

Primordial magnetic field limits from cosmic microwave background bispectrum of magnetic passive scalar modes

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Primordial magnetic fields lead to non-Gaussian signals in the cosmic microwave background (CMB) even at the lowest order, as magnetic stresses and the temperature anisotropy they induce depend quadratically on the magnetic field. In contrast, CMB non-Gaussianity due to inflationary scalar perturbations arises only as a higher order effect. Apart from a compensated scalar mode, stochastic primordial magnetic fields also produce scalar anisotropic stress that remains uncompensated till neutrino decoupling. This gives rise to an adiabaticlike scalar perturbation mode that evolves passively thereafter (called the passive mode). We compute the CMB reduced bispectrum ($B_{L_1 L_2 L_3}$) induced by this passive mode, sourced via the Sachs-Wolfe effect, on large angular scales. For any configuration of bispectrum, taking a partial sum over mode-coupling terms, we find a typical value of $l_1(l_1 + 1)l_3(l_3 + 1)b_{l_1 l_2 l_3} \sim 6 - 9 \times 10^{-16}$, for a magnetic field of $B_0 \sim 3$ nG, assuming a nearly scale-invariant magnetic spectrum. We also evaluate, in full, the bispectrum for the squeezed collinear configuration over all angular mode-coupling terms and find $l_1(l_1 + 1)l_3(l_3 + 1)b_{l_1 l_2 l_3} \approx -1.4 \times 10^{-16}$. These values are more than $\sim 10^6$ times larger than the previously calculated magnetic compensated scalar mode CMB bispectrum. Observational limits on the bispectrum from WMAP7 data allow us to set upper limits of $B_0 \sim 2$ nG on the present value of the cosmic magnetic field of primordial origin. This is over 10 times more stringent than earlier limits on B_0 based on the compensated mode bispectrum.

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I. INTRODUCTION

The origin and evolution of large-scale magnetic fields in the Universe is not yet clearly understood. There is observational evidence for micro-Gauss strength magnetic fields, ordered on kiloparsec to 10 kpc scales, in galaxies and clusters of galaxies [1–5]. There is also evidence for galactic scale fields at higher redshift [6,7]. There are only tentative indications, to date, of fields not associated with individual galaxies or clusters [8]. Further, recent γ -ray observations using *Fermi*/LAT data provide hints of a detection [9] or a lower bound [10] of $B \sim 10^{-16}$ G on intergalactic scales. Constraints on purely cosmological magnetic fields have also been derived from the CMB [11–30], big bang nucleosynthesis, and polarized radiation from extragalactic radio sources (for a review see e.g. [31]).

It is generally accepted that the observed fields require a seed magnetic field, possibly of primordial origin, which

are then amplified by astrophysical processes [3,4,31–34]. A strong enough seed may mainly require amplification due to flux-freezing which arises during the collapse to form structures [4]. On the other hand a weak seed would require considerable dynamo action as well [3,4,32–34]. As yet no compelling theory exists for the origin of strong primordial fields. Equally, dynamo theories are not without their own difficulties, in particular, how large-scale dynamos lead to coherent enough fields on saturation [3]. Thus it is important to keep an open mind on the issue of the origin of magnetic fields.

Large-scale primordial fields could have resulted from phase transitions in the early universe, for example, during the inflationary era [35–39]. An important way to constrain or detect such fields is via their imprints on the cosmic microwave background (CMB) anisotropies. Considerable work has already been done to calculate the magnetic field signals in the power spectrum (both temperature and polarization) of the CMB anisotropies [13–30,40–45]. The possibility of non-Gaussian signals in the CMB temperature anisotropies has drawn increased attention of late. This possibility is particularly relevant when one considers magnetically induced anisotropies in the CMB, for the

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following reason: In standard inflationary models, the Gaussian statistics of the quantum fluctuations in the inflaton field leads to Gaussian statistics for the CMB temperature field. In such models, any non-Gaussian signal in the temperature field arises generically due to higher order effects [46–51].

The situation is different in the case of magnetically induced signals. Magnetic stresses depend quadratically on the field. Hence, even if the fields are assumed to be Gaussian, the corresponding stresses are not. This implies that the CMB temperature anisotropy, which these stresses induce, do not have Gaussian statistics. Unlike the case of CMB non-Gaussianity from inflationary perturbations, in the case of magnetic fields, the CMB non-Gaussianity is a leading order effect. Thus the study of CMB non-Gaussianity has a special significance in the context of probing and detecting primordial cosmological magnetic fields.

Preliminary studies of such non-Gaussian signals in the CMB induced by primordial magnetic fields have begun [52–54], based on earlier calculations of non-Gaussianity in the magnetic stress energy [55]. These have been limited to the scalar mode signals on large scales and further restricted to a component referred to as the compensated scalar mode by [40]. However, the work of Shaw and Lewis [40] has revealed that much larger CMB anisotropies can result from scalar perturbations sourced by the magnetic anisotropic stresses prior to neutrino decoupling (also see [56–58]). This mode is referred to as the passive scalar mode [40]. In this paper we calculate the contribution to the CMB bispectrum from the passive scalar mode. As we show here, this contribution greatly dominates over the contribution calculated earlier by two of us [52] (hereafter referred to as SS09), and also calculated by [53]. The current work allows tighter limits to be placed on primordial magnetic fields.

In the next section we describe the properties of the primordial stochastic magnetic field and in Sec. III we describe the perturbation induced due to the passive scalar mode. The magnetic CMB temperature anisotropy is described in Sec. IV and in its subsections we calculate the three-point correlation of the scalar anisotropic stress in two different configurations. In Sec. V we calculate the passive mode reduced bispectrum for the different configurations and in Sec. VI we use the reduced magnetic bispectra to put constraints on the primordial magnetic field. Our conclusions are summarized in Sec. VII.

II. PRIMORDIAL STOCHASTIC MAGNETIC FIELD

We consider a stochastic magnetic field B which is a Gaussian random field characterized and completely specified by its energy power spectrum $M(k)$. In addition we assume the magnetic field is nonhelical.

On galactic and larger scales, any velocity induced by Lorentz forces is generally so small that it does not lead to

appreciable distortion of the initial field [59,60]. Hence, the magnetic field simply redshifts away as $\mathbf{B}(\mathbf{x}, t) = \mathbf{b}_0(\mathbf{x})/a^2$, where, \mathbf{b}_0 is the magnetic field at the present epoch (i.e. at $z = 0$ or $a = 1$).

We define $\mathbf{b}(\mathbf{k})$ as the Fourier transform of the magnetic field $\mathbf{b}_0(\mathbf{x})$. The energy power spectrum is defined by the relation $\langle b_i(\mathbf{k})b_j^*(\mathbf{q}) \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{q}) P_{ij}(\mathbf{k}) M(k)$, where $P_{ij}(\mathbf{k}) = (\delta_{ij} - k_i k_j / k^2)$ is the projection operator ensuring $\nabla \cdot \mathbf{b}_0 = 0$. This gives $\langle \mathbf{b}_0^2 \rangle = 2 \int (dk/k) \Delta_b^2(k)$, where $\Delta_b^2(k) = k^3 M(k) / (2\pi^2)$ is the power per logarithmic interval in k -space residing in the stochastic magnetic field.

As in [61], we assume a power-law magnetic energy spectra, $M(k) = Ak^n$ that has a cutoff at $k = k_c$, where k_c is the Alfvén-wave damping length scale [59,60]. We fix A by demanding that the variance of the magnetic field smoothed over a “galactic” scale, $k_G = 1 \text{ h Mpc}^{-1}$, (using a sharp k -space filter) is B_0 . This gives (for $n > -3$ and for $k < k_c$)

$$\Delta_b^2(k) = \frac{k^3 M(k)}{2\pi^2} = \frac{B_0^2}{2} (n+3) \left(\frac{k}{k_G}\right)^{3+n}. \quad (1)$$

The magnetic spectral index is restricted to $-3 \leq n < -3/2$. Blue spectral indices are strongly disfavored by many observations like the CMB power spectra [28,29] and especially by the gravitational wave limits of [62].

III. PASSIVE SCALAR MODE

The stress tensor (space-space part of the energy-momentum tensor) for magnetic fields in terms of the present-day magnetic field value \mathbf{b}_0 is

$$T_j^i(\mathbf{x}) = \frac{1}{4\pi a^4} \left(\frac{1}{2} b_0^2(\mathbf{x}) \delta_j^i - b_0^i(\mathbf{x}) b_{0j}(\mathbf{x}) \right). \quad (2)$$

In Fourier space, the product of magnetic fields becomes a convolution

$$S_j^i(\mathbf{k}) = \int b^i(\mathbf{q}) b_j(\mathbf{k} - \mathbf{q}) d^3 \mathbf{q}, \quad (3)$$

$$T_j^i(\mathbf{k}) = \frac{1}{4\pi a^4} \left(\frac{1}{2} S_\alpha^\alpha(\mathbf{k}) \delta_j^i - S_j^i(\mathbf{k}) \right). \quad (4)$$

This can be expressed in terms of the magnetic perturbations to the energy-momentum tensor as

$$T_j^i(\mathbf{k}) = p_\gamma (\Delta_B \delta_j^i + \Pi_{Bj}^i), \quad (5)$$

where Δ_B and Π_{Bj}^i are the perturbations in the energy density and anisotropic stress, respectively, as defined in [40] and p_γ is the radiation pressure.

The magnetic stresses are nonlinear in the field but we assume that they are always small compared to the total energy density and pressure of the photons, baryons, etc., thus allowing a purely linear treatment of the perturbations. Hence the scalar, vector, and tensor perturbations decouple and evolve independently and here we focus on the

anisotropic scalar perturbations $\Pi_B(\mathbf{k})$ which are given by applying the relevant projection operator to $T_j^i(\mathbf{k})$ [55],

$$\Pi_B(\mathbf{k}) = -\frac{3}{2}(\hat{\mathbf{k}}_i \hat{\mathbf{k}}_j - \frac{1}{3}\delta_{ij})\Pi_B^{ij}. \quad (6)$$

Note that $\Pi_B(\mathbf{k})$ of [40] is equal to $-\tau^S(\mathbf{k})$ of [55].

Such an anisotropic stress term also arises due to neutrinos after they decouple. The net anisotropic stress tensor acts as the source term in the scalar Einstein's equations that lead to the Bardeen equation for the potential. Prior to neutrino decoupling, the only source of anisotropic stress is the magnetic field. Once the neutrinos decouple, the anisotropic stress due to neutrinos also contributes but with an opposite sign to that of the magnetic field, thus compensating the contribution from the magnetic field [18]. After compensation there are two modes of perturbation [40]. The first one, called the passive mode is an adiabatic-like mode but has non-Gaussian statistics. It grows logarithmically in amplitude between the epochs of magnetic field generation and neutrino decoupling but then evolves passively after neutrino decoupling. This behavior has also been confirmed in [57] in the context of deriving the magnetic Sachs-Wolfe effect for a causally generated primordial magnetic field. The second, more well-studied perturbation [23–26], is called the compensated mode.

The final curvature perturbation due to the passive mode is given by Eq. 86 of Shaw and Lewis [40],

$$\zeta = \zeta(\tau_B) - \frac{1}{3}R_\gamma \Pi_B \left[\ln\left(\frac{\tau_\nu}{\tau_B}\right) + \left(\frac{5}{8R_\nu} - 1\right) \right] \quad (7)$$

in the conformal Newtonian gauge. The evolution of the curvature perturbation has also been discussed (in synchronous gauge) in [56] for the case of an extra source of anisotropic stress cancelling the neutrino anisotropic stress. The role of anisotropic stresses on CMB has also been discussed by [63]. Here $R_\nu \sim 0.4$ is the fractional contribution of neutrino energy density towards the total energy density of the relativistic component. The fractional contribution of radiation energy density is $R_\gamma = 1 - R_\nu \sim 0.6$. The $\zeta(\tau_B)$ term represents all primordial contributions to the curvature perturbation. The log term shows the growth in the curvature between the epochs of magnetic field generation τ_B and neutrino decoupling τ_ν . In the radiation-dominated era the conformal time τ is inversely proportional to the temperature T so that $\tau_\nu/\tau_B = T_B/T_\nu$. The term $\ln(T_B/T_\nu)$ is always bigger than 10 for the range of choices of τ_B corresponding to the temperature range from $T_B \approx 10^{14}$ GeV of the inflationary epoch to $T_B \approx 10^3$ GeV of the electroweak epoch. Hence, we can neglect the $((5/8R_\nu) - 1)$ term with less than 5% error. After neutrino decoupling, the anisotropic stress Π_B is compensated and there is a remnant adiabatic mode with amplitude given by [40],

$$\zeta \approx -\frac{1}{3}R_\gamma \Pi_B \ln\left(\frac{\tau_\nu}{\tau_B}\right). \quad (8)$$

This so-called passive mode evolves passively like an adiabatic perturbation but the statistics of Π_B are non-Gaussian unlike the standard inflationary adiabatic mode. Also, unlike the compensated mode, the passive mode amplitude depends logarithmically on the epoch of magnetic field generation. Therefore, limits on the primordial magnetic field strength arising from this passive mode will be somewhat sensitive to the assumed model and epoch of magnetic field generation.

IV. MAGNETIC CMB ANISOTROPY AND THREE-POINT CORRELATION

We consider the CMB temperature anisotropies that are sourced by the scalar passive mode. On large angular scales the important contribution is via the magnetically induced Sachs-Wolfe effect

$$\frac{\Delta T}{T}(\mathbf{n}) = \frac{1}{3}\Phi(\mathbf{x}_0 - \mathbf{n}D^*) = \frac{1}{5}\zeta(\mathbf{x}_0 - \mathbf{n}D^*), \quad (9)$$

where the second equality is from, for example, Eq. (10.42) of [64]. The unit vector \mathbf{n} gives the direction of observation and D^* is the (angular diameter) distance to the surface of last scatter. Then employing Eq. (8) we get

$$\begin{aligned} \frac{\Delta T}{T}(\mathbf{n}) &\approx -\frac{1}{15}R_\gamma \ln\left(\frac{\tau_\nu}{\tau_B}\right)\Pi_B(\mathbf{x}_0 - \mathbf{n}D^*) \\ &\approx \mathcal{R}_p \Pi_B(\mathbf{x}_0 - \mathbf{n}D^*), \end{aligned} \quad (10)$$

where we define

$$\mathcal{R}_p = -\frac{1}{15}R_\gamma \ln\left(\frac{\tau_\nu}{\tau_B}\right). \quad (11)$$

In SS09 we had calculated the bispectrum due to the compensated scalar mode for which $\Delta T/T = (R_\gamma/20)\Delta_B$ [65]. Equation (10) for the passive scalar mode leads to a $\Delta T/T$ which is larger by a factor $\mathcal{A} = (4/3)\ln(\tau_\nu/\tau_B) \times (\Pi_B/\Delta_B)$. Assuming that the fractional perturbations Δ_B and Π_B are of the same order, we get \mathcal{A} of order 50 for τ_B corresponding to $T_B \approx 10^{14}$ GeV. As we will see below for the magnetic spectra we consider, this alone leads to an enhanced passive scalar mode contribution to the bispectrum of order $\mathcal{A}^3 \approx 10^5$ and thus stronger limits on the primordial magnetic field by a factor $\mathcal{A}^{0.5}$. We assume instantaneous recombination which is a good approximation for large angular scales. There could also be additional integrated Sachs-Wolfe contributions to $\Delta T/T(\mathbf{n})$.

The magnetic Sachs-Wolfe contribution to $\Delta T/T$ has also been estimated by Bonvin and Caprini [57], although in a different context and with different approximations. They obtain a log term in their Eq. (6.11) which is very similar in magnitude to our estimate here in Eq. (10). This can be seen after taking into account their definition of Π_B which is a factor of $R_\gamma/3$ times the Π_B defined here.

The CMB temperature fluctuation in a direction (\mathbf{n}) at the observer can be expanded in terms of the spherical harmonics to give

$$\frac{\Delta T(\mathbf{n})}{T} = \sum_{lm} a_{lm} Y_{lm}(\mathbf{n}), \quad (12)$$

where, for the passive scalar mode,

$$a_{lm} = 4\pi(-i)^l \int \frac{d^3 k}{(2\pi)^3} \mathcal{R}_p \Pi_B(\mathbf{k}) j_l(kD^*) Y_{lm}^*(\hat{\mathbf{k}}). \quad (13)$$

Here $\Pi_B(\mathbf{k})$ is the Fourier transform of $\Pi_B(\mathbf{x})$.

The non-Gaussianity in the CMB temperature anisotropy can be evaluated by calculating its 3-point correlation function (in harmonic space), called the bispectrum, $B_{l_1 l_2 l_3}^{m_1 m_2 m_3}$. We assume that the magnetic perturbations are uncorrelated with the primary (inflationary) perturbations in the CMB.

In terms of the a_{lm} 's the CMB bispectrum is given by

$$B_{l_1 l_2 l_3}^{m_1 m_2 m_3} = \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle. \quad (14)$$

From Eq. (13) we can express $B_{l_1 l_2 l_3}^{m_1 m_2 m_3}$ as

$$B_{l_1 l_2 l_3}^{m_1 m_2 m_3} = \mathcal{R}_p^3 \int \left[\prod_{i=1}^3 (-i)^{l_i} \frac{d^3 k_i}{2\pi^2} j_{l_i}(k_i D^*) Y_{l_i m_i}^*(\hat{\mathbf{k}}_i) \right] \zeta_{123}, \quad (15)$$

with ζ_{123} defined as

$$\zeta_{123} = \langle \hat{\Pi}_B(\mathbf{k}_1) \hat{\Pi}_B(\mathbf{k}_2) \hat{\Pi}_B(\mathbf{k}_3) \rangle. \quad (16)$$

The function ζ_{123} is the three-point correlation of $\hat{\Pi}_B(\mathbf{k})$ and involves a 6-point correlation function of the magnetic fields. Using Wick's theorem it can be written as a function of magnetic spectra in an analogous manner to [55] and Eq. (7) of SS09, $\zeta_{123} = \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \psi_{123}$, where

$$\psi_{123} = \frac{1}{(4\pi p_\gamma)^3} \int d^3 s M(|\mathbf{k}_1 + \mathbf{s}|) M(\mathbf{s}) M(|\mathbf{s} - \mathbf{k}_3|) \times (\mathcal{F}_{\Pi_B \Pi_B \Pi_B}), \quad (17)$$

where $\mathcal{F}_{\Pi_B \Pi_B \Pi_B}$ is the angular component of the three-point correlation ζ_{123} of the scalar anisotropic stress $\Pi_B(\mathbf{k})$ in mode-coupling integral ψ_{123} .

A. Three-point correlation of scalar anisotropic stress

This angular component $\mathcal{F}_{\Pi_B \Pi_B \Pi_B}$ is given by a 58-term expression in [55] derived by applying relevant projection operators to extract the scalar part of the full bispectrum i.e. the six-point correlation of the magnetic fields. In this particular case, the operator is given by $\mathcal{A}_{ijklmn} = (-1)^3 \mathcal{Q}_{ij}(\mathbf{k}_3) \mathcal{Q}_{kl}(\mathbf{k}_1) \mathcal{Q}_{mn}(\mathbf{k}_2)$ where $\mathcal{Q}_{ab}(\mathbf{k}) \equiv \delta_{ab} - (3/2)P_{ab}(\mathbf{k})$ and $P_{ab}(\mathbf{k}) = \delta_{ab} - \hat{\mathbf{k}}_a \hat{\mathbf{k}}_b$ is the projection operator. We present the full angular component (in our notation we have absorbed a factor of 8 multiplying \mathcal{F} in [55] into our definition of $\mathcal{F}_{\Pi_B \Pi_B \Pi_B}$):

$$\mathcal{F}_{\Pi_B \Pi_B \Pi_B} = \sum_{n=0}^6 \mathcal{F}_{\Pi_B \Pi_B \Pi_B}^n \quad (18)$$

with

$$\begin{aligned} \mathcal{F}_{\Pi_B \Pi_B \Pi_B}^0 &= -9, \\ \mathcal{F}_{\Pi_B \Pi_B \Pi_B}^1 &= 0, \\ \mathcal{F}_{\Pi_B \Pi_B \Pi_B}^2 &= (\bar{\beta}^2 + \bar{\gamma}^2 + \bar{\mu}^2 + 9(\theta_{13}^2 + \theta_{23}^2 + \theta_{12}^2) + 3(\alpha_3^2 + \alpha_1^2 + \alpha_2^2 + \beta_3^2 + \beta_1^2 + \beta_2^2 + \gamma_3^2 + \gamma_1^2 + \gamma_2^2)), \\ \mathcal{F}_{\Pi_B \Pi_B \Pi_B}^3 &= -3(\bar{\mu}(\beta_3 \gamma_3 + \beta_1 \gamma_1 + \beta_2 \gamma_2 + \frac{1}{3} \bar{\beta} \bar{\gamma}) + \bar{\gamma}(\alpha_3 \gamma_3 + \alpha_1 \gamma_1 + \alpha_2 \gamma_2) + \bar{\beta}(\alpha_3 \beta_3 + \alpha_1 \beta_1 + \alpha_2 \beta_2) \\ &\quad + 3\theta_{13}(\alpha_3 \alpha_1 + \beta_3 \beta_1 + \gamma_3 \gamma_1) + 3\theta_{23}(\alpha_3 \alpha_2 + \beta_3 \beta_2 + \gamma_3 \gamma_2) + 3\theta_{12}(\alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2) \\ &\quad + 9\theta_{13} \theta_{23} \theta_{12}), \\ \mathcal{F}_{\Pi_B \Pi_B \Pi_B}^4 &= 3(\bar{\gamma} \bar{\mu} \alpha_3 \beta_3 + \bar{\beta} \bar{\mu} \alpha_1 \gamma_1 + \bar{\beta} \bar{\gamma} \beta_2 \gamma_2 + 3(\bar{\mu} \theta_{13} \beta_3 \gamma_1 + \bar{\gamma} \theta_{23} \alpha_3 \gamma_2 + \bar{\beta} \theta_{12} \alpha_1 \beta_2) \\ &\quad + 3(\alpha_3 \beta_3 (\alpha_1 \beta_1 + \alpha_2 \beta_2) + \alpha_1 \gamma_1 (\alpha_3 \gamma_3 + \alpha_2 \gamma_2) + \beta_2 \gamma_2 (\beta_3 \gamma_3 + \beta_1 \gamma_1)) \\ &\quad + 9(\theta_{13} \theta_{23} \gamma_1 \gamma_2 + \theta_{13} \theta_{12} \beta_3 \beta_2 + \theta_{23} \theta_{12} \alpha_3 \alpha_1)), \\ \mathcal{F}_{\Pi_B \Pi_B \Pi_B}^5 &= -9(\bar{\mu} \alpha_3 \beta_3 \alpha_1 \gamma_1 + \bar{\gamma} \alpha_3 \beta_3 \beta_2 \gamma_2 + \bar{\beta} \alpha_1 \gamma_1 \beta_2 \gamma_2 + 3(\theta_{13} \beta_3 \gamma_1 \beta_2 \gamma_2 + \theta_{23} \alpha_3 \alpha_1 \gamma_1 \gamma_2 \\ &\quad + \theta_{12} \alpha_3 \beta_3 \alpha_1 \beta_2)), \\ \mathcal{F}_{\Pi_B \Pi_B \Pi_B}^6 &= 27\alpha_3 \beta_3 \alpha_1 \gamma_1 \beta_2 \gamma_2, \end{aligned} \quad (19)$$

where

$$\begin{aligned}\bar{\beta} &= (\hat{s} \cdot \widehat{\mathbf{k}_3} - s), & \bar{\gamma} &= (\hat{s} \cdot \widehat{\mathbf{k}_1} + s), \\ \bar{\mu} &= (\widehat{\mathbf{k}_3} - s \cdot \widehat{\mathbf{k}_1} + s),\end{aligned}\quad (20)$$

where the hat on a vector denotes its unit vector. Also,

$$\begin{aligned}\alpha_a &= \hat{\mathbf{k}}_a \cdot s, & \beta_a &= \hat{\mathbf{k}}_a \cdot \widehat{\mathbf{k}_3} - s, \\ \gamma_a &= \hat{\mathbf{k}}_a \cdot \widehat{\mathbf{k}_1} + s,\end{aligned}\quad (21)$$

and

$$\theta_{ab} = \hat{\mathbf{k}}_a \cdot \hat{\mathbf{k}}_b, \quad (22)$$

where our angle definitions are consistent with [55] and slightly different to SS09.

For simplicity, we evaluate this $\mathcal{F}_{\Pi_B \Pi_B \Pi_B}$ expression to find the bispectrum in the following two cases below. We expect that the magnitude of the bispectrum will be of similar order for a general case.

Case I.—Consider any bispectrum configuration but include only the s -independent terms (constant for any particular configuration) in the $\mathcal{F}_{\Pi_B \Pi_B \Pi_B}$ expression. Then the s integral in Eq. (17) is performed without reference to any particular bispectrum configuration.

Case II.—The squeezed collinear configuration where we calculate fully the mode-coupling integral over all angular terms in the $\mathcal{F}_{\Pi_B \Pi_B \Pi_B}$ expression.

B. Case I: Evaluation with s -independent terms in $\mathcal{F}_{\Pi_B \Pi_B \Pi_B}$

The s -independent terms include θ_{12} , θ_{23} , θ_{13} that are constant for a given $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ configuration. For each configuration considered, we calculate the sum of the five s -independent terms in Eq. (19) for $\mathcal{F}_{\Pi_B \Pi_B \Pi_B}$

$$\mathcal{F}_{\Pi_B \Pi_B \Pi_B}^1 = -9 + 9(\theta_{12}^2 + \theta_{23}^2 + \theta_{13}^2) - 27(\theta_{13}\theta_{12}\theta_{23}). \quad (23)$$

We give in Table I the values of $m = \mathcal{F}_{\Pi_B \Pi_B \Pi_B}^1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ for specific configurations $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$. Note that m happens to vanish exactly for the local isosceles configuration. Hence the Case I mode-coupling integral reduces to

$$\psi_{123} = \frac{m}{(4\pi p_\gamma)^3} I = \frac{3^3 m}{(4\pi \rho_0)^3} I, \quad (24)$$

where ρ_0 is the present-day energy density of radiation and

$$\begin{aligned}I &= \int d^3 s M(|\mathbf{k}_1 + s|) M(s) M(|s - \mathbf{k}_3|) \\ &= 2\pi A^3 \int_{-1}^1 d\mu \int_0^\infty ds s^{n+2} (s^2 + k_1^2 + 2sk_1\mu)^{(n/2)} \\ &\quad \times (s^2 + k_3^2 - 2sk_3\mu)^{(n/2)},\end{aligned}\quad (25)$$

where $\nu = \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{s}}$ and $\mu = \hat{\mathbf{k}}_3 \cdot \hat{\mathbf{s}}$.

TABLE I. The sum of s -independent terms $m = \mathcal{F}_{\Pi_B \Pi_B \Pi_B}^1$ in four different configurations $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ for evaluating the Case I bispectrum.

Configuration	$(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$	$(\theta_{12}, \theta_{23}, \theta_{13})$	m
Local	$k_1 \sim k_3$	$(0, 0, -1)$	0
Isosceles	$k_2 \ll k_1, k_3$		
Equilateral	$k_1 \sim k_2 \sim k_3$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	-5.625
Ssqueezed	$k_1 \sim k_3$	$(1, -1, -1)$	-8 ^a
Collinear	$k_2 \ll k_1, k_3$ $\hat{\mathbf{k}}_1 = \hat{\mathbf{k}}_2 = -\hat{\mathbf{k}}_3$		
Midpoint	$k_1 \sim k_2 \sim \frac{k_3}{2}$	$(1, -1, -1)$	-9
Collinear	$\hat{\mathbf{k}}_1 = \hat{\mathbf{k}}_2 = -\hat{\mathbf{k}}_3$		

^aFor the squeezed collinear configuration case, $\mathcal{F}_{\Pi_B \Pi_B \Pi_B}^1$ picks up another term $\bar{\mu}^2 \sim 1$.

We perform the mode-coupling integral using the technique discussed in [16,17,61]. As m vanishes for the local isosceles configuration, ψ_{123} is zero for this configuration. For the equilateral and squeezed collinear configurations we split the s integral into two subranges $0 < s < k_1 \sim k_3$ and $s > k_1 \sim k_3$. We then approximate the integrands by assuming $s \ll k_1 \sim k_3$ and $s \gg k_1 \sim k_3$ for the respective subranges. For the midpoint collinear configuration we split the s integral into two subranges $0 < s < k_1$ and $s > 2k_1 \sim k_3$ while neglecting the very small contribution from the middle subrange $k_1 < s < 2k_1 \sim k_3$. Again, we approximate the integrands by assuming $s \ll k_1$ and $s \gg 2k_1 \sim k_3$ for the respective subranges. To derive numerical estimates of the bispectrum and magnetic field strengths we will focus on nearly scale-invariant spectra (which can be produced by an acausal mechanism like inflation), i.e. $n \rightarrow -3$, which yield

$$I \simeq 4\pi A^3 \left[\frac{k_1^{2n+3} k_3^n}{n+3} - \frac{k_3^{3n+3}}{3n+3} \right] \simeq 4\pi A^3 \frac{k_1^{2n+3} k_3^n}{n+3}. \quad (26)$$

The latter equation is obtained in the limit $n \rightarrow -3$ where we can neglect the terms with $(3n+3)^{-1}$ compared to $(n+3)^{-1}$. Hence the mode-coupling integral for Case I—taking only s -independent angular part—is

$$\psi_{123} = (4)^4 m \frac{\pi^7}{k_G^6} (n+3)^2 \left(\frac{k_1}{k_G}\right)^{2n+3} \left(\frac{k_3}{k_G}\right)^n V_A^6. \quad (27)$$

Here we have defined V_A , the Alfvén velocity in the radiation-dominated era as [60]

$$V_A = \frac{B_0}{(16\pi\rho_0/3)^{1/2}} \approx 3.8 \times 10^{-4} B_{-9}, \quad (28)$$

with $B_{-9} \equiv (B_0/10^{-9}\text{Gauss})$.

C. Case II: Squeezed collinear configuration—all angular terms

For Case II—We take the squeezed collinear bispectrum configuration as $k_1 \sim k_3$ and $k_2 \ll k_1, k_3$ and $\hat{\mathbf{k}}_1 = \hat{\mathbf{k}}_2 = -\hat{\mathbf{k}}_3$. One wavevector (\mathbf{k}_2) is negligibly small compared to

the other two which are also in exactly opposite directions and of a similar magnitude. This affords considerable simplification and reduction of the angular terms given by Eq. (19) in doing the mode-coupling integral. Using $\mathbf{k}_3 \approx -\mathbf{k}_1$ we see that

$$\begin{aligned}\bar{\beta} &= (\hat{\mathbf{s}} \cdot \widehat{\mathbf{k}}_3 - s) \approx (\hat{\mathbf{s}} \cdot -\widehat{\mathbf{k}}_1 - s) \\ &\approx -(\hat{\mathbf{s}} \cdot \widehat{\mathbf{k}}_1 + s) \approx -\bar{\gamma},\end{aligned}\quad (29)$$

$$\begin{aligned}\bar{\mu} &= (\widehat{\mathbf{k}}_3 - s \cdot \widehat{\mathbf{k}}_1 + s) \approx (-\widehat{\mathbf{k}}_1 - s \cdot \widehat{\mathbf{k}}_1 + s) \\ &\approx -(\widehat{\mathbf{k}}_1 + s \cdot \widehat{\mathbf{k}}_1 + s) \approx -1,\end{aligned}\quad (30)$$

$$\alpha_1 = \alpha_2 = -\alpha_3 = \alpha = \nu = -\mu, \quad (31)$$

$$\beta_1 = \beta_2 = -\beta_3 = \beta, \quad (32)$$

$$\gamma_1 = \gamma_2 = -\gamma_3 = \gamma, \quad (33)$$

$$\theta_{12} = 1, \quad \theta_{23} = \theta_{13} = -1. \quad (34)$$

These relations substituted into Eq. (19) reduce the 58 terms to a 10-term angular expression for the squeezed collinear configuration

$$\begin{aligned}\mathcal{F}_{\Pi_B \Pi_B \Pi_B} &= -8 + \bar{\beta}^2 + 9(\mu^2 + 2\gamma^2) + 6\mu\bar{\beta}\gamma \\ &\quad + 3\bar{\beta}^2\gamma^2 - 9\gamma^2(3\mu^2 + \gamma^2) - 18\mu\bar{\beta}\gamma^3 \\ &\quad + 27\mu^2\gamma^4.\end{aligned}\quad (35)$$

We again perform the mode-coupling as discussed above and split the s integral into two subranges $0 < s < k_1 \sim k_3$ and $s > k_1 \sim k_3$. Once again we approximate the integrands by assuming $s \ll k_1 \sim k_3$ and $s \gg k_1 \sim k_3$ for the respective subranges to give

$$\psi_{123} = \frac{1}{(4\pi p_\gamma)^3} I = \left(\frac{3}{4\pi\rho_0}\right)^3 I, \quad (36)$$

where

$$\begin{aligned}I &= \int d^3s M(|\mathbf{k}_1 + \mathbf{s}|)M(s)M(|\mathbf{s} - \mathbf{k}_3|)[-8 + \bar{\beta}^2 \\ &\quad + 9(\mu^2 + 2\gamma^2) + 6\mu\bar{\beta}\gamma + 3\bar{\beta}^2\gamma^2 - 9\gamma^2(3\mu^2 + \gamma^2) \\ &\quad - 18\mu\bar{\beta}\gamma^3 + 27\mu^2\gamma^4].\end{aligned}\quad (37)$$

Performing the integrals for all the terms, with $n \rightarrow -3$, we find that there is considerable though incomplete cancellation between the 10 terms

$$\begin{aligned}I &\simeq 2\pi A^3 \frac{k_1^{2n+3} k_3^n}{n+3} \left[-16 + \frac{2}{3} + 6 + 36 + 4 + 2 - 18 \right. \\ &\quad \left. - 18 - 12 + 18 \right] \simeq 2\pi A^3 \frac{k_1^{2n+3} k_3^n}{n+3} \left[\frac{8}{3} \right].\end{aligned}\quad (38)$$

We draw attention to how the full evaluation gives a result for ψ_{123} that is of opposite sign and one-sixth magnitude

(8/3) of the value if we only consider the constant term (-16) for this squeezed collinear configuration. The sign of the mode-coupling integral is important as the bispectrum is not a positive-definite quantity (unlike the power spectrum) and the observed limits on the bispectrum may also be asymmetric about zero. Then the Case II—squeezed collinear mode-coupling integral is

$$\psi_{123} = \left[\frac{8}{3} \right] 2(4)^3 \frac{\pi^7}{k_G^6} (n+3)^2 \left(\frac{k_1}{k_G}\right)^{2n+3} \left(\frac{k_3}{k_G}\right)^n V_A^6. \quad (39)$$

As the n dependence of ψ_{123} is identical for both Case I and Case II we can write

$$\psi_{123} = \mathcal{K} \left[2(4)^3 \frac{\pi^7}{k_G^6} (n+3)^2 \left(\frac{k_1}{k_G}\right)^{2n+3} \left(\frac{k_3}{k_G}\right)^n V_A^6 \right], \quad (40)$$

where

$$\mathcal{K} = \begin{cases} 2m & \text{Case I} \\ \frac{8}{3} & \text{Case II.} \end{cases} \quad (41)$$

V. PASSIVE SCALAR CMB BISPECTRUM

Using Eq. (40) in Eq. (15) we can evaluate the CMB bispectrum for the passive scalar mode for both Case I and II. The algebraic steps are the same as those for the compensated scalar mode in SS09. We express the delta function present in ξ_{123} in its integral form $\delta(\mathbf{k}) = (1/(2\pi)^3) \int d^3x \exp(i\mathbf{k} \cdot \mathbf{x})$, use the spherical wave expansion of the exponential terms, substitute it into Eq. (15), and integrate over the angular parts of $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{x})$. This algebra is also very similar to that in the calculation of the primordial bispectrum [49]. Then it becomes possible to write the bispectrum $B_{l_1 l_2 l_3}^{m_1 m_2 m_3}$, in terms of a reduced bispectrum $b_{l_1 l_2 l_3}$ (also referred to as the Komatsu-Spergel estimator [47]) as

$$B_{l_1 l_2 l_3}^{m_1 m_2 m_3} = \mathcal{G}_{m_1 m_2 m_3}^{l_1 l_2 l_3} b_{l_1 l_2 l_3}, \quad (42)$$

where

$$b_{l_1 l_2 l_3} = \left(\frac{\mathcal{R}_p}{\pi^2}\right)^3 \int x^2 dx \prod_{i=1}^3 \int k_i^2 dk_i j_{l_i}(k_i x) j_{l_i}(k_i D^*) \psi_{123}, \quad (43)$$

and we have introduced the Gaunt integral

$$\mathcal{G}_{m_1 m_2 m_3}^{l_1 l_2 l_3} = \int d\Omega Y_{l_1 m_1} Y_{l_2 m_2} Y_{l_3 m_3}. \quad (44)$$

We substitute Eq. (40) into Eq. (43) for the reduced bispectrum. The integrals over k_2 can be immediately done using the spherical Bessel function closure relation

$$\int k_2^2 dk_2 j_{l_2}(k_2 x) j_{l_2}(k_2 D^*) = (\pi/2x^2) \delta(x - D^*), \quad (45)$$

and the delta function makes the x -integral trivial. We are then left with integrals over k_1 and k_3 given by

$$\begin{aligned}
 b_{l_1 l_2 l_3} &= \mathcal{K} \left(\frac{\mathcal{R}_p}{\pi^2} \right)^3 \frac{\pi}{2} [2(4)^3 \pi^7 (n+3)^2 V_A^6] \\
 &\times \left[\int \frac{dk_3}{k_3} j_{l_3}^2(k_3 D^*) \left(\frac{k_3}{k_G} \right)^{n+3} \right] \\
 &\times \left[\int \frac{dk_1}{k_1} j_{l_1}^2(k_1 D^*) \left(\frac{k_1}{k_G} \right)^{2(n+3)} \right]. \quad (46)
 \end{aligned}$$

The k_3 and k_1 integrals are similar in form to the usual Sachs-Wolfe term (for power-law magnetic spectra) and can be evaluated analytically for any n (Eq. 6.574.2 of [66]) in terms of gamma functions. For $n \rightarrow -3$ we have

$$\begin{aligned}
 b_{l_1 l_2 l_3} &= \mathcal{K} \left(\frac{\mathcal{R}_p}{\pi^2} \right)^3 \frac{\pi}{2} [2(4)^3 \pi^7 (n+3)^2 V_A^6] \\
 &\times \frac{1}{2l_3(l_3+1)} \frac{1}{2l_1(l_1+1)} \quad (47)
 \end{aligned}$$

or

$$l_1(l_1+1)l_3(l_3+1)b_{l_1 l_2 l_3} = \mathcal{K} \mathcal{R}_p^3 \left(\frac{\pi}{2} \right)^2 (4)^3 (n+3)^2 V_A^6, \quad (48)$$

where a factor of $1/(D^*k_G)^{3(n+3)}$ that also appears has been approximated to unity for the case $n \rightarrow -3$ of a scale-invariant magnetic field index.

For *Case I*—The bispectrum considering only constant angular terms

$$\begin{aligned}
 l_1(l_1+1)l_3(l_3+1)b_{l_1 l_2 l_3} &\approx (-1.35 \times 10^{-16}) \left(\frac{3m}{4} \right) \\
 &\times \left(\frac{n+3}{0.2} \right)^2 \left(\frac{B_{-9}}{3} \right)^6, \quad (49)
 \end{aligned}$$

where we have used $T_B \approx 10^{14}$ GeV corresponding to a possible inflationary epoch for magnetic field generation. This evaluates in different configurations to

$$\begin{aligned}
 l_1(l_1+1)l_3(l_3+1)b_{l_1 l_2 l_3} &\approx \\
 (10^{-16}) \left(\frac{n+3}{0.2} \right)^2 \left(\frac{B_{-9}}{3} \right)^6 &\times \left\{ \begin{array}{l} 0 \quad \text{local isosceles} \\ 5.7 \quad \text{equilateral} \\ 8.1 \quad \text{squeezed collinear} \\ 9.2 \quad \text{midpoint collinear} \end{array} \right\}. \quad (50)
 \end{aligned}$$

We see that the constant-term only evaluation gives a bispectrum of order $6-9 \times 10^{-16}$ independent of the exact configuration. The exception is the local isosceles case (where the constant-term sum happens to vanish) but if a full evaluation over all angular terms is done, it yields a nonzero bispectrum.

For *Case II*—The squeezed collinear bispectrum considering all angular terms

$$\begin{aligned}
 l_1(l_1+1)l_3(l_3+1)b_{l_1 l_2 l_3} &\approx (-1.35 \times 10^{-16}) \left(\frac{n+3}{0.2} \right)^2 \\
 &\times \left(\frac{B_{-9}}{3} \right)^6. \quad (51)
 \end{aligned}$$

The full evaluation of all angular terms for the squeezed collinear case produces two important changes (compared to squeezed collinear in Case I evaluation): the sign of the bispectrum has changed to negative (angular terms in mode-coupling contribute with different signs and change the total sign) and its value had diminished by factor of 6.

Note that the values of the reduced bispectrum given by Eqs. (50) and (51) are $\sim 10^6$ times larger than the values obtained for the compensated scalar mode in SS09. This is essentially due to the large value of \mathcal{A}^3 as discussed in Sec. IV.

A numerical evaluation of the mode-coupling integral involves an integrable singularity at $s = k_3$ and $s = -k_1$. The singularity is integrable (even without a large-scale cutoff) provided we consider the mode-coupling integral over s together with the k_1 and k_3 integrals. We defer this numerical investigation for later work and proceed with our analytic results.

VI. CONSTRAINT ON PRIMORDIAL MAGNETIC FIELD

In order to put constraints on the primordial magnetic field, we compare our magnetic reduced bispectrum with the reduced bispectrum that arises from nonlinear terms in the gravitational potential. These have a value given by

$$l_1(l_1+1)l_3(l_3+1)b_{l_1 l_2 l_3} \sim 4 \times 10^{-18} f_{\text{NL}} \quad (52)$$

characterized by f_{NL} ([50]).

The reduced bispectrum for the magnetic compensated scalar mode [52,53] was a factor of a few times 10^4 smaller than the standard inflationary signal (for models with $f_{\text{NL}} \sim 1$). However, we find that the magnitude of the corresponding reduced bispectrum for the magnetic passive scalar mode, calculated above, is ~ 30 times larger (Case II) or even up to ~ 200 times larger (Case I) than the inflationary signal. Both these magnetic bispectra values are for field strength of 3 nG and $n = -2.8$, close to the scale-invariant spectrum.

Conversely, we can put upper limits on the primordial magnetic field by equating the magnetic bispectrum to the observed upper limit for the inflationary bispectrum. This is evaluated using the latest WMAP7 limits (95% C.L.) on non-Gaussianity in the observed CMB [67], $-10 < f_{\text{NL}}^{\text{loc}} < 74$, taking the appropriate side of these limits for the different signs of bispectrum. Since we have considered only the Sachs-Wolfe contribution to magnetic field-induced temperature anisotropies, our magnetic bispectra will be accurate only on large scales. However, the f_{NL} we are using for comparison to data comes from WMAP data that has a maximum multipole of $l \sim 750$ and therefore the bispectra

comparison is not exact. Yet our B_0 limits are expected to be fairly robust as B_0 depends very weakly ($B_0 \propto f_{\text{NL}}^{1/6}$) on f_{NL} as also discussed in SS09. We obtain the following upper limits for the magnetic field in different cases:

Case I: For any bispectrum configuration considering only s -independent terms

$$B_0 < 3 \text{ nG}. \quad (53)$$

For individual configurations the magnetic field limits are equilateral (2.7 nG), squeezed collinear (2.5 nG), and midpoint collinear (2.5 nG), all for $T_B \approx 10^{14}$ GeV corresponding to the inflationary epoch and using $f_{\text{NL}}^{\text{loc}} < 74$.

Case II: For the squeezed collinear bispectrum configuration, including all angular terms, we obtain an upper limit

$$B_0 < 2 \text{ nG}, \quad (54)$$

where this magnetic field limit value (2.4 nG) is derived using $T_B \approx 10^{14}$ GeV and $-10 < f_{\text{NL}}^{\text{loc}}$.

It is currently unclear what would be an appropriate epoch to adopt for the generation of a large-scale primordial cosmic magnetic field [31,35–39]. We recall that via the logarithmic factor in Eq. (8), there is a weak dependence of B_0 on τ_B , the magnetic field generation epoch. We find a 2–4 nG variation in the B_0 upper limit when τ_B is varied between the inflationary and electroweak epochs (using $T_B \approx 10^{14} \rightarrow 10^3$ GeV), taken as the earliest and latest possible epochs of magnetic field generation. However, as the electroweak transition is not expected to give a scale-invariant magnetic field spectrum, we caution that the choice of electroweak epoch may be too late to generate a field with $n \rightarrow -3$. Hence, the variation in the magnetic field upper limit quoted above may be an overestimate.

VII. CONCLUSION

We have calculated here the CMB bispectrum, on large angular scales, due to the passive scalar mode. This mode is sourced by the magnetic scalar anisotropic stress before neutrino decoupling. The CMB bispectrum due to the passive scalar mode is more than 2 orders of magnitude

larger than the bispectrum due to the primary inflation-induced scalar mode, for $f_{\text{NL}} \sim 1$ and $B_0 \sim 3$ nG. It is also a factor of 10^6 times greater than that previously calculated for the compensated scalar mode [52,53]. This is the first calculation of a type of magnetic contribution to CMB bispectrum that can clearly dominate over the primary bispectrum at large scales. In the CMB power spectrum, by contrast, the passive scalar mode signal is highly subdominant to the inflationary signal [40]. Our work thus shows that the magnetically induced signals, being intrinsically non-Gaussian, could be more easily distinguished from the primary CMB anisotropies, when one considers the bispectrum rather than the power spectrum.

Using the WMAP7 limits for f_{NL} we have placed an upper limit on the strength of the primordial magnetic field $B_0 < 2$ nG which is more than an order of magnitude stronger than the limit from the compensated scalar mode (35 nG) obtained in SS09. The passive mode limit is only weakly (logarithmically) dependent on the epoch of magnetic field generation τ_B . We note that for scale-invariant magnetic spectra $n \rightarrow -3$, the bispectrum and the derived magnetic field limit are independent of both the assumed cosmological model as well as our choice of smoothing scale k_G for the stochastic magnetic field B_0 .

The primordial magnetic field limit is expected to improve if the integrated Sachs-Wolfe effect and full scalar anisotropies are taken into account. We also expect stronger constraints to follow from the tensor and vector mode bispectra that dominate at low and high l , respectively. Better observational limits on CMB non-Gaussianity from PLANCK data are expected to further tighten primordial magnetic field limits.

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