Unitarity analysis of general Born-Infeld gravity theories

İbrahim Güllü,[*] Tahsin Çağrı Şişman,[†] and Bayram Tekin[‡]

Department of Physics, Middle East Technical University, 06531, Ankara, Turkey

(Received 13 October 2010; published 10 December 2010)

We develop techniques of analyzing the unitarity of general Born-Infeld gravity actions in D-dimensional spacetimes. The determinantal form of the action allows us to find a compact expression quadratic in the metric fluctuations around constant curvature backgrounds. This is highly nontrivial since for the Born-Infeld actions, in principle, infinitely many terms in the curvature expansion should contribute to the quadratic action in the metric fluctuations around constant curvature backgrounds, which would render the unitarity analysis intractable. Moreover in even dimensions, unitarity of the theory depends only on finite number of terms built from the powers of the curvature tensor. We apply our techniques to some four-dimensional examples.

DOI: 10.1103/PhysRevD.82.124023 PACS numbers: 04.50.−h

I. INTRODUCTION

Tree-level unitarity analysis, that is tachyon and ghost freedom, of a generic gravity model with arbitrary powers of the curvature tensors around a constant (nonzero) curvature background is a nontrivial problem. On the other hand, for flat backgrounds, only the quadratic terms contribute to the propagators, and therefore the analysis is rather simple. In fact, in four dimensions the only unitary model, apart from the Einstein's gravity, is the $R+\alpha R^2$ theory at the quadratic order. But, this model is not renormalizable without a $\beta R^2_{\mu\nu}$ term which, when augmented to the action, ruins unitarity by introducing a massive ghost mode [1].

Experience from quantum field theory dictates that at high energies Einstein's gravity should be replaced with a theory that has higher powers of various curvature tensors symbolically written in the form

$$I=\int d^4x\left\{\frac{1}{\kappa}(R-2\Lambda_0)+\sum_{n=2}^{\infty}a_n(\mathrm{Riem},\mathrm{Ric},R,\nabla\mathrm{Riem},\ldots)^n\right\}. \tag{1}$$

The main nontrivial question is how to find the correct couplings a_n that yield a viable unitary theory. One might view gravity as a low energy approximation to a microscopic theory such as string theory and thus expect to find a unitary (but not necessarily renormalizable) gravity theory to any desired order in the curvature by perturbatively computing a_n. Of course beyond quadratic order this is a very difficult computational problem. Another approach is the so called asymptotically safe gravity which conjectures that the dimensionless versions of all the coupling constants in (1) have a nontrivial UV fixed point and even for infinitely many coupling constants the theory has predictive power since the critical surface is finite dimensional [2–4]. In this work, encouraged by our recent observation in three dimensions that we briefly summarize below, we take a different route and propose that certain Born-Infeld (BI) type gravity actions might define unitary models to all orders. Unitarity analysis around constant curvature backgrounds is itself a complicated problem when many powers of curvature tensors are involved, here we develop the techniques of carrying out this analysis in detail and provide two nonunitary examples in four dimensions. In subsequent work [5], we will give more examples in three and four dimensions that are unitary.

Let us recapitulate the three-dimensional BI gravity action [6] and its success of reproducing the known viable theories:

$$I_{\mathrm{BINMG}}=-\frac{4m^2}{\kappa^2}\int d^3x\left[\sqrt{-\det\left(g-\frac{1}{m^2}G\right)}-\left(1-\frac{\lambda}{2}\right)\sqrt{-\det g}\right], \tag{2}$$

where the components of the matrix G read as $G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$. In (2), we have referred to this model as the BI extended new massive gravity (BINMG), since in small curvature expansion, this model reproduces the cosmological Einstein-Hilbert theory at the first order, the new massive gravity theory [7], which is unitary [8–11], at the second order and the extended new massive gravity based on the existence of the holographic c functions at the cubic and fourth orders [6,12]. With the help of the techniques we develop below, we have shown that BINMG is a unitary theory at all orders around flat and constant curvature vacua [5]. One of course would like to find analogs of (2) in higher, especially in four, dimensions. To be able to do this, one has to first establish tools for the unitarity analysis which is the purpose of this work. In what follows, for the sake of generality, we will keep the discussion in D dimensions and for generic BI actions with the only

[*]e075555@metu.edu.tr
[†]sisman@metu.edu.tr
[‡]btekin@metu.edu.tr

restriction that they reproduce the (cosmological) Einstein-Hilbert theory at the first order.

The history of the BI-type actions is quite rich and for the nongravitational cases a nice review was given in [13]. As for gravity, BI-type gravitational actions actually precedes a decade their counterparts in electrodynamics. It was Eddington who first proposed that, at least in the absence of matter, using the connection as the independent variable, Einstein-Hilbert action can be replaced by $I = \int d^4x\sqrt{\det R_{\mu\nu}(\Gamma)}$ [14]. (Note that one actually has to dig this result out from Eddington's book, since it is not clearly stated in one place. But, Schrödinger, attributing to Eddington, writes this action explicitly on page 113 of his book [15]). More recently, Eddington's approach (in fact a slight modification of it) was resuscitated in [16] (and the references therein) as an alternative to Big Bang cosmology without an initial singularity and with finite density. In [17], instead of Eddington's Palatini formulation, the metric formulation where the metric is the only independent variable was used in the form $I = \int d^4x\sqrt{\det(g_{\mu\nu} + \alpha R_{\mu\nu} + X_{\mu\nu})}$ and constraints such as ghost freedom on BI-type gravity actions was studied. Our work follows this line of thought and extends the unitarity analysis to constant curvature spaces. We would like to point out to some related works where BI-type gravities, their cosmological and other solutions have been studied [18–22].

The main idea of this work is to find a way to obtain the quadratic action in the *metric perturbation* of a generic BI gravity around its constant curvature vacuum, and this can be achieved either by explicitly calculating the $O(h_{\mu\nu}^2)$ action or by finding the equivalent quadratic action in the *curvature* that has the same propagator with the original action. Once the equivalent quadratic theory is known unitarity analysis follows with the conventional methods described in [10]. To facilitate understanding and show what is to be expected, let us give one of our results here. Let $A_{\mu\nu}$ be an *arbitrary* (0, 2) tensor built from the curvature tensors, then we will show that, in four dimensions, at $O(h_{\mu\nu})$ and $O(h_{\mu\nu}^2)$, the action

$$I = \frac{2}{\kappa\alpha}\int d^4x\Big[\sqrt{-\det(g_{\mu\nu} + A_{\mu\nu})} - (\alpha\Lambda_0 + 1)\sqrt{-\det g}\Big] \tag{3}$$

is equivalent to the simpler action

$$I_{O(A^2)} = \frac{1}{\kappa\alpha}\int d^4x\sqrt{-g}\Big(A - 2\alpha\Lambda_0 + \frac{1}{4}A^2 - \frac{1}{2}A_{\mu\nu}^2\Big), \tag{4}$$

where A is the trace of $A_{\mu\nu}$. Once this is done unitarity analysis can be carried out with the known methods which we shall not repeat in this work.

The layout of the paper is as follows: In Sec. II, second order expansions of the relevant tensors in the metric perturbation $h_{\mu\nu}$ are given. Section III is the bulk of the paper which contains our general analysis of BI gravities and the corresponding equivalent quadratic actions. We also give two examples in four dimensions in this section. Some technical details are delegated to the Appendices.

II. SECOND ORDER EXPANSIONS OF CURVATURE TENSORS

In order to study the fluctuations of generic BI actions around constant curvature backgrounds, we will need to expand various tensors up to second order in the metric perturbation $h_{\mu\nu}$ which is defined as

$$g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \tau h_{\mu\nu}, \tag{5}$$

where we introduced a small (dimensionless) parameter τ and a background metric $\bar{g}_{\mu\nu}$ which is quite generic at this stage (i.e. not necessarily constant curvature). [Taking the risk of being pedantic, let us note that (5) is exact, and that there of course does not exist a natural dimensionless parameter in gravity at all scales. So, what one actually means by (5) is that in some frame $h_{\mu\nu}$ is small compared to $\bar{g}_{\mu\nu}$ for all points in the spacetime, and since there will be another expansion, that is the curvature expansion, τ is introduced to keep track of the $h_{\mu\nu}$ orders.] Some of the computations in this section are actually somewhat tedious but straightforward. They could also be found in the literature, albeit somewhat scattered, and probably not in the form we present here which proved quite handy in our calculations that follow in the remainder of this work. The inverse metric $g^{\mu\nu}$ can be found as

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \tau h^{\mu\nu} + \tau^2 h^{\mu\rho}h_\rho^\nu + O(\tau^3). \tag{6}$$

The trace of the metric perturbation is given as $h = \bar{g}^{\mu\nu}h_{\mu\nu}$. By using these results, the second order expansion of the Christoffel connection becomes

$$\Gamma^\rho_{\mu\nu} = \bar{\Gamma}^\rho_{\mu\nu} + \tau(\Gamma^\rho_{\mu\nu})_L - \tau^2 h^\rho_\beta(\Gamma^\beta_{\mu\nu})_L + O(\tau^3), \tag{7}$$

where $\bar{\Gamma}^\rho_{\mu\nu}$ is a background metric compatible connection $\bar{\nabla}_\rho\bar{g}_{\mu\nu} = 0$ and the linearized connection $(\Gamma^\rho_{\mu\nu})_L$ is defined as

$$(\Gamma^\rho_{\mu\nu})_L \equiv \tfrac{1}{2}\bar{g}^{\rho\lambda}(\bar{\nabla}_\mu h_{\nu\lambda} + \bar{\nabla}_\nu h_{\mu\lambda} - \bar{\nabla}_\lambda h_{\mu\nu}). \tag{8}$$

The main object to consider is the Riemann tensor from which all the other curvature tensors and scalars follow. Hence, substitution of $\Gamma^\rho_{\mu\nu} = \bar{\Gamma}^\rho_{\mu\nu} + \delta\Gamma^\rho_{\mu\nu}$ to the Riemann tensor $R^\mu{}_{\nu\rho\sigma} \equiv \partial_\rho\Gamma^\mu_{\sigma\nu} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu} - \rho \leftrightarrow \sigma$ yields

$$R^\mu{}_{\nu\rho\sigma} = \bar{R}^\mu{}_{\nu\rho\sigma} + \bar{\nabla}_\rho(\delta\Gamma^\mu_{\sigma\nu}) - \bar{\nabla}_\sigma(\delta\Gamma^\mu_{\rho\nu}) + \delta\Gamma^\mu_{\rho\lambda}\delta\Gamma^\lambda_{\sigma\nu} - \delta\Gamma^\mu_{\sigma\lambda}\delta\Gamma^\lambda_{\rho\nu}, \tag{9}$$

where $\delta\Gamma^{\rho}_{\mu\nu} = \tau(\Gamma^{\rho}_{\mu\nu})_L - \tau^2 h^{\rho}_{\beta}(\Gamma^{\beta}_{\mu\nu})_L$ at this order. Therefore, the Riemann tensor becomes

$$R^{\mu}{}_{\nu\rho\sigma} = \bar{R}^{\mu}{}_{\nu\rho\sigma} + \tau(R^{\mu}{}_{\nu\rho\sigma})_L - \tau^2 h^{\mu}_{\beta}(R^{\beta}{}_{\nu\rho\sigma})_L - \tau^2 \bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}[(\Gamma^{\gamma}_{\rho\alpha})_L(\Gamma^{\beta}_{\sigma\nu})_L - (\Gamma^{\gamma}_{\sigma\alpha})_L(\Gamma^{\beta}_{\rho\nu})_L] + O(\tau^3). \tag{10}$$

Note that raising and lowering is done by $\bar{g}_{\mu\nu}$, but in the above expression, for the sake of notational clarity, we do not raise and lower the indices of the linearized Christoffel connection. Here, the linearized Riemann tensor $(R^{\mu}{}_{\nu\rho\sigma})_L$ is defined as

$$(R^{\mu}{}_{\nu\rho\sigma})_L \equiv \frac{1}{2}(\bar{\nabla}_{\rho}\bar{\nabla}_{\sigma}h^{\mu}_{\nu} + \bar{\nabla}_{\rho}\bar{\nabla}_{\nu}h^{\mu}_{\sigma} - \bar{\nabla}_{\rho}\bar{\nabla}^{\mu}h_{\sigma\nu} - \bar{\nabla}_{\sigma}\bar{\nabla}_{\rho}h^{\mu}_{\nu} - \bar{\nabla}_{\sigma}\bar{\nabla}_{\nu}h^{\mu}_{\rho} + \bar{\nabla}_{\sigma}\bar{\nabla}^{\mu}h_{\rho\nu}). \tag{11}$$

With this result, the second order expansion of the Ricci tensor and the scalar curvature, respectively, take the following forms:

$$R_{\nu\sigma} = \bar{R}_{\nu\sigma} + \tau(R_{\nu\sigma})_L - \tau^2 h^{\mu}_{\beta}(R^{\beta}{}_{\nu\mu\sigma})_L - \tau^2 \bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}[(\Gamma^{\gamma}_{\mu\alpha})_L(\Gamma^{\beta}_{\sigma\nu})_L - (\Gamma^{\gamma}_{\sigma\alpha})_L(\Gamma^{\beta}_{\mu\nu})_L] + O(\tau^3), \tag{12}$$

$$R = \bar{R} + \tau R_L + \tau^2\{\bar{R}^{\rho\lambda}h_{\alpha\rho}h^{\alpha}_{\lambda} - h^{\nu\sigma}(R_{\nu\sigma})_L - \bar{g}^{\nu\sigma}h^{\mu}_{\beta}(R^{\beta}{}_{\nu\mu\sigma})_L - \bar{g}^{\nu\sigma}\bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}[(\Gamma^{\gamma}_{\mu\alpha})_L(\Gamma^{\beta}_{\sigma\nu})_L - (\Gamma^{\gamma}_{\sigma\alpha})_L(\Gamma^{\beta}_{\mu\nu})_L]\} + O(\tau^3), \tag{13}$$

where the linearized Ricci tensor and the linearized scalar curvature are defined, respectively, as

$$R^{L}_{\nu\sigma} \equiv \tfrac{1}{2}(\bar{\nabla}_{\mu}\bar{\nabla}_{\sigma}h^{\mu}_{\nu} + \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}h^{\mu}_{\lambda} - \Box h_{\sigma\nu} - \bar{\nabla}_{\sigma}\bar{\nabla}_{\nu}h), \tag{14}$$

$$R_L = \bar{g}^{\alpha\beta}R^{L}_{\alpha\beta} - \bar{R}^{\alpha\beta}h_{\alpha\beta}. \tag{15}$$

Note again that the above formulae work for *any* background space including constant curvature spaces which we shall concentrate below.

III. BI-TYPE ACTIONS AT $O(h^2_{\mu\nu})$

A. General analysis

A generic Born-Infeld type action which reproduces the Einstein-Hilbert theory with a bare cosmological constant (Λ_0) at the first order in small *curvature* expansion is of the form

$$I = \frac{2}{\kappa\alpha}\int d^D x[\sqrt{-\det(g_{\mu\nu} + A_{\mu\nu})} - (\alpha\Lambda_0 + 1)\sqrt{-\det g}], \tag{16}$$

where $A_{\mu\nu}$ should read as $A_{\mu\nu} = \alpha(R_{\mu\nu} + \beta\tilde{R}_{\mu\nu}) + O(R^2)$ with the definition $\tilde{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{D}g_{\mu\nu}R$. The $O(R^2)$ terms may involve rank (0, 2) combinations of the Riemann and the Ricci tensors, the metric and the scalar curvature. It could also involve the derivatives of these tensors, but we will not explicitly consider such actions, and we will demand parity invariance, so we do not use the $\epsilon^{\mu\nu\lambda\sigma\ldots\theta}$ tensor in the construction of $A_{\mu\nu}$. Of course, all these technical restrictions can be removed and the following discussion can be extended without much difficulty to cover the type of actions used in [23]. Here, the dimensionful parameter α with a $(\text{mass})^{-2}$ dimension appears only beyond the Einstein-Hilbert theory, and κ is related to the Newton's constant. Note that in the Born-Infeld extension of Maxwell's theory, $\sqrt{-\det(g_{\mu\nu} + bF_{\mu\nu})}$, one *must* introduce a dimensionful parameter b, since Maxwell's theory is scale invariant, but the BI theory cannot be. On the other hand, gravity is not scale invariant and in principle one need not introduce a new scale, one can simply use the already existing two scales κ and Λ_0. Nevertheless, introducing a new scale α gives more flexibility to the theory.

To study the unitarity of (16), one should consider the quadratic fluctuations around a critical point of the action. Assuming that $\bar{g}_{\mu\nu}$ is the critical point and $h_{\mu\nu}$ is the fluctuation, we should compute the $O(h^2)$ terms in the action. To do this by just pulling out the volume density, it is convenient to write the action in the form

$$I = \frac{2}{\kappa\alpha}\int d^D x\sqrt{-g}[\sqrt{-\det(\delta^{\rho}_{\nu} + g^{\rho\mu}A_{\mu\nu})} - (\alpha\Lambda_0 + 1)]. \tag{17}$$

Using the second order expansion of the inverse metric, (6), and assuming an expansion of $A_{\mu\nu}$ in the metric perturbation as

$$A_{\mu\nu} \equiv \bar{A}_{\mu\nu} + \tau A^{(1)}_{\mu\nu} + \tau^2 A^{(2)}_{\mu\nu} + O(\tau^3), \tag{18}$$

one has

$$g^{\rho\mu}A_{\mu\nu} = \bar{g}^{\rho\mu}\bar{A}_{\mu\nu} + \tau(\bar{g}^{\rho\mu}A^{(1)}_{\mu\nu} - h^{\rho\mu}\bar{A}_{\mu\nu}) + \tau^2(\bar{g}^{\rho\mu}A^{(2)}_{\mu\nu} - h^{\rho\mu}A^{(1)}_{\mu\nu} + h^{\rho\sigma}h^{\mu}_{\sigma}\bar{A}_{\mu\nu}). \tag{19}$$

In order to find the second order action in metric perturbation, let us separate the background part of $g^{\rho\mu}A_{\mu\nu}$ and define $\tau B^{\rho}_{\nu} \equiv g^{\rho\mu}A_{\mu\nu} - \bar{g}^{\rho\mu}\bar{A}_{\mu\nu}$, whose introduction will make the expansion more transparent. For a maximally symmetric constant curvature background, one has $\bar{A}_{\mu\nu} \equiv a\bar{g}_{\mu\nu}$ where a is a dimensionless constant *fixed* in the theory in terms of the dimensionful parameters such as Λ_0, α, etc. The effective cosmological constant Λ will also be fixed by the dimensionful parameters. For complicated actions, even finding Λ is a nontrivial problem. The obvious and the conventional method is to find the equations of motion and insert the maximally symmetric solution. But, finding the equations of motion for these actions is simply too complicated. Therefore, we will give a method which bypasses this. Then, B^ρ_ν becomes

$$B^\rho_\nu = (\bar{g}^{\rho\mu} A^{(1)}_{\mu\nu} - a h^\rho_\nu) + \tau(\bar{g}^{\rho\mu} A^{(2)}_{\mu\nu} - h^{\rho\mu} A^{(1)}_{\mu\nu} + a h^{\rho\sigma} h_{\sigma\nu}). \tag{20}$$

Now, we can re-express the BI action with the help of the B^ρ_ν tensor

$$\begin{aligned} I &= \frac{2}{\kappa\alpha} \int d^D x \sqrt{-g} \{ \sqrt{-\det[(1+a)\delta^\rho_\nu + \tau B^\rho_\nu]} - (\alpha\Lambda_0 + 1)\} \\ &= \frac{2}{\kappa\alpha}(1+a)^{((D-4)/(2))} \int d^D x \sqrt{-g} \\ &\quad \times \left\{ (1+a)^2 \sqrt{-\det\left[\delta^\rho_\nu + \frac{\tau}{(1+a)} B^\rho_\nu\right]} - (1+a)^{((4-D)/(2))}(\alpha\Lambda_0 + 1) \right\}, \end{aligned} \tag{21}$$

where $a \neq -1$ which is required in order to have a well-defined leading order: if this requirement is not put, then the flat space limit cannot be reproduced in the limit of vanishing cosmological constant. (For example, if one had fixed $\alpha = -\frac{1}{\Lambda_0}$ with $A_{\mu\nu} = \alpha R_{\mu\nu}$, then one would not have a proper flat space limit.) Here, the factor $(1+a)^2$ is left in front of the determinantal part in order not to introduce a factors in the second order terms coming from the expansion of the determinant. To find the second order expansion of the action in the metric perturbation, let us Taylor expand the determinant in terms of traces up to the order that we shall need

$$\begin{aligned} [\det(1+M)]^{1/2} = 1 &+ \tfrac{1}{2}\mathrm{Tr}M + \tfrac{1}{8}(\mathrm{Tr}M)^2 - \tfrac{1}{4}\mathrm{Tr}(M^2) \\ &+ \tfrac{1}{6}\mathrm{Tr}(M^3) - \tfrac{1}{8}\mathrm{Tr}(M^2)\,\mathrm{Tr}M \\ &+ \tfrac{1}{48}(\mathrm{Tr}M)^3 + O(M^4). \end{aligned} \tag{22}$$

With this formula, the second order expansion of $\sqrt{-g}$ becomes

$$\begin{aligned} \sqrt{-\det g_{\mu\nu}} &= \sqrt{-\det(\bar{g}_{\mu\nu} + \tau h_{\mu\nu})} \\ &= \sqrt{-\bar{g}}\left[1 + \frac{\tau}{2}h + \frac{1}{8}\tau^2(h^2 - 2h^2_{\mu\nu}) + O(\tau^3)\right]. \end{aligned} \tag{23}$$

Then, after using the expansions of the Lagrangian and $\sqrt{-g}$ in (21) one obtains up to $O(\tau^3)$

$$\begin{aligned} I = \frac{2}{\kappa\alpha}(1+a)^{((D-4)/(2))} \int d^D x \sqrt{-\bar{g}} \Big\{ &[(1+a)^2 - (1+a)^{((4-D)/(2))}(\alpha\Lambda_0+1)] + \frac{\tau}{2}[(1+a)B^\rho_\rho \\ &+ [(1+a)^2 - (1+a)^{((4-D)/(2))}(\alpha\Lambda_0+1)]h] \\ &+ \frac{\tau^2}{8}[(B^\rho_\rho)^2 - 2B^\rho_\nu B^\nu_\rho + 2(1+a)hB^\rho_\rho + [(1+a)^2 \\ &- (1+a)^{((4-D)/(2))}(\alpha\Lambda_0+1)](h^2 - 2h^2_{\mu\nu})] \Big\}. \end{aligned} \tag{24}$$

$O(\tau^0)$ term just gives the value of the action for the vacuum solution and it will not be relevant anymore. But, it gives us some crucial information about the BI-type actions, that is for even dimensions the value of the constant curvature is not bounded by the action; however, for odd dimensions $a > -1$ is required for the reality of the action. Now, we would like to go back to our original tensor $A_{\mu\nu}$. First, we write the $O(\tau)$ term in the above expression in terms of $A_{\mu\nu}$. This term gives the nonlinear equation of motion for the constant curvature background. One needs to find what the zeroth order of B^ρ_ρ is in terms of $A_{\mu\nu}$. This is given as

$$B^\rho_\rho = \bar{g}^{\rho\mu} A^{(1)}_{\mu\rho} - ah + O(\tau). \tag{25}$$

Then, the action at the first order reads

$$\begin{aligned} I_{O(h)} = \frac{(1+a)^{((D-4)/(2))}}{\kappa\alpha} \int d^D x \sqrt{-\bar{g}} [(1+a)(\bar{g}^{\rho\mu} A^{(1)}_{\mu\rho} + h) \\ - (1+a)^{((4-D)/(2))}(\alpha\Lambda_0 + 1)h]. \end{aligned} \tag{26}$$

After removing possible boundary terms, taking the variation with respect to $h_{\mu\nu}$ or more concisely looking at the coefficient of $h^{\mu\nu}$ and equating it to zero yields the source-free nonlinear equation of motion for a constant curvature background, namely, the equation of motion that relates Λ to Λ_0 and the other parameters of the theory. Hence, to get the vacuum of the theory, one need not explicitly find the equations of motion which is straightforward but quite tedious.

Now, let us find the quadratic action in $h_{\mu\nu}$ in terms of $A_{\mu\nu}$. The $(B^\rho_\rho)^2 - 2B^\rho_\nu B^\nu_\rho + 2(1+a)hB^\rho_\rho$ terms in (24) can be written in terms of $A_{\mu\nu}$ as

$$\begin{aligned} &(B^\rho_\rho)^2 - 2B^\rho_\nu B^\nu_\rho + 2(1+a)hB^\rho_\rho \\ &\quad = (\bar{g}^{\mu\nu} A^{(1)}_{\mu\nu})^2 - 2A^{(1)}_{\mu\nu} A^{\mu\nu}_{(1)} + h^{\mu\nu}[4aA^{(1)}_{\mu\nu} \\ &\qquad + 2\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma} A^{(1)}_{\rho\sigma} - 2a^2 h_{\mu\nu} - a(2+a)\bar{g}_{\mu\nu}h]. \end{aligned} \tag{27}$$

Contribution coming from the $\tau(1+a)B^\rho_\rho$ term in (24) is

$$B^\rho_\rho = O(\tau^0) + \tau[\bar{g}^{\mu\nu} A^{(2)}_{\mu\nu} - h^{\mu\nu}(A^{(1)}_{\mu\nu} - ah_{\mu\nu})]. \tag{28}$$

In all together, the quadratic action in $h_{\mu\nu}$ in terms of $A_{\mu\nu}$ boils down to

$$I_{O(h^2)} = -\frac{(1+a)^{((D-4)/(2))}}{\kappa\alpha}\int d^Dx\sqrt{-\bar{g}}\Big\{\frac{1}{2}A^{(1)}_{\mu\nu}A^{\mu\nu}_{(1)} - \frac{1}{4}(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu})^2 - (1+a)\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu} + h^{\mu\nu}\Big(A^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}\Big) - \frac{1}{4}[1-(1+a)^{((4-D)/(2))}(\alpha\Lambda_0+1)](h^2-2h^2_{\mu\nu})\Big\}. \tag{29}$$

To remove a possible confusion coming from the notation, we should note what is represented by the term $A^{(1)}_{\mu\nu}A^{\mu\nu}_{(1)}$: It is basically $A^{(1)}_{\mu\nu}A^{\mu\nu}_{(1)} \equiv \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}A^{(1)}_{\mu\nu}A^{(1)}_{\alpha\beta}$, that is $A^{\mu\nu}_{(1)}$ *does not* represent the first order of $A^{\mu\nu}$. If required, we show the first order of $A^{\mu\nu}$ as $(A^{\mu\nu})_{(1)}$. Equation (29) is our main formula which can be applied to any BI-type action for any value of the constant curvature [i.e. we have not done a small curvature expansion, that is, the formula at $O(h^2)$ takes care of all the contributions coming from all powers of the curvature]. Let us summarize what one needs to do to analyze the unitarity of a given BI gravity: One computes $A^{(1)}_{\mu\nu}$ and $A^{(2)}_{\mu\nu}$, and using (26) one finds the vacuum of the theory, and finally computes the $O(h^2)$ action via (29). Then, this action can be studied using conventional techniques that were discussed in [10]. Of course, as we shall see below with some examples, depending on the complexity of $A_{\mu\nu}$, explicit computation of (29) could be a very cumbersome problem in generic dimensions. But, a close scrutiny of it reveals remarkable simplifications in even dimensions, higher than two, and especially in four dimensions. Such simplifications, in four dimensions, will provide us with another method of analyzing the unitarity of the BI gravities, namely, the method of Hindawi *et al.* [24] that leads to the construction of an equivalent quadratic action (in curvature) whose unitarity has been already studied by conventional methods. Let us concentrate on $D=4$ first whose action is

$$I_{O(h^2)} = -\frac{1}{\kappa\alpha}\int d^4x\sqrt{-\bar{g}}\Big\{\frac{1}{2}A^{(1)}_{\mu\nu}A^{\mu\nu}_{(1)} - \frac{1}{4}(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu})^2 - (1+a)\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu} + h^{\mu\nu}\Big(A^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}\Big) + \frac{1}{4}\alpha\Lambda_0(h^2-2h^2_{\mu\nu})\Big\}. \tag{30}$$

By examining this action, one can figure out an interesting relation between the metric perturbation expansion that led to this action and the $A_{\mu\nu}$ expansion of (16). Remember that $A_{\mu\nu}$ is dimensionless, so assuming proper convergence, a Taylor series expansion over $A_{\mu\nu}$ is legitimate. If $A_{\mu\nu}$ involves terms of $O(R^2)$ and/or any other higher curvature terms, the $A_{\mu\nu}$ expansion is not simply equal to the curvature expansion in which the expansion is over the nondimensional quantity αR. Let us write symbolically the expansion of (16) in $A_{\mu\nu}$ as

$$I = \frac{2}{\kappa\alpha}\int d^4x[\sqrt{-\det(g_{\mu\nu}+A_{\mu\nu})} - (\alpha\Lambda_0+1)\times\sqrt{-\det g}] \sim \frac{2}{\kappa\alpha}\int d^4x\sqrt{-g}\Big[\sum_{n=0}^{\infty}c_nA^n - (\alpha\Lambda_0+1)\Big] = \int d^4x\sqrt{-g}\Big[\frac{1}{\kappa}(R-2\Lambda_0) + \frac{2}{\kappa\alpha}\sum_{n=2}^{\infty}c_nA^n\Big], \tag{31}$$

where the last equality follows from our assumption that Einstein-Hilbert action is reproduced at the lowest order. Note that, up to $n=3$, this expansion can be obtained with help of (22), and the nth order term represented with A^n involves terms like A^n, $A^{n-2}A^2_{\mu\nu}$, $A^{n-3}A^\mu_\rho A^\rho_\nu A^\nu_\mu$, etc. In principle, each order in (31) contributes to the quadratic action in the metric perturbation given in (30), but we will see that this is not the case in four dimensions. The $O(h^2)$ contributions coming from the $O(A^n)$ terms where $n\geq 2$ have the form

$$I^{(n)}_{O(h^2)} = \int d^4x\sqrt{-\bar{g}}c_n\{\bar{A}^{n-2}[d_{n1}A^{(1)}_{\mu\nu}A^{\mu\nu}_{(1)} + d_{n2}(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu})^2] + \bar{A}^{n-1}[d_{n3}A^{(2)}_{\mu\nu} + d_{n4}h^{\mu\nu}A^{(1)}_{\mu\nu} + d_{n5}h\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}] + \bar{A}^n[d_{n6}h^2_{\mu\nu} + d_{n7}h^2]\}, \tag{32}$$

where $\bar{A}$ is defined as $\bar{A}_{\mu\nu}\equiv a\bar{g}_{\mu\nu}$ as above, and the coefficients d_n are just numbers. Therefore, the $O(h^2)$ contributions coming from the $O(A^n)$ terms are in the form of $[e_{n2}(h)a^{n-2} + e_{n1}(h)a^{n-1} + e_{n0}(h)a^n]$. Hence, one expects that if each order in the $A_{\mu\nu}$ expansion of (16) contributes to the quadratic action in metric fluctuations, then that action will be composed of the seven terms specified in (32) with a coefficient which is a power series in a. With this result, one can trace the contribution coming from each order in (31) to the $O(h^2)$ action (30). Let us investigate each term in (30) in order to find which orders in the $A_{\mu\nu}$ expansion contributes. The first two terms in (30), which are quadratic in $A_{\mu\nu}$, have coefficients that do not depend on a. Therefore, these two terms involve $O(h^2)$ contributions only coming from the second order terms in the $A_{\mu\nu}$ expansion of (16). The coefficient of the third term in (30) is $(1+a)$, so it is composed of contributions coming from $O(A)$ and $O(A^2)$ terms in the $A_{\mu\nu}$ expansion (31). The fourth term has a coefficient which does not depend on a, so it comes from the first order of the $A_{\mu\nu}$ expansion. Thus, all the $O(h^2)$ contributions coming from $O(A^n)$ terms with $n>2$ are identically zero for four-dimensional BI-type actions, and as we will see this curious case has a generalization to higher even dimensions. With these observations, one can deduce the fact that in four dimensions (30) can be obtained first by making an expansion in $A_{\mu\nu}$ up to third order via (22), and then by finding the quadratic action in metric fluctuations. In other words, remarkably the free theory of the following actions are exactly the same:

$$I = \frac{2}{\kappa\alpha}\int d^4x\Big[\sqrt{-\det(g_{\mu\nu}+A_{\mu\nu})} - (\alpha\Lambda_0+1)\sqrt{-\det g}\Big], \tag{33}$$

and

$$I_{O(A^2)} = \frac{2}{\kappa\alpha}\int d^4x\sqrt{-g}\Big[\frac{1}{2}g^{\mu\nu}A_{\mu\nu} - \alpha\Lambda_0 + \frac{1}{8}(g^{\mu\nu}A_{\mu\nu})^2 - \frac{1}{4}A^2_{\mu\nu}\Big], \tag{34}$$

which was obtained by expanding (33). Here, note that we truncated the $A_{\mu\nu}$ expansion at the second order, *but we do not require $A_{\mu\nu}$ to be small*. This truncation can be done and the equality of the above two actions at the free level can be achieved merely due to the fact that contributions of the higher order terms in the $A_{\mu\nu}$ expansion to the quadratic action in metric fluctuations are identically zero. Such a remarkable cancellation in four dimensions is related to the fact that we have the square root of the determinant of a linear combination of matrix functions one of which is expanded around a constant curvature space and it would not work for a generic background. Let us verify this result by explicitly calculating the quadratic action in metric fluctuations for (34). However, to be as general as possible and to see some cancellations, let us work in D dimensions where only measure in (34) changes to d^Dx. Then, expanding each term in (34) by using (23) and (19) with $\bar{A}_{\mu\nu} \equiv a\bar{g}_{\mu\nu}$ one has

$$g^{\mu\nu}A_{\mu\nu} = aD + \tau(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu} - ah) + \tau^2(\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu} - h^{\mu\nu}A^{(1)}_{\mu\nu} + ah^2_{\mu\nu}), \tag{35}$$

and all together up to quadratic order, the action reads

$$I_{O(A^2)} = \frac{1}{\kappa\alpha}\int d^Dx\sqrt{-\bar{g}}\Big\{\Big[aD - \frac{a^2D}{2} + \frac{a^2D^2}{4} - \alpha\Lambda_0\Big] + \tau\Big[\Big(1+\frac{aD}{2}-a\Big)\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu} + \frac{a(D-2)}{2}\Big(1+\frac{(D-4)a}{4}\Big)h - \alpha\Lambda_0 h\Big] - \tau^2\Big[\frac{1}{2}A^{\mu\nu}_{(1)}A^{(1)}_{\mu\nu} - \frac{1}{4}(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu})^2 - \Big(1+\frac{aD}{2}-a\Big)\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu} + \Big(1+\frac{aD}{2}-2a\Big)h^{\mu\nu}\Big(A^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}\Big) - \frac{(D-4)}{8}\Big(a+\frac{D-6}{4}a^2\Big)(h^2 - 2h^2_{\mu\nu}) + \frac{\alpha\Lambda_0}{4}(h^2-2h^2_{\mu\nu})\Big]\Big\}. \tag{36}$$

In obtaining this result, one should rewrite $(A^{\mu\nu})_{(2)}$ and $(A^{\mu\nu})_{(1)}$ coming from $A^2_{\mu\nu}$ in (34) in terms of $A^{(2)}_{\mu\nu}$ and $A^{(1)}_{\mu\nu}$ as

$$\begin{aligned}(A^{\mu\nu})_{(2)} &= (g^{\mu\alpha}g^{\nu\beta}A_{\alpha\beta})_{(2)} \\ &= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}A^{(2)}_{\alpha\beta} + 3ah^{\mu}_{\rho}h^{\rho\nu} - \bar{g}^{\mu\alpha}h^{\nu\beta}A^{(1)}_{\alpha\beta} - \bar{g}^{\nu\beta}h^{\mu\alpha}A^{(1)}_{\alpha\beta},\end{aligned} \tag{37}$$

$$(A^{\mu\nu})_{(1)} = (g^{\mu\alpha}g^{\nu\beta}A_{\alpha\beta})^{(1)} = \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}A^{(1)}_{\alpha\beta} - 2ah^{\mu\nu}. \tag{38}$$

Let us now concentrate only on the $O(\tau^2)$ terms:

$$I^{O(A^2)}_{O(h^2)} = -\frac{1}{\kappa\alpha}\int d^Dx\sqrt{-\bar{g}}\Big\{\frac{1}{2}A^{\mu\nu}_{(1)}A^{(1)}_{\mu\nu} - \frac{1}{4}(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu})^2 - \Big(1+\frac{aD}{2}-a\Big)\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu} + \Big(1+\frac{aD}{2}-2a\Big)h^{\mu\nu} \times\Big(A^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}\Big) - \frac{(D-4)}{8}\Big[a + \frac{(D-6)a^2}{4}\Big] \times(h^2 - 2h^2_{\mu\nu}) + \frac{\alpha\Lambda_0}{4}(h^2 - 2h^2_{\mu\nu})\Big\}. \tag{39}$$

In four dimensions, (39) reduces to (30) as it was promised. In Appendix A, we give a simple example with two-dimensional matrix functions that shows the connection between the $A_{\mu\nu}$ expansion and the metric perturbation expansion. In generic even dimensions with $D = 2n+2$, if one wants to carry out a similar analysis, then one has to expand up to $O(A^{n+1})$ with $n \geq 1$. But, again we should stress that the compact formula (29) works all the time without recourse to such an expansion. However, depending on the complexity of $A_{\mu\nu}$, one can choose to use either the expansion method or the compact expression. As for odd dimensions, because of the nonpolynomial prefactor $(1+a)^{((D-4)/(2))}$ in (29), all the terms in the $A_{\mu\nu}$ expansion (or the small curvature expansion) contribute. The most efficient way to get the quadratic fluctuations for odd dimensions is to use (29).

A similar analysis can be done for the $O(h)$ action in four dimension which is

$$I_{O(h)} = \frac{1}{\kappa\alpha}\int d^4x\sqrt{-\bar{g}}[(1+a)\bar{g}^{\rho\mu}A^{(1)}_{\mu\rho} + (a - \alpha\Lambda_0)h]. \tag{40}$$

This action involves contributions coming only from the second order expansion of (16) in $A_{\mu\nu}$ just as the $O(h^2)$ action. In order to understand this behavior, let us first look at the $O(h)$ contributions coming from the $O(A^n)$ term for $n \geq 2$

$$I^{(n)}_{O(h)} = \int d^4x\sqrt{-\bar{g}}c_n[\bar{A}^{n-1}d_{n1}(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}) + d_{n2}\bar{A}^n h], \tag{41}$$

where d_n coefficients are just numbers. Therefore, the $O(h)$ contributions coming from the $O(A^n)$ terms are simply in the form of $[e_{n1}(h)a^{n-1} + e_{n0}(h)a^n]$, since $\bar{A} \sim a$. Hence, one expects that if each order in the $A_{\mu\nu}$ expansion of (16) contributes to the linear action in $h_{\mu\nu}$, then it will be composed of the two terms specified in (41) with coefficients that are of the form a^n. With this result, one can trace the contribution coming from each order in (31) to the $O(h)$ action (40). Let us investigate each term in (40) in order to find which orders in the $A_{\mu\nu}$ expansion contribute. The first term in (40) has a coefficient of $(1 + a)$. Therefore, this term involves $O(h)$ contributions coming from the second order terms in the $A_{\mu\nu}$ expansion of (16). The coefficient of h in (40) is also first order in a, but this time it implies that only the first order of $A_{\mu\nu}$ expansion contributes. Therefore, the vacuum of (16) and (34) are the same. One can verify this result explicitly from $O(h)$ of the $O(A^2_{\mu\nu})$ action which can be read from (36) as

$$I^{O(A^2)}_{O(h)} = \frac{1}{\kappa\alpha}\int d^D x\sqrt{-\bar{g}}\bigg[\bigg(1 + \frac{aD}{2} - a\bigg)\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu} + \frac{a(D-2)}{2}\bigg(1 + \frac{(D-4)a}{4}\bigg)h - \alpha\Lambda_0 h\bigg].$$

This action reduces to (40) in four dimensions. Just like the analysis of $O(h^2)$, for generic even dimensions $D = 2n + 2$ one has to expand (16) to $O(A^{n+1})$ with $n \geq 1$, then find the vacuum of the theory. For odd dimensions, since all the powers of A^n contribute, the most efficient way to find the vacuum of the theory is to use (26).

B. An example

To apply our tools, for the sake of simplicity, let us consider the following model which we know to be non-unitary even around the flat space:

$$I = \frac{2}{\kappa\alpha}\int d^4x\Big[\sqrt{-\det(g_{\mu\nu} + \alpha R_{\mu\nu})} - (\alpha\Lambda_0 + 1)\sqrt{-\det g}\Big]. \tag{42}$$

Here, according to our results above, one expects (which we shall verify below with several different techniques) that the second order action in the metric perturbation $h_{\mu\nu}$ involves contributions only coming from the $O[(\alpha R)^2]$ expansion:

$$I_{O(R^2)} = \frac{2}{\kappa\alpha}\int d^4x\sqrt{-g}\bigg[\frac{\alpha}{2}(R - 2\Lambda_0) - \frac{\alpha^2}{4}\bigg(R^2_{\mu\nu} - \frac{1}{2}R^2\bigg)\bigg]. \tag{43}$$

Therefore, the $O[(\alpha R)^3]$, $O[(\alpha R)^4]$, and etc. terms should vanish at $O(h^2)$. Hence, at $O(h)$ and $O(h^2)$ (42) and (43) are equivalent. Let us explicitly show this by analyzing the linearized free theory of (42) around the extremum of it by using (30).

1. Analyzing the BI action formed by the Ricci tensor via second order perturbations in $h_{\mu\nu}$

Let us define $A_{\mu\nu} \equiv \alpha R_{\mu\nu}$. Then, $\bar{A}_{\mu\nu} = \alpha\Lambda\bar{g}_{\mu\nu} \Rightarrow a \equiv \alpha\Lambda$, where Λ will be determined in terms of Λ_0. Then, $A^{(1)}_{\mu\nu}$ is given as

$$A^{(1)}_{\mu\nu} = \alpha R^L_{\mu\nu}, \tag{44}$$

and $A^{(2)}_{\mu\nu} = \alpha R^{(2)}_{\nu\sigma}$ and referring the details to Appendix B, we have

$$\alpha\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} = \alpha h^{\mu\nu}\bigg(\frac{1}{2}R^L_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}R_L - \frac{\Lambda}{4}\bar{g}_{\mu\nu}h\bigg). \tag{45}$$

First of all, let us determine the nonlinear equations of motion for the constant curvature background which will relate Λ to Λ_0 by using (26)

$$\begin{aligned} I_{O(h)} &= \frac{1}{\kappa\alpha}\int d^4x\sqrt{-\bar{g}}[(1+a)(\bar{g}^{\rho\mu}A^{(1)}_{\mu\rho} + h) - (\alpha\Lambda_0 + 1)h] \\ &= \frac{1}{\kappa\alpha}\int d^4x\sqrt{-\bar{g}}[\alpha(1+\alpha\Lambda)\bar{\nabla}_\mu(\bar{\nabla}_\nu h^{\mu\nu} - \bar{\nabla}^\mu h) \\ &\quad + \alpha(\Lambda - \Lambda_0)h]. \end{aligned} \tag{46}$$

Note that this first order correction should be zero around the extremum, therefore, after dropping the first term which is a boundary term, one has $\Lambda = \Lambda_0$. As for the second order action, one has (30)

$$\begin{aligned} I_{O(h^2)} = -\frac{1}{\kappa\alpha}\int d^4x\sqrt{-\bar{g}}\bigg\{&\frac{\alpha^2}{2}R^L_{\mu\nu}R^{\mu\nu}_L - \frac{\alpha^2}{4}(R_L + \Lambda h)^2 \\ &- (\alpha + \alpha^2\Lambda)h^{\mu\nu}\bigg(\frac{1}{2}R^L_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}R_L - \frac{\Lambda}{4}\bar{g}_{\mu\nu}h\bigg) \\ &+ \alpha h^{\mu\nu}\bigg[R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}(R_L + \Lambda h)\bigg] \\ &+ \frac{\alpha}{4}\Lambda_0(h^2 - 2h^2_{\mu\nu})\bigg\}, \end{aligned} \tag{47}$$

where $\int d^4x\sqrt{-\bar{g}}R^L_{\mu\nu}R^{\mu\nu}_L$ is calculated in Appendix B as

$$\begin{aligned} \int d^4x&\sqrt{-\bar{g}}R^{\mu\nu}_L R^L_{\mu\nu} \\ &= -\frac{1}{2}\int d^4x\sqrt{-\bar{g}}h^{\mu\nu}\bigg[(\bar{g}_{\mu\nu}\Box - \bar{\nabla}_\mu\bar{\nabla}_\nu + \Lambda\bar{g}_{\mu\nu})R_L \\ &\quad + \bigg(\Box\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\bigg) - \frac{14\Lambda}{3}R^L_{\mu\nu} + \frac{\Lambda}{3}\bar{g}_{\mu\nu}R_L \\ &\quad + \frac{8\Lambda^2}{3}h_{\mu\nu}\bigg]. \end{aligned} \tag{48}$$

Then, after some algebra the quadratic action reduces to

$$I_{O(h^2)} = -\frac{1}{\alpha\kappa}\int d^4x\sqrt{-\bar{g}}\Big\{h^{\mu\nu}\Big[\Big(\frac{\alpha}{2}+\frac{2\alpha^2\Lambda}{3}\Big)\mathcal{G}^L_{\mu\nu} - \frac{\alpha^2}{4}\Big(\Box\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\Big)\Big] - \frac{\alpha}{4}(\Lambda-\Lambda_0)(h^2-2h^2_{\mu\nu})\Big\}, \tag{49}$$

where we kept the background gauge noninvariant term (the last part) just to show an intermediate step of the computation. Once $\Lambda = \Lambda_0$ is used, one ends up with

$$I_{O(h^2)} = -\frac{1}{\alpha\kappa}\int d^4x\sqrt{-\bar{g}}h^{\mu\nu}\Big[\Big(\frac{\alpha}{2}+\frac{2\alpha^2\Lambda_0}{3}\Big)\mathcal{G}^L_{\mu\nu} - \frac{\alpha^2}{4}\Big(\Box\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda_0}{3}\bar{g}_{\mu\nu}R_L\Big)\Big]. \tag{50}$$

This action is exactly equivalent to the linearized action one obtains from the $O[(\alpha R)^2]$ action (43). [Note that the linearized version of (43) has been worked out in several places [10,25], and we also reproduce it below.] The fact that (50) has at most α^2 terms show that the contributions coming from all $O[(\alpha R)^{n+2}]$ vanish. We stress once again that this is a highly nontrivial cancellation brought by the determinantal structure of the action. It is worth to study explicitly how this cancellation takes place at $O[(\alpha R)^3]$ which we do now. At this order the action reads

$$I_{O(R^3)} = \frac{2}{\kappa\alpha}\int d^4x\sqrt{-g}\Big[\frac{\alpha}{2}(R-2\Lambda_0) - \frac{\alpha^2}{4}\Big(R^2_{\mu\nu}-\frac{1}{2}R^2\Big) + \frac{\alpha^3}{48}(8R_{\mu\rho}R^\rho_\nu R^{\mu\nu} - 6R^2_{\mu\nu}R + R^3)\Big], \tag{51}$$

and defining

$$K \equiv R^2_{\mu\nu} - \frac{1}{2}R^2, \qquad S \equiv 8R^{\mu\nu}R_{\mu\alpha}R^\alpha{}_\nu - 6RR^2_{\mu\nu} + R^3, \tag{52}$$

one has

$$I_{O(R^3)} = \frac{1}{\kappa}\int d^4x\sqrt{-g}\Big[(R-2\Lambda_0) - \frac{\alpha}{2}K + \frac{\alpha^2}{24}S\Big]. \tag{53}$$

Finding the $O(h^2)$ action of this theory is a very cumbersome problem. To somewhat simplify this, one can first find the equations of motion then linearize the equations of motion and then do the reverse calculus of variations procedure to get the action. Of course in this process boundary terms are dropped and one has to be careful with an overall sign that can be fixed by coupling the gravity action to matter. The equations of motion follow as

$$\frac{\kappa\alpha}{4}\tau_{\mu\nu} = -\frac{\alpha}{4}\Big[(R-2\Lambda_0) - \frac{\alpha}{2}K + \frac{\alpha^2}{24}S\Big]g_{\mu\nu} + \frac{\alpha}{2}R_{\mu\nu} + \frac{\alpha^2}{4}\Big[RR_{\mu\nu} - 2R_{\lambda\nu\alpha\mu}R^{\lambda\alpha} - \Box\Big(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\Big)\Big] + \frac{\alpha^3}{4}(2R^\rho_\mu R_{\rho\alpha}R^\alpha_\nu + [g_{\mu\nu}\nabla_\alpha\nabla_\beta(R^{\beta\rho}R^\alpha_\rho) + \Box(R^\rho_\nu R_{\mu\rho}) - 2\nabla_\alpha\nabla_\mu(R^\rho_\nu R^\alpha_\rho)]) + \frac{\alpha^3}{8}([2\nabla_\alpha\nabla_\mu(RR^\alpha_\nu) - g_{\mu\nu}\nabla_\alpha\nabla_\beta(RR^{\alpha\beta}) - \Box(RR_{\mu\nu})] - 2RR^\rho_\nu R_{\mu\rho}) - \frac{\alpha^3}{8}[(g_{\mu\nu}\Box - \nabla_\nu\nabla_\mu) + R_{\mu\nu}]\Big(R^2_{\alpha\beta}-\frac{1}{2}R^2\Big), \tag{54}$$

where we defined the energy-momentum tensor as $\tau_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta I_{\text{matter}}}{\delta g^{\mu\nu}}$. Constant curvature background ($\bar{R}_{\mu\nu} = \Lambda\bar{g}_{\mu\nu}$) should satisfy source-free equations of motion with the results

$$\bar{K} = \bar{R}^2_{\mu\nu} - \tfrac{1}{2}\bar{R}^2 = -4\Lambda^2, \qquad \bar{S} = 8\bar{R}^{\mu\nu}\bar{R}_{\mu\alpha}\bar{R}^\alpha_\nu - 6\bar{R}\bar{R}^2_{\mu\nu} + \bar{R}^3 = 0, \tag{55}$$

Then, the equations are satisfied if $\Lambda = \Lambda_0$. Now, let us linearize (54) around its vacuum (defining $T_{\mu\nu}(h) \equiv \delta(\frac{\tau_{\mu\nu}}{2})$) by use of the formulae in Appendix C and

$$\delta K = -2\Lambda R_L, \qquad \delta S = 0. \tag{56}$$

The linearized equations of motion after using the source-free equation of motion for constant curvature background becomes

$$T_{\mu\nu}(h) = \Big(\frac{1}{\kappa}+\frac{4\alpha\Lambda_0}{3\kappa}\Big)\mathcal{G}^L_{\mu\nu} - \frac{\alpha}{2\kappa}\Big(\Box\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda_0}{3}\bar{g}_{\mu\nu}R_L\Big), \tag{57}$$

which exactly matches the equations that result from the matter coupled version of the action (43) as promised. This shows explicitly that $O[(\alpha R)^3]$ terms cancel each other. This cancellation will work for any arbitrary order beyond this, as we will show with a different method below.

2. *Another method for unitarity analysis*

Hindawi *et al.* [24] gave another method of analyzing a generic higher derivative gravity model by reducing it to the equivalent quadratic theory in the sense that it has the same free Lagrangian as the original higher derivative theory. Here, we will review their approach and apply it to our example (42). Before we describe their method, we should note that unlike our method which led to the compact formula (29) that works in all cases, the method of Hindawi *et al.* works only when one deals with not matrices but scalar objects or one has a finite number of curvature terms. Keeping this caveat in mind, which will be better understood below, when Hindawi *et al.* method works, it provides a fast algorithm in getting the equivalent quadratic action.

To understand the essence of the Hindawi *et al.* method let us consider the following simplified problem. Suppose we have a function $f(x(t))$, and we would like to find the ϵ^2 order of $f(x(t_0+\epsilon))$. But, instead of doing this, we can find a function $g(x(t)) = a_0 + a_1 x(t) + a_2 x^2(t)$ whose second order expansion around t_0 yields the same second order expansion of $f(x(t))$ around the same point. After some straightforward analysis, one can show that $g(x(t))$ can be obtained by expanding $f(x(t))$ around $x_0 = x(t_0)$ up to and including $O[(x(t)-x_0)^2]$, since $O[(x(t)-x_0)^{2+n}]$ gives ϵ^{2+n} corrections with $n \geq 1$. Hence, one can read the coefficients for the correct $g(x(t))$ to be

$$a_0 = f(x_0) - \left[\frac{df}{dx}\right]_{x_0} x_0 + \frac{1}{2}\left[\frac{d^2 f}{dx^2}\right]_{x_0} x_0, \\ a_1 = \left[\frac{df}{dx}\right]_{x_0} - x_0\left[\frac{d^2 f}{dx^2}\right]_{x_0}, \qquad a_2 = \frac{1}{2}\left[\frac{d^2 f}{dx^2}\right]_{x_0}. \tag{58}$$

Note that if one just wants the "equations of motion," then one carries out the above procedure at $O(\epsilon)$. In this example, f represents the Lagrangian, x any curvature tensor or scalar, and ϵ represents the metric perturbation $h_{\mu\nu}(x)$. Similarly, t_0, x_0 are used in analogy with the background metric $\bar{g}_{\mu\nu}$, etc.

a. Cubic theory

Now, let us turn to our example (42) and to specifically its third order expansion in curvature given in (51). In order not to introduce the metric or its inverse during the expansion around $(\bar{R}, \bar{R}^\mu_\nu)$, let us take the Lagrangian density of (51) to be a function of R and R^μ_ν as

$$f(R, R^\mu_\nu) \equiv \frac{\alpha}{2}(R - 2\Lambda_0) - \frac{\alpha^2}{4}\left(R^\mu_\nu R^\nu_\mu - \frac{1}{2}R^2\right) + \frac{\alpha^3}{48}(8R^\mu_\rho R^\rho_\nu R^\nu_\mu - 6R^\mu_\nu R^\nu_\mu R + R^3). \tag{59}$$

Expanding $f(R, R^\mu_\nu)$ around $(\bar{R}, \bar{R}^\mu_\nu)$ with the assumption of small fluctuations about the background yields

$$f(R, R^\mu_\nu) = f(\bar{R}, \bar{R}^\mu_\nu) + \left[\frac{\partial f}{\partial R}\right]_{(\bar{R}, \bar{R}^\mu_\nu)}(R - \bar{R}) + \left[\frac{\partial f}{\partial R^\alpha_\beta}\right]_{(\bar{R}, \bar{R}^\mu_\nu)} \times (R^\alpha_\beta - \bar{R}^\alpha_\beta) + \frac{1}{2}\left[\frac{\partial^2 f}{\partial R^2}\right]_{(\bar{R}, \bar{R}^\mu_\nu)}(R - \bar{R})^2 + \left[\frac{\partial f}{\partial R \partial R^\alpha_\beta}\right]_{(\bar{R}, \bar{R}^\mu_\nu)}(R - \bar{R})(R^\alpha_\beta - \bar{R}^\alpha_\beta) + \frac{1}{2}\left[\frac{\partial^2 f}{\partial R^\rho_\sigma \partial R^\alpha_\beta}\right]_{(\bar{R}, \bar{R}^\mu_\nu)}(R^\alpha_\beta - \bar{R}^\alpha_\beta)(R^\rho_\sigma - \bar{R}^\rho_\sigma). \tag{60}$$

Computing the relevant derivatives one ends up with

$$f(R, R^\mu_\nu) = f(\bar{R}, \bar{R}^\mu_\nu) + \left[\frac{\alpha}{2} + \frac{\alpha^2}{4}\bar{R} + \frac{\alpha^3}{16}(\bar{R}^2 - 2\bar{R}^\mu_\nu \bar{R}^\nu_\mu)\right](R - \bar{R}) + \left[-\frac{\alpha^2}{2}\bar{R}^\beta_\alpha + \frac{\alpha^3}{4}(2\bar{R}^\beta_\nu \bar{R}^\nu_\alpha - \bar{R}^\beta_\alpha \bar{R})\right](R^\alpha_\beta - \bar{R}^\alpha_\beta) + \frac{1}{2}\left(\frac{\alpha^2}{4} + \frac{\alpha^3}{8}\bar{R}\right)(R - \bar{R})^2 + \left(-\frac{\alpha^3}{4}\bar{R}^\beta_\alpha\right)(R - \bar{R})(R^\alpha_\beta - \bar{R}^\alpha_\beta) + \frac{1}{2}\left[-\frac{\alpha^2}{2}\delta^\beta_\rho \delta^\sigma_\alpha + \frac{\alpha^3}{4}(2\delta^\beta_\rho \bar{R}^\sigma_\alpha + 2\bar{R}^\beta_\rho \delta^\sigma_\alpha - \delta^\beta_\rho \delta^\sigma_\alpha \bar{R})\right](R^\alpha_\beta - \bar{R}^\alpha_\beta)(R^\rho_\sigma - \bar{R}^\rho_\sigma). \tag{61}$$

For constant curvature backgrounds, the corresponding quadratic action becomes

$$I = \frac{2}{\kappa\alpha}\int d^4x\sqrt{-g}\left[\frac{\alpha}{2}(R - 2\Lambda_0) + \frac{\alpha^2}{8}R^2 - \frac{\alpha^2}{4}R^2_{\mu\nu}\right], \tag{62}$$

which once again shows that the cubic term in (59) does not contribute to the free theory. We should stress that if one takes arbitrary coefficients instead of the ones we have which are $(8, -6, 1)$ at the cubic order (59), then one would get a different quadratic action that does not follow from the $A_{\mu\nu}$ (in this case it is just $\alpha R_{\mu\nu}$) expansion of (42).

Now, let us also obtain the source-free nonlinear equations of motion for a constant curvature background by finding the equivalent action at $O(R)$. Similar steps lead to

$$f(R, R^\mu_\nu) = f(\bar{R}, \bar{R}^\mu_\nu) + \left[\frac{\partial f}{\partial R}\right]_{(\bar{R}, \bar{R}^\mu_\nu)}(R - \bar{R}) + \left[\frac{\partial f}{\partial R^\alpha_\beta}\right]_{(\bar{R}, \bar{R}^\mu_\nu)}(R^\alpha_\beta - \bar{R}^\alpha_\beta), \tag{63}$$

and to the action

$$I = \int d^4x\sqrt{-g}\left[\frac{(1 + \alpha\Lambda)}{\kappa}\left(R - 2\frac{\Lambda_0 + \alpha\Lambda^2}{1 + \alpha\Lambda}\right) + O(R^2)\right], \tag{64}$$

where we used $\bar{R}_{\mu\nu} = \Lambda \bar{g}_{\mu\nu}$. Then, identifying $\Lambda = \frac{\Lambda_0 + \alpha\Lambda^2}{1 + \alpha\Lambda}$, one obtains $\Lambda = \Lambda_0$.

b. Full nonlinear action

We mentioned above that Hindawi *et al.* method does not work when one deals directly with matrices. Let us show this with

$$f(R_{\mu\nu}) = \sqrt{\det(g_{\mu\nu} + \alpha A_{\mu\nu})}, \tag{65}$$

and try to find df which is needed for this analysis. Defining $M_{\mu\nu} \equiv g_{\mu\nu} + \alpha A_{\mu\nu}$, one has

$$df = d(\sqrt{\det M}) = \frac{\sqrt{\det M}}{2}\,\mathrm{Tr}[M^{-1}dM], \tag{66}$$

where $M^{-1}dM$ is an ordinary matrix multiplication. Here, the basic problem is to find M^{-1} which cannot be done in exact form for a general $A_{\mu\nu}$ and even when $A_{\mu\nu} = R_{\mu\nu}$. But, one can always expand the determinant in terms of traces and apply the Hindawi *et al.* method. Even though this is the case, for a complicated $A_{\mu\nu}$ the determinant will yield many terms in generic dimensions and as we show below even for four dimensions. Let us consider the action (42) and use the exact formula

$$\det M = \tfrac{1}{24}\{(\mathrm{Tr}M)^4 - 6\,\mathrm{Tr}(M^2)(\mathrm{Tr}M)^2 + 3[\mathrm{Tr}(M^2)]^2 + 8\,\mathrm{Tr}(M^3)\mathrm{Tr}M - 6\,\mathrm{Tr}(M^4)\}. \tag{67}$$

for $M = \delta^\mu_\nu + \alpha R^\mu_\nu$, one gets

$$\begin{aligned}\det(\delta^\mu_\nu + \alpha R^\mu_\nu) &= 1 + \alpha R + \frac{\alpha^2}{2}R^2 + \frac{\alpha^3}{6}R^3 + \frac{\alpha^4}{24}R^4 - \frac{\alpha^2}{2}R^\mu_\nu R^\nu_\mu \\ &\quad - \frac{\alpha^3}{2}RR^\mu_\nu R^\nu_\mu + \frac{\alpha^3}{3}R^\mu_\rho R^\rho_\nu R^\nu_\mu - \frac{\alpha^4}{4}R^2 R^\mu_\nu R^\nu_\mu \\ &\quad + \frac{\alpha^4}{3}RR^\mu_\rho R^\rho_\nu R^\nu_\mu + \frac{\alpha^4}{8}R^\mu_\nu R^\nu_\mu R^\rho_\sigma R^\sigma_\rho - \frac{\alpha^4}{4}R^\mu_\rho R^\rho_\sigma R^\sigma_\nu R^\nu_\mu.\end{aligned} \tag{68}$$

Defining $f(R, R^\mu_\nu) \equiv \sqrt{\det(\delta^\mu_\nu + \alpha R^\mu_\nu)}$ and with the help of (60) and the formulae in Appendix D, we have the corresponding quadratic action as

$$I = \frac{2}{\kappa\alpha}\int d^4x\sqrt{-g}\left[\frac{\alpha}{2}(R - 2\Lambda_0) + \frac{\alpha^2}{8}R^2 - \frac{\alpha^2}{4}R^2_{\mu\nu}\right]. \tag{69}$$

Once again we have proven that the $O(R^{2+n})$ with $n \geq 1$ terms do not contribute to the free theory for the exact BI action (42) around its constant curvature vacuum. Note that with the help of an equivalent action at the linear level as we have done before,

$$I = \int d^4x\sqrt{-g}\left\{\frac{1}{\kappa}(1 + \alpha\Lambda)\left[R - 2\left(\frac{\Lambda_0 + \alpha\Lambda^2}{1 + \alpha\Lambda}\right)\right]\right\}, \tag{70}$$

setting $\Lambda = \frac{\Lambda_0 + \alpha\Lambda^2}{1+\alpha\Lambda}$, one has $\Lambda = \Lambda_0$. We should stress that to get this result with the conventional method of finding the field equations and looking for a solution of the form $\bar{R}_{\mu\nu} = \Lambda\bar{g}_{\mu\nu}$ is highly cumbersome for an action which is given as the square root of $\det(\delta^\mu_\nu + \alpha R^\mu_\nu)$ (68).

C. Unitarity of the theory proposed by Deser and Gibbons

While constructing the BI-type gravity actions, among various criteria, one of the easiest to realize is the unitarity of the model around flat space. This means when small curvature expansion is carried out at the quadratic order in four dimensions, one should get the unique theory $\frac{1}{\kappa}(R - 2\Lambda_0) + \alpha R^2 + \gamma(R^2 - 4R^2_{\mu\nu} + R^2_{\mu\nu\rho\sigma})$ which is free of ghosts. Deser and Gibbons [17] suggested that at the quadratic order, one should get, dropping the αR^2 term, only the Einstein plus the Gauss-Bonnet combination (the γ term). We will study such actions in a separate work, but here let us consider an example (the one suggested by Deser and Gibbons) of these models in which one does not have quadratic terms when expanded around small curvature:

$$I = \frac{2}{\kappa\alpha}\int d^4x\left[\sqrt{-\det\left[g_{\mu\nu} + \alpha R_{\mu\nu} + \frac{\alpha^2}{2}\left(R_{\mu\rho}R^\rho_\nu - \frac{1}{2}RR_{\mu\nu}\right)\right]} - (\alpha\Lambda_0 + 1)\sqrt{-\det g}\right]. \tag{71}$$

It is easy to see that the lowest order correction to the Einstein-Hilbert theory goes like $O(R^3)$, which means around flat space the graviton propagator is the same as that of Einstein-Hilbert theory. (Note that for flat space to be the vacuum, one also sets $\Lambda_0 = 0$.) But, around its constant curvature vacuum unitarity of this model has not been checked before, since it is a highly nontrivial computation without the tools we have developed above. To carry out the analysis, we can find the $O(A^2_{\mu\nu})$ action which has the same $O(h^2)$ action as (71). Here, $A_{\mu\nu} = \alpha R_{\mu\nu} + \frac{\alpha^2}{2} \times (R_{\mu\rho}R^\rho_\nu - \frac{1}{2}RR_{\mu\nu})$ and let us stress again that $O(A^2_{\mu\nu})$ action is not equivalent to $O[(\alpha R)^4]$ action. If one naively does the latter expansion, one will simply get an inconclusive result since one would have neglected the $O[(\alpha R)^{4+n}]$ corrections. But, an expansion in $A_{\mu\nu}$ takes care of all the relevant terms and cancellations. Therefore, using (22) we have

$$\begin{aligned}I_{O(A^2)} &= \frac{2}{\kappa\alpha}\int d^4x\sqrt{-g}\left[\frac{\alpha}{2}(R - 2\Lambda_0) + \frac{\alpha^3}{4}\left(RR^\mu_\nu R^\nu_\mu - R^\mu_\rho R^\rho_\nu R^\nu_\mu - \frac{1}{4}R^3\right)\right] + \frac{\alpha^4}{32}\left(R^\mu_\nu R^\nu_\mu R^\rho_\sigma R^\sigma_\rho \right. \\ &\quad \left. - \frac{3}{2}R^2 R^\mu_\nu R^\nu_\mu + \frac{1}{4}R^4 - 2R^\mu_\rho R^\rho_\nu R^\nu_\sigma R^\sigma_\mu + 2RR^\mu_\rho R^\rho_\nu R^\nu_\mu\right)\Bigg].\end{aligned} \tag{72}$$

Now, let us just concentrate on the higher curvature terms and define

$$f(R, R^{\mu}_{\nu}) \equiv \frac{\alpha^3}{4}\Big(RR^{\mu}_{\nu}R^{\nu}_{\mu} - R^{\mu}_{\rho}R^{\rho}_{\nu}R^{\nu}_{\mu} - \frac{1}{4}R^3\Big) + \frac{\alpha^4}{32}\Big(R^{\mu}_{\nu}R^{\nu}_{\mu}R^{\rho}_{\sigma}R^{\sigma}_{\rho} - \frac{3}{2}R^2R^{\mu}_{\nu}R^{\nu}_{\mu} + \frac{1}{4}R^4 - 2R^{\mu}_{\rho}R^{\rho}_{\nu}R^{\nu}_{\sigma}R^{\sigma}_{\mu} + 2RR^{\mu}_{\rho}R^{\rho}_{\nu}R^{\nu}_{\mu}\Big). \quad (73)$$

The first thing we should find is the correct Λ which can be found by using the first order expansion (63) of $f(R, R^{\mu}_{\nu})$ around $(\bar{R}, \bar{R}^{\mu}_{\nu}) = (4\Lambda, \Lambda\delta^{\mu}_{\nu})$. This procedure leads to the equivalent linear action

$$I = \frac{2}{\kappa\alpha}\int d^4x\sqrt{-g}\Big[\Big(\frac{\alpha}{2} - \frac{3\alpha^3\Lambda^2}{4} + \frac{\alpha^4\Lambda^3}{4}\Big) \times\Big[R - 2\frac{(\frac{\alpha\Lambda_0}{2} - \alpha^3\Lambda^3 + \frac{3\alpha^4\Lambda^4}{8})}{(\frac{\alpha}{2} - \frac{3\alpha^3\Lambda^2}{4} + \frac{\alpha^4\Lambda^3}{4})}\Big] + O(R^2)\Big], \quad (74)$$

from which one can get the equation that determines Λ

$$-\Lambda_0 + \Lambda + \frac{\alpha^2\Lambda^3}{2} - \frac{\alpha^3\Lambda^4}{4} = 0, \quad (75)$$

which has real roots, but they are not particularly illuminating to display here. (One thing we can note is that even for $\Lambda_0 = 0$, there are two real roots one of which is nonzero with a value $\Lambda \approx 2.59/\alpha$.) Now, we can employ the Hindawi *et al.* method to get the equivalent quadratic action using (60) and the relevant results of Appendix D:

$$I = \frac{2}{\kappa\alpha}\int d^4x\sqrt{-g}\Big[\Big(-\alpha\Lambda_0 - \alpha^3\Lambda^3 + \frac{3\alpha^4\Lambda^4}{4}\Big) + \Big(\frac{\alpha}{2} + \frac{3\alpha^3\Lambda^2}{4} - \frac{\alpha^4\Lambda^3}{2}\Big)R - \frac{\alpha^3\Lambda}{4}\Big(1 - \frac{\alpha\Lambda}{2}\Big)R^2 + \frac{\alpha^3\Lambda}{4}\Big(1 - \frac{\alpha\Lambda}{2}\Big)R^2_{\mu\nu}\Big]. \quad (76)$$

For generic α, this theory is plagued with a massive ghost [1,10]. Thus, the action proposed by Deser and Gibbons [17] does not yield a unitary spin-2 theory around its constant curvature background for any choice of the curvature except the flat space. But, setting $\alpha = \frac{2}{\Lambda}$ one can get rid of the "bad" $R^2_{\mu\nu}$ term, and hope to obtain a unitary theory. However, this turns out to be not true, since in this case setting $\Lambda = \Lambda_0$ which follows from (75), one ends up with

$$I = \int d^4x\sqrt{-g}\Big[-\frac{1}{\kappa}(R - 2\Lambda_0)\Big], \quad (77)$$

which has the opposite sign of the Einstein-Hilbert action. That means as long as one has $\kappa > 0$ (which we must have for the unitary in flat space), the small fluctuations will have negative kinetic energy and even for the tuned value of α, (71) defines a nonunitary theory. We should note in passing that this result does not necessarily imply negative energy for the *exact* nonvacuum solutions such as black holes of (71). We have not yet found the black hole solutions of this action, but we can give an example in which small fluctuations around the vacuum have negative energy yet the exact solutions have positive energy. This example is the Einstein-Gauss-Bonnet theory whose exact spherically symmetric solution was given in [26] and whose energy was computed in [25]. As discussed in the latter work, this energy is positive, even though the linearized action of the Einstein-Gauss-Bonnet theory around its constant curvature vacuum is opposite to that of Einstein's theory [just like (77)]. This is because the spherically symmetric Schwarzschild-de Sitter solution goes (say in five dimensions) as $-g_{00} = g^{rr} \sim 1 + \frac{m}{r^2} + \Lambda r^2$ unlike the usual Schwarzschild solution which goes like $-g_{00} = g^{rr} \sim 1 - \frac{m}{r^2}$, the two minus signs take care of each other.

IV. CONCLUSION

We have developed techniques of analyzing the unitarity of Born-Infeld gravity actions around their constant curvature vacua. The special determinantal form of the action gave rise to remarkable simplifications that allow one to write a compact expression for the free, that is $O(h^2)$, theory. To summarize our result, let us note the following: One needs to find the $O(h^2)$ action of

$$\mathcal{L} = \frac{2}{\kappa\alpha}\big[\sqrt{-\det(\delta^{\mu}_{\nu} + \alpha R^{\mu}_{\nu} + \beta(\mathrm{Riem}, \mathrm{Ric}, R, \ldots)^{\mu}_{\nu})} - (\alpha\Lambda_0 + 1)\big] \quad (78)$$

to study its tree-level unitarity. In this work what we have done is to give a method to determine the parameters K, Λ, a, b, c in the following Lagrangian whose $O(h^2)$ expansion equals that of (78)

$$\mathcal{L}_{\text{equivalent}} = \frac{1}{K(\kappa, \alpha, \beta, \Lambda_0, \ldots)}[R - 2\Lambda(\kappa, \alpha, \beta, \Lambda_0, \ldots)] + a(\kappa, \alpha, \beta, \Lambda_0, \ldots)R^2 + b(\kappa, \alpha, \beta, \Lambda_0, \ldots)R^2_{\mu\nu} + c(\kappa, \alpha, \beta, \Lambda_0, \ldots)R^2_{\mu\nu\rho\sigma}$$

to *all orders* in the curvature expansion. Once this equivalent quadratic Lagrangian is obtained, unitarity analysis proceeds with the standard methods as discussed in [10]. We have also presented two examples one of which was proposed as a unitary theory in flat space [17], but turned out to be nonunitary in curved space according to our computation above. The other simpler example was considered to show the details of our method.

Let us give a recipe of how one should check the tree-level unitarity of a given Born-Infeld gravity in generic dimension D around its constant curvature vacuum. First to find the effective cosmological constant Λ, one has to expand the action up to $O(h)$ around the constant curvature vacuum using (26), or one should find the equivalent linear, that is $O(R)$, action and read the cosmological constant from it. Then, one should find the $O(h^2)$ action using (29) or alternatively one should construct the equivalent quadratic action $[O(R^2)]$. The method we have presented (29) works just as good in odd and even dimensions. But, the second method, as discussed in detail in the text, which proceeds by construction of an equivalent $O(R)$ and $O(R^2)$ actions should be done with great care depending on the number of dimensions and on the complexity of $A_{\mu\nu}$. The original BI action cannot simply be expanded in small curvature to get these equivalent actions via (22). What always works, in principle, is that the determinant can be expanded exactly in terms of traces within the square root, then one can do the expansions (63) and (60), and use the Hindawi *et al.* technique. But, the exact expansion of the determinant in terms of traces can generate quite a large number of terms especially for $D \geq 4$. [For example, in (71) doing such an exact expansion is not advised to the reader.] Therefore, to get the equivalent action one should proceed as follows in generic even dimensions $D = 2n + 2$: one has to expand the BI action up to $O(A^{n+1})$ with $n \geq 1$ using (22), if the resultant action is not already quadratic in the curvature, then using (60), the equivalent quadratic action should be constructed. For generic odd dimensions, the best way is to use (29), but for $D = 3$ and for not so complicated $A_{\mu\nu}$, exact trace expansion can also be employed. In this work we have laid out the details of checking unitarity of BI gravities, in a separate work we will provide examples of unitary models around flat and constant curvature backgrounds [5].

ACKNOWLEDGMENTS

This work is supported by the TÜBİTAK Grant No. 110T339, and METU Grant No. BAP-07-02-2010-00-02. Some of the calculations in this paper were either done or checked with the help of the computer package CADABRA [27,28].

APPENDIX A: A TWO-DIMENSIONAL EXAMPLE

In order to understand why in even dimensions finite number of terms in the $A_{\mu\nu}$ expansion of the BI-type actions contribute to $O(h)$ and $O(h^2)$ expansions, let us study a simple two-dimensional determinantal function

$$f(\tau, \gamma) = \sqrt{\det\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \gamma \begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix}\right]}, \tag{A1}$$

where τ and γ are two independent variables. The τ, γ expansions of $f(\tau, \gamma)$ represent the metric perturbation expansion and the $A_{\mu\nu}$ expansion, respectively, for the BI-type actions. What we will show in this Appendix is that $f(\tau, \gamma)$ and the function $g(\tau, \gamma)$ defined as

$$g(\tau, \gamma) \equiv 1 + \tfrac{1}{2}\gamma[a(\tau) + d(\tau)], \tag{A2}$$

have the same the $O(\tau)$ expansion around $\tau = 0$ only if

$$\left[\begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix}\right]_{\tau=0} = \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix}, \tag{A3}$$

which is the analog of the maximally symmetric constant curvature background in the BI-type gravity. Here, the important point about $g(\tau, \gamma)$ is that it is just the $O(\gamma)$ expansion of $f(\tau, \gamma)$ obtained by using (22), but note that we exactly define $g(\tau, \gamma)$ in this way and *do not assume that γ is small*. Thus, staying at first order in τ expansion requires just the first order in γ, while one naively expects that first order in τ expansion should involve each order in γ. Let us understand this in more detail by considering a generic function $\phi(\tau, \gamma)$ and expand it in τ as a Taylor series around $\tau = 0$

$$\phi(\tau, \gamma) = \phi(\tau = 0, \gamma) + \left(\frac{\partial \phi}{\partial \tau}\right)_{\tau=0} \tau + O(\tau^2) + \ldots, \tag{A4}$$

where $(\frac{\partial \phi}{\partial \tau})_{\tau=0}$ is a function of γ only. One can write the power series expansion of $(\frac{\partial \phi}{\partial \tau})_{\tau=0}$ in γ by assuming $\phi(\tau, \gamma) = \sum_{i=0}^{\infty} \psi_i(\tau)\gamma^i$ and expanding each $\psi_i(\tau)$ to the first order in τ. Then, one has

$$\phi(\tau, \gamma) = \sum_{i=0}^{\infty} \psi_i(\tau = 0)\gamma^i + \left[\sum_{i=0}^{\infty} \left(\frac{\partial \psi_i}{\partial \tau}\right)_{\tau=0} \gamma^i\right]\tau + \ldots$$
$$\Rightarrow \left(\frac{\partial \phi}{\partial \tau}\right)_{\tau=0} = \sum_{i=0}^{\infty} \left(\frac{\partial \psi_i}{\partial \tau}\right)_{\tau=0} \gamma^i. \tag{A5}$$

For the determinantal function $f(\tau, \gamma)$, the terms $(\frac{\partial \psi_i}{\partial \tau})_{\tau=0}$, $i \geq 2$ are all zero. Let us observe this for the $i = 2$ term explicitly. First, one can have the $O(\gamma^2)$ expansion of $f(\tau, \gamma)$ by using (22) as

$$\sqrt{\det\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \gamma \begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix}\right]}$$
$$= 1 + \frac{1}{2}\gamma[a(\tau) + d(\tau)] + \frac{1}{8}\gamma^2[a(\tau) + d(\tau)]^2 - \frac{1}{4}\gamma^2[a^2(\tau) + 2b(\tau)c(\tau) + d^2(\tau)] + O(\gamma^3). \tag{A6}$$

Assuming

$$\begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix} = \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix} + \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\tau + O(\tau^2), \tag{A7}$$

one has

$$
\begin{aligned}
f(\tau,\gamma) &= 1+\tfrac{1}{2}\gamma[(a_0+\tau a_1)+(a_0+\tau d_1)]+\tfrac{1}{8}\gamma^2[(a_0+\tau a_1)+(a_0+\tau d_1)]^2-\tfrac{1}{4}\gamma^2[(a_0+\tau a_1)^2+2(\tau b_1)(\tau c_1)\\
&\quad +(a_0+\tau d_1)^2]+O(\gamma^3)\\
&=(1+\gamma a_0)+\tfrac{1}{2}\gamma\tau(a_1+d_1)+\tfrac{1}{8}\gamma^2[4a_0^2+4\tau a_0(a_1+d_1)+O(\tau^2)]-\tfrac{1}{4}\gamma^2[2a_0^2+2\tau a_0(a_1+d_1)+O(\tau^2)]+O(\gamma^3)\\
&=(1+\gamma a_0)+\tfrac{1}{2}\gamma\tau(a_1+d_1)+O(\tau^2)+O(\gamma^3). \qquad \text{(A8)}
\end{aligned}
$$

Thus, $O(\tau)$ contributions coming from the two $O(\gamma^2)$ terms cancel each other because of the specific coefficients in (22) and the assumption (A3). Now, let us verify our proposal by explicitly calculating $O(\tau)$ expansions of $f(\tau,\gamma)$ and $g(\tau,\gamma)$. By using (A7) in $f(\tau,\gamma)$, one obtains

$$
\begin{aligned}
&\sqrt{\det\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\gamma\begin{pmatrix}a_0+\tau a_1 & \tau b_1\\ \tau c_1 & a_0+\tau d_1\end{pmatrix}+O(\tau^2)\right]}\\
&=(1+\gamma a_0)\sqrt{\det\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{\gamma\tau}{(1+\gamma a_0)}\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}+O(\tau^2)\right]}, \qquad \text{(A9)}
\end{aligned}
$$

and it is possible to make the $O(\tau)$ expansion by using (22);

$$
\begin{aligned}
f(\tau,\gamma) &= (1+\gamma a_0)\left[1+\frac{1}{2}\frac{\gamma\tau}{(1+\gamma a_0)}(a_1+d_1)+O(\tau^2)\right]\\
&=(1+\gamma a_0)+\frac{1}{2}\gamma\tau(a_1+d_1)+O(\tau^2). \qquad \text{(A10)}
\end{aligned}
$$

Therefore, the $O(\gamma^{2+n})$, $n\geq 1$ terms in γ expansion of $f(\tau,\gamma)$ do not contribute to the $O(\tau)$ terms, only if (A3) holds. On the other hand, the $O(\tau)$ expansion of $g(\tau,\gamma)$ can be simply found as

$$
\begin{aligned}
g(\tau,\gamma) &= 1+\tfrac{1}{2}\gamma[2a_0+\tau(a_1+d_1)]\\
&=(1+\gamma a_0)+\tfrac{1}{2}\gamma\tau(a_1+d_1). \qquad \text{(A11)}
\end{aligned}
$$

As a result, if one wants to consider $O(\tau)$ behavior of $f(\tau,\gamma)$, then one can equally work with just $g(\tau,\gamma)$ which is simply equal to the $O(\gamma)$ expansion of $f(\tau,\gamma)$.

APPENDIX B: ANALYZING EINSTEIN-HILBERT ACTION AND QUADRATIC CURVATURE GRAVITY WITH SECOND ORDER PERTURBATIONS

In this Appendix, second order expansions of the curvature tensors are used in the well-known cases of the Einstein-Hilbert theory, and the quadratic actions including the Einstein-Gauss-Bonnet theory. This will help us construct the following $O(h^2)$ actions that frequently appear in the computations:

$$
\begin{aligned}
&\int d^4x\sqrt{-\bar g}R_{(2)}, \qquad \int d^4x\sqrt{-\bar g}\bar g^{\mu\nu}R^{(2)}_{\mu\nu},\\
&\int d^4x\sqrt{-\bar g}R_L^{\mu\nu}R^L_{\mu\nu}, \qquad \int d^4x\sqrt{-\bar g}(R^2_{\mu\rho\sigma\lambda})^{(2)}, \qquad \text{(B1)}\\
&\int d^4x\sqrt{-\bar g}\bar g^{\sigma\nu}\bar g^{\lambda\gamma}(R^{\mu}{}_{\rho\sigma\lambda})^{(1)}(R^{\rho}{}_{\mu\gamma\nu})^{(1)}
\end{aligned}
$$

in terms of the building blocks appearing in Eq. (25) of [25].

1. Analysis of the Einstein-Hilbert action

First, let us find the second order in metric perturbation for Einstein-Hilbert action:

$$
I=\frac{1}{\kappa}\int d^4x\sqrt{-g}(R-2\Lambda_0), \qquad \text{(B2)}
$$

and expanding up to third order in $h_{\mu\nu}$ yields

$$
\begin{aligned}
I&=\frac{1}{\kappa}\int d^4x\sqrt{-\bar g}\left[1+\frac{\tau}{2}h+\frac{1}{8}\tau^2(h^2-2h^2_{\mu\nu})+O(\tau^3)\right]\\
&\quad\times[(\bar R-2\Lambda_0)+\tau R_L+\tau^2R_{(2)}+O(\tau^3)]\\
&=\frac{1}{\kappa}\int d^4x\sqrt{-\bar g}\left\{(\bar R-2\Lambda_0)+\tau\left[\frac{1}{2}h(\bar R-2\Lambda_0)+R_L\right]\right.\\
&\quad+\tau^2\left[\frac{1}{8}(\bar R-2\Lambda_0)(h^2-2h^2_{\mu\nu})+\frac{1}{2}hR_L+R_{(2)}\right]\\
&\quad\left.+O(\tau^3)\right\}. \qquad \text{(B3)}
\end{aligned}
$$

One can find the nonlinear equation of motion for constant curvature background by investigating the first order term in τ of the above action as

$$
I_{O(h)}=\frac{1}{\kappa}\int d^4x\sqrt{-\bar g}\left[\frac{1}{2}h(\bar R-2\Lambda_0)+R_L\right], \qquad \text{(B4)}
$$

after putting the explicit form of R_L and dropping out a boundary term one can get

$$
I_{O(h)}=\frac{1}{\kappa}\int d^4x\sqrt{-\bar g}h(\Lambda-\Lambda_0), \qquad \text{(B5)}
$$

from which it follows that $(\Lambda-\Lambda_0)\bar g_{\mu\nu}=0$ upon taking variation with respect to $h_{\mu\nu}$.

One can read the second order action as

$$
\begin{aligned}
I_{O(h^2)}&=\frac{1}{\kappa}\int d^4x\sqrt{-\bar g}\left\{h^{\mu\nu}\left[\frac{1}{2}\left(\Lambda-\frac{1}{2}\Lambda_0\right)(\bar g_{\mu\nu}h-2h_{\mu\nu})\right.\right.\\
&\quad\left.\left.+\frac{1}{2}\bar g_{\mu\nu}R_L\right]+R_{(2)}\right\}, \qquad \text{(B6)}
\end{aligned}
$$

where $R_{(2)}$ can be read from (13) as

$$
\begin{aligned}
R_{(2)}&=\bar R^{\rho\lambda}h_{\alpha\rho}h^{\alpha}_{\lambda}-h^{\mu\nu}R^L_{\mu\nu}-\bar g^{\nu\sigma}h^{\mu}_{\beta}(R^{\beta}{}_{\nu\mu\sigma})_L\\
&\quad-\bar g^{\nu\sigma}\bar g^{\mu\alpha}\bar g_{\beta\gamma}[(\Gamma^{\gamma}_{\mu\alpha})_L(\Gamma^{\beta}_{\sigma\nu})_L-(\Gamma^{\gamma}_{\sigma\alpha})_L(\Gamma^{\beta}_{\mu\nu})_L]. \qquad \text{(B7)}
\end{aligned}
$$

Let us concentrate on $\int d^4x\sqrt{-\bar{g}}R_{(2)}$ part of the action and work out the integration by parts;

$$\int d^4x\sqrt{-\bar{g}}R_{(2)} = \int d^4x\sqrt{-\bar{g}}\{\Lambda h^2_{\mu\nu} - h^{\mu\nu}R^L_{\mu\nu} - \bar{g}^{\nu\sigma}h^{\mu}_{\beta}(R^{\beta}{}_{\nu\mu\sigma})_L - \bar{g}^{\nu\sigma}\bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}[(\Gamma^{\gamma}_{\mu\alpha})_L(\Gamma^{\beta}_{\sigma\nu})_L - (\Gamma^{\gamma}_{\sigma\alpha})_L(\Gamma^{\beta}_{\mu\nu})_L]\}. \tag{B8}$$

One can find $\bar{g}^{\nu\sigma}h^{\mu}_{\beta}(R^{\beta}{}_{\nu\mu\sigma})_L$ as

$$\bar{g}^{\nu\sigma}h^{\mu}_{\beta}(R^{\beta}{}_{\nu\mu\sigma})_L = h^{\mu\nu}\left(R^L_{\mu\nu} - \frac{4\Lambda}{3}h_{\mu\nu} + \frac{\Lambda}{3}\bar{g}_{\mu\nu}h\right). \tag{B9}$$

By using the definition of the linearized Christoffel connection in (8) and doing integration by parts, the last two terms in $\int d^4x\sqrt{-\bar{g}}R_{(2)}$ can be found as

$$\int d^4x\sqrt{-\bar{g}}\bar{g}^{\nu\sigma}\bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}(\Gamma^{\gamma}_{\mu\alpha})_L(\Gamma^{\beta}_{\sigma\nu})_L = \int d^4x\sqrt{-\bar{g}}\left[-\frac{1}{2}h^{\mu\nu}\left(\bar{\nabla}^{\sigma}\bar{\nabla}_{\mu}h_{\nu\sigma} + \bar{\nabla}^{\sigma}\bar{\nabla}_{\nu}h_{\mu\sigma} - \frac{3}{2}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}h\right) + h^{\mu\nu}\left(\frac{4\Lambda}{3}h_{\mu\nu} - \frac{\Lambda}{12}\bar{g}_{\mu\nu}h\right) + \frac{1}{4}h^{\mu\nu}\bar{g}_{\mu\nu}R_L\right], \tag{B10}$$

$$\int d^4x\sqrt{-\bar{g}}\bar{g}^{\nu\sigma}\bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}(\Gamma^{\gamma}_{\sigma\alpha})_L(\Gamma^{\beta}_{\mu\nu})_L = \int d^4x\sqrt{-\bar{g}}\left[-\frac{1}{4}h^{\mu\nu}(3\Box h_{\mu\nu} - \bar{\nabla}^{\sigma}\bar{\nabla}_{\mu}h_{\sigma\nu} - \bar{\nabla}^{\sigma}\bar{\nabla}_{\nu}h_{\mu\sigma})\right]. \tag{B11}$$

Finally, $\int d^4x\sqrt{-\bar{g}}R_{(2)}$ becomes

$$\int d^4x\sqrt{-\bar{g}}R_{(2)} = \int d^4x\sqrt{-\bar{g}}h^{\mu\nu}\left(-\frac{1}{2}R^L_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}R_L + \Lambda h_{\mu\nu} - \frac{\Lambda}{4}\bar{g}_{\mu\nu}h\right), \tag{B12}$$

and putting this result in (B6) yields

$$I_{O(h^2)} = -\frac{1}{2\kappa}\int d^4x\sqrt{-\bar{g}}h^{\mu\nu}\left[\mathcal{G}^L_{\mu\nu} + \frac{1}{2}(\Lambda_0 - \Lambda)(\bar{g}_{\mu\nu}h - 2h_{\mu\nu})\right], \tag{B13}$$

and since $\Lambda = \Lambda_0$ is found from equations of motion for constant curvature background;

$$I_{O(h^2)} = -\frac{1}{2\kappa}\int d^4x\sqrt{-\bar{g}}h^{\mu\nu}\mathcal{G}^L_{\mu\nu}. \tag{B14}$$

2. Analysis of the quadratic action

Now, let us consider the quadratic actions in the form

$$I = \int d^4x\sqrt{-g}\left[\frac{1}{\kappa}(R - 2\Lambda_0) + \alpha R^2 + \beta R^2_{\mu\nu}\right], \tag{B15}$$

and calculate the second order action in metric perturbations. Then, up to third order, the expansion of the action is

$$I = \int d^4x\sqrt{-\bar{g}}\left\{\left[\frac{1}{\kappa}(\bar{R} - 2\Lambda_0) + \alpha\bar{R}^2 + \beta\bar{R}^2_{\mu\nu}\right] + \tau\left[\frac{1}{2}h\left(\frac{1}{\kappa}(\bar{R} - 2\Lambda_0) + \alpha\bar{R}^2 + \beta\bar{R}^2_{\mu\nu}\right) + \left(\frac{1}{\kappa}R_L + 2\alpha\bar{R}R_L + \beta\bar{R}^{\mu\nu}R^L_{\mu\nu} + \beta(R^{\mu\nu})_{(1)}\bar{R}_{\mu\nu}\right)\right] + \tau^2\left[\frac{1}{8}(h^2 - 2h^2_{\mu\nu})\left(\frac{1}{\kappa}(\bar{R} - 2\Lambda_0) + \alpha\bar{R}^2 + \beta\bar{R}^2_{\mu\nu}\right) + \frac{1}{2}h\left(\frac{1}{\kappa}R_L + 2\alpha\bar{R}R_L + \beta\bar{R}^{\mu\nu}R^L_{\mu\nu} + \beta(R^{\mu\nu})_{(1)}\bar{R}_{\mu\nu}\right) + \left(\frac{1}{\kappa}R_{(2)} + 2\alpha\bar{R}R_{(2)} + \alpha R^2_L + \beta\bar{R}^{\mu\nu}R^{(2)}_{\mu\nu} + \beta(R^{\mu\nu})_{(1)}R^L_{\mu\nu} + \beta(R^{\mu\nu})_{(2)}\bar{R}_{\mu\nu}\right)\right]\right\}. \tag{B16}$$

Here, note that $(R^{\mu\nu})_{(1)}$ and $(R^{\mu\nu})_{(2)}$ are the first and the second order terms in the metric perturbation expansion of $R^{\mu\nu}$. First of all, in order to find the nonlinear equation of motion for constant curvature background, one needs to study the $O(\tau)$ term in the above action. After using the definitions of $R^L_{\mu\nu}$, R_L and dropping out the boundary terms one can get

$$I_{O(h)} = \frac{1}{\kappa}\int d^4x\sqrt{-\bar{g}}h(\Lambda - \Lambda_0), \tag{B17}$$

which yields the equation of motion $\Lambda = \Lambda_0$. Then, let us move to the second order term in metric perturbation. After using the result given in (B12), one can obtain

$$I_{O(h^2)} = -\frac{1}{2}\int d^4x\sqrt{-\bar{g}}\bigg\{\Big(\frac{1}{\kappa}+8\alpha\Lambda+4\beta\Lambda\Big)h^{\mu\nu}\mathcal{G}^L_{\mu\nu} - \frac{1}{2}h^2\Big[\frac{1}{\kappa}(\Lambda-\Lambda_0)+2\beta\Lambda^2\Big]+h^2_{\mu\nu}\Big[\frac{1}{\kappa}(\Lambda-\Lambda_0) + 6\beta\Lambda^2\Big]+2\alpha h^{\mu\nu}(\bar{g}_{\mu\nu}\bar{\Box}-\bar{\nabla}_\mu\bar{\nabla}_\nu+\Lambda\bar{g}_{\mu\nu})R_L - 2\beta(\Lambda\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}+R^{\mu\nu}_L R^L_{\mu\nu}+R^{\mu\nu}_{(2)}\Lambda\bar{g}_{\mu\nu})\bigg\}. \quad \text{(B18)}$$

Here, let us first handle $\int d^4x\sqrt{-\bar{g}}R^{\mu\nu}_L R^L_{\mu\nu}$. Using the definition of $R^L_{\mu\nu}$ and using the linearized Bianchi identity (and also its covariant derivative) which is

$$\bar{\nabla}^\mu\mathcal{G}^L_{\mu\nu}=0, \qquad \mathcal{G}^L_{\mu\nu}\equiv R^L_{\mu\nu}-\tfrac{1}{2}\bar{g}_{\mu\nu}R_L-\Lambda h_{\mu\nu}, \quad \text{(B19)}$$

one can find the following result after use of integration by parts

$$\int d^4x\sqrt{-\bar{g}}R^{\mu\nu}_L R^L_{\mu\nu} = -\frac{1}{2}\int d^4x\sqrt{-\bar{g}}h^{\mu\nu}\Big[(\bar{g}_{\mu\nu}\Box-\bar{\nabla}_\mu\bar{\nabla}_\nu+\Lambda\bar{g}_{\mu\nu})R_L + \Big(\Box\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\Big)-\frac{14\Lambda}{3}R^L_{\mu\nu}+\frac{\Lambda}{3}\bar{g}_{\mu\nu}R_L + \frac{8\Lambda^2}{3}h_{\mu\nu}\Big]. \quad \text{(B20)}$$

Second, $(R^{\mu\nu})_{(2)}$ is related to $R^{(2)}_{\mu\nu}$ in the following way:

$$\bar{g}_{\mu\nu}(R^{\mu\nu})_{(2)} = \bar{g}_{\mu\nu}(g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta})^{(2)} = \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}-2h^{\mu\nu}R^L_{\mu\nu}+3\Lambda h^2_{\mu\nu}. \quad \text{(B21)}$$

The $\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}$ term can be given in terms of $R_{(2)}$ with

$$R_{(2)} = (g^{\mu\nu}R_{\mu\nu})_{(2)} = \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}-h^{\mu\nu}R^L_{\mu\nu}+\Lambda h^2_{\mu\nu}. \quad \text{(B22)}$$

Then, $\int d^4x\sqrt{-\bar{g}}\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}$ becomes

$$\int d^4x\sqrt{-\bar{g}}\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} = h^{\mu\nu}\Big(\frac{1}{2}R^L_{\mu\nu}-\frac{1}{4}\bar{g}_{\mu\nu}R_L-\frac{\Lambda}{4}\bar{g}_{\mu\nu}h\Big), \quad \text{(B23)}$$

with the help of (B12). By use of these results and the equation of motion for constant curvature background which is $\Lambda=\Lambda_0$ in $I_{O(h^2)}$, one can get

$$I_{O(h^2)} = -\frac{1}{2}\int d^4x\sqrt{-\bar{g}}h^{\mu\nu}\Big[\Big(\frac{1}{\kappa}+8\alpha\Lambda+\frac{4}{3}\beta\Lambda\Big)\mathcal{G}^L_{\mu\nu} + (2\alpha+\beta)(\bar{g}_{\mu\nu}\Box-\bar{\nabla}_\mu\bar{\nabla}_\nu+\Lambda\bar{g}_{\mu\nu})R_L + \beta\Big(\Box\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\Big)\Big], \quad \text{(B24)}$$

which is same as Eq. (25) of [25].

Finally, let us analyze the Einstein-Gauss-Bonnet theory,

$$I = \int d^4x\sqrt{-g}\Big[\frac{1}{\kappa}(R-2\Lambda_0)+\gamma(R^2_{\mu\rho\sigma\lambda}-4R^2_{\mu\nu}+R^2)\Big], \quad \text{(B25)}$$

just to check the consistency of our construction. Here, the only remaining part that we have not analyzed is the $R^2_{\mu\rho\sigma\lambda}$ term. First, let us use the previous result in order to obtain the second order action in metric perturbations for the terms other than $R^2_{\mu\rho\sigma\lambda}$:

$$I_{O(h^2)} = -\frac{1}{2}\int d^4x\sqrt{-\bar{g}}h^{\mu\nu}\Big[\Big(\frac{1}{\kappa}+\frac{8}{3}\gamma\Lambda\Big)\mathcal{G}^L_{\mu\nu} - 2\gamma(\bar{g}_{\mu\nu}\Box-\bar{\nabla}_\mu\bar{\nabla}_\nu+\Lambda\bar{g}_{\mu\nu})R_L - 4\gamma\Big(\bar{\Box}\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\Big)\Big]. \quad \text{(B26)}$$

Then, up to third order, expansion of the last term becomes

$$I = \gamma\int d^4x\sqrt{-\bar{g}}\bigg\{\bar{R}^2_{\mu\rho\sigma\lambda}+\tau\Big[(R^2_{\mu\rho\sigma\lambda})^{(1)}+\frac{1}{2}h\bar{R}^2_{\mu\rho\sigma\lambda}\Big] + \tau^2\Big[(R^2_{\mu\rho\sigma\lambda})^{(2)}+\frac{1}{2}h(R^2_{\mu\rho\sigma\lambda})^{(1)} + \frac{1}{8}\bar{R}^2_{\mu\rho\sigma\lambda}(h^2-2h^2_{\mu\nu})\Big]\bigg\}. \quad \text{(B27)}$$

First of all, it should be shown that first order part is a boundary term such that it should not give a contribution to equation of motion for constant curvature background:

$$I_{O(h)} = \int d^4x\sqrt{-\bar{g}}\Big[(R^2_{\mu\rho\sigma\lambda})^{(1)}+\frac{1}{2}h\bar{R}^2_{\mu\rho\sigma\lambda}\Big], \quad \text{(B28)}$$

where

$$\bar{R}^2_{\mu\rho\sigma\lambda}=\frac{8\Lambda^2}{3}, \qquad (R^2_{\mu\rho\sigma\lambda})^{(1)}=\frac{4\Lambda}{3}R_L. \quad \text{(B29)}$$

Then,

$$I_{O(h)} = \int d^4x\sqrt{-\bar{g}}[(\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu}-\Box h)], \quad \text{(B30)}$$

and since the remaining part is a boundary term, no contribution comes to the constant curvature background equation of motion from the square of the Riemann tensor. Then, moving to the part that is second order in metric perturbation

$$I_{O(h^2)} = \gamma \int d^4x\sqrt{-\bar{g}}\Big[(R^2_{\mu\rho\sigma\lambda})^{(2)} + \frac{2\Lambda}{3}hR_L + \frac{\Lambda^2}{3}(h^2 - 2h^2_{\mu\nu})\Big], \qquad \text{(B31)}$$

where

$$\begin{aligned}(R^2_{\mu\rho\sigma\lambda})^{(2)} &= (R^\mu{}_{\rho\sigma\lambda}R^\rho{}_{\mu\gamma\nu}g^{\sigma\nu}g^{\lambda\gamma})^{(2)}\\ &= 2\bar{R}^\rho{}_\mu{}^{\lambda\sigma}(R^\mu{}_{\rho\sigma\lambda})^{(2)} + 2\bar{R}^\mu{}_{\rho\sigma\lambda}\bar{R}^\rho{}_\mu{}^\lambda{}_\nu g^{\sigma\nu}_{(2)}\\ &\quad + \bar{g}^{\sigma\nu}\bar{g}^{\lambda\gamma}(R^\mu{}_{\rho\sigma\lambda})^{(1)}(R^\rho{}_{\mu\gamma\nu})^{(1)}\\ &\quad + 2[\bar{R}^\rho{}_\mu{}^\lambda{}_\nu(R^\mu{}_{\rho\sigma\lambda})^{(1)} + \bar{R}^\mu{}_{\rho\sigma}{}^\gamma(R^\rho{}_{\mu\gamma\nu})^{(1)}]g^{\sigma\nu}_{(1)}\\ &\quad + \bar{R}^\mu{}_{\rho\sigma\lambda}\bar{R}^\rho{}_{\mu\gamma\nu}g^{\sigma\nu}_{(1)}g^{\lambda\gamma}_{(1)},\end{aligned} \qquad \text{(B32)}$$

Using $\bar{R}_{\mu\nu\rho\sigma} = \frac{\Lambda}{3}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho})$ and $R_{(2)} = \bar{g}^{\rho\sigma}R^{(2)}_{\rho\sigma} + g^{\rho\sigma}_{(1)}R^{(1)}_{\rho\sigma} + \bar{R}_{\rho\sigma}g^{\rho\sigma}_{(2)}$;

$$(R^2_{\mu\rho\sigma\lambda})^{(2)} = \frac{4\Lambda}{3}R_{(2)} + \bar{g}^{\sigma\nu}\bar{g}^{\lambda\gamma}(R^\mu{}_{\rho\sigma\lambda})^{(1)}(R^\rho{}_{\mu\gamma\nu})^{(1)} - \frac{4\Lambda}{3}\bar{g}^{\rho\lambda}(R^\mu{}_{\rho\sigma\lambda})^{(1)}h^\sigma_\mu + \frac{2\Lambda^2}{9}(h^2 - h^2_{\mu\nu}), \qquad \text{(B33)}$$

and using (B9)

$$(R^2_{\mu\rho\sigma\lambda})^{(2)} = \frac{4\Lambda}{3}R_{(2)} - \bar{g}^{\sigma\nu}\bar{g}^{\lambda\gamma}(R^\mu{}_{\rho\lambda\sigma})^{(1)}(R^\rho{}_{\mu\gamma\nu})^{(1)} - \frac{4\Lambda}{3}h^{\mu\nu}R^L_{\mu\nu} + \frac{14\Lambda^2}{9}h^2_{\mu\nu} - \frac{2\Lambda^2}{9}h^2. \qquad \text{(B34)}$$

Now, let us consider $\int d^4x\sqrt{-\bar{g}}(R^2_{\mu\rho\sigma\lambda})^{(2)}$. The $\int d^4x\sqrt{-\bar{g}}\bar{g}^{\sigma\nu}\bar{g}^{\lambda\gamma}(R^\mu{}_{\rho\sigma\lambda})^{(1)}(R^\rho{}_{\mu\gamma\nu})^{(1)}$ term can be found as

$$\begin{aligned}&\bar{g}^{\mu\nu}\bar{g}^{\rho\alpha}(R^\lambda{}_{\sigma\rho\mu})^{(1)}(R^\sigma{}_{\lambda\alpha\nu})^{(1)}\\ &= h^{\mu\nu}\Big[2\Big(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\Big) + (\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu\\ &\quad + \Lambda\bar{g}_{\mu\nu})R_L\Big] - \frac{\Lambda}{9}h^{\mu\nu}(30R^L_{\mu\nu} - 9\bar{g}_{\mu\nu}R_L - 32\Lambda h_{\mu\nu}\\ &\quad + 2\Lambda\bar{g}_{\mu\nu}h),\end{aligned} \qquad \text{(B35)}$$

after a somewhat lengthy calculation where the definition of the linearized Riemann tensor is used and the terms are rearranged by using integration by parts. Using this result with (B12), one get

$$\begin{aligned}&\int d^4x\sqrt{-\bar{g}}(R^2_{\mu\rho\sigma\lambda})^{(2)}\\ &= \int d^4x\sqrt{-\bar{g}}h^{\mu\nu}\Big\{-\Big[2\Big(\Box\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\Big)\\ &\quad + (\bar{g}_{\mu\nu}\Box - \bar{\nabla}_\mu\bar{\nabla}_\nu + \Lambda\bar{g}_{\mu\nu})R_L\Big] + \frac{\Lambda}{3}(8\mathcal{G}^L_{\mu\nu} - 4R^L_{\mu\nu}\\ &\quad + 6\Lambda h_{\mu\nu} - \Lambda\bar{g}_{\mu\nu}h)\Big\},\end{aligned} \qquad \text{(B36)}$$

and plugging it in the action:

$$I_{O(h^2)} = -\frac{1}{2}\gamma\int d^4x\sqrt{-\bar{g}}h^{\mu\nu}\Big[-\frac{8\Lambda}{3}\mathcal{G}^L_{\mu\nu} + 2(\bar{g}_{\mu\nu}\Box - \bar{\nabla}_\mu\bar{\nabla}_\nu + \Lambda\bar{g}_{\mu\nu})R_L + 4\Big(\Box\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\Big)\Big], \qquad \text{(B37)}$$

and considering this result with the part of the action coming from γR^2 and $-4\gamma R^2_{\mu\nu}$ terms given in (B26), one finds that all the γ terms vanish, and the Gauss-Bonnet term does not contribute to the equation of motion.

APPENDIX C: LINEARIZATION OF THE $O(R^3)$ ACTION

The following formulae are needed for the linearization of the $O[(\alpha R)^3]$ equations. The quadratic parts below already appeared in [25], we reproduce them here for the sake of completeness, the cubic parts are new.

$$\begin{aligned}\delta(R_{\lambda\nu\alpha\mu}R^{\lambda\alpha}) &= \frac{2\Lambda}{3}R^L_{\mu\nu} + \frac{\Lambda}{3}\bar{g}_{\mu\nu}R_L + \frac{\Lambda^2}{3}h_{\mu\nu},\\ \delta(\Box R_{\mu\nu}) &= \Box R^L_{\mu\nu} - \Lambda\Box h_{\mu\nu},\\ \delta(\nabla_\mu\nabla_\nu R) &= \bar{\nabla}_\mu\bar{\nabla}_\nu R_L,\\ \delta(\Box R) &= \Box R_L,\\ \delta(R^\rho_\mu R_{\rho\alpha}R^\alpha_\nu) &= 3\Lambda^2R^L_{\mu\nu} - 2\Lambda^3h_{\mu\nu},\\ \delta(R_{\mu\nu}R^2_{\alpha\beta}) &= 4\Lambda^2R^L_{\mu\nu} + 2\Lambda^2\bar{g}_{\mu\nu}R_L,\\ \delta(R_{\mu\nu}R^2) &= 16\Lambda^2R^L_{\mu\nu} + 8\Lambda^2\bar{g}_{\mu\nu}R_L,\\ \delta(RR^\rho_\nu R_{\mu\rho}) &= 8\Lambda^2R^L_{\mu\nu} + \Lambda^2\bar{g}_{\mu\nu}R_L - 4\Lambda^3h_{\mu\nu},\\ \delta(\nabla_\alpha\nabla_\mu R^\alpha_\nu) &= \frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu R_L + \frac{4\Lambda}{3}R^L_{\mu\nu} - \frac{\Lambda}{3}\bar{g}_{\mu\nu}R_L - \frac{4\Lambda^2}{3}h_{\mu\nu},\\ \delta(\nabla_\mu\nabla_\nu R_{\alpha\beta}) &= \bar{\nabla}_\mu\bar{\nabla}_\nu R_{\alpha\beta} - \Lambda\bar{\nabla}_\mu\bar{\nabla}_\nu h_{\alpha\beta}.\end{aligned} \qquad \text{(C1)}$$

Here, last two equations can related by using linearized Bianchi identity:

$$\bar{\nabla}^\mu\mathcal{G}^L_{\mu\nu} = 0, \qquad \mathcal{G}^L_{\mu\nu} \equiv R^L_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}R_L - \Lambda h_{\mu\nu}. \qquad \text{(C2)}$$

APPENDIX D: COEFFICIENTS FOR THE $R - \bar{R}$ EXPANSION

Coefficients in the expansion of the square root of (68) are

$$\begin{aligned}
\left[\frac{\partial f}{\partial R}\right]_{(\bar{R},\bar{R}^{\mu}_{\nu})} &= \frac{\alpha}{2(1+\alpha\Lambda)^2}(1+4\alpha\Lambda+6\alpha^2\Lambda^2+4\alpha^3\Lambda^3),\\
\left[\frac{\partial f}{\partial R^{\alpha}_{\beta}}\right]_{(\bar{R},\bar{R}^{\mu}_{\nu})} &= -\frac{\alpha\delta^{\beta}_{\alpha}}{2(1+\alpha\Lambda)^2}(\alpha\Lambda+3\alpha^2\Lambda^2+3\alpha^3\Lambda^3),\\
\left[\frac{\partial^2 f}{\partial R^2}\right]_{(\bar{R},\bar{R}^{\mu}_{\nu})} &= \frac{\alpha^2}{2(1+\alpha\Lambda)^2}(1+4\alpha\Lambda+6\alpha^2\Lambda^2) - \frac{\alpha^2}{4(1+\alpha\Lambda)^6}(1+4\alpha\Lambda+6\alpha^2\Lambda^2+4\alpha^3\Lambda^3)^2,\\
\left[\frac{\partial^2 f}{\partial R^{\rho}_{\sigma}\partial R^{\alpha}_{\beta}}\right]_{(\bar{R},\bar{R}^{\mu}_{\nu})} &= -\frac{\alpha^2}{2(1+\alpha\Lambda)^2}[(1+\alpha\Lambda)^2\delta^{\beta}_{\rho}\delta^{\sigma}_{\alpha} - \alpha^2\Lambda^2\delta^{\beta}_{\alpha}\delta^{\sigma}_{\rho}] - \frac{\alpha^2\delta^{\beta}_{\alpha}\delta^{\sigma}_{\rho}}{4(1+\alpha\Lambda)^6}(\alpha\Lambda+3\alpha^2\Lambda^2+3\alpha^3\Lambda^3)^2,\\
\left[\frac{\partial f}{\partial R\partial R^{\alpha}_{\beta}}\right]_{(\bar{R},\bar{R}^{\mu}_{\nu})} &= -\frac{\alpha^2\delta^{\beta}_{\alpha}}{2(1+\alpha\Lambda)^2}(\alpha\Lambda+3\alpha^2\Lambda^2) + \frac{\alpha^2\delta^{\beta}_{\alpha}}{4(1+\alpha\Lambda)^6}(1+4\alpha\Lambda+6\alpha^2\Lambda^2+4\alpha^3\Lambda^3)(\alpha\Lambda+3\alpha^2\Lambda^2+3\alpha^3\Lambda^3).
\end{aligned} \tag{D1}$$

Coefficients in the expansion of the (73) are

$$\begin{aligned}
f(\bar{R},\bar{R}^{\mu}_{\nu}) &= -\alpha^3\Lambda^3\left(1-\frac{\alpha\Lambda}{4}\right), & \left[\frac{\partial f}{\partial R}\right]_{(\bar{R},\bar{R}^{\mu}_{\nu})} &= -2\alpha^3\Lambda^2\left(1-\frac{3\alpha\Lambda}{8}\right),\\
\left[\frac{\partial f}{\partial R^{\alpha}_{\beta}}\right]_{(\bar{R},\bar{R}^{\mu}_{\nu})} &= \frac{5\alpha^3\Lambda^2}{4}\delta^{\beta}_{\alpha} - \frac{\alpha^4\Lambda^3}{2}\delta^{\beta}_{\alpha}, & \left[\frac{\partial^2 f}{\partial R^2}\right]_{(\bar{R},\bar{R}^{\mu}_{\nu})} &= -\frac{3\alpha^3}{2}\Lambda + \frac{9\alpha^4\Lambda^2}{8},\\
\left[\frac{\partial f}{\partial R\partial R^{\alpha}_{\beta}}\right]_{(\bar{R},\bar{R}^{\mu}_{\nu})} &= \frac{\alpha^3\Lambda}{2}\delta^{\beta}_{\alpha} - \frac{9\alpha^4\Lambda^2}{16}\delta^{\beta}_{\alpha}, & \left[\frac{\partial^2 f}{\partial R^{\rho}_{\sigma}\partial R^{\alpha}_{\beta}}\right]_{(\bar{R},\bar{R}^{\mu}_{\nu})} &= \frac{\alpha^3\Lambda}{2}\delta^{\beta}_{\rho}\delta^{\sigma}_{\alpha} - \frac{\alpha^4\Lambda^2}{4}\delta^{\beta}_{\rho}\delta^{\sigma}_{\alpha} + \frac{\alpha^4\Lambda^2}{4}\delta^{\sigma}_{\rho}\delta^{\beta}_{\alpha}.
\end{aligned} \tag{D2}$$

[1] K. S. Stelle, Phys. Rev. D **16**, 953 (1977).
[2] S. Weinberg, in *Understanding the Fundamental Constituents of Matter*, edited by A. Zichichi (Plenum Press, New York, 1977).
[3] M. R. Niedermaier, Phys. Rev. Lett. **103**, 101303 (2009).
[4] S. Weinberg, Phys. Rev. D **81**, 083535 (2010).
[5] I. Gullu, T. C. Sisman, and B. Tekin (work in progress).
[6] I. Gullu, T. C. Sisman, and B. Tekin, Classical Quantum Gravity **27**, 162001 (2010); Phys. Rev. D **82**, 024032 (2010).
[7] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, Phys. Rev. Lett. **102**, 201301 (2009); Phys. Rev. D **79**, 124042 (2009).
[8] S. Deser, Phys. Rev. Lett. **103**, 101302 (2009).
[9] M. Nakasone and I. Oda, Prog. Theor. Phys. **121**, 1389 (2009).
[10] I. Gullu and B. Tekin, Phys. Rev. D **80**, 064033 (2009).
[11] I. Gullu, T. C. Sisman, and B. Tekin, Phys. Rev. D **81**, 104017 (2010).
[12] A. Sinha, J. High Energy Phys. 06 (2010) 61.
[13] G. W. Gibbons, Rev. Mex. Fis. **49S1**, 19 (2003).
[14] A. Eddington, *The Mathematical Theory of General Relativity* (Cambridge University Press, Cambridge, England, 1924).
[15] E. Schrödinger, *Space-Time Structure* (Cambridge University Press, Cambridge, England, 1950).
[16] M. Banados and P. G. Ferreira, Phys. Rev. Lett. **105**, 011101 (2010).
[17] S. Deser and G. W. Gibbons, Classical Quantum Gravity **15**, L35 (1998).
[18] M. N. R. Wohlfarth, Classical Quantum Gravity **21**, 1927 (2004); **21**, 5297(E) (2004).
[19] J. A. Nieto, Phys. Rev. D **70**, 044042 (2004).
[20] D. Comelli and A. Dolgov, J. High Energy Phys. 11 (2004) 062.
[21] D. Comelli, Phys. Rev. D **72**, 064018 (2005).
[22] R. Ferraro and F. Fiorini, Phys. Lett. B **692**, 206 (2010).
[23] I. Gullu, T. C. Sisman, and B. Tekin, Phys. Rev. D **81**, 104018 (2010).
[24] A. Hindawi, B. A. Ovrut, and D. Waldram, Phys. Rev. D **53**, 5597 (1996).
[25] S. Deser and B. Tekin, Phys. Rev. D **67**, 084009 (2003).
[26] D. G. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985).
[27] K. Peeters, Comput. Phys. Commun. **176**, 550 (2007).
[28] K. Peeters, arXiv:hep-th/0701238.