form the Vlasov equation reads [19]

$$\mathcal{L}f = 0,$$

where the Liouville operator is

$$\mathcal{L} \equiv \partial_t + (c/p^0)p^ie_i + [F^i_{\rm EM} - \Gamma^i{}_{ab}p^ap^bc/p^0]\partial_{p^i},$$

and the electromagnetic force responsible for geodesic deviation is $F^i_{\rm EM} \equiv q(E^i + \epsilon^{ijk}p_jB_k/\gamma m)$. In vector notation the Vlasov equation reads

$$\partial_t f + \frac{\mathbf{p}\cdot\nabla f}{\gamma m} + \left[q\left(\mathbf{E} + \frac{\mathbf{p}\times\mathbf{B}}{\gamma m}\right) - \mathbf{G}\right]\cdot\nabla_{\mathbf{p}}f = 0, \tag{12}$$

where $\nabla_{\mathbf{p}} \equiv (\partial_{p_1}, \partial_{p_2}, \partial_{p_3})$ and $\gamma = \sqrt{1 + p_ip^i/(mc)^2}$. In the absence of gravitational waves, the Vlasov equation has the following spatially homogeneous (thermodynamical) equilibrium solution, the Synge-Jüttner distribution, e.g. [34],

$$f_{\rm SJ} = \frac{n_0\mu}{4\pi(mc)^3K_2(\mu)}e^{-\mu\gamma}, \tag{13}$$

where n_0 is the spatial particle number density, $\mu \equiv mc^2/k_BT$, k_B is the Boltzmann constant, T the temperature, and $K_2(\mu)$ is a modified Bessel function of second kind. The Synge-Jüttner distribution is a straightforward extension of a Maxwellian distribution, obtained by replacing the classical kinetic energy with its relativistic counterpart. However, for the remainder of this paper we will consider a slightly more general isotropic distribution $f_0 = f_0(p^2)$, as a background distribution rather than the specific case given in Eq. (13).

The gravitational force like term $\mathbf{G}$ has components $G^i \equiv \Gamma^i{}_{ab}p^ap^b/\gamma m$, where $\Gamma^i{}_{ab}$ are the Ricci rotation coefficients. For a linearized GW propagating in the z direction in Minkowski space this becomes

$$G_1 = \tfrac{1}{2}(v_{\rm ph} - p_3/\gamma m)[p_1\dot{h}_+ + p_2\dot{h}_\times], \quad (14)$$

$$G_2 = \tfrac{1}{2}(v_{\rm ph} - p_3/\gamma m)[-p_2\dot{h}_+ + p_1\dot{h}_\times], \quad (15)$$

$$G_3 = \tfrac{1}{2}(\gamma m)^{-1}[(p_1^2 - p_2^2)\dot{h}_+ + 2p_1p_2\dot{h}_\times], \quad (16)$$

where $v_{\rm ph}$ is the phase velocity of the gravitational wave, which we allow to deviate slightly from c.

III. SOLUTION OF THE VLASOV EQUATION

Next we need to solve the linearized Vlasov equation. Dividing the magnetic field and the distribution function into its perturbed and unperturbed parts, $f = f_0(p^2) + f_1(\mathbf{p})\exp[i(kz - \omega t)]$, $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1\exp[i(kz - \omega t)]$), i.e. looking for plane wave solutions induced by the GW propagating at an angle to the constant magnetic field $\mathbf{B}_0$, this reads

$$-i\omega f_1 + \frac{i\mathbf{p}\cdot\mathbf{k}f_1}{\gamma m} + \frac{q\mathbf{p}\times\mathbf{B}_0}{\gamma m}\cdot\nabla_{\mathbf{p}}f_1 = -[q\mathbf{E} - \mathbf{G}]\cdot\nabla_{\mathbf{p}}f_0, \quad (17)$$

where the term proportional to $\mathbf{p}\times\mathbf{B}_1$ of the right-hand side has been dropped, which is valid since we limit ourselves to an isotropic background distribution function. In order to solve Eq. (17) we adapt cylindrical coordinates in momentum space $(p_\perp, \phi, p_\parallel)$, with the cylinder axis directed along the unperturbed magnetic field. In this coordinate system Eq. (17) can be written

$$-i\left(\omega - \frac{\mathbf{p}\cdot\mathbf{k}}{\gamma m} - \omega_c\partial_\phi\right)f_1 = -[q\mathbf{E} - \mathbf{G}]\cdot\nabla_{\mathbf{p}}f_0. \quad (18)$$

where $\omega_c = qB_0/m$ is the cyclotron frequency. Equation (18) can be solved using an expansion in the eigenfunctions

$$\psi_n(\phi, p_\perp) = \frac{1}{\sqrt{2\pi}}\exp[-i(n\phi - k_\perp p_\perp\sin\phi/\gamma m\omega_c)], \quad (19)$$

where $k_\perp = k\sin\theta$. Useful properties of these eigenfunctions are the orthogonality relation

$$\int_0^{2\pi}\psi_n\psi_m^* d\phi = \delta_{nm}, \quad (20)$$

where $*$ denotes complex conjugate, and the Bessel function expansion

$$\psi_m(\phi, p_\perp) = \sum_{n=-\infty}^{\infty} J_n\left(\frac{k_\perp p_\perp}{\gamma m\omega_c}\right)\exp[i(n - m)\phi]. \quad (21)$$

Next we divide $f_1 = f_{em} + f_{gw}$ and use the orthogonality property to expand f_{em} and f_{gw} as $f_{em} = \sum W_n\psi_n$ and $f_{gw} = \sum G_n\psi_n$, where the electromagnetically induced perturbation W_n is

$$-i\left(\omega - \frac{p_\parallel k_\parallel}{\gamma m} - \frac{n\omega_c}{\gamma}\right)W_n = \int_0^{2\pi}\psi_n^* q\mathbf{E}\cdot\nabla_{\mathbf{p}}f_0 d\phi, \quad (22)$$

where $k_\parallel = k\cos\theta$. Dividing W_n further as $W_n = W_n^i E_i$, Eq. (22) gives W_n^i from the components of

$$\mathbf{W}_n \equiv \frac{-i4\pi q}{(\omega - \frac{p_\parallel k_\parallel}{\gamma m} - \frac{n\omega_c}{\gamma})}\frac{\partial f_0}{\partial p^2}\mathbf{V}_{\rm op}J_n(\xi), \quad (23)$$

where $\xi = k_\perp p_\perp/m\omega_c$ and the vector operator $\mathbf{V}_{\rm op}$ is

$$\mathbf{V}_{\rm op} \equiv \begin{pmatrix} p_\parallel\sin\theta + \frac{np_\perp\cos\theta}{\xi} \\ -ip_\perp\frac{\partial}{\partial\xi} \\ p_\parallel\cos\theta - \frac{np_\perp\sin\theta}{\xi} \end{pmatrix}. \quad (24)$$

The gravitationally induced perturbation G_n can be further divided into its two GW polarizations as $G_n = g_n^+ h_+ + g_n^\times h_\times$ with

$$-i\left(\omega - \frac{p_\parallel k_\parallel}{\gamma m} - \frac{n\omega_c}{\gamma}\right)g_n^+ h_+ \equiv \int_0^{2\pi}\psi_n^*\mathbf{G}^+\cdot\nabla_{\mathbf{p}}f_0 d\phi, \quad (25)$$

which after integration gives

$$g_n^+ = \frac{\pi\omega}{(\omega - \frac{p_\parallel k_\parallel}{\gamma m} - \frac{n\omega_c}{\gamma})}\frac{\partial f_0}{\partial p^2}\Big\{(2p_\parallel^2 - p_\perp^2)\sin^2\theta J_n(\xi) + \frac{4n}{\xi}p_\parallel p_\perp\sin\theta\cos\theta J_n(\xi) + p_\perp^2(1 + \cos^2\theta)\times\left(2\frac{\partial^2 J_n(\xi)}{\partial\xi^2} + J_n(\xi)\right)\Big\}, \quad (26)$$

and

$$-i\left(\omega - \frac{p_\parallel k_\parallel}{\gamma m} - \frac{n\omega_c}{\gamma}\right)g_n^\times h_\times \equiv \int_0^{2\pi}\psi_n^*\mathbf{G}^\times\cdot\nabla_{\mathbf{p}}f_0 d\phi \quad (27)$$

which leads to

$$g_n^\times = \frac{4\pi i\omega}{(\omega - \frac{p_\parallel k_\parallel}{\gamma m} - \frac{n\omega_c}{\gamma})}\frac{\partial f_0}{\partial p^2}\Big\{p_\perp p_\parallel\sin\theta\frac{\partial J_n(\xi)}{\partial\xi} + \frac{n}{\xi}p_\perp^2\cos\theta\left(\frac{\partial J_n(\xi)}{\partial\xi} - \frac{J_n(\xi)}{\xi}\right)\Big\}, \quad (28)$$

where $\mathbf{G} = \mathbf{G}^+ + \mathbf{G}^\times$ in Eqs. (25) and (27) has been divided into its two polarization components, proportional to h_+ and $h_\times$, respectively, as defined from (14)–(16). The solutions of the integrals over the azimuthal angle have been performed using the Bessel expansion (21).

IV. THE COUPLED WAVE EQUATIONS

Next we use Maxwell's equations and the expressions for the perturbed distribution functions from the previous section. Carrying out integrations over the azimuthal angles to find the current densities induced by the EMW and the GW, we arrive at the wave equation for the EM field with GW source terms

$$[(\omega^2 - k^2c^2)\delta^{ij} + k^ik^jc^2 + i\omega\mu_0\sigma^{ij}]E_j = (D^i_+ + I^i_+)h_+ + (D^i_\times + I^i_\times)h_\times, \qquad (29)$$

where σ^{ij} is the standard conductivity tensor for a relativistic Vlasov plasma (see e.g. Ref. [35]), $D^i_{+,\times}$, which is due to the effective current densities in (11) and (10), is given by

$$D^i_+ = \omega k B_0 \sin\theta \delta^i_2, \qquad (30)$$

$$D^i_\times = -\omega k B_0 \sin\theta \delta^i_1, \qquad (31)$$

whereas $I^i_{+,\times}$ (due to the current induced by the GW-force density) can be written in vector form as

$$\mathbf{I}_{+,\times} = \sum_{\mathrm{PS}} 2\pi \int \frac{dp_\perp dp_\parallel p_\perp}{\gamma m} \sum_n g_n^{+,\times} \mathbf{V}^*_{\mathrm{op}} J_n(\xi), \qquad (32)$$

where $\sum_{\mathrm{PS}}$ denotes a sum over particle species. Note that the same vector operator $\mathbf{V}_{\mathrm{op}}$ that determined the electromagnetic part of the distribution function in (23) appears in the GW source (albeit here with a conjugate).

In principle one could proceed from (29) by multiplying with the adjoint of the electromagnetic dispersion matrix of the left-hand side to solve for the electromagnetic field amplitudes of the various components. Since σ^{ab} is Hermitian (at least to a good approximation, in case the pole contribution associated with wave-particle interaction processes are small), the adjoint of the matrix is built up from the eigenvectors divided by the determinant of the original matrix. However, due to the complex mode structure in a magnetized plasma, such a procedure would not be very illuminating in the general case, and thus we will wait with further simplifications of (29) until a particular parameter regime is chosen.

Next we consider the EM sources in the wave equation for the GW. From (1) we obtain

$$(\omega^2 - c^2k^2)h_+ = -c^2\kappa(\delta T_{11} - \delta T_{22}), \qquad (33)$$

$$(\omega^2 - c^2k^2)h_\times = -2c^2\kappa\delta T_{12}. \qquad (34)$$

The energy-momentum tensor has sources from the perturbed distribution function, due to the EM- and GW-force density, which is found from the solution for the Vlasov equation. Furthermore, there are sources due to the perturbations of the electromagnetic field tensor. Including these three sources, Eqs. (33) and (34) are written

$$(\omega^2 - c^2k^2)h_+ = -c^2\kappa(F^i_+E_i + 2Fh_+ + H^+_+h_+ + H^\times_+h_\times + H^i_+E_i), \qquad (35)$$

$$(\omega^2 - c^2k^2)h_\times = -2c^2\kappa(F^i_\times E_i + Fh_\times + H^+_\times h_+ + H^\times_\times h_\times + H^i_\times E_i), \qquad (36)$$

where

$$F^i_+ = -4\frac{k}{\mu_0\omega}B_0\sin\theta\delta^i_2, \qquad (37)$$

$$F^i_\times = 2\frac{k}{\mu_0\omega}B_0\sin\theta\delta^i_1, \qquad (38)$$

$$F = \frac{B_0^2}{\mu_0}\sin^2\theta, \qquad (39)$$

$$H^{+,\times}_+ = \sum_{\mathrm{PS}}\pi\int\frac{dp_\perp dp_\parallel}{\gamma m}\sum_n g_n^{+,\times}Q_+J_n(\xi), \qquad (40)$$

$$H^{+,\times}_\times = \sum_{\mathrm{PS}}i\pi\int\frac{dp_\perp dp_\parallel}{\gamma m}\sum_n g_n^{+,\times}Q_\times J_n(\xi), \qquad (41)$$

$$H^i_+ = \sum_{\mathrm{PS}}\pi\int\frac{dp_\perp dp_\parallel}{\gamma m}\sum_n W^i_nQ_+J_n(\xi), \qquad (42)$$

$$H^i_\times = \sum_{\mathrm{PS}}i\pi\int\frac{dp_\perp dp_\parallel}{\gamma m}\sum_n W^i_nQ_\times J_n(\xi), \qquad (43)$$

$$Q_+ \equiv p_\perp\left(2p_\parallel^2 + p_\perp^2\cos^2\theta + \frac{4np_\perp p_\parallel}{\xi}\sin\theta\cos\theta\right) + 2p_\perp^3(1+\cos^2\theta)\frac{\partial^2}{\partial\xi^2}, \qquad (44)$$

$$Q_\times \equiv 2p_\parallel p_\perp^2\sin^2\theta\frac{\partial}{\partial\xi} + \frac{n}{\xi}p_\perp^3\cos\theta\left(\frac{\partial}{\partial\xi} - \frac{1}{\xi}\right). \qquad (45)$$

Equations (35) and (36) are GW wave equations with electromagnetic source terms, which together with the counterpart (29) constitute a closed set. Solving for $h_{+,\times}$ using (35) and (36) and substituting into (29) gives a single dispersion relation for coupled GW-EMWs propagating in a magnetized plasma. Our coupled set has the advantage having solved the integration over the azimuthal angle everywhere, and allowing for a fully relativistic temperature. However, due to the complexity of the eigenmodes of a magnetized plasma, the properties of the dispersion relation will be very far from explicit, until specific assumptions about the parameter regime are made. This will be explored in the next section.

V. SPECIAL CASES

In principle the general result (29) combined with (35) and (36) could be simplified further, by multiplying (29) with the adjoint of the matrix $(\omega^2 - k^2c^2)\delta^{ij} + k^ik^jc^2 + i\mu_0\omega\sigma^{ij}$, to solve for the electric field with the gravitational terms as source terms for EM radiation. However, in practice this would not be particularly helpful when evaluating specific cases. This is related to the asymmetry between the GWs and the EMWs that comes from the necessity to use the short-wave approximation.

Because of the short-wave approximation the influence of the energy-momentum tensor on the GWs is limited, and the properties of the GWs are known to a good approximation, in particular, the polarization is approximately the same as in vacuum. This is in contrast to the EMWs, where the wave polarization may differ a lot, depending on the parameter regime. Further simplifications of the system (29), (35), and (36) are preferably done after a parameter regime is chosen, and the wave polarization of free EMWs thereby is known.

For moderate energy density in the background state, the modification of the GWs from the pure vacuum case is typically quite small. Hence the case of most interest is when the matter contribution to the GW dispersion relation is enhanced due to some sort of resonance. The physics of this can often be contained in only a few of the various GW-EMW coupling terms, which can simplify the full dispersion relation significantly. Several different possibilities of this kind exist, which is best illustrated by a set of examples.

A. Parallel propagation—GW cyclotron resonances

First we consider the case of wave propagation parallel to the external magnetic field, i.e. we put $\theta = 0$ in all formulas. This leads to large simplifications, as all the GW-source terms in the EM wave equation (29) are seen to vanish. Inspection of Eqs. (35) and (36) then shows that all contributions to the GW dispersion relation comes from the perturbation of the distribution function directly induced by the GW, which is encoded in the H coefficients (40)–(43). Evaluating these coefficients for $\theta = 0$, the argument of the Bessel function becomes zero, and as a consequence we find that the only terms in the sum over n that survive are $n = \pm 2$. Furthermore, from (35) and (36) we see that the h_+ and $h_\times$ are coupled, such as to make circularly polarized modes with combinations $h_+ \pm i h_\times$ the natural variables. Specifically we obtain from (35) and (36)

$$(\omega^2 - k^2c^2 + M_\pm)(h_+ \pm i h_\times) = 0, \qquad (46)$$

with

$$M_\pm = \frac{\pi\kappa c^2 k^2}{\omega}\sum_{\mathrm{PS}}\int \frac{p_\perp^3 dp_\perp dp_\parallel}{2m(kp_\parallel/m - \gamma\omega \mp 2\omega_c)} \times\left[\left(\frac{\omega}{k} - \frac{p_\parallel}{\gamma m}\right)\frac{\partial}{\partial p_\perp} + \frac{p_\perp}{\gamma m}\frac{\partial}{\partial p_\parallel}\right] f_0. \qquad (47)$$

The dispersion relation $(\omega^2 - k^2c^2 + M_\pm) = 0$, where the $+(-)$ stands for left- (right-) hand circular polarization, agrees with Eqs. (24) and (25) in Ref. [19], where it has been thoroughly studied. In the nonrelativistic limit we also have agreement with Ref. [18]. We will not repeat a detailed analysis of Eq. (46) here, but just point out a few of the main features of the dispersion relation.

(i) Because of the pole contributions, obtained when the denominators $kp_\parallel/m - \gamma\omega \mp 2\omega_c = 0$, the waves are typically cyclotron damped. For a low (nonrelativistic) temperature the damping is most pronounced when $\omega \simeq 2\omega_c$. The resonances occur for all particle species present in the plasma, which may be electrons, ions, or positrons. Electron and positron contributions are identical, if we just interchange right- and left-hand circular polarizations, whereas naturally the ion contribution is much different, as the ion-cyclotron time resonance is lower by orders of magnitude.

(ii) The real part of the dispersion relation also has relatively sharp peaks at the cyclotron resonances for low temperatures. However, these resonances are smoothened considerably when the temperature is increased to approach the relativistic regime.

(iii) If we let the background distribution function deviate from that of thermodynamic equilibrium, it is possible that the damping turns into wave growth, due to the presence of free energy. The necessary condition of this is discussed in some detail in Sec. V of Ref. [19].

B. Perpendicular propagation—Alfvén wave resonance

For perpendicular propagation the GW coupling to EMWs differs a lot depending on whether the plasma motion is quasineutral or not. If the gravitational wave frequency is much lower than the lowest cyclotron frequency (normally due to ions, except for an electron-positron plasma), the particle motion is essentially $\mathbf{E}\times\mathbf{B}$ drifts to leading order for all particle species, in which case the current (due to the polarization drift) is a small correction term, and the excited EMWs become (compressional) Alfvén waves. This low-frequency case is what we will consider in this section. For a strong magnetic field, we can expand the Bessel functions to the lowest order approximation. Noting that compressional Alfvén waves have only the E_2 component nonzero (to the leading approximation), and that the part originating from the GW-distribution function does not contribute significantly in the short Larmor radius limit, we obtain from (29)

$$(\omega^2 - k^2C_A^2)E_2 = \frac{\omega^3 B_0 h_+}{k}, \qquad (48)$$

where $C_A = c/\sqrt{1 + \sum_{\mathrm{PS}}\omega_p^2/\omega_c^2}$ is the relativistic Alfvén velocity. From (35), again noting that the kinetic terms do not contribute significantly in the short Larmor radius limit, we next obtain

$$(\omega^2 - k^2c^2)h_+ = \kappa\frac{kB_0E_2}{\omega\mu_0}. \qquad (49)$$

Combining (48) and (49) we thus obtain

$$(\omega^2 - k^2c^2) = \kappa \frac{B_0^2}{\mu_0} \frac{\omega^2}{(\omega^2 - k^2C_A^2)}. \quad (50)$$

Clearly the GW-EM coupling is much enhanced (by a factor $\sim c^2/|c^2 - C_A^2|$) in case the relativistic Alfvén velocity matches the speed of light in vacuum. Such a matching occurs for strongly magnetized plasmas of modest densities, in particular, regions around (double) pulsars can be of interest. Applications of Alfvén wave resonances have been considered e.g. by Refs. [16,17], where nonlinearities also have been taken into account.

C. Oblique propagation—plasma frequency resonance

We next consider an electron-ion plasma, and let the wave frequency ω be much higher than the ion-plasma frequency. For propagation at an arbitrary angle θ, this allows us to neglect the ion dynamics when the EM-wave properties are evaluated. The specific case of excitation of high-frequency plasma waves when the GW propagates at an arbitrary angle to the external magnetic field has not been considered previously, as far as we know. To obtain simple formulas, we now focus on the low-temperature limit, $(k_BT/m_e)^{1/2} \ll \omega/k_z$ and $k_\perp v_t/\omega_c \ll 1$. From (33) evaluated in the low-temperature limit we obtain

$$(\omega^2 - k^2c^2)h_+ = \kappa \frac{\sin\theta B_0 k}{\omega \mu_0} E_2. \quad (51)$$

From (29) and still considering the low-temperature limit, we obtain

$$(\omega^2 - \omega_p^2)E_2 = \omega_c^2 h_+ c B_0 \sin^3\theta h_+. \quad (52)$$

In deriving (52), we have assumed that the coupled GW-EMW has most of its energy in the gravitational degrees of freedom (i.e. that we consider a wave that basically is a GW inducing EM perturbation, rather than the other way around), which is fulfilled if

$$|\kappa B_0^2 \omega_c^2 \sin^4\theta/(\omega^2 - \omega_p^2)| \ll |(\omega^2 - \omega_p^2)|. \quad (53)$$

It may seem somewhat unexpected that the transverse electric field component E_2 experiences a resonance of the same type as electrostatic and longitudinal plasma oscillations. However, this property is a basic consequence of the EM conductivity tensor [35]. It follows from the coupling between the longitudinal and transverse components of the electric field induced by the magnetic field, together with the fact that in a (cold) magnetized plasma for $\theta \neq 0$, high-frequency EMWs have a phase velocity $\omega/k = c$, precisely when $\omega = \omega_p$. Combining Eqs. (51) and (52) we obtain the dispersion relation

$$\omega^2 - k^2c^2 = \kappa \frac{B_0^2}{\mu_0} \frac{\omega_c^2 \sin^4\theta}{(\omega^2 - \omega_p^2)}. \quad (54)$$

The coupling in Eq. (54) is strong close to the resonance $\omega^2 \simeq \omega_p^2$, but as pointed out above, we cannot approach the resonance too close while applying this dispersion relation, due to the condition (53). Furthermore, for a given distance from the exact resonance, the coupling strength increases with ω_c^2/ω_p^2. As long as $\omega \gtrsim \omega_{ci}$, the omission of ion dynamics in comparison with the electron dynamics can be justified, and hence we can apply formula (54) until ω_c^2/ω_p^2 is of the order of m_i^2/m_e^2. The main limitation of the applicability of (54) comes from the fact that it requires a constant density over several wavelengths, to benefit from the resonance $\omega^2 \simeq \omega_p^2$. Close to suitable GW sources, however, most plasmas are strongly inhomogeneous. This is somewhat unfortunate, as the strong scaling with B_0 in the dispersion relation (to the fourth power) makes the region close to astrophysical objects like pulsars and/or magnetars the most interesting ones when it comes to applying (54). For the interstellar medium, where the density might be approximately constant for long distances, the typical magnitude of the external magnetic field is normally too weak to make the GW modification described by (54) very significant.

VI. SUMMARY AND DISCUSSION

We have derived a set of coupled wave equations for EMWs driven by GWs and vice versa, in a magnetized relativistic Vlasov plasma. The coupling terms contain integrations in momentum space over various moments of the distribution function. Although integration of the azimuthal angles in momentum space has been carried out, we note that the remaining integrals can only be done numerically. Focusing on different special cases, many of the properties of the coupled equations can be deduced analytically, and explicit forms of the dispersion relations can be derived governing the coupled GW-EMW propagation. Writing the GW dispersion relation as $\omega^2 - k^2c^2 = S(\omega, \mathbf{k})$, we note that typically the modification from the vacuum case can be estimated as $S \sim c^2R^{-2}$, where R is the characteristic radius of curvature associated with the energy density of the unperturbed plasma. However, there are several types of resonance phenomena that can enhance the influence of the plasma on the GW dispersion relation far beyond this generic estimate. These includes cyclotron resonances, Alfvén wave resonances, and plasma frequency resonances. We also note that these resonances can be strongly dependent on the GW polarization. While resonances can affect the GW dispersion relation, as described for three different cases in Sec. V, it is not obvious that it is enough to have observable consequences, due to the small value of the coupling parameter κ. Let us illustrate this with a few examples. Specifically, let us consider the case of merging of a compact binary, where at least one of the objects is strongly magnetized, i.e. a pulsar or a magnetar. A GW-EMW interaction can then reveal itself in case the arrivals of different parts of the frequency spectrum are slightly modified, compared to what we obtain from a pure vacuum dispersion relation.

For a small matter contribution to the dispersion relation, we have $\Delta t(\omega)/T_p = \Delta v_g(\omega)/c$, where $\Delta t(\omega)$ is the modification in arrival time (compared to the pure vacuum dispersion relation) as a function of frequency, T_p is the propagation time in the region which contributes to the time delay, and $\Delta v_g(\omega)$ is the modification of the group velocity due to the plasma interaction. For matter effects to be visible, we estimate that $\Delta t(\omega)$ must be larger or comparable to a resolution time $T_{\rm res}$ that is of the order of the characteristic time scale for the frequency increase, i.e. characteristic $\Delta t \gtrsim T_{\rm res} \sim \omega/(d\omega/dt)$, where $d\omega/dt$ is the frequency increase rate of the binary due to the loss of gravitational wave energy. Calculating the group velocity from Eq. (54), assuming that we are close to the plasma resonance (i.e. $|\omega^2 - \omega_p^2| \ll \omega^2$), the condition of a detectable time delay becomes

$$\frac{T_{\rm res}}{0.01\ {\rm s}} \lesssim \left(\frac{B_0}{2\times 10^{-5}\ {\rm T}}\right)^4 \left(\frac{1({\rm rad/s})^2}{\omega^2 - \omega_p^2}\right)^2 \left(\frac{T_p}{10^{14}\ {\rm s}}\right). \qquad (55)$$

The normalization factors picked here for illustrative purposes show that GW detection through a slowly accumulated time delay which varies with the GW frequency is not possible through the considered coupling mechanism. While a propagation time $T_p = 10^{14}$ corresponding to 100×10^6 ly might be possible, and the frequency mismatch factor (measuring the deviation from precise resonance $\omega^2 = \omega_p^2$) and time resolution factors are not out of the question, the trouble fulfilling the condition comes from the magnetic field factor. Over intergalactic distances a value $B_0 \sim 10^{-12}$ T could be possible, rather than 2×10^{-5} T that we need to fulfill the detection condition (unless all the other factors can be improved by several orders of magnitude). Another regime of detection that might be possible in principle is a much stronger GW-EMW interaction, where the background energy density is very high. Naturally in such a case we cannot benefit from a large accumulation distance. Furthermore, for strong magnetic fields we cannot use Eq. (54), since the condition for neglecting ion dynamics cannot be fulfilled. Instead we consider the dispersion relation (50) applicable for $\omega \ll \omega_{ci}$, and calculate the group velocity under the condition that $C_A^2 \simeq c^2$, applicable for pulsar or magnetar magnetic field strengths. Applying our detection estimate for this regime, we obtain

$$\frac{T_{\rm res}}{0.01\ {\rm s}} \lesssim \left(\frac{B_0}{10^{10}\ {\rm T}}\right)\left(\frac{1000\ {\rm rad/s}}{\omega}\right)\left(\frac{T_p}{10\ {\rm s}}\right). \qquad (56)$$

Here the considered normalization factor for $T_{\rm res}$ is still reasonable, and the value for B_0 is certainly possible, since it corresponds to observed magnetar field strengths. However, for a propagation time $T_p = 10$ s to be applicable, the magnetic field must fill a region of distance 3×10^9 m with the same high field strength, which is far too much, since the really strong field value picked in the example is applicable only to the region closest to the magnetar surface. Naturally we could get a favorable subfactor by picking a GW frequency lower than 1000 rad/s, but unfortunately $T_{\rm res}$ is rapidly increasing in case such a choice is made [36]. Thus it is difficult to fulfill the detection condition also in this regime. This does not necessarily mean that all possibilities of detectable GW-EMW interaction have been exhausted, as the general formulas presented in Sec. IV contain many other possibilities. It should be pointed out that Ref. [10] has made the case that the cyclotron type of resonances may be strong enough such as to have observable consequences, although there has been some debate over this [19]. Finally we stress that issues of this type will need development of high precision GW detectors [2,37] to be settled.

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[1] L. P. Grishchuk and A. G. Polnarev, *General Relativity and Gravitation*, edited by A. Held (Plenum Press, New York, 1980), Vol. 2, pp. 416–430.
[2] M. Maggiore, Phys. Rep. **331**, 283 (2000).
[3] M. Marklund, G. Brodin, and P. Dunsby, Astrophys. J. **536**, 875 (2000).
[4] G. Brodin, M. Marklund, and M. Servin, Phys. Rev. D **63**, 124003 (2001).
[5] Yu G. Ignat'ev, Phys. Lett. A **230**, 171 (1997).
[6] G. Brodin and M. Marklund, Phys. Rev. Lett. **82**, 3012 (1999).
[7] G. Brodin, M. Marklund, and P. K. S. Dunsby, Phys. Rev. D **62**, 104008 (2000).
[8] D. Papadopoulos, N. Stergioulas, L. Vlahos, and J. Kuijpers, A&A International **377**, 701 (2001).
[9] A. B. Balakin, V. R. Kurbanova, and W. Zimdahl, J. Math. Phys. (N.Y.) **44**, 5120 (2003).
[10] L. Vlahos, G. Voyatzis, and D. Papadopoulos, Astrophys. J. **604**, 297 (2004).
[11] M. D. Duez, Y. T. Liu, S. L. Shapiro, and B. C. Stephens, Phys. Rev. D **72**, 024028 (2005).
[12] M. D. Duez, Y. T. Liu, S. L. Shapiro, and B. C. Stephens, Phys. Rev. D **72**, 024029 (2005).
[13] H. Isliker, I. Sandberg, and L. Vlahos, Phys. Rev. D **74**, 104009 (2006).

[14] B. D. Farris, T. K. Li, Y. T. Liu, and S. L. Shapiro, Phys. Rev. D **78**, 024023 (2008).
[15] M. Servin and G. Brodin, Phys. Rev. D **68**, 044017 (2003).
[16] H. Isliker, I. Sandberg, and L. Vlahos, Phys. Rev. D **74**, 104009 (2006).
[17] M. Forsberg and G. Brodin, Phys. Rev. D **77**, 024050 (2008).
[18] P. G. Macedo and H. Nelson, Phys. Rev. D **28**, 2382 (1983).
[19] M. Servin, G. Brodin, and M. Marklund, Phys. Rev. D **64**, 024013 (2001).
[20] J. Moortgat and J. Kuijpers, Astron. Astrophys. **402**, 905 (2003).
[21] J. Moortgat and J. Kuijpers, Phys. Rev. D **70**, 023001 (2004).
[22] M. Marklund, P. K. S. Dunsby, and G. Brodin, Phys. Rev. D **62**, 101501(R) (2000).
[23] C. Tsagas, Classical Quantum Gravity **22**, 393 (2005).
[24] M. Giovannini, Classical Quantum Gravity **23**, R1 (2006).
[25] J. D. Barrow, R. Maartens, and C. G. Tsagas, Phys. Rep. **449**, 131 (2007).
[26] E. Fenu and R. Durrer, Phys. Rev. D **79**, 024021 (2009).
[27] H. J. M. Cuesta, Phys. Rev. D **65**, 064009 (2002).
[28] J. Moortgat and J. Kuijpers, Mon. Not. R. Astron. Soc. **368**, 1110 (2006).
[29] F. Y. Li and M. X. Tang, Int. J. Mod. Phys. D **11**, 1049 (2002).
[30] R. Ballantini *et al.*, Classical Quantum Gravity **20**, 3505 (2003).
[31] M. Servin, G. Brodin, M. Marklund, and M. Bradley, Phys. Rev. E **62**, 8493 (2000).
[32] G. F. R. Ellis and H. van Elst, *Cosmological Models, Theoretical and Observational Cosmology*, edited by M. Lachieze-Rey (Kluwer, Dordrecht, 1999).
[33] Note that EMW fields are induced by the GWs, hence terms such as $h_{+,\times}\partial_x E_i$ are quadratic in $h_{+,\times}$.
[34] R. L. Liboff, *Kinetic Theory* (Wiley, New York, 1977).
[35] D. G. Swanson, *Plasma Waves* (Taylor & Francis, London, 2003).
[36] The estimate $T_{\rm res} \sim \omega/(d\omega/dt)$ implies that $T_{\rm res}$ is a function of the binary frequency and decreases the closer the system is to the final merging. Picking $T_{\rm res}$ as 0.01 s means that we consider the radiation emitted during the last stage before the collapse when the system still has a clear peak in the GW spectrum.
[37] See e.g. the LISA (laser interferometer space antenna) home page: http://lisa.jpl.nasa.gov/.