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Hot conformal gauge theories

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We compute the nonzero temperature free energy up to the order $g^6 \ln(1/g)$ in the coupling constant for vectorlike $SU(N)$ gauge theories featuring matter transforming according to different representations of the underlying gauge group. The number of matter fields, i.e. flavors, is arranged in such a way that the theory develops a perturbative stable infrared fixed point at zero temperature. Because of large distance conformality we trade the coupling constant with its fixed point value and define a reduced free energy which depends only on the number of flavors, colors, and matter representation. We show that the reduced free energy changes sign, at the second, fifth, and sixth order in the coupling, when decreasing the number of flavors from the upper end of the conformal window. If the change in sign is interpreted as a signal of an instability of the system then we infer a critical number of flavors. Surprisingly this number, if computed to the order g^2, agrees with previous predictions for the lower boundary of the conformal window for nonsupersymmetric gauge theories. The higher order results tend to predict a higher number of critical flavors. These are universal properties, i.e. they are independent of the specific matter representation.

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I. INTRODUCTION

Non-Abelian gauge theories are expected to exist in a number of different phases which can be classified according to the force measured between two static sources. The knowledge of this phase diagram is relevant for the construction of extensions of the standard model (SM) that invoke dynamical electroweak symmetry breaking [1,2]. An up-to-date review is [3] while earlier reviews are [4,5]. The phase diagram is also useful in providing ultraviolet completions of unparticle [6] models [7,8] and it has been investigated recently using different analytical methods [9–19].

Here we wish to understand, in a rigorous way, the dynamics of gauge theories lying in the conformal window at nonzero temperature. The physical applications are numerous ranging from the above mentioned models of dynamical electroweak symmetry breaking to cosmology [20–24]. Thermodynamical properties of these gauge theories were also investigated in the literature using holographic models [17,25,26].

Our starting point are asymptotically free vectorlike gauge theories near the Banks-Zaks infrared stable fixed point [27]. The presence of such a perturbative fixed point allows a controllable computation of the free energy for these theories which we carry till order g^6 in the gauge coupling. The absence of an intrinsic scale in the theory is evident in having a free energy directly proportional to the fourth power of the temperature, for any temperature. We will trade the value of the coupling with its value at the infrared fixed point turning the coefficient of T^4 into an algebraic expression of the number of flavors, colors, and matter representation, encoding a great deal of information of the underlying gauge theory.

Despite the fact that we can keep the coupling constant small it is a fact that perturbation theory breaks down, at finite temperature, due to the loss of analyticity in the coupling associated to the presence of infrared singularities. For the free energy this problem sets in at $\mathcal{O}(g^6)$, or four-loop order [28,29]. At best one can assume that the free energy is computable to $\mathcal{O}(g^6)$, though not via loop diagrams. The highest order one can achieve using Feynman diagrams is $\mathcal{O}(g^6 \ln(1/g))$ and was recently determined in [30]. We adapt their results for the case of gauge theories featuring large distance conformality while generalizing the discussion to any matter representation, different number of colors, and flavors.

We discover a number of surprising features when plotting the free energy, for a given matter representation, as a function of the number of flavors: (i) there is a change in sign of the free energy at a critical number of flavors whose value depends on the representation where it belongs and the order to which the computations were carried, and (ii) this number is smaller than the one for which asymptotic freedom is lost.

It is tempting to identify it with the critical number of flavors below which, at zero temperature, conformality is lost. The obvious caveat is that as we decrease the number of flavors away from the point when asymptotic freedom is lost perturbation theory ceases to be reliable and therefore we interpret this phenomenon only as a *strong* indication that the finite temperature free energy is aware of the nontrivial underlying gauge dynamics.

Another amusing feature is that at the two-loops level, when the results are scheme independent, the change in sign of the free energy occurs for a given number of flavors

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which is surprisingly close to the one predicted using the Schwinger-Dyson results [10] as well as the Ryttov-Sannino β function [12]. This value becomes larger when going to the fifth and sixth order in the coupling.

II. REVIEW OF THE FREE ENERGY COMPUTATION

The perturbative free energy at finite temperatures was computed to order $\mathcal{O}(g^6 \ln(1/g))$ in [30] and in the $\overline{\text{MS}}$ scheme. The result was derived using an effective field theory approach utilizing matching of coefficients in the effective theory expression with the dimensionally regularized perturbative expansion. An in-depth presentation of the method can be found in [31], where the order $\mathcal{O}(g^5)$ was recomputed. In order to present the results in a relatively self-contained way we briefly review the method here.

The perturbative free energy for a massless asymptotically free gauge theory receives contributions from the following three mass scales: $2\pi T$, gT, and g^2T. They are, respectively, the particle momentum in the plasma, the onset of the color-electric (Debye) screening, and finally the onset of color-magnetic screening. The idea is then to construct an effective field theory that reproduce static observables at the different scales. This is done by the method of dimensional reduction, where the static properties of a $3+1$-dimensional field theory at high temperatures are expressed in terms of an effective field theory in three space dimensions [29,32].

The free energy density expressing the static equilibrium properties of the plasma is given by the usual logarithm of the partition function,

$$F = -\frac{T}{V}\ln\mathcal{Z}, \tag{1}$$

$$\mathcal{Z} = \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\int_0^\beta d\tau \int d^d x \mathcal{L}\right), \tag{2}$$

$$\mathcal{L} = \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}\not{D}\psi. \tag{3}$$

It is customary to introduce finite temperature by Euclideanizing the time dimension $t = -i\tau$ and compactifying it with a period $\beta = 1/T$, with T the temperature, and the bosonic (fermionic) fields respecting periodic (antiperiodic) boundary conditions. In regimes when the gauge coupling, g is small the free energy density can be computed perturbatively.

On the other hand, by dimensional reduction one can compute the free energy density using an effective field theory in three space dimensions, by separating the electro- and magnetostatic parts of the Lagrangian. The free energy density is expressed as

$$F = T[f_E(T, g; \Lambda_E) + f_M(m_E^2, g_E, \lambda_E^{(i)}, \ldots; \Lambda_E, \Lambda_M) + f_G(g_M, \ldots; \Lambda_M)], \tag{4}$$

where the effective free energy densities f_E, f_M, and f_G represent the contributions from the three scales, i.e. f_E gives the contribution from the momentum scale by effectively integrating out the fermions and the high momentum degrees of freedom down to the scale Λ_E corresponding to a distance of order $1/(gT)$. At greater distances the fields are replaced by electrostatic and magnetostatic gauge fields $A_0^a(\mathbf{x})$ and $A_i^a(\mathbf{x})$, which are proportional to the zero-frequency modes of the gauge fields $A_\mu(\tau, \mathbf{x})$ that are the only fields able to propagate over such distances [33]. f_E is then the normalization for this transition, i.e.

$$\mathcal{Z} = e^{-f_E(\Lambda_E)V}\int^{\Lambda_E} \mathcal{D}A_\mu^a(\mathbf{x}) \exp\left(-\int d^d x \mathcal{L}_E\right), \tag{5}$$

where the Lagrangian is now an effective electrostatic Lagrangian,

$$\mathcal{L}_E = \frac{1}{2}\mathrm{Tr}F_{ij}^2 + \mathrm{Tr}[D_i, A_0]^2 + m_E^2 \mathrm{Tr}A_0^2 + \lambda_E^{(1)}[\mathrm{Tr}(A_0^2)]^2 + \lambda_E^{(2)}\mathrm{Tr}A_0^4 + \delta\mathcal{L}_E. \tag{6}$$

$\delta\mathcal{L}_E$ represents higher order interaction terms which contribute beyond the order $\mathcal{O}(g^6)$. Shorthand notation has been used for the gauge fields, i.e. $A_\mu = T^a A_\mu^a$ and the corresponding gauge coupling is denoted by g_E. This Lagrangian defines f_M which is dependent on the still lower momentum scale Λ_M, corresponding to the distance $1/(g^2T)$. Again at greater distances, only the magnetostatic gauge fields play a role,

$$\mathcal{Z} = e^{-f_E(\Lambda_E)V}e^{-f_M(\Lambda_E,\Lambda_M)V}\int^{\Lambda_M} \mathcal{D}A_i^a(\mathbf{x}) \times \exp\left(-\int d^d x \mathcal{L}_M\right),$$

with the effective magnetostatic Lagrangian

$$\mathcal{L}_M = \frac{1}{2}\mathrm{Tr}(F_{ij}^2) + \delta\mathcal{L}_M, \tag{7}$$

containing the gauge coupling g_M. Note that the direct identification of the normalization functions f_E and f_M with the free energy densities of the respective Lagrangians is strictly true only when using dimensional regularization to cut off the ultraviolet and infrared divergences in the perturbation expansions. In the same sense, f_G is identified with the integral expression in the above partition function.

The Lagrangian (7) defines a confining theory when the higher order terms are not considered and is thus nonperturbative. However, f_G can still be expressed as a power series in g [33]. The leading order is proportional to $(g^2T)^3$ and the coefficient can be determined by lattice computations only. However, the logarithmic ultraviolet divergence coming from the scale Λ_M can be evaluated exactly by matching the coefficient with that in f_M, since the expression must be scale invariant. In this way one proceeds backward and matches all coefficients with appropriate tuning to get rid of the somewhat arbitrary momentum scales. Finally one ends up with an exact expression for the free energy to order $\mathcal{O}(g^6\ln(1/g))$, while the order $\mathcal{O}(g^6)$ coefficient remains unknown and uncomputable from perturbation methods. One also shifts the scale of the coupling constant from the dimensional regularization scale to an arbitrary renormalization scale μ by using the renormalization group equation for the running of the coupling constant.

We here express the leading order magnitudes only and refer to [30] for the full result,

$$f_E \sim T^4, \qquad m_E^2 \sim g^2T^2, \qquad g_E^2 \sim g^2T,$$
$$\lambda_E^{(1)} \sim g^4T, \qquad \lambda_E^{(2)} \sim g^4T, \qquad g_M^2 \sim g^2T.$$

Note that the perturbation expansion parameter in $\mathcal{L}_E$ is g_E^2/m_E which is of order g. Thus the perturbation expansion by this method is an expansion in g rather than g^2.

III. HOT CONFORMAL FREE ENERGY AT $\mathcal{O}(g^2)$

Our starting point is a generic asymptotically free gauge theory with N_f Dirac flavors transforming according to the representation r of the underlying gauge group. We will consider, to this order, also the case of $\mathcal{N}=1$ supersymmetric gauge theories for reasons which will become clearer shortly.

To be more specific, throughout this paper we will consider matter transforming according to four different but single representations, i.e. the adjoint representation (denoted G) under the gauge group $SU(2)$, the fundamental representation under the gauge groups $SU(3)$ and $SU(2)$, the two-index symmetric representation under the gauge group $SU(3)$, and the two-index antisymmetric representation under the gauge group $SU(4)$.

The relevant group normalization factors are

$$\mathrm{Tr}[T_r^aT_r^b] = T[r]\delta^{ab}, \qquad T_r^aT_r^a = C_2[r]\mathbf{1}, \qquad (8)$$

where T_r^a is the a-th group generator in the representation r and $a = 1, \ldots, d[G]$. We denote with $d[r]$ the dimension of the representation. $T[r]$ and $C_2[r]$ are related via the identity $C_2[r]d[r] = T[r]d[G]$. In Table I, for completeness, we list the normalization used for the group factors in the different representations. We list in the last column also the number of colors which will be considered. The normalizations were taken from [12].

TABLE I. Normalization of the relevant group factors for the representations used throughout this paper.

r	$T[r]$	$C_2[r]$	$d[r]$	N
G	N	N	N^2-1	2
$\square$	$\frac{1}{2}$	$\frac{N^2-1}{2N}$	N	2, 3
$\square\square$	$\frac{N+2}{2}$	$\frac{(N-1)(N+2)}{N}$	$\frac{N(N+1)}{2}$	3
$\begin{matrix}\square\\\square\end{matrix}$	$\frac{N-2}{2}$	$\frac{(N+1)(N-2)}{N}$	$\frac{N(N-1)}{2}$	4

The β function up to four-loop order

$$\beta(g) = -\frac{\beta_0}{(4\pi)^2}g^3 - \frac{\beta_1}{(4\pi)^4}g^5 - \frac{\beta_2}{(4\pi)^6}g^7 - \frac{\beta_3}{(4\pi)^8}g^9 + \mathcal{O}(g^{11}) \qquad (9)$$

was computed in [34]. As for the free energy expression the four-loop β function is also computed in the $\overline{\mathrm{MS}}$ scheme, thus no ambiguities in the scheme-dependence of the expressions arise. Only β_0 and β_1 are scheme-independent and read

$$\beta_0 = \frac{11}{3}C_2[G] - \frac{4}{3}T[r]N_f, \qquad (10)$$

$$\beta_1 = \frac{34}{3}C_2^2[G] - \left(\frac{20}{3}C_2[G] + 4C_2[r]\right)T[r]N_f. \qquad (11)$$

Asymptotic freedom is lost when the lowest order coefficient β_0 changes sign. This occurs at

$$N_f^{\mathrm{AF}} := \frac{11}{4}\frac{C_2[G]}{T[r]}. \qquad (12)$$

For a given fermion representation, the second coefficient β_1 is negative above this critical number of flavors and an infrared-stable fixed point develop which is known as the Banks-Zaks fixed point [27]. Such a theory display large distance conformality.

The Banks-Zaks fixed point disappears when β_1 changes sign. This occurs at

$$N_f^{\mathrm{Lost}} := \frac{17C_2[G]}{10C_2[G] + 6C_2[r]}\frac{C_2[G]}{T[r]}. \qquad (13)$$

We are now equipped to investigate the *conformal* free energy by starting with the nonsupersymmetric case to the order g^2,

$$\frac{F}{\pi^2T^4} = -\frac{d[G]}{9}\left[\frac{1}{5} + \frac{7}{20}\frac{d[r]}{d[G]}N_f - \left(C_2[G] + \frac{5}{2}T[r]N_f\right)\frac{g^2(\mu)}{(4\pi)^2}\right]. \qquad (14)$$

For the supersymmetric case we have

$$\beta_0^{\mathrm{SUSY}} = 3C_2[G] - 2T[r]N_f, \qquad (15)$$

$$\beta_1^{\mathrm{SUSY}} = 6C_2^2[G] - 4(C_2[G] + 2C_2[r])T[r]N_f, \qquad (16)$$

leading to

$$N_{f,\mathrm{SUSY}}^{\mathrm{AF}} = \frac{3}{2}\frac{C_2[G]}{T[r]}, \qquad (17)$$

$$N^{\mathrm{Lost}}_{f,\mathrm{SUSY}} = \frac{3C_2[G]}{2C_2[G] + 4C_2[r]} \frac{C_2[G]}{T[r]}. \qquad (18)$$

We obtain for the supersymmetric free energy [35]

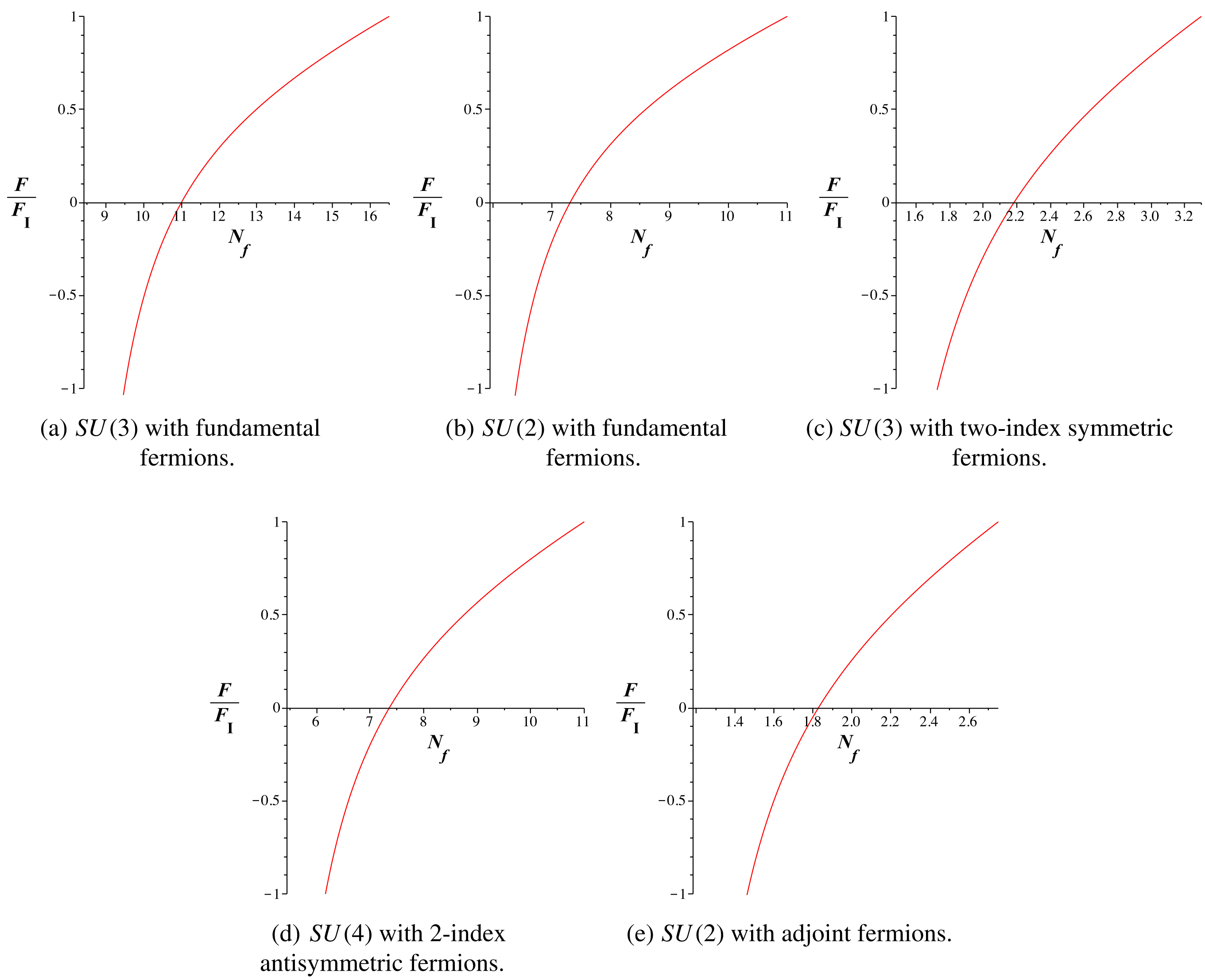
$$\frac{F_{\mathrm{SUSY}}}{\pi^2 T^4} = -\frac{d[G]}{24}\left[1 + 2\frac{d[r]}{d[G]}N_f - 6(C_2[G] + 6T[r]N_f)\frac{g^2(\mu)}{(4\pi)^2}\right]. \qquad (19)$$

To determine the free energy dependence on the number of flavors and colors in the perturbative regime of the conformal window we replace the coupling constant with the Banks-Zaks fixed point value g^* at two-loop order, given in Appendix A. The free energies read

$$\frac{F^*}{\pi^2 T^4} = -\frac{d[G]}{9}\left[\frac{1}{5} + \frac{7}{20}\frac{d[r]}{d[G]}N_f + \frac{(C_2[G] + \frac{5}{2}T[r]N_f)(11C_2[G] - 4T[r]N_f)}{34C_2^2[G] - (20C_2[G] + 12C_2[r])T[r]N_f}\right],$$

$$\frac{F^*_{\mathrm{SUSY}}}{\pi^2 T^4} = -\frac{d[G]}{24}\left[1 + 2\frac{d[r]}{d[G]}N_f + \frac{3(C_2[G] + 6T[r]N_f)(3C_2[G] - 2T[r]N_f)}{3C_2^2[G] - 2(C_2[G] + 2C_2[r])T[r]N_f}\right].$$

We observe immediately that due to the *conformal* large distance nature of our theories the free energy dependence on the energy scale is only via the temperature which factors out leaving behind, as expected, a numerical factor containing information on the specific theory studied. These coefficients are *universal*, i.e. independent on renormalization schemes.



(a) $SU(3)$ with fundamental fermions.
(b) $SU(2)$ with fundamental fermions.
(c) $SU(3)$ with two-index symmetric fermions.
(d) $SU(4)$ with 2-index antisymmetric fermions.
(e) $SU(2)$ with adjoint fermions.

FIG. 1 (color online). Normalized free energy in the conformal window for different theories.

TABLE II. Comparison of the different critical number of flavors obtained via the g^2 free energy ($\bar{N}_f$), the Ryttov-Sannino β function N_f^{RS} (obtained for the parameter $\gamma = 1$), and the ladder approximation N_f^{Ladder}.

r	N	$\bar{N}_f$	$N_f^{\mathrm{RS}}(\gamma = 1)$	Discrepancy	N_f^{Ladder}
$\square$	3	11.001 55	11	0.000 14	11.914
$\square$	2	7.311 00	7.33…	0.003 04	7.859
$\square\square$	3	2.184 63	2.2	0.006 99	2.502
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	4	7.357 12	7.33…	0.003 24	8.104
G	2	1.827 39	1.833…	0.003 24	2.075

We can now plot the free energies as a function of number of flavors for different number of colors and gauge theories. We choose to normalize our results to the free energy obtained, at order g^2, when replacing the number of flavors with the one for which asymptotic freedom is lost for any given underlying gauge theory (defined as F_I in the plots). The results are shown in Fig. 1. A generic feature emerging from the plots is that there is always a critical number of flavors $\bar{N}_f$ for which the normalized free energy vanishes. We interpret the change in the sign of the free energy as an indication of an instability of the system which identifies with the point where large distance conformality is lost. In Table II we compare this value with the expected critical number of flavors obtained using the Ryttov-Sannino β function N_f^{RS} (obtained for the parameter $\gamma = 1$) as well as the one obtained via the ladder approximation and indicated with N_f^{Ladder}.

The agreement among these numbers is surprisingly good as can be seen from the column indicated by *discrepancy* defined as $|\bar{N}_f - N_f^{\mathrm{RS}}|/N_f^{\mathrm{RS}}$.

We still do not have a deep understanding of why these different methods agree so well among each other however we speculate that this agreement might be due to the fact that all these approaches make use of the two-loop universal coefficients of the β function.

Once we noticed such an agreement we asked ourselves: *How about supersymmetry?* We find, also in the supersymmetric case, the existence of a critical number of flavors below which the free energy changes sign. In Table III we compare $\bar{N}_f$ with the critical number of flavors obtained using the supersymmetric all-orders β function when setting to zero its numerator for both $\gamma = -1$ and $\gamma = -\frac{1}{5}$. In this case we find a reasonable agreement when taking as critical number of flavors the one for which the anomalous dimension of the chiral superfield γ is around $-1/5$. This is not the preferred value obtained from Seiberg's results [36] which, however, were tested via dualities only for the case of the fundamental representation.

TABLE III. Comparison of $\bar{N}_f$ for supersymmetric gauge theories with the critical number of flavors obtained using the supersymmetric all-orders β function when setting to zero its numerator for both $\gamma = -1$ and $\gamma = -\frac{1}{5}$.

r	N	$\bar{N}_f$	$N_f(\gamma = -1(-\frac{1}{5}))$	Discrepancy
$\square$	3	7.6062	4.5 (7.5)	0.69 (0.014)
$\square$	2	5.1137	3.0 (5)	0.70 (0.023)
$\square\square$	3	1.4365	0.9 (1.5)	0.59 (0.042)
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	4	4.9794	3.0 (5)	0.77 (0.004)
G	2	1.2071	0.75 (1.25)	0.61 (0.034)

IV. CONFORMAL FREE ENERGY TO THE LAST PERTURBATIVE ORDER

Having at hand a perturbative expansion it is natural to go beyond the g^2 order. To determine the free energy at any given order, in perturbation theory, we have consistently solved for the value of the coupling evaluated at the infrared fixed value and in the same renormalization scheme. The expressions of the fixed point value of the coupling to the highest order computed here are given in Appendix A.

The four-loop β function was computed in [34] up to a normalization constant for the fourth-order Casimir. In [34] the explicit expressions for all the coefficients were given for the fundamental and adjoint representations while we derive in Appendix B the expressions for any totally (anti)symmetric representation for $SU(N)$, $SO(N)$, and $Sp(N)$ gauge groups.

Beyond the g^3 order one notices the emergence of logarithms of the ratio of the renormalization to the temperature scale. Since we assumed, in our computations, the temperature scale to be such that the gauge theory coupling constant, at zero temperature, has (quasi) reached the fixed point value it is therefore natural to evaluate the coupling at the renormalization scale point $2\pi T$. We have, however, checked by direct evaluation of the free energy at the renormalization scale point of $g^2 T$ that, due to the logarithmic dependence, the results are rather insensitive to the choice of the reference scale as it is clear from Fig. 2 where we show the results for fermions in the fundamental representation for the two choices of the renormalization scale point, $2\pi T$ (left panel) and $g^2 T$ (right panel).

It is for this reason that we show in Fig. 3 the results for the remaining theories evaluated at the scale $2\pi T$. We observe the following universal behaviors:

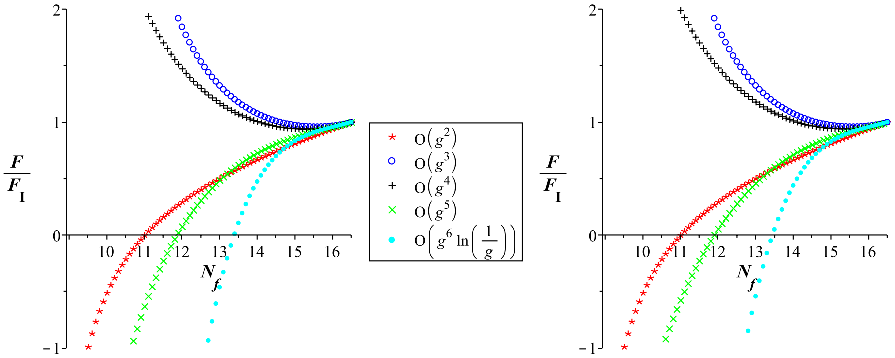
(i) The free energy to the lowest interesting scheme-independent order in perturbation theory (i.e. g^2) changes sign at a critical number of flavors ($\bar{N}_f$),

(ii) This critical value increases at the order g^5 and increases further at the order g^6,

(iii) The free energy does not change sign if truncated at the order g^3 or g^4.

V. CONCLUSIONS

We unveiled the finite temperature structure of gauge theories of fundamental interactions featuring a perturba-



(a) $\mu = 2\pi T$. (b) $\mu = g^2 T$.

FIG. 2 (color online). Normalized free energy for different orders in g computed at the renormalization scale μ with fermions in the fundamental representation of $SU(3)$.

tive infrared stable fixed point to the last computable order in perturbation theory. Differently from gauge theories assumed to generate a nonperturbative renormalization-invariant scale at zero temperature, like QCD, our results are perturbative in the entire energy range (i.e. for any choice of the temperature) since we can use as a control parameter the number of flavors to tune the theory near the perturbative stable infrared fixed point.

We discovered a number of universal properties, i.e. independent of the matter representation and the supersymmetric structure of the underlying gauge theory, suggesting that asymptotically free gauge theories featuring large distance conformality share very similar dynamics.

If we were to take the point of view [30] that having exhausted the perturbative results we determined the full result for the free energy at nonzero temperature, we would then have *discovered* that there is a critical number of flavors below which the free energy changes sign signaling the onset of an instability which we interpret as the end of the conformal window.

APPENDIX A: BANKS-ZAKS FIXED POINTS UP TO FOUR LOOPS

Here we give the exact expression for the Banks-Zaks infrared fixed point [27] to different orders in g.

The two-loop expression is

$$\frac{\alpha^*}{4\pi} = -\frac{\beta_0}{\beta_1}. \tag{A1}$$

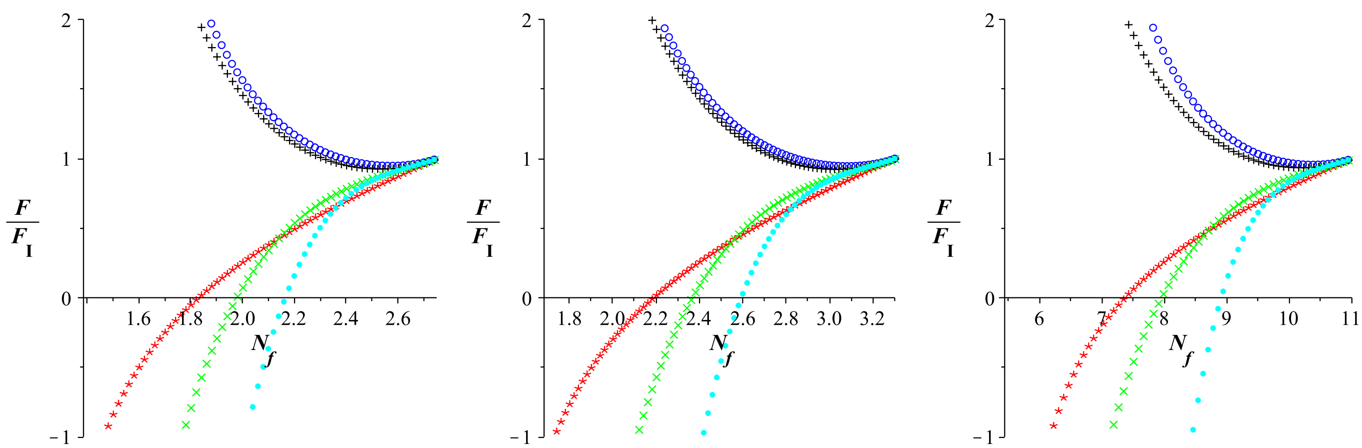
The three-loop expression is

$$\frac{\alpha^*}{4\pi} = -\frac{\beta_1 + \sqrt{\beta_1^2 - 4\beta_2\beta_3}}{\beta_2}. \tag{A2}$$

The four-loop expression is

$$\frac{\alpha^*}{4\pi} = -\frac{b_2^2(1 + i\sqrt{3}) - b_1(1 + i\sqrt{3}) + 2\beta_2 b_2}{12\beta_3 b_2}, \tag{A3}$$

where



(a) $SU(2)$ with adjoint fermions. (b) $SU(3)$ with two-index symmetric fermions. (c) $SU(4)$ with two-index antisymmetric fermions.

FIG. 3 (color online). Normalized free energy for different orders in g computed at the renormalization scale $\mu = 2\pi T$ with fermions in different representation. The legend (color code) is the same as in Fig. 2.

$$b_1 = 12\beta_1\beta_3 - 4\beta_2^2,$$

$$b_2 = (36\beta_1\beta_2\beta_3 - 108\beta_0\beta_3^2 - 8\beta_2^3 + 12\sqrt{3}\sqrt{4\beta_1^3\beta_3 - \beta_1^2\beta_2^2 - 18\beta_1\beta_2\beta_3\beta_0 + 27\beta_0^2\beta_3^2 + 4\beta_0\beta_2^3\beta_3})^{1/3},$$

with the β_i's given in [34].

APPENDIX B: GENERALIZATION FOR THE FOURTH-ORDER CASIMIR

The general fourth-order Casimir invariants for simple Lie groups were derived in [37,38]. We will use the results therein to generalize the expression for the four-loop β function in [34] to any representation of the groups $SU(N)$, $SO(N)$, $Sp(N)$. We must however keep track of the different normalization of the Killing form in the literature; thus we define an overall normalization constant b as

$$\mathrm{Tr}\, f^a_{cd} f^{bcd} = bh\delta^{ab} = \eta I_2[G]\delta^{ab}, \tag{B1}$$

where f^{abc} are the structure constants and h is the dual Coxeter number. The equation defines the second-order Casimir invariant I_2 with eigenvalue $I_2[r]$ as given in [37] and $\eta = b/b'$ relates the results therein to the arbitrary normalization b, with b' chosen in [37,38] to be $b' = \{2, 1, 2\}$ for the groups $\{SU(N), SO(N), Sp(N)\}$. In this paper, b is set to 1. In this appendix we are following the notation introduced by Okubo [37] when naming the second-order Casimir invariants, i.e. $I_2[r]$, corresponding to $C_2[r]$ in the more recent literature and the one used in the main text.

The fourth-order Casimir invariant is related to the symmetrized fourth-order trace,

$$d_r^{abcd} = \frac{1}{4!}\sum_P \mathrm{Tr}[T^aT^bT^cT^d], \tag{B2}$$

where the sum is over all permutations of the generator indices. We let T^a be any representation of the generators for a simple Lie group. It is well-known that the fourth-order Casimir invariant is not unique, in fact the square of I_2 is also a fourth-order Casimir invariant, leading to the ambiguity

$$I_4' = I_4 + C(I_2)^2, \tag{B3}$$

where I_4' defines a new fourth-order Casimir invariant, with C being an arbitrary constant. To get rid of the ambiguity one defines a *modified* Casimir invariant J_4 with a specific metric found by Okubo and for short indicated with δ^{abcd},

$$J_4 = \eta^2\delta_{abcd}T^aT^bT^cT^d. \tag{B4}$$

Then requiring the identity for irreducible representations,

$$\delta_{abcd}\,\mathrm{Tr}[T^aT^bT^cT^d] = \eta^2 d[r]J_4[r], \tag{B5}$$

where $J_4[r]$ is the eigenvalue of J_4 in the representation r, it follows that J_4 satisfies similar sum rules as I_2 and I_3, hence it is the appropriate fourth-order Casimir invariant to work with [37].

It now follows that for a general representation, one can write

$$d_r^{abcd} = c_1\delta^{abcd} + \frac{1}{3}c_2(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}), \tag{B6}$$

where c_i are some constants dependent on the representation r. The two terms are orthogonal. We will only be concerned with the first term, as the second term was given for any representation in [34]. From [37] one finds that

$$c_1 = \eta^2\frac{d[r]J_4[r]}{d[\lambda]J_4[\lambda](2 + d[G])}, \tag{B7}$$

where λ is the defining representation, which we will take to be the fundamental one. Then, contracting Eq. (B6) with δ^{abcd} one finds

$$\delta^{abcd}\delta_{abcd} = \frac{\eta^2 d[r]J_4[r]}{c_1} = d[\lambda]J_4[\lambda](2 + d[G]). \tag{B8}$$

Hence, we derive that

$$\begin{aligned} c_1^2\delta^{abcd}\delta_{abcd} &= \left[\frac{d[r]J_4[r]}{d[\lambda]J_4[\lambda]}\eta^2\right]^2\frac{d[\lambda]J_4[\lambda]}{(2 + d[G])} \\ &= \left[\frac{d[r]J_4[r]}{d[\lambda]J_4[\lambda]}b^2\right]^2\frac{d[\lambda]J_4[\lambda]}{b'^4(2 + d[G])}. \end{aligned} \tag{B9}$$

Writing the expression in this form, we exactly get the definitions of the normalization constant $\tilde{I}_4[r]$ and the traceless tensor d^{abcd} used in the four-loop β function paper [34] (note that the normalization constant was defined without the tilde, but is used here in order not to confuse it with $I_4[r]$ defined in [37]), i.e.

$$\tilde{I}_4[r] = \frac{d[r]J_4[r]}{d[\lambda]J_4[\lambda]}b^2, \tag{10}$$

$$d^{abcd}d_{abcd} = \frac{d[\lambda]J_4[\lambda]}{b'^4(2 + d[G])}. \tag{B11}$$

As noted in [34], d^{abcd} is representation-independent, and the contracted product can be written as

$$d^{abcd}d_{abcd}[SU(N)] = \frac{d[G](d[G] - 3)(d[G] - 8)}{16\cdot 6(2 + d[G])}, \tag{B12}$$

$$d^{abcd}d_{abcd}[SO(N)] = \frac{d[G](d[G]-1)(d[G]-3)}{12(2+d[G])}, \quad \text{(B13)}$$

$$d^{abcd}d_{abcd}[Sp(N)] = \frac{d[G](d[G]-1)(d[G]-3)}{16\cdot 12(2+d[G])}, \quad \text{(B14)}$$

where the fundamental representation is taken as the defining representation λ, and with

	$SU(N)$	$SO(N)$	$Sp(N)$
$d[G]$	N^2-1	$N(N-1)/2$	$N(N+1)/2$

Correspondingly $\tilde{I}_4[r]$ can be derived from [37], where all invariants are given. For completion, we give the expressions here.

Denote the fundamental representations with $\{\Lambda_j\}$ corresponding to completely antisymmetric tensor representations, while $\{k\Lambda_1\}$ are the completely symmetric representations in the sense of Young's tableaux, i.e.

$$\Lambda_j \sim \begin{array}{c}\boxed{1}\\ \boxed{2}\\ \vdots\\ \boxed{j}\end{array}, \qquad k\Lambda_1 \sim \boxed{1}\boxed{2}\cdots\boxed{k}$$

To simplify the expressions, we define

$$\zeta_j = N(N+1) - 6j(N-j),$$

$$\kappa_k = N(N-1) + 6k(N+k).$$

Then, we find for $SU(N)$:

$$d[\Lambda_j] = \frac{N(N-1)\ldots(N-j+1)}{j!} \qquad 1 \le j \le N-1$$

$$d[k\Lambda_1] = \frac{N(N+1)\ldots(N+k-1)}{k!} \qquad k \ge 1$$

$$\tilde{I}_4[\Lambda_j] = \frac{(N-4)!}{N!}\frac{N-j}{(j-1)!}\zeta_j\prod_{r=1}^{j}(N-r+1)b^2$$

$$\tilde{I}_4[k\Lambda_1] = \frac{(N-1)!}{(N+3)!}\frac{N+k}{(k-1)!}\kappa_k\prod_{r=1}^{k}(N+r-1)b^2.$$

$SO(N)$:

$$d[\Lambda_j] = \frac{N!}{j!(N-j)!} \qquad 1 \le j \le \frac{N-3}{2}$$

$$d[k\Lambda_1] = \frac{N+2k-2}{k!}\frac{(N+k-3)!}{(N-2)!} \qquad k \ge 1$$

$$\tilde{I}_4[\Lambda_j] = \frac{(N-4)!}{(j-1)!(N-j-1)!}\zeta_j b^2$$

$$\tilde{I}_4[k\Lambda_1] = \frac{(N-2+2k)(N-2+k)!}{(k-1)!(N+2)!} \times [N^2-3N+8+6k(N-2+k)]b^2,$$

for N odd:

$$d[\Lambda_{(N-1)/2}] = 2^{(N-1)/2}, \qquad \tilde{I}_4[\Lambda_{(N-1)/2}] = -2^{(N-9)/2},$$

for N even:

$$d[\Lambda_{N/2}] = 2^{(N-2)/2}, \qquad \tilde{I}_4[\Lambda_{N/2}] = -2^{(N-10)/2}.$$

$Sp(N)$:

$$d[\Lambda_j] = \frac{N+2-2j}{j!}\frac{(N+1)!}{(N+2-j)!} \qquad 1 \le j \le N/2$$

$$d[k\Lambda_1] = \frac{(N+k-1)!}{k!(N-1)!} \qquad k \ge 1$$

$$\tilde{I}_4[\Lambda_j] = \frac{(N+2-2j)(N-3)!}{(j-1)!(N-j+1)!} \times [N^2+3N+8-6j(N+2-j)]b^2$$

$$\tilde{I}_4[k\Lambda_1] = \frac{(N+k)!}{(k-1)!(N+3)!}\kappa_k b^2.$$

In particular, we list the coefficients for the relevant groups in this paper, where b was taken as 1,

$r \setminus I_4[r]$	$SU(N)$	$SO(N)$	$Sp(N)$
G	$2Nb^2$	$(N-8)b^2$	$(N+8)b^2$
$\square$	b^2	b^2	b^2
$\square\square$	$(N+8)b^2$	$(N+8)b^2$	$(N+8)b^2$
$\begin{array}{c}\square\\ \square\end{array}$	$(N-8)b^2$	$(N-8)b^2$	$(N-8)b^2$

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