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Semiclassical computation of three-point functions of closed string vertex operators in $AdS_5 \times S^5$

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We consider the leading large string tension correction to correlation functions of three vertex operators of particular massive string states in $\mathrm{AdS}_5 \times S^5$ string theory. We assume that two of these states are "heavy" carrying large spins (of order string tension) and thus can be treated semiclassically while the third state is "light" having fixed quantum numbers. We study several examples. In the case when the "heavy" states are described by a folded string with large-spin in AdS_5 the 3-point function scales as a semiclassical spin parameter of the "heavy" state in power of the string level of the "light" massive string state. We observe similar behavior in the case of "heavy" states which admit a small angular momentum limit, which may thus represent creatures of three quantum massive string states.

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I. INTRODUCTION

According to the AdS/CFT duality [1] between the $\mathcal{N} = 4$ SYM (supersymmetric Yang-Mills) theory and the superstring theory in $\mathrm{AdS}_5 \times S^5$, the planar correlators of single-trace conformal primary operators in gauge theory should be related to the correlation functions of the corresponding closed-string vertex operators on a world sheet with the two-sphere topology. The integrated vertex operators may be parametrized by a point x^m on the AdS_5 boundary $V(\mathrm{x}) = \int d^2\xi \mathrm{V}(x(\xi) - x; \ldots)$. They depend on quantum numbers $Q_i = (S, J, \ldots)$ (such as spins and orbital momenta) and the 4d dimension (or AdS energy) Δ of the string states they represent. The dimension Δ is related to the quantum numbers Q_i and the string tension $T = \frac{\sqrt{\lambda}}{2\pi}$ by the marginality condition on the vertex operator V.

As we will review below in Sec. II, the vertex operators have generically an exponential dependence on the dimension Δ and the charges Q_i of the corresponding string states. Thus, when these quantum numbers are as large as the string tension, the vertex operators effectively scale exponentially with the string tension. It is then natural to expect that the leading large $\sqrt{\lambda}$ contribution to correlation functions of such operators is determined by a semiclassical string trajectory with sources provided by the vertex operators. This observation may lead to a prediction for the strong-coupling behavior of the corresponding gauge theory correlators for the dual (BPS) operators.

Such semiclassical approach was developed successfully for the calculation of two-point functions in [2–6] and also for the calculation of correlators involving Wilson loops [7–11]. A generalization to certain three-point functions was discussed in [6,12] and more recently addressed in [13,14].

More generally, one may consider a correlation function of some number of "heavy" (or "semiclassical") vertex operators V_H with $\Delta \sim Q_i \sim \sqrt{\lambda} \gg 1$ and some number of "light" (or "quantum") operators V_L with $Q_i \sim 1$ and $\Delta \sim \sqrt[4]{\lambda}$ (or $\Delta \sim 1$ for "massless" or BPS states). In this case one may again expect that, in a large $\sqrt{\lambda}$ expansion, the leading-order contribution to

$$\mathrm{K}_{H_1 \ldots H_n L_1 \ldots L_m} = \langle V_{H_1}(\mathrm{x}_1) \ldots V_{H_n}(\mathrm{x}_n) \times V_{L_1}(\mathrm{x}_{n+1}) \ldots V_{L_m}(\mathrm{x}_{n+m}) \rangle \quad (1.1)$$

will be given by the semiclassical string trajectory determined by the "heavy" operator insertions. To compute $\mathrm{K}_{H_1 \ldots H_n L_1 \ldots L_m}$ one should first construct the classical string solution that determines the leading large $\sqrt{\lambda}$ contribution to $\mathrm{K}_{H_1 \ldots H_n} = \langle V_{H_1}(\mathrm{x}_1) \ldots V_{H_n}(\mathrm{x}_n) \rangle$ and then compute (1.1) by simply evaluating the product of "light" vertex operators $V_{L_1}(\mathrm{x}_{n+1}) \ldots V_{L_m}(\mathrm{x}_{n+m})$ on this solution.

One may understand this procedure as a limit of the general semiclassical computation for the correlator of $n + m$ "heavy" operators, all of which have large quantum numbers. In this case the classical trajectory should be determined by a solution of the string equations with source terms provided by *all* the $n + m$ operators. Finding such surface appears to be hard in general, but if we formally assume that the charges of m of the $n + m$ sources are much smaller than the other n, then the semiclassical trajectory will be dominated by the contribution of the n large charges (the effect of the m small-charge sources may then be included perturbatively). Thus the leading contribution to the correlator will then be computed as suggested above for $\mathrm{K}_{H_1 \ldots H_n L_1 \ldots L_m}$. We will return to the discussion of the validity of this approach and the computation of quantum ($\frac{1}{\sqrt{\lambda}}$) corrections to the leading approximation in the concluding Sec. V.

Three-point correlation functions are the first nontrivial examples where these considerations become relevant.

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While finding a semiclassical trajectory controlling the leading contribution to $\langle V_{H_1}(x_1)V_{H_2}(x_2)V_{H_3}(x_3)\rangle$ is so far an unsolved problem [12], the discussion above suggests that one can use the semiclassical trajectory for the correlation function of *two* "heavy" operators $\langle V_{H_1}(x_1)V_{H_2}(x_2)\rangle$, which is straightforward to find [4,6], to compute the leading contribution to a correlator containing two "heavy" and one "light" state $\langle V_{H_1}(x_1)V_{H_2}(x_2)V_L(x_3)\rangle$. Examples of such computations, with V_H corresponding to a semiclassical string state with large-spin in S^5 and V_L representing a BPS state corresponding to a massless (supergravity) scalar or dilaton mode, were recently presented in [13,14].

The aim of the present paper is to consider more general cases when V_L may represent a massive string mode.[1] We shall study few explicit examples, attempting to clarify the general structure of such three-point functions. We shall also consider several choices for the "heavy" operator V_H. One will be the physically interesting case when V_H represents a folded string with large-spin S in AdS_5 dual to twist J operator. We will also try to shed light on the correlation function of three massive string states from the first excited string level by choosing V_H to represent a "small" semiclassical string that admits a smooth fixed-spin limit as proposed in [17].

The two-point and three-point correlation functions are special in that their dependence on the position of the operators is controlled by the target space conformal invariance[2]

$$\langle V_1(x_1)V_2(x_2)\rangle = \frac{C_{12}\delta_{\Delta_1,\Delta_2}}{|x_1-x_2|^{2\Delta_1}}, \tag{1.2}$$

$$\langle V_1(x_1)V_2(x_2)V_3(x_3)\rangle = \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}}. \tag{1.3}$$

Here in (1.2) $V_1 = V_2^*$. The two-point function coefficient C_{12} may be set to unity by a choice of normalization of vertex operators. The three-point function coefficient C_{123} may be extracted by setting x_1, x_2, x_3 to specific values. As we shall see below, in the case of $\langle V_H(x_1)V_H(x_2)V_L(x_3)\rangle$ a natural choice will be $|x_1| = |x_2| = 1$ and $x_3 = 0$.

To isolate the issue of normalization of operators one may consider ratios of particular three-point correlators with different operators or different values of quantum numbers of the same operator. Combining such correlators one may hope to extract information about normalization-independent data, like factors involving quantum numbers of the different types of the vertex operators at the same time. For example, in the combined ratio

$$\frac{\langle V_H(x_1)V_H(x_2)V_L(x_3)\rangle}{\langle V_H(x_1)V_H(x_2)V_{L'}(x_3)\rangle}\times\frac{\langle V_{H'}(x_1)V_{H'}(x_2)V_{L'}(x_3)\rangle}{\langle V_{H'}(x_1)V_{H'}(x_2)V_L(x_3)\rangle} \tag{1.4}$$

the normalization factors of both "heavy" and "light" states cancel out. Here H and H' and well as L and L' may differ by, e.g., choice of charges. This ratio is determined completely by terms in the three-point function $\langle V_H(x_1)V_H(x_2)V_L(x_3)\rangle$ which depend in a nontrivial way on the charges of both the "heavy" and the "light" states.

The structure of the rest of the paper is as follows. In Sec. II we review the structure of the bosonic part of some closed-string vertex operators of the $AdS_5\times S^5$ superstring. We consider several examples which will be used in later sections: the "massless" operators representing dilaton and the superconformal primary state of charge J, the massive state with spin S on the leading Regge trajectory and a special singlet string state existing on massive string levels.

In Sec. III we review the semiclassical calculation of two-point correlation functions of large charge operators. We discuss in detail the string states dual to large-spin twist-two operators and to large twist J operators.

We then proceed in Sec. IV to discuss the three-point functions of one "light" and two "heavy" operators discussed in Sec. II and III. We also use the same approach to construct the three-point functions in the case when the "heavy" operators are described by a classical trajectory admitting a small-spin limit. In all cases we will identify the normalization-independent features of the three-point function coefficient.

Some concluding remarks including comments on the validity of our approach and on the calculation of quantum corrections to the three-point functions are made in Sec. V.

II. EXAMPLES OF STRING VERTEX OPERATORS

Let us start with a review of the structure of relevant vertex operators following [4,17]. Their form is perhaps most transparent in the 6 + 6 embedding coordinates.[3] In these coordinates the action of the $AdS_5\times S^5$ superstring sigma model has the following structure:

[1]Correlation functions of (non-near-BMN) massive string states were not discussed in the past, apart from not directly related study of decay of semiclassical spinning string in [15,16].

[2]Here we assume for simplicity that the primary operators are scalar. In the case of primary with spin operators there are extra kinematic factors (see, e.g., [18]). For example, from a three-point function of two scalar operators and one spin s operator $V_{m_1\ldots m_s}(x_3)$ (which is a symmetrized traceless tensor) we get an extra factor $d_{m_1}\ldots d_{m_s}$, where $d_m = \frac{(x_3-x_1)_m}{(x_3-x_1)^2} - \frac{(x_3-x_2)_m}{(x_3-x_2)^2}$. In the case of "heavy" operators with spins such factors may be ignored as we will consider ratios of three-point functions to their two-point functions. In the case when the "light" operator corresponds to spin s operator like $\mathrm{tr}(\bar{Z}D_+^sZ)$ we shall implicitly assume that the corresponding extra factor $(d_+)^s = [\frac{(x_3-x_1)_+}{(x_3-x_1)^2} - \frac{(x_3-x_2)_+}{(x_3-x_2)^2}]^s$ is included.

[3]We shall follow the notation of [6]. The relation to the notation for the coordinates of AdS_5 and S^5 in [4] is: $Y_M\to N_M$, $X_k\to n_k$.

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\xi (\partial Y_M \bar{\partial} Y^M + \partial X_k \bar{\partial} X_k + \text{ fermions}), \quad (2.1)$$

$$Y_M Y^M = -Y_0^2 - Y_5^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1,$$
$$X_k X_k = X_1^2 + \ldots + X_6^2 = 1. \quad (2.2)$$

For a world sheet with Minkowski signature the 2d derivatives are $\partial = \partial_+$, $\bar{\partial} = \partial_-$. In general, the vertex operators are constructed in terms of Y_M, X_k and fermions and correspond to the highest-weight states of $SO(2,4) \times SO(6)$ representations. They are (exactly) marginal operators of dimension two, i.e. are particular linear combinations of products of Y_M, X_k and their derivatives that are eigenvectors of the 2d anomalous dimension operator.[4] Fermions render the $AdS_5 \times S^5$ sigma model UV finite; since we will be interested in the leading-order of the semiclassical expansion we may nevertheless ignore them both in the action and the vertex operators. Being interested in the leading large string tension approximation we may also ignore all $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections to the bosonic part of the vertex operators (see also Sec. V).

Let us recall the basic relation between the embedding coordinates and the global and Poincaré coordinates in AdS_5 that we will use below:

$$Y_5 + iY_0 = \cosh\rho e^{it}, \qquad Y_1 + iY_2 = \sinh\rho \cos\theta e^{i\phi_1},$$
$$Y_3 + iY_4 = \sinh\rho \sin\theta e^{i\phi_2}, \qquad Y_m = \frac{x_m}{z},$$
$$Y_4 = \frac{1}{2z}(-1 + z^2 + x^m x_m), \qquad Y_5 = \frac{1}{2z}(1 + z^2 + x^m x_m), \quad (2.3)$$

where $x^m x_m = -x_0^2 + x_i x_i$ ($m = 0, 1, 2, 3$; $i = 1, 2, 3$). If a highest-weight state of an $SO(2,4)$ representation is labeled by the three Cartan generators (E, S_1, S_2) corresponding to rotations in the planes (5, 0), (1, 2) and (3, 4), a wave function or a vertex operator representing a state with AdS energy E should contain a factor $(Y_5 + iY_0)^{-E} = (\cosh\rho)^{-E} e^{-iEt}$. This is just the AdS analog of the flat-space energy dependent plane wave factor e^{-iEt}. If, equivalently, the representation is labeled by the $SO(1,1)$ generator in the (5, 4) plane, then the corresponding factor is $(Y_5 + Y_4)^{-\Delta}$, where Δ is the eigenvalue of the dilatation generator (acting as $z \to kz$, $x_m \to kx_m$).

For the construction of the classical string solution describing the semiclassical approximation of the two-point function of "heavy" vertex operators it is useful, as described in [4,6], to consider the Euclidean continuation

$$t_e = it, \qquad Y_{0e} = iY_0, \qquad x_{0e} = ix_0, \quad (2.4)$$

[4] Even though the sigma model of the Green-Schwarz type is not a factorized CFT, the (anti)holomorphy of the two components of the stress tensor guarantees that the left and right dimensions are well-defined quantities.

so that $Y^M Y_M = -Y_5^2 + Y_{0e}^2 + Y_i Y_i + Y_4^2 = -1$. The SO (2, 4) symmetry is then replaced by SO(1, 5), which contains the discrete transformation $Y_{0e} \leftrightarrow Y_4$, $E \leftrightarrow \Delta$ that relates the factors $(Y_5 + iY_0)^{-E}$ and $(Y_5 + Y_4)^{-\Delta}$. Up to a normalization factor, we shall sometimes denote this factor by K in the following:

$$K(x, z) = k_\Delta (Y_+)^{-\Delta} = k_\Delta (z + z^{-1} x^m x_m)^{-\Delta},$$
$$Y_+ \equiv Y_5 + Y_4. \quad (2.5)$$

As is well known, $K(x - x', z) = k_\Delta [z + z^{-1}(x - x')^2]^{-\Delta}$ is a solution of the scalar Laplace equation in AdS_5 with mass $m^2 = \Delta(\Delta - 4)$; the normalization constant k_Δ can be chosen such that $K(x - x', z \to 0) = \delta^{(4)}(x - x')$.

In general, an unintegrated vertex operator will have the structure

$$\text{V} \sim (Y_+)^{-\Delta} [(\partial^s Y)^r \ldots (\bar{\partial}^m X)^n + \ldots]$$
$$\equiv (Y_+)^{-\Delta} U(Y, X, \ldots). \quad (2.6)$$

To construct an integrated vertex operator parametrized by the four coordinates of a point on the boundary of the Euclidean Poincaré patch of AdS_5 space, we should shift $x_m = (x_{0e}, x_i)$ by a constant vector x_m (translations in x_m are part of global conformal symmetry)

$$V(\text{x}) = \int d^2\xi \text{V}(x(\xi) - \text{x}; \ldots)$$
$$= \int d^2\xi K(x(\xi) - \text{x}, z(\xi)) U[x(\xi) - \text{x}, z(\xi), X(\xi)]. \quad (2.7)$$

Let us now discuss some examples of such vertex operators which we shall use as "heavy" or "light" factors in the three-point correlation functions below.

A. Dilaton operator

The 10-d dilaton field is decoupled from the metric perturbation in the Einstein frame [19], i.e. it satisfies the free massless 10-d Laplace equation in $AdS_5 \times S^5$. Keeping nonzero value of S^5 momentum (corresponding to a higher KK harmonic of the 10-d dilaton), the corresponding massless string vertex operator representing a highest-weight state of $SO(2,4) \times SO(6)$ is simply proportional to the world sheet Lagrangian

$$\text{V}_J^{(\text{dil})} = (Y_+)^{-\Delta} (\text{X}_x)^J (\partial Y_M \bar{\partial} Y^M + \partial X_k \bar{\partial} X_k + \text{ fermions}),$$
$$\text{X}_x \equiv X_1 + iX_2 = \cos\vartheta e^{i\varphi}. \quad (2.8)$$

Here and below in this section we shall ignore the fermionic terms and overall normalization factors in the vertex operators. The marginality condition is $2 = 2 - \frac{1}{2\sqrt{\lambda}} \times [\Delta(\Delta - 4) - J(J + 4)] + \mathcal{O}(\frac{1}{(\sqrt{\lambda})^2})$, so that to the leading-order in the large $\sqrt{\lambda}$ expansion $\Delta = 4 + J$. Inclusion of fermions should guarantee that all higher-order corrections vanish as this should be a BPS state. The corresponding dual gauge theory operator should be $\sim\mathrm{tr}(F^2_{mn}Z^J + \ldots)$ or, for $J = 0$, just the SYM Lagrangian.

The form of the resulting integrated dilaton operator (2.7) and (2.8), can be understood as follows. On string side, 10-d dilaton couples to string action as $\int d^2\xi e^{(1/2)\Phi(X)} g_{IJ}(X)\partial X^I \bar\partial X^J + \ldots$ where $g_{IJ}(X)$ is Einstein-frame metric and X^I are 10-d coordinates. To get the on shell (marginal) vertex operator one is to linearize in Φ and restrict Φ to be a solution of the corresponding wave equation. In the AdS/CFT context we should then have (ignoring KK momentum dependence) $\Phi(x, z) = \int d^4\mathrm{x}\, K(x - \mathrm{x}, z)\phi(\mathrm{x})$, where the "4-d dilaton" $\phi(x) = \Phi(x, z \to 0)$ is an arbitrary boundary source function. The corresponding (D3-brane) coupling on the gauge theory side is $\int d^4\mathrm{x}\,\mathrm{tr}[e^{-\phi(x)}F^2_{mn}(\mathrm{x}) + \ldots]$. The string theory and gauge theory correlation functions are then obtained by taking functional derivatives over $\phi(x)$; insertion of the gauge theory Lagrangian into a gauge theory correlator corresponds to insertion of $V^{(\mathrm{dil})}$, i.e. the string theory Lagrangian *multiplied* by the function $K \sim (Y_+)^{-\Delta}$, into the string theory correlator.

Note that the constant part of the dilaton appears in the string action in the same way as the string tension factor $\sqrt{\lambda}$ and in the gauge theory action as the gauge coupling λ. Taking the derivative $\lambda\frac{\partial}{\partial\lambda}$ of a gauge theory correlator corresponds to the insertion of the gauge theory action; applying $\lambda\frac{\partial}{\partial\lambda}$ to a string theory correlator corresponds to the insertion of the string theory action. The two are indeed related as the "zero-momentum dilaton" corresponds to the dilaton operator ($\Delta = 4$) integrated over the four-space,

$$V^{(0-\mathrm{dil})} \equiv \int d^4\mathrm{x}\, V^{(\mathrm{dil})}(\mathrm{x}) \to \int d^4\mathrm{x} \int d^2\xi (z + z^{-1}|x(\xi) - \mathrm{x}|^2)^{-4}(\partial Y_M \bar\partial Y^M + \ldots). \quad (2.9)$$

Doing first the integral over x one finds that the $K \sim (z + z^{-1}|x - \mathrm{x}|^2)^{-4}$ factor goes away and we end up just with the string action, i.e. $V^{(0-\mathrm{dil})} \sim \int d^2\xi(\partial Y_M \bar\partial Y^M + \ldots)$.

This implies, in particular, the following "zero-momentum dilaton" relation,

$$\langle V(\mathrm{x}_1)V^*(\mathrm{x}_2)V^{(0-\mathrm{dil})}\rangle = \lambda\frac{\partial}{\partial\lambda}\langle V(\mathrm{x}_1)V^*(\mathrm{x}_2)\rangle = -\lambda\frac{\partial\Delta}{\partial\lambda}\frac{1}{|\mathrm{x}_1 - \mathrm{x}_2|^{2\Delta}}\ln|\mathrm{x}_1 - \mathrm{x}_2|^2, \quad (2.10)$$

i.e. the insertion into a two-point function of the dilaton operator integrated over four-space (i.e. of the gauge theory action on the gauge theory side or the string theory action on the string theory side) is proportional to the λ-derivative of the dimension (see [14] for a closely related discussion).

As a result, the C_{123} corresponding to $\langle V(\mathrm{x}_1)\times V^*(\mathrm{x}_2)V^{(\mathrm{dil})}(\mathrm{x}_3)\rangle$ should be proportional to $\lambda\frac{\partial}{\partial\lambda}\Delta$. Indeed, taking $\Delta_1 = \Delta_2 = \Delta$ and $\Delta_3 = 4$ in (1.3) and integrating over x_3 one gets $C_{123}|\mathrm{x}_1 - \mathrm{x}_2|^{-2\Delta+4}\int d^4\mathrm{x}_3 \frac{1}{|\mathrm{x}_3-\mathrm{x}_1|^4|\mathrm{x}_3-\mathrm{x}_2|^4}$. The latter is proportional to $C_{123}|\mathrm{x}_1 - \mathrm{x}_2|^{-2\Delta}\ln(\epsilon|\mathrm{x}_1 - \mathrm{x}_2|)$ (ϵ is a cutoff) and should be compared with (2.10).

B. Superconformal primary scalar operator

This scalar represents the superconformal primary state and is the highest-weight state of the SO(6) representation $[0, J, 0]$, $J \geq 2$. The corresponding dimension is $\Delta = J$. The dual gauge theory operator is the BMN operator $\mathrm{tr}Z^J$. The dilaton operator is the supersymmetry descendant of this operator.

The corresponding massless string state originates from the trace of the graviton in S^5 directions that induces also the components of the graviton in AdS_5 directions and mixes with the RR five-form [19,20]. As discussed in [7,13], the bosonic part of corresponding vertex operator can be taken in the form (ignoring derivative terms that will not contribute to the computation done in Sec. IV)

$$\mathrm{V}^{(\mathrm{scal})}_J = (Y_+)^{-\Delta}\mathrm{X}^J_x[z^{-2}(\partial x^m \bar\partial x_m - \partial z \bar\partial z) - \partial X_k \bar\partial X_k]. \quad (2.11)$$

The two-derivative factor here can also be written as $[z^{-2}(\partial x^m \bar\partial x_m - \partial Z_k \bar\partial Z_k)]$, with $Z_k = zX_k$, $Z_kZ_k = z^2$, so this is just the string Lagrangian with the 4d and 6d parts taken with opposite sign.[5]

C. Operators with spin on leading Regge trajectory

In flat-space (bosonic) string theory a spin S state on the leading Regge trajectory is represented by $V_S = e^{-iEt}(\partial \mathrm{x}_x \bar\partial \mathrm{x}_x)^{(S/2)}$, $\mathrm{x}_x = x_1 + ix_2$, with the marginality condition being $2 = S - \frac{1}{2}\alpha' E^2$, i.e. $E = \sqrt{\frac{2}{\alpha'}(S - 2)}$. By analogy, in $AdS_5 \times S^5$, candidate operators for states on the leading Regge trajectory are (after the Euclidean continuation and $E \to \Delta$ flip)

$$V_S = (Y_+)^{-\Delta}(\partial Y_x \bar\partial Y_x)^{(S/2)} + \ldots, \quad Y_x = Y_1 + iY_2, \quad (2.12)$$

$$V_J = (Y_+)^{-\Delta}(\partial \mathrm{X}_x \bar\partial \mathrm{X}_x)^{(J/2)} + \ldots, \quad \mathrm{X}_x = X_1 + iX_2, \quad (2.13)$$

[5]Note that a similar factor would appear if one would start with a near-horizon limit of the D3-brane metric $ds^2 = H^{-1/2}(z)dx_m dx^m + H^{1/2}(z)dZ_k dZ_k$, $H = \frac{Q}{z^4}$ and formally consider a local deformation of the Q parameter. A similar deformation (but with different coefficients for the "4d" and "6d" parts of the metric) corresponds to a fixed scalar dual to $\mathrm{tr}F^4 + \ldots$ operator which is a supersymmetry descendant of the $\mathrm{tr}Z^4$ operator [21].

where the ellipsis stand for terms resulting from the diagonalization of the 2d anomalous dimension operator. In general, ignoring fermions, the operator $(\partial X_x \bar\partial X_x)^{(J/2)}$ in the SO(6) sigma model may mix with

$$(X_x)^{2p+2q}(\partial X_x)^{(J/2)-2p}(\bar\partial X_x)^{(J/2)-2q}(\partial X_\ell \partial X_\ell)^p(\bar\partial X_k \bar\partial X_k)^q, \tag{2.14}$$

where $p, q = 0, \ldots, \frac{J}{4}$; $l, k = 1, \ldots, 6$. The operator $(Y_+)^{-\Delta}(\partial Y_x \bar\partial Y_x)^{(S/2)}$ in the SO(2, 4) sigma model may mix with

$$(Y_+)^{-E-p-q}Y_x^{p+q}(\partial Y_+)^p(\partial Y_x)^{(S/2)-p}(\bar\partial Y_+)^q(\bar\partial Y_x)^{(S/2)-q} + O(\partial Y_M \partial Y^M \bar\partial Y_K \bar\partial Y^K), \tag{2.15}$$

where $p, q = 0, \ldots, \frac{S}{4}$; $M, K = 0, 1, \ldots 5$. The true vertex operators are eigenvectors of the anomalous dimension matrix, i.e. they are particular linear combinations of the above structures determined, e.g., by solving Laplace (or Lichnerowitz) type equation for the corresponding tensor wave function, e.g., $\hat\gamma\Psi = [2 - S + \frac{1}{2}\alpha'\nabla^2 + \sum c_k \alpha'^k (R\ldots)^n \ldots \nabla^p]\Psi = 0$.

Since all operators in Eqs. (2.14) and (2.15) have the same classical dimension, their mixing is not suppressed by $\alpha' \sim \frac{1}{\sqrt{\lambda}}$. However, considering such operators as the "heavy" ones in a correlation function (i.e. treating them semiclassically assuming that their dimension Δ and spins are as large as $\sqrt{\lambda}$) makes it unnecessary to consider explicitly the effects of the mixing. Indeed, all that is required is that the classical solution they source should have a definite energy or Δ, thus effectively representing an eigenvector of the 2d anomalous dimension operator [4,6].

D. Singlet scalar operators on higher string levels

There exist special massive string state vertex operators with finite quantum numbers for which the leading-order bosonic part is known explicitly and thus they can be used as candidates for "light" vertex operators in the semiclassical computation of the correlation functions discussed in the introduction. These are singlet operators that do not mix with other operators to leading nontrivial order in $\frac{1}{\sqrt{\lambda}}$ [4,17].

Consider, e.g., an operator built out of derivatives of S^5 coordinates X_k. An example of a scalar operator carrying no spins is[6]

$$V_q = (Y_+)^{-\Delta}[(\partial X_k \bar\partial X_k)^q + \ldots]. \tag{2.16}$$

This operator corresponds to a scalar string state at level $n = q - 1$ so that the fermionic contributions should make the $q = 1$ state massless (BPS), with $\Delta = 4$ following from the marginality condition. The $q = 2$ choice corresponds to a scalar state on the first excited string level.[7]

The number of $(\partial X_k \bar\partial X_k)$ factors in an operator cannot increase due to renormalization [4]; thus if an operator does not contain any such factors, they cannot be induced by renormalization. This leads to an example of another scalar operator which is a true singlet and is known explicitly at the leading-order

$$V_r = (Y_+)^{-\Delta}(\partial X_k \partial X_k \bar\partial X_\ell \bar\partial X_\ell)^{r/2}, \qquad r = 2, 4, \ldots . \tag{2.17}$$

Ignoring fermionic contributions, its dimension is determined from $0 = \hat\gamma = 2 - 2r + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) + 8r] + \mathcal{O}(\frac{1}{(\sqrt{\lambda})^2})$, i.e. $\Delta = 2\sqrt{r-1}\sqrt[4]{\lambda} + 2 - \frac{2r-1}{\sqrt{r-1}\sqrt[4]{\lambda}} + \mathcal{O}(\frac{1}{(\sqrt[4]{\lambda})^3})$. While the contribution of fermions may, of course, change the subleading terms, cf. [17,22], they cannot alter the form of the fermion-independent part of the vertex operator. As this operator represents a singlet scalar, the corresponding field should satisfy a simple Laplace-type equation[8] $(-\nabla^2 + M^2)\Phi = 0$, $M^2 = \Delta(\Delta - 4) = 4(r-1)\sqrt{\lambda} + \ldots$. Thus adding S^5 KK momentum is straightforward by simply including a factor of X_x^J as in (2.8).

We may also consider the AdS_5 counterpart of the singlet operator (2.17), namely

$$V_k = (Y_+)^{-\Delta}(\partial Y_M \partial Y^M \bar\partial Y_K \bar\partial Y^K)^{k/2}, \qquad k = 2, 4, \ldots . \tag{2.18}$$

The operators V_r in (2.17) and V_k in (2.18) have a very special structure: their derivative factor is constructed out of chiral components $T_{++} = T$ and $T_{--} = \bar T$ of the stress tensor of the S^5 or AdS_5 sigma models, respectively, i.e. $V_r = (Y_+)^{-\Delta}(T\bar T)^{r/2}$. Thus, when evaluated on a classical string solutions[9] the factor $(T\bar T)^{r/2}$ will simply be a constant in power $r/2$. Up to this constant the contribution of this singlet operator to a three-point correlator with two "heavy" operators will then be the same as that of the "naive" scalar operator $(Y_+)^{-\Delta}X_x^J$.

The simplest example of the operator (2.17) is $r = 2$ representing a massive state on the first excited string level, which should be dual to a member of Konishi multiplet (see [17]). We may thus use it not only to evaluate three-point correlators of a singlet massive string mode with two "heavy" modes represented by large-spin operators like (2.12), but also with two "heavy" modes corresponding to

[6]The marginality condition for this operator is $0 = \hat\gamma = 2 - 2q + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta-4) + 2q(q-1)] + \frac{1}{(\sqrt{\lambda})^2}[\frac{2}{3}q(q-1)(q-\frac{7}{2}) + 4q] + \mathcal{O}(\frac{1}{(\sqrt{\lambda})^3})$.

[7]Then [17] $\Delta(\Delta - 4) = 4\sqrt{\lambda} - 4 + \mathcal{O}(\frac{1}{\sqrt{\lambda}})$, so that $\Delta = 2\sqrt[4]{\lambda} + 2 + \frac{0}{\sqrt[4]{\lambda}} + \mathcal{O}(\frac{1}{(\sqrt[4]{\lambda})^3})$. Here the subleading terms should not, however, be trusted as fermions are expected to change the Δ independent terms in the one-loop anomalous dimension.

[8]To leading-order in large $\sqrt{\lambda}$ we may ignore a constant shift in Δ, i.e. ignore position of that scalar in a supermultiplet.

[9]In conformal gauge the classical stress tensors of the bosonic AdS_5 and S^5 sigma models are separately conserved and traceless so that their holomorphic components can be chosen to be constant; the Virasoro condition equates their sums to zero.

a "small" semiclassical string which may also be used to represent string modes on the first excited level as discussed in [17]. As we shall discuss below, such a calculation may then determine the leading term in the correlation function of there "light" massive string modes like three Konishi-type states.

III. REVIEW OF SEMICLASSICAL APPROXIMATION FOR TWO-POINT CORRELATOR OF LARGE-SPIN STATES

The semiclassical calculation of the correlation function of two "heavy" string states represented by large orbital momentum (2.8) and (2.11) or large-spin (2.12) and (2.13) vertex operators was described in [6] (see also [10,12] for related discussions). Here we review the main points of this calculation.

The classical solution describing a pointlike string with large orbital momentum in S^5 corresponding, e.g., to BMN-type states with vertex operators as in (2.8) or (2.11) with Δ, $J \sim \sqrt{\lambda}$ is $t = \kappa\tau$ (in AdS_5) and $\varphi = \kappa\tau$ (in S^5). It represents a massive geodesic in AdS_5, running through the center of the space and never reaching the boundary. After a Euclidean continuation the geodesic reaches the boundary: in Poincaré coordinates it is (cf. (2.4))[10]

$$z = \frac{1}{\cosh(\kappa\tau_e)}, \qquad x_{0e} = \tanh(\kappa\tau_e), \qquad x_i = 0, \qquad \varphi = -i\kappa\tau_e, \qquad \tau_e = i\tau. \tag{3.1}$$

The radial coordinate z vanishes in the limits $\tau_e \to \pm\infty$, implying indeed that the Euclidean trajectory reaches the boundary at the two points: $x_{0e} = -1$, $x_i = 0$ and $x_{0e} = 1$, $x_i = 0$.[11] These points are the locations of the two vertex operators sourcing the classical trajectory.

Quite generally, the two vertex operators whose two-point function we are computing are placed at $\tau_e = \pm\infty$ on the Euclidean 2d world sheet cylinder. Their positions may be mapped to arbitrary positions ξ_1 and ξ_2 on the ξ complex plane [4,6] by the transformation:

$$e^{\tau_e + i\sigma} = \frac{\xi - \xi_2}{\xi - \xi_1}. \tag{3.2}$$

Given a classical solution with given global charges on a Lorentzian 2d cylinder, its analytically continued form mapped onto the complex plane should then be the stationary trajectory of the path integral representing the two-point correlation function of the vertex operators with the given global charges. The "delta-function" sources representing the vertex operators for the ("semiclassical") string states are placed at positions ξ_1 and ξ_2. The role of matching onto source terms is to relate the parameters of the semiclassical solution to the quantum numbers $(\Delta, J, \ldots)$ that label the vertex operators.[12] Then, using the massless scalar operators like (2.8) or (2.11) with $J = \sqrt{\lambda}\kappa \gg 1$, $\Delta = J$, the four-dimensional and two-dimensional conformal invariances imply that, in general, the two-point function should have the form

$$\langle V_J(\mathrm{x}_1) V_J^*(\mathrm{x}_2)\rangle \sim \frac{1}{|\mathrm{x}_1 - \mathrm{x}_2|^{2\Delta}} \int \frac{d^2\xi_1 d^2\xi_2}{|\xi_1 - \xi_2|^4}. \tag{3.3}$$

The semiclassical trajectory (3.1) is consistent with the special choice of $\mathrm{x} = (-1, 0, 0, 0)$, $\mathrm{x}' = (1, 0, 0, 0)$; the divergent 2d "Möbius" factor should cancel against the standard normalization of the string path integral [6].

To apply this method to the two-point function $\langle V_S(\mathrm{x}_1) V_S^*(\mathrm{x}_2)\rangle$ of operators (2.12) and (2.7)

$$V_S(\mathrm{x}) = c \int d^2\xi [z(\xi) + z(\xi)^{-1}(x(\xi) - \mathrm{x})^2]^{-\Delta} \times [\partial Y_x(x(\xi) - \mathrm{x})\bar{\partial} Y_x(x(\xi) - \mathrm{x})]^{S/2}$$
$$Y_x(x) = Y_1(x) + iY_2(x) = \frac{x_1 + ix_2}{z}, \tag{3.4}$$

we should consider the limit of $\Delta \sim S \sim \sqrt{\lambda} \gg 1$, with $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$ being large. As was demonstrated in [6] (see also [4,5]) the semiclassical trajectory saturating this two-point correlator is equivalent to the conformally transformed (3.2) Euclidean continuation of the asymptotic large-spin limit [23,24] of the spinning folded string solution in AdS_3, i.e.

$$t = \kappa\tau, \qquad \phi \equiv \phi_1 = \kappa\tau, \qquad \rho = \mu\sigma, \qquad \kappa = \mu \approx \frac{1}{\pi}\ln\mathcal{S} \gg 1. \tag{3.5}$$

The background (3.5) approximates the exact elliptic function solution [3] in the limit κ, $\mu \gg 1$ on the interval $\sigma \in [0, \frac{\pi}{2}]$; to obtain the formal periodic solution on $0 < \sigma \le 2\pi$ one needs to combine together four stretches $\rho = \mu\sigma$ of the folded string.

In the embedding coordinates, the formal Euclidean continuation of this solution is[13]

$$Y_5 = \cosh(\kappa\tau_e)\cosh(\mu\sigma),$$
$$Y_{0e} = \sinh(\kappa\tau_e)\cosh(\mu\sigma), \qquad Y_4 = 0,$$
$$Y_1 = \cosh(\kappa\tau_e)\sinh(\mu\sigma),$$
$$Y_2 = -i\sinh(\kappa\tau_e)\sinh(\mu\sigma), \qquad Y_3 = 0. \tag{3.6}$$

[10]The Euclidean stationary-point solution for the coordinates of S^5 is, in general, complex (see also [2,4,5,8,9]) but there is no *a priori* condition that such solution should be real.

[11]By a dilatation and translation, the position of the two end points may be chosen to be $x_{0e} = 0$ and $x_{0e} = a$; the corresponding solution is then [6]: $z = \frac{a}{2\cosh(\kappa\tau_e)}$, $x_{0e} = \frac{a}{2}[\tanh(\kappa\tau_e) + 1]$, $x_i = 0$.

[12]Note that the transformation from a cylinder to the complex plane is not essential if we are interested only in the value of a correlator of integrated vertex operators.

[13]Here we depart from the notation in [6] in that we do not change the sign of ϕ at the same time as doing the Euclidean continuation.

In Poincaré coordinates (2.3) this becomes

$$z = \frac{1}{\cosh(\kappa\tau_e)\cosh(\mu\sigma)}, \qquad x_{0e} = \tanh(\kappa\tau_e), \quad (3.7)$$

$$x_1 = \tanh(\mu\sigma), \qquad x_2 = -i\tanh(\mu\sigma)\tanh(\kappa\tau_e), \quad (3.8)$$

$$x_\pm \equiv x_1 \pm ix_2 = re^{\pm i\phi} = \frac{\tanh(\mu\sigma)}{\cosh(\kappa\tau_e)}e^{\pm\kappa\tau_e}, \quad (3.9)$$

$$z^2 + x_{0e}^2 + x_1^2 + x_2^2 = 1. \quad (3.10)$$

While in Poincaré coordinates in Lorentzian signature the string moves towards the center of AdS, rotating and stretching, after the Euclidean continuation the resulting complex world surface described by (3.6) approaches the boundary ($z \to 0$) at $\tau_e \to \pm\infty$ at $x_{0e}(\pm\infty) = \pm 1$ and "lightlike" lines in the (complex) (x_1, x_2) plane:

$$\tau_e \to +\infty: z \to 0, \qquad x_{0e} \to 1,$$
$$x_+ \to 2\tanh(\mu\sigma), \qquad x_- \to 0, \quad (3.11)$$

$$\tau_e \to -\infty: z \to 0, \qquad x_{0e} \to -1,$$
$$x_+ \to 0, \qquad x_- \to 2\tanh(\mu\sigma). \quad (3.12)$$

The radius $r = \sqrt{1 - x_{0e}^2 - z^2}$ in the (x_1, x_2) plane goes to zero at the boundary while the angle ϕ in (3.9) goes to $\pm i\infty$.

Note that the fact that this classical solution does not simply end at two points at the boundary does not represent a problem. In general, we are supposed to start with two vertex operators (2.7) parametrized by some arbitrary points x_1 and x_2 (which are also the points where dual gauge theory operators are inserted in the SYM correlator corresponding to (1.2)) and then find the classical string trajectory "sourced" by such operators. As was shown in [6], doing this for the choice of $\mathrm{x}_1 = (1, 0, 0, 0)$ and $\mathrm{x}_2 = (-1, 0, 0, 0)$ (or similar choice related by rescaling and translation, see footnote 11) leads to the stationary-point solution (3.7), (3.8), (3.9), and (3.10). Thus positions of the boundary values of the classical string coordinates need not, in general, coincide with the positions of the vertex operators x_1 and x_2 (though that does happen for simple string solutions which are pointlike in AdS_5, cf. [12,13]).

The discussion above generalizes straightforwardly to the large-spin operator carrying also large orbital momentum $J = \sqrt{\lambda}\mathcal{J}$ in S^5,

$$V_{S,J}(0) = \int d^2\xi (Y_+)^{-\Delta}(\mathrm{X}_x)^J(\partial Y_x \bar{\partial} Y_x)^{S/2}. \quad (3.13)$$

The corresponding Euclidean semiclassical solution [23] is given by a generalization of the Euclidean continuation of (3.5)

$$t_e = \kappa\tau_e, \qquad \phi = -i\kappa\tau_e,$$
$$\rho = \mu\sigma, \qquad \varphi = -i\nu\tau_e, \quad (3.14)$$

$$\kappa = \sqrt{\mu^2 + \nu^2}, \quad \mu \approx \frac{1}{\pi}\ln\mathcal{S} \gg 1, \quad \nu = \mathcal{J}. \quad (3.15)$$

Its energy is

$$E - S = \sqrt{J^2 + \frac{\lambda}{\pi^2}\ln^2\mathcal{S}} = \frac{\sqrt{\lambda}}{\pi}\sqrt{\ell^2 + 1}\ln\mathcal{S}, \qquad \ell \equiv \frac{\nu}{\mu}. \quad (3.16)$$

Written in Poincaré coordinates, it is the same as in (3.7), (3.8), (3.9), and (3.10) with $\kappa^2 = \mu^2 + \nu^2$ and $\varphi = -i\nu\tau_e$. Note that in the formal limit of $\mu \to 0$ we recover the geodesic solution (3.1). We will use this observation to test some of the calculations described in the next section.

IV. SEMICLASSICAL COMPUTATION OF THREE-POINT FUNCTIONS OF TWO "HEAVY" AND ONE "LIGHT" STATES

Let us now apply the strategy described in the introduction to the computation of the leading semiclassical contribution to the correlators like $\langle V_{H_1}(\mathrm{x}_1)V_{H_2}(\mathrm{x}_2)V_L(\mathrm{x}_3)\rangle$ where the "heavy" and "light" vertex operators are among the operators discussed in Sec. II. Again, since the quantum numbers of the "light" operators are much smaller than those of the "heavy" ones (assumed to be order $\sqrt{\lambda}$) the "light" source terms in the string equations determining the stationary-point trajectory can be ignored, so that this trajectory should be the same as for the two-point correlator $\langle V_{H_1}(\mathrm{x}_1)V_{H_2}(\mathrm{x}_2)\rangle$. Then to compute the above three-point function we just need to evaluate it on a classical string solution carrying the same quantum numbers as the two heavy operators (assumed to be of the same type up to opposite signs of spins or momenta, i.e. conjugate to each other).

Given that the x_i dependence of the correlators like (1.2) and (1.3) is determined by the conformal invariance, it is sufficient to consider a special choice of the points, fixing, e.g., the position of the "light" operator to be at zero, $\mathrm{x}_3 = (0, 0, 0, 0)$. In this case the contribution of the "light" vertex operator will be given by (see (2.7) and (2.5)

$$V_L(0) = \int d^2\xi (Y_+)^{-\Delta_L} U[x(\xi), z(\xi), X(\xi)]. \quad (4.1)$$

Furthermore, for all simple classical string solutions associated with the "heavy" operators we will consider below will have the following property (cf. (2.3), (3.1), and (3.6):

$$z^2 + x_m x^m = 1, \quad \text{i.e.} \quad Y_4 = 0, \quad Y_5 = Y_+ = z^{-1}. \quad (4.2)$$

Then the leading semiclassical contribution to the three-point function will be given simply by (we assume $\Delta_{H_1} = \Delta_{H_2} \gg \Delta_L \equiv \Delta$)

$$\langle V_{H_1}(\mathrm{x}_1)V_{H_2}(\mathrm{x}_2)V_L(0)\rangle \sim \int d^2\xi z_{\rm cl}^{\Delta} U[x(\xi), z_{\rm cl}(\xi), X_{\rm cl}(\xi)], \tag{4.3}$$

where the subscript cl on the arguments of U emphasizes that they are given by the classical solution saturating the two-point correlator of the "heavy" operators.[14]

Also, the solutions we shall consider below will be such that they approach the boundary points x_1 and x_2 which have $|\mathrm{x}_1| = 1$, $|\mathrm{x}_2| = 1$.[15] In this case to extract the (normalized) structure coefficients $C_{123} \sim \frac{C_{123}}{C_{12}}$ in (1.3) we should consider

$$C_{123} = \frac{\langle V_{H_1}(\mathrm{x}_1)V_{H_2}(\mathrm{x}_2)V_L(0)\rangle}{\langle V_{H_1}(\mathrm{x}_1)V_{H_2}(\mathrm{x}_2)\rangle} = c_\Delta \int d^2\xi z_{\rm cl}^{\Delta} U[x_{\rm cl}(\xi), z_{\rm cl}(\xi), X_{\rm cl}(\xi)], \tag{4.4}$$

where c_Δ depends only on the normalization of the "light" operator. In such a ratio the (divergent) "Möbius" factor (3.3) cancels out as well, guaranteeing a finite result.

In what follows we shall omit the subscript "cl" on the coordinates of the classical solution. Also, since we are interested just in the value of the integral in (4.4) we may compute it directly on the 2d cylinder, i.e. before doing the conformal transformation (3.2), so that $\int d^2\xi \to \int_{-\infty}^{\infty} d\tau_e \int_0^{2\pi} d\sigma$.

Let us now consider some specific examples corresponding to different choices of the "heavy" and "light" operators, i.e. the choices of the classical solution and of V_L (4.1) or U in (4.4).

A. V_H corresponding to folded string with large-spin in AdS_5

Let us start with the case when the two "heavy" operators are $V_{S,J}$ and $V_{-S,-J}$ in (3.4) and (3.13) with $S = \sqrt{\lambda}\mathcal{S}$, $J = \sqrt{\lambda}\mathcal{J}$ and $\ln\mathcal{S} \gg 1$, so that the corresponding semiclassical trajectory is directly related to the large-spin solution (3.6), (3.7), (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), and (3.14).

1. V_L as dilaton operator

If we choose V_L to be the dilaton operator (2.8) then the three-point correlator (4.4) takes the form

$$C_{123} = c_\Delta \int_{-\infty}^{\infty} d\tau_e \int_0^{2\pi} d\sigma z^\Delta U, \tag{4.5}$$

$$U = (\mathrm{X}_x)^j[z^{-2}(\partial x_m \bar\partial x^m + \partial z \bar\partial z) + \partial X_k \bar\partial X_k], \quad \Delta = 4 + j. \tag{4.6}$$

Here we denoted the (fixed) KK momentum of the dilaton by j to distinguish it from the (large) angular momentum J of the "heavy" operators. In this case the momenta of the two "heavy" operators should be, in fact, J and $-J - j$ to satisfy the momentum conservation but as in [13] we shall formally ignore this as $J \gg j$. The normalization constant c_Δ of the dilaton vertex operator was computed in [7]:

$$c_\Delta = c_{j+4} = \frac{2^{-j/2}}{2\pi^2}(j + 3). \tag{4.7}$$

Let us note that in the simplest case of the "heavy" operators represented by scalar BPS operators corresponding to supergravity modes when the classical trajectory is given by (3.1) we find that $U = e^{j\kappa\tau_e} \times (\kappa^2 - \kappa^2) = 0$ so that the three-point function vanishes identically. This agrees with the absence of the three-point couplings containing an odd number of dilatons in the NS-NS sector[16] of the type IIB supergravity in the Einstein frame (a similar statement is true also in weak-coupling expansion of the dual gauge theory).

Evaluating U on the large-spin folded string classical solution in (3.14) we get

$$U = e^{j\nu\tau_e}(\kappa^2\cosh^2\rho + \mu^2 - \kappa^2\sinh^2\rho - \nu^2) = 2\mu^2 e^{j\nu\tau_e}, \tag{4.8}$$

so that the integral in (4.6) becomes

$$C_{123} = 4c_\Delta \int_{-\infty}^{\infty} d\tau_e \int_0^{(\pi/2)} d\sigma \frac{2\mu^2 e^{j\nu\tau_e}}{[\cosh(\mu\sigma)\cosh(\kappa\tau_e)]^\Delta}, \quad \kappa^2 = \mu^2 + \nu^2, \tag{4.9}$$

where we used that the expression for ρ in (3.5) approximates the exact folded solution for $\mu \gg 1$ on the interval $(0, \frac{\pi}{2})$ and should be combined four times to correspond to a 2π periodic solution. While we should eventually take μ large we shall formally keep it finite at intermediate steps.

Doing the integral over σ and τ_e we get[17]

$$C_{123} = c_\Delta 2^{j+8}\frac{\mu}{\kappa} C(j, \mu) B(J, \frac{\nu}{\kappa}), \tag{4.10}$$

$$C(j, \mu) = \sinh\Big(\frac{\pi}{2}\mu\Big) {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}(5 + j), \frac{3}{2}, -\sinh^2\Big(\frac{\pi}{2}\mu\Big)\Big), \tag{4.11}$$

[14] The corresponding world sheet can be pictured as connecting the x_1 and x_2 points with the role of the third "small" operator being to connect it also to the point $\mathrm{x}_1 = 0$ (see [13]).

[15] For example, for (3.6), (3.7), (3.8), (3.9), and (3.10) the boundary points are $x(\tau_e = \infty) = (1, \tanh\mu\sigma, -i\tanh\mu\sigma, 0)$, and $x(\tau_e = -\infty) = (-1, \tanh\mu\sigma, i\tanh\mu\sigma, 0)$ so that $|x(\tau_e = +\infty)|^2 = |x(\tau_e = -\infty)|^2 = 1$.

[16] We ignore fermions and so do not consider the RR scalar operators.

[17] Note that the integral over τ_e is convergent as $(4 + j)\kappa > j\nu$.

$$B\left(j,\frac{\nu}{\kappa}\right)=\frac{{}_2F_1(4+j,b_+,b_++1,-1)}{b_+}+\frac{{}_2F_1(4+j,b_-,b_-+1,-1)}{b_-},\qquad(4.12)$$

$$b_\pm\equiv 4+j\left(1\pm\frac{\nu}{\kappa}\right).\qquad(4.13)$$

Note that in the formal $\mu\to 0$ limit corresponding to the case when the classical trajectory (3.14) degenerates into a geodesic we recover the vanishing of the three-point coupling mentioned above

$$C_{123}|_{\mu\to 0}=\frac{c_\Delta 2^{j+5}\pi}{(j+2)(j+3)}\frac{\mu^2}{\nu}+\mathcal{O}(\mu^3).\qquad(4.14)$$

Considering the large-spin or large $\mu=\frac{1}{\pi}\ln\mathcal{S}$ limit with fixed $\ell=\frac{\nu}{\mu}$ (cf. (3.16)) we may express (4.10) in terms of $\mathcal{S}$ and ℓ in the following factorized form

$$C_{123}=c_\Delta 2^{j+8}C(j,\mathcal{S})\tilde{B}(j,\ell),\qquad \tilde{B}(j,\ell)=\frac{1}{\sqrt{\ell^2+1}}B\left(j,\frac{\ell}{\sqrt{\ell^2+1}}\right),\qquad(4.15)$$

$$C(j,\mathcal{S})=\frac{1}{2}\mathcal{S}^{1/2}(1-\mathcal{S}^{-1})\,{}_2F_1\left(\frac{1}{2},\frac{1}{2}(j+5),\frac{3}{2},-\frac{1}{4}(\mathcal{S}+\mathcal{S}^{-1}-2)\right).\qquad(4.16)$$

Explicitly, for $j=0$ or $\Delta=4$ we get (recalling that $\nu=\mathcal{J}$)

$$C_{123}=\frac{32c_\Delta[2+\cosh(\mu\pi)]\sinh(\mu\pi/2)}{9\cosh^3(\pi\mu/2)\sqrt{1+\ell^2}}=\frac{64c_\Delta(\mathcal{S}-1)(\mathcal{S}^2+4\mathcal{S}+1)\ln\mathcal{S}}{9\pi(\mathcal{S}+1)^3\sqrt{\mathcal{J}^2+\frac{1}{\pi^2}\ln^2\mathcal{S}}}.\qquad(4.17)$$

In the large $\mathcal{S}$ limit this becomes

$$C_{123}\sim\frac{\ln S}{\sqrt{J^2+\frac{\lambda}{\pi^2}\ln^2 S}}.\qquad(4.18)$$

Thus if $J\gg\frac{\sqrt{\lambda}}{\pi}\ln S$ the three-point coupling again vanishes, in agreement with the above argument that the dilaton does not couple to BPS states. If $J\ll\frac{\sqrt{\lambda}}{\pi}\ln S$ the three-point coupling approaches a constant, which is consistent with the expectation that the dilaton should generically couple to massive string modes, e.g., via their mass term in a string field theory action. For example, adding a massive scalar to a string effective action one gets an exponential dilaton coupling in the mass term in the Einstein frame, $S=\int d^{10}x\sqrt{g}(\partial^\mu\Psi\partial_\mu\Psi+M^2e^{\gamma\Phi}\Psi^2+\ldots)$. Starting with such action the three-point function may be computed using standard methods [1,25], e.g. as in the case when all three modes are supergravity modes.

The expression in (4.18) resembles the λ-derivative of the strong-coupling limit of the dimension of the large-spin twist J operator (equal to the energy of the string solution in (3.16)):

$$\lambda\frac{\partial\Delta_{S,J}}{\partial\lambda}=\frac{\lambda\ln^2 S}{2\pi^2\sqrt{J^2+\frac{\lambda}{\pi^2}\ln^2 S}}+\ldots,\qquad \Delta_{S,J}=S+\sqrt{J^2+\frac{\lambda}{\pi^2}\ln^2 S}+\ldots.\qquad(4.19)$$

Indeed, this relation should be expected in view of the discussion in Sec. II A (see (2.9) and (2.10)). Comparing to (4.18), there is, however, a mismatch in one factor of $\ln S$. This appears to be depending on how one regularizes divergent integrals appearing in the case of insertion of the dilaton operator integrated over four-space (2.9). Indeed, to repeat the above computation with the dilaton vertex operator replaced by its zero-momentum version, i.e. by the string action, we should omit the z^Δ factor in (4.5). Then for $j=0$ we get from (4.9) $C_{123}=8\mu^2c_4\int_{-\infty}^{\infty}d\tau_e\int_0^{(\pi/2)}d\sigma$. Since from the form of the classical solution (3.6), (3.7), and (3.8) it is clear that the space-time coordinates depend on τ_e through $\bar{\tau}_e=\kappa\tau_e$ it is natural to introduce a cutoff L on $\bar{\tau}_e$. That gives $C_{123}\sim\frac{\mu^2}{\kappa}L$, which is indeed the same function of S and J as $\lambda\frac{\partial\Delta_{S,J}}{\partial\lambda}$ in (4.19).[18]

The above relation between the value of the classical action on a solution (with time integral cut off using $t=\kappa\tau$ variable) and the derivative of the corresponding classical energy over string tension for fixed spins appears to be quite general (and can be argued for using thermodynamical arguments as in [26]). We shall see it applying also in the example discussed in Sec. IV B 1.

Let us comment also on the formal limit of large j which is easy to analyze by evaluating the integral in (4.9) over τ_e in a saddle-point approximation (see [13] for a similar discussion).[19] Rescaling τ_e by κ first we end up with (cf. (4.15) and (4.16))

$$C_{123}=\frac{8c_\Delta}{\pi(1+\ell^2)^{5/2}}C(j,\mathcal{S})e^{jh(\ell)},\qquad(4.20)$$

[18] Note that if we consider unintegrated vertex operators before dividing over Möbius group factor then in the analog of (1.3) we will get like in (3.3) an extra factor of world sheet distance powers: $\langle V(x_1,\xi_1)V^*(x_2,\xi_2)V_3(x_3,\xi_3)\rangle\sim\frac{1}{|\xi_1-\xi_2|^2|\xi_1-\xi_3|^2|\xi_2-\xi_3|^2}$ where $V_3=V^{(0-\text{dil})}$ here is the string Lagrangian. Integrating this over ξ_3 to get insertion of $V^{(0-\text{dil})}$ or the string action produces $\frac{1}{|\xi_1-\xi_2|^4}\ln(a|\xi_1-\xi_2|)$ where a is a world sheet cutoff. The integral of this factor then cancels against the normalization to the two-point function.

[19] Unlike the τ_e integral, the σ integral in (4.9) does not possess a real saddle point—the integrand is an increasing function, so that we evaluate this integral exactly.

$$h(\ell) = -\frac{1}{2}\left[\ln(1+\ell^2) + \frac{\ell}{\sqrt{1+\ell^2}}\ln\frac{\sqrt{1+\ell^2}-\ell}{\sqrt{1+\ell^2}+\ell}\right], \tag{4.21}$$

where

$$h(\ell) = \begin{cases} \frac{1}{2}\ell^2 - \frac{5}{12}\ell^4 + \ldots, & \ell \to 0 \\ \ln 2 - \frac{1}{2}\left(\frac{1}{2} + \ln 2 + \ln\ell\right)\frac{1}{\ell^2} + \ldots, & \ell \to \infty \end{cases}. \tag{4.22}$$

Taking $\mathcal{S} \to \infty$ in $C(j, \mathcal{S})$ given in (4.16) and then expanding it at large j we find[20]

$$C(j, \mathcal{S})|_{\mathcal{S}\to\infty} = \frac{2^{j+2}[\Gamma((j+4)/2)]^2}{\Gamma(j+4)} + \mathcal{O}(\mathcal{S}^{-1}) = \sqrt{\frac{\pi}{2j}} + \mathcal{O}(j^{-3/2}, \mathcal{S}^{-1}). \tag{4.23}$$

Since this j dependence is not exponential, it is subject to corrections coming from fluctuations around the saddle point of the τ_e integral.

2. V_L as superconformal primary scalar operator

Let us now turn to the case when the "light" operator is another massless scalar vertex operator in (2.11). In the case when the classical solution is a BMN geodesic or a folded spinning string in S^5 representing a massive string mode with spins (J_1, J_2) similar computation was done recently in [13]. Here we will consider the case of the large-spin folded string solution in AdS_5.

In the case of (2.11) the factor U in (4.4) evaluated on the large-spin solution (3.7), (3.8), and (3.14) takes the form (we again use j for the S^5 momentum of the "light" operator so that here $\Delta = j$; cf. (4.8))

$$U = e^{j\nu\tau_e}\left[\kappa^2\left(\frac{2}{\cosh^2(\kappa\tau_e)} - 1\right) + \mu^2\left(\frac{2}{\cosh^2(\mu\sigma)} - 1\right) + \nu^2\right]. \tag{4.24}$$

Then the integral in (4.6) becomes

$$C_{123} = 4c_\Delta \int_{-\infty}^{\infty} d\tau_e \int_0^{(\pi/2)} d\sigma \frac{2e^{j\nu\tau_e}}{[\cosh(\mu\sigma)\cosh(\kappa\tau_e)]^\Delta} \times \left[\frac{\kappa^2}{\cosh^2(\kappa\tau_e)} - \mu^2\tanh^2(\mu\sigma)\right]. \tag{4.25}$$

In each term the τ_e and σ integrals factorize. Even in the large μ limit the result is a relatively complicated function of ℓ and j which can be analyzed in various limits.

In the large $\ell = \frac{\nu}{\mu} = \frac{\pi\mathcal{J}}{\ln\mathcal{S}}$ limit we find

$$C_{123} = c_\Delta 2^{j+2}\sqrt{\pi}\frac{j-1}{j}\frac{\Gamma(j/2)}{\Gamma((j+3)/2)}\ell + \mathcal{O}(\ell^{-1}), \tag{4.26}$$

i.e. the three-point function scales proportionally to J.

The leading term in the small ℓ expansion for general μ is

$$C_{123} = 8c_\Delta\sqrt{\pi}\frac{\Gamma((j+2)/2)}{j\Gamma((j+3)/2)}\left[\frac{1}{\cosh^{(j+1)/2}(\mu\pi/2)} + (j-1)\,{}_2F_1\left(\frac{1}{2}, \frac{j+1}{2}, \frac{3}{2}, -\sinh^2(\mu\pi/2)\right)\right] \times \sinh(\mu\pi/2) + \mathcal{O}(\ell). \tag{4.27}$$

Taking $\mu = \frac{1}{\pi}\ln\mathcal{S}$ large, the leading term here is

$$C_{123} = 8c_\Delta\sqrt{\pi}\frac{\Gamma((j+2)/2)}{j\Gamma((j+3)/2)} \times \left[\sqrt{\pi}\frac{(j-1)\Gamma(j/2)}{2\Gamma((j+1)/2)} + \frac{2^j}{j\mathcal{S}^{j/2}} + \ldots\right], \tag{4.28}$$

i.e. the three-coupling approaches a constant.

Note that if we formally take the small μ limit for fixed ℓ and j we get

$$C_{123} = \mu\frac{2^{j+4}\pi(1+\ell^2)c_\Delta}{4\ell^2(1+j)+(2+j)^2} \times \left[(\sqrt{1+\ell^2}(2+j)+\ell j)\,{}_2F_1\left(2+j, \frac{1}{2}\left(2+j-\frac{\ell j}{\sqrt{1+\ell^2}}\right), \frac{1}{2}\left(4+j-\frac{\ell j}{\sqrt{1+\ell^2}}\right), -1\right) + (\sqrt{1+\ell^2}(2+j)-\ell j)\,{}_2F_1\left(2+j, \frac{1}{2}\left(2+j+\frac{\ell j}{\sqrt{1+\ell^2}}\right), \frac{1}{2}\left(4+j+\frac{\ell j}{\sqrt{1+\ell^2}}\right), -1\right)\right]. \tag{4.29}$$

Taking then $\ell \to 0$ we find a nonvanishing result:

$$C_{123} = \mu\left[\frac{8c_\Delta\pi^{3/2}\Gamma((j+4)/2)}{(j+2)\Gamma((j+3)/2)} + \mathcal{O}(\ell)\right]. \tag{4.30}$$

This term arises entirely from the contributions that would vanish if the limit $\mu \to 0$ were taken directly in the integrand of Eq. (4.25). The limit $\ell \to \infty$ of (4.29) leads to

$$C_{123} = \frac{2^{j+3}\pi c_\Delta}{j+1}\mu\ell[1 + \mathcal{O}(\ell^{-2})]. \tag{4.31}$$

Since the normalization constant c_Δ of the BPS operator is [13]

[20]Numerical analysis suggests that the result below holds also at finite $\mathcal{S}$.

$$c_\Delta = c_j = \frac{(j+1)\sqrt{j}}{2^{j+3}\pi N}\sqrt{\lambda}\,, \qquad (4.32)$$

it follows that in this limit

$$C_{123} \to \frac{1}{N} J\sqrt{j}\,. \qquad (4.33)$$

We thus formally recover, as in a similar computation in [13], the result [20] for the three-point function of the three BMN-type operators (here with $j_1 = j_2 = J$, $j_3 = j$).

3. V_L as fixed-spin operator on leading Regge trajectory

To explore the structure of the three-point functions with the "light" state being a massive string state let us now consider the insertion of an operator on the leading Regge trajectory, i.e. V_s in (2.12). We change the notation ($S \to s$, $\Delta \to \Delta_s$) assuming now a fixed value of spin s and dimension $\Delta_s = \sqrt{2(s-2)}\sqrt[4]{\lambda} + \ldots$, which are much smaller than the semiclassical parameters (S, $\Delta_S \sim \sqrt{\lambda}$) of the two "heavy" operators which are taken again to correspond to the large-spin folded string solution (3.6), (3.7), (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), and (3.14). We shall ignore the "mixing" terms indicated by dots in (2.12) so that the result for C_{123} below will be qualitative.

In this case the value of U in (4.4) is (cf. (4.8) and (4.24))

$$U = (\partial Y_x \bar{\partial} Y_x)^{s/2} = e^{2s\kappa\tau_e}\Big[\mu^2\cosh^2(\mu\sigma) + \kappa^2\sinh^2(\mu\sigma)\Big]^{s/2}. \qquad (4.34)$$

For $\nu = 0$ (i.e. $\kappa = \mu$) this becomes

$$U = \mu^s e^{2s\mu\tau_e}[\cosh(2\mu\sigma)]^{s/2}. \qquad (4.35)$$

Doing the integral in (4.4) we conclude that for large μ the three-point coefficient scales as

$$C_{123} \sim \mu^{s-2} \sim (\ln\mathcal{S})^{s-2}. \qquad (4.36)$$

It is interesting to note that the 2d operator mixing discussed in Sec. II C does not alter this behavior. Indeed, it is not hard to see that each derivative in the vertex operator brings in a factor of μ while the integration measure cancels two such factors. Since all operators in Eq. (2.15) have s derivatives, each of them yields an overall μ^{s-2} factor.

Let us now estimate the large s behavior of this correlator. In the large $\mu = \frac{1}{\pi}\ln\mathcal{S}$ limit, this can be easily done by evaluating the σ and τ_e integrals in the saddle-point approximation[21]

$$C_{123} \approx \mu^{s-2} c_{\Delta_s}\int_{-\infty}^{+\infty}\frac{d\tau_e e^{2s\tau_e}}{\cosh^{\Delta_s}\tau_e}\int_0^\infty\frac{d\sigma\cosh^{s/2}(2\sigma)}{\cosh^{\Delta_s}\sigma} = \frac{c_{\Delta_s}}{\pi^{s-2}}e^{H(\mathcal{S},s)}, \qquad (4.37)$$

[21] Here we first rescale the 2d coordinates by μ and then take $\mu \to \infty$; we choose the real saddle point for the σ integral.

$$H = (s-2)\ln\ln\mathcal{S} + h_{\tau_e}(s) + h_\sigma(s), \qquad (4.38)$$

$$h_{\tau_e} = \Big(\frac{1}{2}\Delta_s - s\Big)\ln\Big(1 - \frac{2s}{\Delta_s}\Big) + \Big(\frac{1}{2}\Delta_s + s\Big)\ln\Big(1 + \frac{2s}{\Delta_s}\Big), \qquad (4.39)$$

$$h_\sigma = \frac{1}{2}\Delta_s\ln 2 + \frac{1}{2}\Delta_s\ln\Big(1 - \frac{s}{\Delta_s}\Big) - s\ln\Big(\frac{\Delta_s}{s} - 1\Big). \qquad (4.40)$$

If we further formally assume that s is as large as $\sqrt{\lambda}$, then $\Delta_s = \sqrt{2(s-2)}\sqrt[4]{\lambda} + \ldots$ will also scale as $\sqrt{\lambda}$, so that the function H in the exponent will be proportional to the string tension, as should be expected in a semiclassical limit.

4. V_L as singlet massive scalar operator

Let us now show that a similar result to (4.36) is found if we choose as the "light" operator the singlet scalar operator (2.17) representing a string state at level $r-1$. The advantage over the previous case is that here the leading bosonic part of the operator (with dimension $\Delta_r = 2\sqrt{(r-1)\sqrt{\lambda}} + \ldots$) is known explicitly. We find that the corresponding factor U in (4.4) evaluated on the large-spin solution (3.14) here is (cf. (4.34))

$$U = (\partial X_k \partial X_k \bar{\partial} X_\ell \bar{\partial} X_\ell)^{r/2} = \nu^{2r}. \qquad (4.41)$$

The simplicity of this result is a consequence of the special structure of the singlet operator (2.17) already mentioned in Sec. II D: it is built out of chiral components of the S^5 sigma model stress tensor which enters the Virasoro conditions. This means that the same constant expression (4.41) will be found for any classical solution describing a string moving nontrivially in AdS_5 with its center of mass orbiting big circle in S^5. If instead of (2.17) we consider the AdS_5 counterpart of this operator given in (2.18) we get the same result as in (4.41)

$$U = (\partial Y_M \partial Y^M \bar{\partial} Y_K \bar{\partial} Y^K)^{r/2} = \nu^{2r}, \qquad (4.42)$$

since the Virasoro condition relates the AdS_5 and S^5 components of the string stress tensor.

Doing the integral in (4.4) we find (cf. (4.15))

$$C_{123} = c_{\Delta_r} C(r,\mathcal{S})\hat{B}(r,\ell), \qquad (4.43)$$

$$C(r,\mathcal{S}) = (\ln\mathcal{S})^{2r-2}(\mathcal{S}^{1/2} - \mathcal{S}^{-1/2}) \times {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}(\Delta_r + 1), \frac{3}{2}, -\frac{1}{4}(\mathcal{S} - 2 + \mathcal{S}^{-1})\Big), \qquad (4.44)$$

$$\hat{B}(r,\ell) = \frac{2^{\Delta_r - 2}[\Gamma(\Delta_r/2)]^2}{\pi^{2r-2}\Gamma(\Delta_r)}\frac{\ell^{2r}}{\sqrt{1+\ell^2}}. \qquad (4.45)$$

In the large-spin limit with fixed ℓ we get

$$C_{123} \sim (\ln\mathcal{S})^{2r-2}\frac{\ell^{2r}}{\sqrt{1+\ell^2}} \sim \frac{J^{2r}}{\ln S\sqrt{J^2+\frac{\lambda}{\pi^2}\ln^2 S}}, \quad (4.46)$$

where we ignored an overall Δ_r dependent factor $\left(\frac{\sqrt{\pi}2^{\Delta_r-2}\Gamma^3(\Delta_r/2)}{\pi^{2r-2}\Gamma((\Delta_r+1)/2)\Gamma(\Delta_r)}\right)$ that may be cancelled against the normalization c_{Δ_r} of the "light" vertex operator.

For fixed ℓ the resulting behavior of the three-point function with large-spin S is thus the same as in (4.36): $\ln S$ in power of the value of the string level. For example, for the singlet operator from the first excited level $r=2$ (which should be dual to a member of Konishi multiplet [17]) we get $C_{123} \sim \ln^2 S\frac{\ell^4}{\sqrt{1+\ell^2}}$. At the same time, this three-point correlator vanishes in the $\ell \to 0$ limit (i.e. if $\frac{\sqrt{\lambda}}{\pi}\ln S \gg J$), which was not the case for the fixed-spin s operator in (4.35) and (4.36) (this vanishing follows directly from the special structure of the singlet operator in (4.41)).

B. V_H corresponding to "small" circular string solution in S^5 with $J_1 = J_2 \neq J_3$

Let us now consider the case of a "heavy" state for which the semiclassical approximation to the two-point function is dominated by the rigid circular string solution in S^5 with three spins $J_1 = J_2$ and J_3 [27]. We shall consider the "small-string" branch of this solution which admits the small-spin limit (see also [17])

$$t=\kappa\tau, \quad X_1+iX_2 = ae^{iw\tau+i\sigma}, \quad X_3+iX_4 = ae^{iw\tau-i\sigma},$$
$$X_5+iX_6=\sqrt{1-2a^2}e^{i\nu\tau} \quad w=\sqrt{1+\nu^2},$$
$$\kappa=\sqrt{4a^2+\nu^2}, \quad J_1=J_2=J=\sqrt{\lambda}a^2w,$$
$$J_3=\sqrt{\lambda}(1-2a^2)\nu. \quad (4.47)$$

Transforming $t=\kappa\tau$ into Poincaré coordinates and rotating to Euclidean signature as in (3.1)

$$z=\frac{1}{\cosh(\kappa\tau_e)}, \quad x_{0e}=\tanh(\kappa\tau_e), \quad \tau_e=i\tau, \quad (4.48)$$

we get a (complex) background for the X_k coordinates in terms of τ_e and σ which should then be substituted into the integrand in (4.4).[22]

[22]Let us mention that a similar semiclassical computation of the three-point string amplitude involving two states corresponding to $J_1 = J_2 \gg 1$ circular spinning string and a graviton as a light operator was first considered in flat-space in [28]. There it was checked that the result of the semiclassical calculation agrees with the large-spin limit of the exact correlation three-point correlation function.

1. V_L as dilaton operator

In this case the integral in (4.6) is found to be (here $\Delta = 4+j$ and the integral over σ is trivial as z in (4.48) depends only on τ_e)

$$C_{123} = 2\pi c_\Delta\int_{-\infty}^{\infty} d\tau_e\frac{(1-2a^2)^{j/2}e^{j\nu\tau_e}}{[\cosh(\kappa\tau_e)]^\Delta}\times 4a^2. \quad (4.49)$$

This expression vanishes as it should in the $a\to 0$ limit when the "heavy" state becomes a BMN state. The integral over τ_e is convergent since $(4+j)\kappa > j\nu$. Explicitly, we get

$$C_{123} = c_\Delta 8\pi a^2(1-2a^2)^{j/2}\int_{-\infty}^{\infty} d\tau_e\frac{e^{j\nu\tau_e}}{[\cosh(\kappa\tau_e)]^\Delta}. \quad (4.50)$$

For $\nu=0$, i.e. for the "small" 2-spin classical trajectory for which $J=\sqrt{\lambda}\mathcal{J}$, $\mathcal{J}=a^2$, $\Delta_J=\sqrt{\lambda}\kappa = 2\sqrt{J}\sqrt[4]{\lambda}$, we get, using that $\Delta = j+4$,

$$(C_{123})_{\nu=0} = c_\Delta 8\pi^{3/2}\frac{\Gamma((j+6)/2)}{(j+4)\Gamma((j+5)/2)}a(1-2a^2)^{j/2}$$
$$\sim \sqrt{J}\left(1-2\frac{J}{\sqrt{\lambda}}\right)^{j/2}. \quad (4.51)$$

For $\nu\neq 0$ the result is:

$$C_{123} = 2^{j+7}\pi c_\Delta\frac{a^2(1-2a^2)^{j/2}}{\sqrt{\nu^2+4a^2}}$$
$$\times\left[\frac{{}_2F_1(4+j,\frac{b_-}{2},1+\frac{b_-}{2},-1)}{b_-} + \frac{{}_2F_1(4+j,\frac{b_+}{2},1+\frac{b_+}{2},-1)}{b_+}\right] \quad (4.52)$$
$$b_\pm = 4+j\pm\frac{j\nu}{\sqrt{\nu^2+4a^2}}.$$

Setting here $j=0$ we get

$$(C_{123})_{j=0} = \frac{32}{3}\pi c_\Delta\frac{a^2}{\sqrt{4a^2+\nu^2}}. \quad (4.53)$$

For $a=\frac{1}{\sqrt{2}}$ when the solution (4.47) reduces to the "large" circular solution with $J_1=J_2$, $J_3=0$ and $\Delta_J=\sqrt{4J^2+\lambda}$ [27] we find[23]

$$C_{123} = \frac{16}{3}\pi c_\Delta\frac{1}{\sqrt{1+w^2}} = \frac{16}{3}\pi c_\Delta\frac{\sqrt{\lambda}}{\sqrt{4J^2+\lambda}}. \quad (4.54)$$

We observe that like in (4.18) and (4.19) (and in agreement with the general discussion in Sec. II A) this expression is proportional to the λ-derivative of the dimension Δ_J of the "heavy" state, $\lambda\frac{\partial}{\partial\lambda}\Delta_J = \frac{\lambda}{2\sqrt{4J^2+\lambda}}$. The same result was found in [14] using somewhat different approach.

[23]For $j\neq 0$ the limit $a\to\frac{1}{\sqrt{2}}$ in (4.52) yields vanishing result.

2. V_L as singlet massive scalar operator

In the case of the operator in (2.17) the value of U in (4.41) is κ^{2r} and thus the analog of the integral in (4.49) is

$$C_{123} = 2\pi c_\Delta \kappa^{2r} \int_{-\infty}^{\infty} d\tau_e \frac{1}{[\cosh(\kappa\tau_e)]^{\Delta_r}} \sim \kappa^{2r-1}. \quad (4.55)$$

Then for the "small" string solution with $\mathcal{J}_1 = \mathcal{J}_2 \equiv \mathcal{J}$ and $\mathcal{J}_3 \to 0$ for which $\kappa = \sqrt{2\mathcal{J}}$ we find that

$$C_{123} \sim (\sqrt{J})^{2r-1} \sim (\Delta_J)^{2r-1}. \quad (4.56)$$

We conclude again that the three-point function scales as a power of the level number of the "light" string state.

As discussed in [17], the small-string or small $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ limit of the solution (4.47) may be used to approximate a string state with fixed quantum number J. Then, e.g., for $r = 2$ representing a state on the first excited string level we get $C_{123} \sim (\frac{J}{\sqrt{\lambda}})^{3/2} \sim \lambda^{-3/4}$, i.e. the three-point function of such three massive string states is constant for fixed quantum numbers.

V. CONCLUDING REMARKS

The general correlation functions of local gauge-invariant operators in planar $\mathcal{N} = 4$ SYM theory expanded at strong coupling are given by perturbative string theory correlators of vertex operators of the dual string states. Standard arguments suggest that, for states with large quantum numbers, a semiclassical approach should give reliable results. A semiclassical limit of a correlation function should be determined by a stationary-point of the classical world sheet action with sources corresponding to the relevant vertex operators. Introducing additional vertex operators for states with small quantum numbers may then be treated as a perturbation of a lower-point correlation function. To leading-order, the evaluation of an $(n + m)$-point correlation functions with n "heavy" states and m "light" states amounts to evaluating the product of m "light" vertex operator factors on the classical world sheet surface saturating the n-point correlation function of the "heavy" operators.

Using this strategy we analyzed several examples of three-point functions in which dimension of the two operators is much larger than that of the third. We considered the case when the "heavy" vertex operators correspond to the large folded spinning string in AdS_5 and also the case when they correspond to the "small" three-spin circular string on S^5. We have found that if the "light" vertex operator represents a BPS state, the three-point function approaches a constant as the charges of the "heavy" states are scaled to infinity. We have also discussed certain excited string states as "light" states; in particular, we considered states on the leading Regge trajectory as well as special singlet states. In all such cases we found that the three-point function depends on the semiclassical parameter raised to a power related to the string level of the "light" state.[24]

Let us now discuss possible sources of quantum string ($\frac{1}{\sqrt{\lambda}}$) corrections to the three-point function coefficients C_{123} in (1.3) or C_{123} in (4.4). One source of corrections to $\langle V_H(x_1)V_H(x_2)V_L(x_3)\rangle$ are corrections to the vertex operators which are of two types: (1) corrections to the dimensions of the operators, and (2) corrections to the form of the vertex operators due to mixing at higher orders. For the "heavy" operators the former corrections alter only the world sheet configuration saturating the two-point function. They can be accounted for by simply replacing the semiclassical parameters of the classical solution in the expressions derived at the leading-order by their quantum-corrected counterparts.

As for the higher-loop mixing terms of 2d operators, they are suppressed by a factor of $\frac{1}{\sqrt{\lambda}}$ without additional dependence on the semiclassical parameters. Thus, for the purpose of finding the leading terms in the string semiclassical expansion, such additional mixings may be ignored. Note also that in the expressions in the previous section the dependence on the charges of the classical solution is decoupled from the dependence on the charges of the "light" vertex operator, so that such corrections will drop out in ratios of correlation functions that are independent of the normalizations of vertex operators.

These quantum corrections may be given a simple 2d Feynman diagram interpretation. Nontrivial contributions come from contractions involving fields from different types of operators. The Wick contractions between two "heavy" operators scale quadratically with some large charge while the Wick contractions between one "heavy" and one "light" operator scale linearly with a large charge. In a semiclassical approach the contributions of the first type are already included in the renormalization of the classical solution describing the two-point functions of the "heavy" operators. Thus the relevant one-loop corrections to the three-point function coefficients C_{123} arise from Wick contractions between one "heavy" and one "light" vertex operator and they scale linearly with the charges of the "heavy" vertex operators.

Let us now comment on the perturbative calculations of such correlation functions in dual gauge theory. Weak coupling calculations of some simple correlation functions provided some early tests of the AdS/CFT correspondence. While early calculations focused on three-point correlators of BPS operators which may be extrapolated to strong coupling, recent one-loop calculations involving non-BPS operators [29] (see also [30,31]) suggest an interesting relation between the three-point coefficients and anomalous dimensions. Indeed, the one-loop correction to the correlation function of two BPS and one Konishi operator

[24]This behavior is consistent also with that of the correlators involving BPS states which belong to the string ground state.

[32] is proportional to the anomalous dimension of the Konishi operator. If this pattern persists at higher orders, this three-point function may provide an independent determination of the anomalous dimension of the Konishi operator at strong coupling.

Note that an algebraic Bethe ansatz approach to the diagonalization of the spin chain Hamiltonian provides sufficient information to evaluate perturbatively the three-point function coefficient, without directly resorting to Feynman diagram approach [33] (for a related approach using open spin chains see [30]). For operators dual to ''fast'' strings, described by Landau-Lifshitz type models, it is possible to do better. In this case a useful representation for the eigenvectors of the dilatation operator may be given in terms of coherent states which are, in turn, determined by solutions of the equations of motion of the LL model.[25] As the Landau-Lifshitz model arises as the fast-string limit of the string sigma model [34], such an approach may provide a relation to the semiclassical methods used in this paper. For example, a successful extrapolation to strong coupling of non-BPS correlation functions may expose nonrenormalization theorems akin to those governing the behavior of certain leading terms in the anomalous dimensions of ''long'' operators dual to ''fast'' strings. It may also suggest how to use integrability methods to tackle the problem of three-point function with all three operators being ''heavy''.

As outlined in the Introduction, a similar semiclassical approach may be attempted also for the calculation of higher-point correlation functions. Unlike three-point functions, higher-point functions should have a nontrivial 4d position dependence. Some of their general features like dependence on large quantum numbers may still be possible to analyze. A semiclassical contribution computed according to the recipe used here to the correlation function of two ''heavy'' and two ''light'' vertex operators appears to be given simply by the value of the product of ''light'' vertex operators on the world surface saturating the two-point function of the ''heavy'' operators. It remains to be seen if it does capture a dominant (in large charge, large λ) part of such four-point functions.[26] Study of such four-point functions combined with their expected factorization properties may also provide information about other three-point functions.

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[25] One-loop corrections to the correlation functions of $SU(2)$-sector operators dual to spinning strings, captured by the expectation value of the nonplanar dilatation operator, have been discussed in [16]. While extremal correlation functions where the classical dimension of one operator equals the sum of the classical dimensions of the other two are particularly easy to evaluate (all correlation functions of operators in the $SU(2)$ sector are of this type), this approach is not restricted to such correlators.

[26] It is interesting to note that if the two ''light'' vertex operators are integrated dilaton vertex operators, the four-point function computed along these lines reproduces the behavior suggested (cf. Sec. II A) by the gauge theory analysis, $K_{1234} \propto (\partial_\lambda \Delta)^2$, where $\Delta \gg 1$ is the dimension of the heavy state (we used that $(\partial_\lambda \Delta)^2 \gg \partial_\lambda^2 \Delta$).

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