

Renormalon subtraction from the average plaquette and the gluon condensate

Taekoon Lee*

Department of Physics, Kunsan National University, Kunsan 573-701, Korea

(Received 28 February 2010; published 22 December 2010)

A Borel summation scheme of subtracting the perturbative contribution from the average plaquette is proposed using the bilocal expansion of the Borel transform. It is shown that the remnant of the average plaquette, after subtraction of the perturbative contribution, scales as a dim-4 condensate. A critical review of the existing procedure of renormalon subtraction is presented.

DOI: 10.1103/PhysRevD.82.114021

PACS numbers: 12.38.-t, 12.38.Gc, 12.38.Cy

I. INTRODUCTION

An old problem in lattice gauge theory is extracting the gluon condensate from the average plaquette, which in pure SU(3) Yang-Mills theory has the formal expansion

$$P(\beta) \equiv \left\langle 1 - \frac{1}{3} \text{Tr} U_{\square} \right\rangle \\ = \sum_{n=1} \frac{c_n}{\beta^n} + \frac{\pi^2}{36} Z(\beta) \left\langle \frac{\alpha_s}{\pi} GG \right\rangle a^4 + O(a^6), \quad (1)$$

where β denotes the lattice coupling and a the lattice spacing. The difficulty of extracting the gluon condensate is that the average plaquette is dominated by the perturbative contribution and it is necessary to subtract it to an accuracy better than one part in 10^4 . The perturbative coefficients c_n were computed to 10-loop order using the stochastic perturbation theory [1], but this alone does not achieve the required accuracy. Therefore, any attempt to extract the gluon condensate using the perturbative expansion must involve extrapolation of the perturbative coefficients to higher orders and, the perturbative expansion being asymptotic, proper handling of them. Since the large order behavior of perturbative expansion is determined by the renormalon singularity of the Borel transform, a natural extrapolation scheme would be based on the renormalon singularity. A program along this line was implemented by Burgio *et al.*, and the authors obtained a surprising result of power correction that scales as a dim-2 condensate [2]. This is in contradiction with the operator product expansion (1) that demands the leading power correction scale as a dim-4 condensate.

The claim of the dim-2 condensate was since then reexamined by several authors. In obtaining the perturbative contribution, Horsley *et al.* employed an extrapolation scheme based on the power law and truncation of the perturbative series at the minimal element [3], and Rakow used stochastic perturbation with boosted coupling to accelerate convergence [4], and Meurice employed extrapolations based on assumed singularity of the plaquette in the complex β plane as well as the renormalon

singularity, with truncation at the minimal element [5]. All these studies did not see any evidence of a dim-2 condensate but found the plaquette data were consistent with a dim-4 condensate.

To help settle these conflicting views on the dim-2 condensate we present in this paper a critical review of the renormalon-based approach of [2], and reveal a serious flaw in the program of renormalon subtraction, and show that the plaquette data, when properly handled, are consistent with a dim-4 condensate.

Specifically, we shall show that the continuum scheme employed for renormalon subtraction in [2] is not at all a scheme where the perturbative coefficients follow a renormalon pattern, and therefore the claimed dim-2 condensate is severely contaminated by the perturbative contribution and cannot be interpreted as a power correction. We then introduce a renormalon subtraction scheme based on the bilocal expansion of the Borel transform, and show that the plaquette data can be fitted well by the sum of a dim-4 condensate and the Borel summed perturbative contribution.

II. RENORMALON SUBTRACTION BY MATCHING LARGE ORDER BEHAVIORS

In this section we give a critical review on the renormalon subtraction procedure of [2]. The perturbative coefficients c_n of the average plaquette at large orders are expected to exhibit the large order behavior of the infrared renormalon associated with the gluon condensate, but the computed coefficients using stochastic perturbation theory turn out to grow much more rapidly than a renormalon behavior. This implies that the coefficients are not yet in the asymptotic regime, which is expected to set in at around order $\bar{n} = \beta z_0$ [z_0 given below in (5)], which gives $\bar{n} \sim 30$ for $\beta \sim 6$, far higher than the computed levels. It therefore appears all but impossible to extract the gluon condensate directly from using the stochastic perturbation theory, since the perturbative contribution must be subtracted, at least, to orders in the asymptotic regime.

In Ref. [2] this problem was approached by introducing a continuum scheme in which the renormalon contribution is subtracted by matching the large order behavior in the continuum scheme to the computed coefficients in the

*tlee@kunsan.ac.kr

TABLE I. The perturbative coefficients of the average plaquette in the continuum scheme.

c_1^{cont}	c_2^{cont}	c_3^{cont}	c_4^{cont}	c_5^{cont}	c_6^{cont}	c_7^{cont}	c_8^{cont}
2.0	-4.9792	10.613	-10.200	-44.218	316.34	-1096.	1947.

lattice scheme. Specifically, in order to relate c_n of the lattice scheme with the renormalon behavior, the average plaquette is written, essentially, as

$$P(\beta) = P^{\text{ren}}(\beta_c) + \delta P(\beta_c) + P_{\text{NP}}(\beta), \quad (2)$$

where

$$P^{\text{ren}}(\beta_c) = \int_0^{b_{\text{max}}} e^{-\beta_c b} \frac{\mathcal{N}}{(1 - b/z_0)^{1+\nu}} db \quad (3)$$

with β_c denoting the coupling in the continuum scheme defined by

$$\beta_c = \beta - r_1 - \frac{r_2}{\beta} \quad (4)$$

and

$$z_0 = \frac{16\pi^2}{33}, \quad \nu = \frac{204}{121}. \quad (5)$$

In Eq. (2) the plaquette is divided into perturbative contributions, made up of the renormalon contribution P^{ren} and the rest of the perturbative contribution δP , and non-perturbative power correction P_{NP} . In this splitting, the asymptotically divergent behavior of the perturbative contribution is contained in P^{ren} , and δP denotes the rest that can be expressed as a convergent series. (Here, the renormalons other than that associated with the gluon condensate and the subleading singularities at $b = z_0$ are ignored, which, if necessary, can be incorporated in P^{ren} .)

We now define $P_{\text{NP}}^{(N)}$ with

$$P_{\text{NP}}^{(N)}(\beta) \equiv P(\beta) - P^{\text{ren}}(\beta_c) - \sum_{n=1}^N (c_n - C_n^{\text{ren}}) \beta^{-n}, \quad (6)$$

where C_n^{ren} denotes the perturbative coefficients of P^{ren} in power expansion in $1/\beta$. Note that $P_{\text{NP}}^{(N)}$ is free of perturbative coefficients to order N . The constants r_1, r_2 that define the continuum scheme and the normalization constant \mathcal{N} are determined so that C_n^{ren} converges to c_n at large orders. In the continuum scheme with

$$r_1 = 3.1, \quad r_2 = 2.0 \quad (7)$$

and an appropriate value for \mathcal{N} , it was observed that C_n^{ren} converge to c_n at the orders computed in stochastic perturbation theory. The last term in (6) being a converging series $P_{\text{NP}}^{(N)}$ will be well-defined at $N \rightarrow \infty$, and this is precisely the quantity that was assumed to represent the power correction, and it was $P_{\text{NP}}^{(8)}$ that was shown to scale as a dim-2 condensate.

The essence of this procedure is that the isolation of the renormalon contribution is obtained by matching the large order behaviors in the lattice and continuum schemes, in which the matching does not involve the low order coefficients. Although the renormalon-caused large order behaviors of any two schemes can be matched, independently of the low order coefficients, it must be noted that the matching would work only when the known coefficients in both schemes exhibit renormalon behavior. Since, however, the computed coefficients in the lattice scheme are far from being in the asymptotic regime and do not follow the renormalon pattern, the matching cannot be performed reliably; therefore, the conclusion of a dimension-2 condensate based on it should be reexamined.

That the above matching has a serious flaw can be easily shown by mapping the perturbative coefficients in the lattice scheme to the continuum scheme (7). If the latter is indeed a good scheme for renormalon subtraction, the mapped coefficients should exhibit a renormalon behavior. However, as can be seen in Table I, which is obtained by mapping the central values of c_n from the stochastic perturbation theory, the coefficients are alternating in sign and far from being of a renormalon behavior. This shows that when mapping the perturbative coefficients between the lattice scheme and (7), the relatively high order coefficients (say, 7- to 10-loop orders) are still very sensitive on the low order coefficients. Therefore, the above large order matching cannot be performed reliably with the computed coefficients, and (7) cannot be the right scheme where one can isolate and subtract the renormalon contribution.

Checking the internal consistency of the subtraction scheme based on the matching of large order behavior also shows the underlying problem. The nonperturbative term in (2) can be written using (6) as

$$P_{\text{NP}}(\beta) = P_{\text{NP}}^{(N)}(\beta) - \left\{ \delta P(\beta_c) - \sum_{n=1}^N (c_n - C_n^{\text{ren}}) \beta^{-n} \right\}. \quad (8)$$

For $P_{\text{NP}}^{(N)}$ to represent the power correction it is clear that

$$\left| \delta P(\beta_c) - \sum_{n=1}^N (c_n - C_n^{\text{ren}}) \beta^{-n} \right| \ll P_{\text{NP}}^{(N)}(\beta) \quad (9)$$

must be satisfied. Since $\delta P(\beta_c)$ is by definition a convergent quantity, it can be written in a series expansion

$$\delta P(\beta_c) \equiv \sum_{n=1}^{\infty} D_n \beta_c^{-n}, \quad (10)$$

where D_n can be computed up to the order c_n are known, and (9) can be written approximately as

$$\frac{|\sum_{n=1}^N D_n \beta c^{-n} - \sum_{n=1}^N (c_n - C_n^{\text{ren}}) \beta^{-n}|}{P_{\text{NP}}^{(N)}(\beta)} \ll 1. \quad (11)$$

Now, in the scheme of (7), and at $N = 8$ and $\beta = 6.0, 6.2$, and 6.4 , for example, the ratios are 69, 59, and 42, respectively: a severe violation of the consistency condition. This again confirms that (7) cannot be a scheme suited for renormalon subtraction.

III. RENORMALON SUBTRACTION BY BOREL SUMMATION

It is now clear that one cannot subtract the perturbative contribution in the plaquette by mapping the renormalon-based coefficients in a continuum scheme to the lattice scheme, and then matching them with the computed high order coefficients. On the other hand, the lesson of our review suggests that one must map the known coefficients in the lattice scheme to a continuum one and look for a scheme where the mapped coefficients follow a renormalon behavior.

Once such a scheme is found one can perform Borel summation to subtract perturbative contribution to isolate the power correction. Borel summation is especially suited for this purpose, since it allows a precise definition of the power corrections in operator product expansion [6–8]. The nature of the renormalon singularity, hence of the large order behavior of perturbation, was obtained through the cancellation of the ambiguities in Borel summation and power corrections [9]. An extensive review of renormalons can be found in [10].

In this paper we shall assume that such a scheme exists and perform Borel summation using the scheme of bilocal expansion of the Borel transform [11]. To Borel-sum the divergent perturbation to a sufficient accuracy for the extraction of the power correction, one must have an accurate description of the Borel transform in the domain that contains the origin as well as the first renormalon singularity in the Borel plane. Bilocal expansion is a scheme of reconstructing the Borel transform in this domain, utilizing the known perturbative coefficients and properties of the first renormalon singularity. After Borel-summing the perturbative contribution, the sum of the Borel summation and a dim-4 power correction can be fitted to the plaquette data. A good fit would suggest then the power correction be of the dim-4 type.

The Borel summation using the first N -loop perturbations of the plaquette in bilocal expansion in a continuum scheme is given in the form

$$P_{\text{BR}}^{(N)}(\beta) = \int_0^\infty e^{-\beta_c b} \left[\sum_{n=0}^{N-1} \frac{h_n}{n!} b^n + \frac{\mathcal{N}}{(1 - b/z_0)^{1+\nu}} \right] db, \quad (12)$$

where the integration over the renormalon singularity is performed with principal value prescription. The essential idea of the bilocal expansion is to interpolate the two perturbative expansions about the origin and about the renormalon singularity to rebuild the Borel transform. By incorporating the renormalon singularity explicitly in the expansion, it can extend the applicability of the ordinary weak coupling expansion to beyond the renormalon singularity, and this scheme was shown to work well with static interquark potential or heavy quark pole mass [11,12]. Here, \mathcal{N} denotes the normalization constant of the large order behavior and the coefficients h_n are determined so that the Borel transform in (12) reproduces the perturbative coefficients in the continuum scheme when expanded at $b = 0$; thus h_n depends on the continuum perturbative coefficients as well as \mathcal{N} . By definition, $P_{\text{BR}}^{(N)}(\beta)$, when expanded in $1/\beta$, reproduces the perturbative coefficients of the average plaquette to N -loop order that were employed in building the Borel transform. For details of the bilocal expansion of the Borel transform we refer the reader to [11,12].

The power correction can then be defined by

$$P_{\text{NP}}^{(N)}(\beta) \equiv P(\beta) - P_{\text{BR}}^{(N)}(\beta), \quad (13)$$

which, by definition, has vanishing perturbative expansions to order N .

Using the perturbation to 10-loop order of the plaquette we compute $P_{\text{BR}}^{(10)}(\beta)$ in the continuum scheme parameterized by Eq. (4). Although \mathcal{N} can be computed perturbatively, using the perturbations of the average plaquette, it is still difficult to obtain a reliable result using the known coefficients, so here it will be treated as a fitting parameter. Thus in our scheme, as in [2], the fitting parameters are \mathcal{N} and r_1, r_2 of Eq. (4).

Using the plaquette data for $6.0 \leq \beta \leq 6.8$ from [13] and the relation between the lattice spacing a and β from static quark force simulation [14]

$$\log(a/r_0) = -1.6804 - 1.7331(\beta - 6) + 0.7849(\beta - 6)^2 - 0.4428(\beta - 6)^3, \quad (14)$$

the fit gives $\mathcal{N} = 165$ and

$$r_1 = 1.611, \quad r_2 = 0.246, \quad (15)$$

values which are substantially different from those in (7). The result of the fit is shown in Fig. 1, which shows that the power correction is consistent with a dim-4 condensate. The agreement improves as β increases, albeit with larger uncertainties; the deviation at low β ($\beta < 6$) may be attributed to a dim-6 condensate, which may be seen, though not presented here, by that adding a dim-6 power

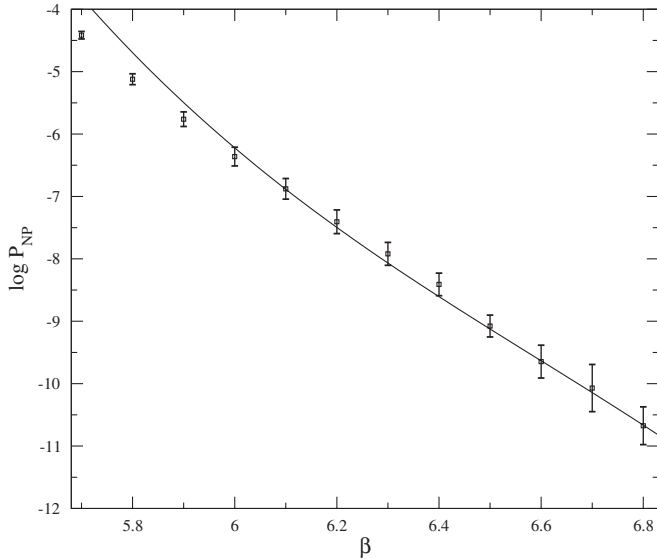


FIG. 1. $\log P_{\text{NP}}$ vs β . The solid line is for $4 \log(a/r_0) + 0.5$. The plot shows the power correction should be of the dim-4 type.

correction in the fit improves the agreement in the whole range of the plot. The error bars are from the uncertainty in the simulated perturbative coefficients of the plaquette. The uncertainty in the normalization constant does not appear to be large; for example, a variation of 20% in \mathcal{N} causes less than a quarter of those by the perturbative coefficients.

From the fit we obtain a dim-4 power correction of $P_{\text{NP}} \approx 1.6(a/r_0)^4$. Because of the asymptotic nature of the perturbative series the power correction of the plaquette is dependent on the subtraction scheme of the perturbative contribution, and thus our result may not be directly compared to those from other subtraction schemes. Nevertheless, it is still interesting to observe that the result is roughly consistent with $0.4(a/r_0)^4$ of [4] and $0.7(a/r_0)^4$ of [5]. Our result turns out to be a little larger than those estimates; this may be partly accounted for by the fact that the existing results were from the fit in the low β range of $\beta \lesssim 6$, in which range the data are below our fitted curve.

IV. SUMMARY

We have reexamined the claim of the dim-2 condensate in the average plaquette, and shown that the renormalon subtraction procedure of [2] that gave rise to the dim-2 condensate fails consistency checks and cannot be reliably implemented with the known results of stochastic perturbation theory. We then introduced a renormalon subtraction scheme based on the bilocal expansion of the Borel transform and found that the plaquette data are consistent with a dim-4 condensate.

ACKNOWLEDGMENTS

This work was supported in part by Korea Research Foundation Grant No. (KRF-2008-313-C00168).

-
- [1] F. Di Renzo and L. Scorzato, *J. High Energy Phys.* **10** (2001) 038.
 - [2] G. Burgio, F. Di Renzo, G. Marchesini, and E. Onofri, *Phys. Lett. B* **422**, 219 (1998).
 - [3] R. Horsley, P. E. L. Rakow, and G. Schierholz, *Nucl. Phys. B, Proc. Suppl.* **106-107**, 870 (2002).
 - [4] P. E. L. Rakow, *Proc. Sci., LAT2005* (2006) 284.
 - [5] Y. Meurice, *Phys. Rev. D* **74**, 096005 (2006).
 - [6] F. David, *Nucl. Phys.* **B234**, 237 (1984).
 - [7] F. David, *Nucl. Phys.* **B263**, 637 (1986).
 - [8] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Nucl. Phys.* **B249**, 445 (1985).
 - [9] A. H. Mueller, *Nucl. Phys.* **B250**, 327 (1985).
 - [10] M. Beneke, *Phys. Rep.* **317**, 1 (1999).
 - [11] T. Lee, *Phys. Rev. D* **67**, 014020 (2003).
 - [12] T. Lee, *J. High Energy Phys.* **10** (2003) 044.
 - [13] G. Boyd *et al.*, *Nucl. Phys.* **B469**, 419 (1996).
 - [14] S. Necco and R. Sommer, *Nucl. Phys.* **B622**, 328 (2002).