Transverse structure of the QCD string

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The characterization of the transverse structure of the QCD string is discussed. We formulate a conjecture as to how the stress-energy tensor of the underlying gauge theory couples to the string degrees of freedom. A consequence of the conjecture is that the energy density and the longitudinal-stress operators measure the distribution of the transverse position of the string, to leading order in the string fluctuations, whereas the transverse-stress operator does not. We interpret recent numerical measurements of the transverse size of the confining string and show that the difference of the energy and longitudinal-stress operators is a particularly natural probe at next-to-leading order. Second, we derive the constraints imposed by open-closed string duality on the transverse structure of the string. We show that a total of three independent ''gravitational'' form factors characterize the transverse profile of the closed string, and obtain the interpretation of recent effective string theory calculations: the square radius of a closed string of length β defined from the slope of its gravitational form factor, is given by $\frac{d-1}{2\pi\sigma}\log\frac{\beta}{4r_0}$ in d space dimensions. This is to be compared with the well-known result that the width of the open string at midpoint grows as $\frac{d-1}{2\pi\sigma}\log\frac{r}{r_0}$. We also obtain predictions for transition form factors among closed-string states.

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I. INTRODUCTION

The area law of Wilson loops in lattice gauge theories [1] has long been interpreted in terms of a string formation by the flux lines. In $SU(3)$ gauge theory the area law, $\langle W\rangle \sim e^{-\sigma A}$, signals the linear confinement of heavy quarks Q and $\bar{Q}$: the static potential takes the form $V(r) \sim \sigma r$, where σ is identified with the string tension. Once the quark interdistance r is significantly larger than the confinement scale $\sqrt{\sigma}$, it was realized a long time ago that the corrections to the static potential, as well as the low-energy excitations of the $Q\bar{Q}$ system, could be described by an effective two-dimensional theory [2]. This ''world sheet'' theory of the $d-1$ [3] massless degrees of freedom $\boldsymbol{h}$, namely, the transverse fluctuations of the string, led to two important predictions: first, the linear potential receives $1/r$ corrections, the Lüscher term [4], and its excitations are spaced by $\frac{\pi}{r}$ gaps. Second, the amplitude of the transverse string fluctuations grows logarithmically with the length of the string [5],

$$w_{lo}^2 \equiv \langle \boldsymbol{h}^2\rangle = \frac{d-1}{2\pi\sigma}\log\frac{r}{r_0}. \tag{1}$$

It is this second aspect of the low-energy string dynamics that is the focus of this paper.

Recently, highly accurate numerical results have been obtained in the $d=2$ $SU(2)$ gauge theory for the expectation value of local operators in the presence of a static $Q\bar{Q}$ pair [6]. Measured as a function of the distance $|\boldsymbol{y}|$ from the $Q\bar{Q}$ axis, it defines a distribution whose second moment was successfully compared to the effective-theory prediction (1). In view of these results and of the prospect of pushing the comparison to next-to-leading order (NLO), the first issue we wish to address is the precise connection between the profile probed by a local gauge-theory operator and the world sheet expectation value of $\boldsymbol{h}^2$.

II. COUPLING OF THE STRESS-ENERGY TENSOR TO THE CONFINING STRING

With the classical picture of a fluctuating ''thin'' string in mind, the stress and energy stored in the flux lines is entirely carried by the string. If only the transverse component of a string element's motion contributes to the string energy, then the Hamiltonian is $H = \int \frac{dm}{\sqrt{1-v_\perp^2}}$, where dm is the rest mass of an element of the string. For the Nambu-Goto string, $dm = \sigma ds$, where ds is the length of the string element, but more sophisticated possibilities, such as a curvature term, should be kept in mind [7,8]. Retaining only the simplest rest mass contribution, the expression for the string energy density in Minkowski space reads, in the static gauge,

$$T_{00}(t, y_1, \boldsymbol{y}) = \sigma \frac{1 + (\partial_1\boldsymbol{h})^2}{\sqrt{1 + (\partial_1\boldsymbol{h})^2 - (\partial_t\boldsymbol{h})^2 - (\partial_t\boldsymbol{h})^2(\partial_1\boldsymbol{h})^2 + (\partial_t\boldsymbol{h}\cdot\partial_1\boldsymbol{h})^2}} \times \delta^{d-1}(\boldsymbol{y} - \boldsymbol{h}(t, y_1)). \tag{2}$$

Here $\boldsymbol{h}$ is the world sheet field; it has $d-1$ components. The world sheet is parametrized by t and y_1, the coordinate that runs along the $Q\bar{Q}$ axis, and $\boldsymbol{y}$ contains the $d-1$ coordinates transverse to the string. The world sheet indices are denoted generically by $a, b, \ldots$ We thus expect the energy-density operator of the underlying gauge theory to couple to the world sheet operator appearing in this expression, understood as an expansion in $\partial_a \boldsymbol{h}$. At zeroth order, we have simply $T_{00}(t, y_1, \boldsymbol{y}) = \sigma\delta^{d-1}(\boldsymbol{y} - \boldsymbol{h}(t, y_1))$, which means that the distribution measured by the energy density operator coincides with the distribution in $\boldsymbol{h}$. In particular, the second moments in $\boldsymbol{y}$ of the transverse distribution obtained from T_{00} is expected to match the expression for $\langle \boldsymbol{h}^2\rangle$ calculated in the world sheet theory. Expanding Eq. (2), we obtain

$$T_{00} = \sigma\delta(\boldsymbol{y} - \boldsymbol{h})[1 + \tfrac{1}{2}((\partial_t\boldsymbol{h})^2 + (\partial_1\boldsymbol{h})^2) - \tfrac{1}{8}[(\partial_1\boldsymbol{h})^2]^2 + \tfrac{3}{8}[(\partial_t\boldsymbol{h})^2]^2 + \tfrac{1}{4}(\partial_1\boldsymbol{h})^2(\partial_t\boldsymbol{h})^2 - \tfrac{1}{2}(\partial_t\boldsymbol{h}\cdot\partial_1\boldsymbol{h})^2]. \quad (3)$$

This expression suggests that at leading order in the fluctuations, $\int d^{d-1}\boldsymbol{y}\boldsymbol{y}^2 T_{00}(t, y_1, \boldsymbol{y})$ measures the world sheet expectation value of the operator

$$\boldsymbol{h}^2(1 + \tfrac{1}{2}(\partial_t\boldsymbol{h})^2 + \tfrac{1}{2}(\partial_1\boldsymbol{h})^2). \quad (4)$$

Thus, when comparing Monte Carlo data for $\int d^{d-1}\boldsymbol{y}\boldsymbol{y}^2 T_{00}(t, y_1, \boldsymbol{y})$ with the effective theory, the world sheet expectation value of $\boldsymbol{h}^2$ needs to computed to NLO, a tour de force achieved very recently [9], but also the leading-order expectation value of the tensor operator $\boldsymbol{h}^2\frac{1}{2}((\partial_t\boldsymbol{h})^2 + (\partial_1\boldsymbol{h})^2)$ needs to be calculated. It is probably simpler to work with the operator $T_{00} - T_{11}$, for which we will see that the undesirable contribution of the quadratic fluctuations cancels out [Eq. (12)]. We note that the effect of generalizing the operator $\boldsymbol{h}$ to $\boldsymbol{h} + \alpha\Box\boldsymbol{h}$ was taken into account in [9], where α is a free "low-energy" parameter, but the expectation value of its square turn out to be independent of α.

It is instructive to note that the energy-density expression (2) derived from geometric considerations coincides with the form of the canonical energy density derived from the Lüscher-Weisz world sheet action with the standard Noether procedure. Indeed the NLO Lagrangian reads

$$\mathcal{L}^{\rm ws} = \tfrac{1}{2}\partial_c\boldsymbol{h}\cdot\partial^c\boldsymbol{h} + c_2(\partial_a\boldsymbol{h}\cdot\partial^a\boldsymbol{h})(\partial_b\boldsymbol{h}\cdot\partial^b\boldsymbol{h}) + c_3(\partial_a\boldsymbol{h}\cdot\partial_b\boldsymbol{h})(\partial^a\boldsymbol{h}\cdot\partial^b\boldsymbol{h}) + \ldots \quad (5)$$

with *a priori* free coefficients c_2 and c_3, and the stress-energy tensor

$$T^{\rm ws}_{ab} = \partial_a\boldsymbol{h}\cdot\partial_b\boldsymbol{h} + 4c_2(\partial_c\boldsymbol{h}\cdot\partial^c\boldsymbol{h})(\partial_a\boldsymbol{h}\cdot\partial_b\boldsymbol{h}) + 4c_3(\partial_a\boldsymbol{h}\cdot\partial_c\boldsymbol{h})(\partial_b\boldsymbol{h}\cdot\partial^c\boldsymbol{h}) - g_{ab}\mathcal{L}^{\rm ws}, \quad (6)$$

with in particular

$$T^{\rm ws}_{00}(t, y_1) = \tfrac{1}{2}((\partial_t\boldsymbol{h})^2 + (\partial_1\boldsymbol{h})^2) + (c_2 + c_3)([(\partial_1\boldsymbol{h})^2]^2 - 3[(\partial_t\boldsymbol{h})^2]^2) + 2c_2(\partial_t\boldsymbol{h})^2(\partial_1\boldsymbol{h})^2 + 2c_3(\partial_t\boldsymbol{h}\cdot\partial_1\boldsymbol{h})^2. \quad (7)$$

Expressions (7) and (2) are consistent for $c_2 = \frac{1}{8}$ and $c_3 = -\frac{1}{4}$, which are the Nambu-Goto values [10]. This agreement suggests that the n-point functions of the gauge-theory stress-energy tensor in the presence of the confining string are generically mapped onto those of the world sheet stress-energy tensor. The unit operator appearing in Eq. (3) must be included in the diagonal components; obviously this term does not affect the conservation equations of the world sheet stress-energy tensor.

In $d = 2$ space dimensions, it was shown in [11] that the Nambu-Goto values for the 'low-energy constants'' c_2 and c_3 are the only ones compatible with open-closed string duality. It was subsequently shown that this requirement is also equivalent to requiring that closed string have a relativistic dispersion relation, in other words requiring Poincaré invariance [12]. If one requires that the effective string theory also describes a situation where the world sheet itself is a torus in a way that is consistent with the open- and closed-string spectral representations, then these values are the only ones possible in any dimension [13]. In view of the geometric interpretation of the energy-density operator, these results show that only a string that is "immaterial," i.e. for which only transverse motion of an element of the string contributes to the string energy, yields a spectrum that is consistent with open-closed string duality. Were it not for this fact, the fraction in Eq. (2) would have been replaced by $\sqrt{\frac{1+\partial_1\boldsymbol{h}^2}{1-(\partial_t\boldsymbol{h})^2}}$, which, in particular, does not yield mixed term $(\partial_t\boldsymbol{h}\cdot\partial_1\boldsymbol{h})^2$. In other words, numerical evidence that the open-string spectrum requires c_2 and c_3 to take up their respective Nambu-Goto values really confirms the "immaterial" nature of the confining string.

It is also of interest to write out the expressions for the longitudinal stress operator T_{11} explicitly [see Eq. (6)],

$$T_{11}(t, y_1, \boldsymbol{y}) = \sigma\delta(\boldsymbol{y} - \boldsymbol{h}(t, y_1))[-1 + \tfrac{1}{2}((\partial_t\boldsymbol{h})^2 + (\partial_1\boldsymbol{h})^2) + (c_2 + c_3)(3[(\partial_1\boldsymbol{h})^2]^2 - [(\partial_t\boldsymbol{h})^2]^2) - 2c_2(\partial_t\boldsymbol{h})^2(\partial_1\boldsymbol{h})^2 - 2c_3(\partial_t\boldsymbol{h}\cdot\partial_1\boldsymbol{h})^2]. \quad (8)$$

This formula implies that the transverse string profiles obtained with T_{00} and T_{11} differ at quadratic order in $\boldsymbol{h}$. There is a specific reason why T_{11} is an interesting probe of the string profile. The transverse profile of the open string depends in general at what point y_1 along the string it is measured. It is easy to see that if one uses T_{11}, then the total longitudinal stress inside a transverse spatial slice, $\int {\rm d}^{d-1}\boldsymbol{y}T_{11}(y_1, \boldsymbol{y})$, does not depend on the position y_1 of the slice along the string. This is simply because from the closed-string point of view, T_{11} plays the role of the energy-density operator, and therefore its forward matrix elements are diagonal in an energy-eigenstate basis. Evaluated on the ground state, this integrated longitudinal stress yields the static force,

$$\int {\rm d}^{d-1}\boldsymbol{y}\langle T_{11}(y_1, \boldsymbol{y})\rangle_{Q\bar{Q}} = -\frac{\partial E_0(r)}{\partial r}, \qquad (\forall\ y_1). \quad (9)$$

Because of this distinguishing property, the integrated longitudinal stress is conserved along the open string, and it is natural to ask how the transverse distribution of longitudinal stress changes as one moves along the string.

It is however not true that any operator tracks the movement of the string at leading order. Take for instance the transverse operator T_{22}. There is no corresponding operator on the world sheet, since it is a two-dimensional field theory. One can show that

$$\int dy_1 \int d^{d-1}\boldsymbol{y}\langle T_{22}(y_1, \boldsymbol{y})\rangle_{Q\bar{Q}} = 0, \tag{10}$$

when correlated with the pair of Polyakov loops. The physical reason why the three-point function of T_{22} vanishes is that the string does not, on average, exert any stress along the transverse directions. Because the sum rule of this operator does not yield a term proportional to the length of the string, this operator is not measuring, to leading order in the fluctuations $\boldsymbol{h}$, the position of the string. Therefore one cannot define a transverse distribution of the string with a probabilistic interpretation based on this operator. Instead this operator is sensitive in leading order to the expectation value of higher-derivative world sheet operators.

We have followed the approach of Lüscher and Weisz [11] and worked in the static gauge. The point of view adopted by Polchinski and Strominger [14] puts more emphasis on the conformal symmetry of the world sheet theory, which severely constrains the class of actions they consider. It is therefore worthwhile to investigate the fate of conformal symmetry in the static gauge as well. This issue is left for a future study. We simply note that the trace of the canonical energy-momentum tensor

$$\begin{aligned} T_a^{\mathrm{ws},a} &= 2c_2(\partial_a \boldsymbol{h}\cdot\partial^a\boldsymbol{h})^2 + 2c_3(\partial_a\boldsymbol{h}\cdot\partial_b\boldsymbol{h})(\partial^a\boldsymbol{h}\cdot\partial^b\boldsymbol{h}) + \ldots \\ &= 2\mathcal{L}^{(4)} + \ldots \end{aligned} \tag{11}$$

no longer vanishes at the quartic order. However it is well-known that the canonical energy-momentum tensor is in general not traceless even when the field theory is conformally invariant. It can however be improved [15] in the sense that terms Δ_{ab} that satisfy $\partial^a\Delta_{ab} = 0$ and do not modify the conserved charges can be added in such a way that T_{ab} is traceless when the theory is conformal. See [16] for a discussion in two-dimensional field theory. It would be interesting to see whether the line of low-energy constants $c_3 = -2c_2$ [11] plays a special role in this respect.

An observation of ''practical'' importance is that the linear combination

$$(T_{00} - T_{11})(t, y_1, \boldsymbol{y}) = 2\sigma\delta(\boldsymbol{y} - \boldsymbol{h}(t, y_1))(1 + \mathrm{O}(\partial\boldsymbol{h})^4) \tag{12}$$

is a scalar from the world sheet point of view, which makes it an adequate operator to measure the mean square amplitude of string fluctuations at next-to-leading order [9].

The rest of this paper is structured as follows: We start by studying the structure of the confining string as seen by the energy-momentum tensor in Sec. III. We then work out the constraints on three-point correlation functions imposed by the open-closed string duality in Sec. IV. The leading-order string formula (1), generalized to contain the contributions of excited states in the three-point function, turns out to be consistent with the functional form in r imposed by the closed-string spectral representation, and we thereby identify the effective theory prediction for the form factors of the closed strings. In particular, we find that the square radius of the ground state closed string, defined in the standard way from the slope of its form factors at the origin, grows logarithmically with the length of the string β. In Sec. V we give the explicit form of the energy-momentum tensor on the lattice in $d + 1$ dimensions. This allows us to interpret a recent high-accuracy calculation of the string width in numerical lattice gauge theory in terms of matrix elements of the energy-momentum tensor. In the rest of this paper, we work in Euclidean space, and our sign conventions are as follows. In Minkowski space, the thermal expectation values of the diagonal components are $\langle T_{00}\rangle = e$ and $\langle T_{11}\rangle = p$ (respectively the energy density and pressure), while in Euclidean space $\langle T_{00}\rangle = e$ and $\langle T_{11}\rangle = -p$.

III. GRAVITATIONAL FORM FACTORS OF CLOSED STRINGS

In this section, we analyze how the transverse size of closed strings can be characterized. In the pure $SU(N)$ gauge theory, the only conserved charges are energy and momentum. Therefore, it is natural to measure the width of the string in terms of the distribution of these charges. While for the open string, the width can be probed directly in x space, it has to be defined initially in momentum space through a form factor for the closed string: the form factors with respect to the energy-momentum tensor $T_{\mu\nu}$ are the Fourier transforms of the energy and distributions. This simple relation between form factors and charge distribution applies because of the nonrelativistic kinematics of the closed string, by which we mean that their transverse size is parametrically larger than their inverse mass. By contrast, the electromagnetic form factors of the proton only correspond to the Fourier transform of charge and magnetization in the infinite-momentum frame [17].

Here we will restrict ourselves to studying the form factors of states that contribute to the Polyakov loop two-point function. These states are translationally invariant in the longitudinal direction, therefore we restrict the momentum transfer to the transverse directions. Furthermore, the closed-string states are rotationally invariant, hence they have spin zero in $(d - 1)$-dimensional space.

In order to exhaustively list the relevant form factors, we decompose the full $(d + 1)$-dimensional energy-momentum tensor into irreducible representations of

d-dimensional space. The closed strings are stretched around a cycle of length β in a spatial direction labeled z, while the other spatial directions are labeled by $k, l = 1, \ldots, (d-1)$. Schematically, the decomposition takes the form

$$\begin{pmatrix} T_{00} & T_{0k} & T_{0z} \\ T_{k0} & T_{kl} & T_{kz} \\ T_{z0} & T_{zk} & T_{zz} \end{pmatrix}. \tag{13}$$

In the following, we choose the normalization of states such that

$$\langle \psi, P' | \psi, P \rangle = (2\pi)^{d-1} \delta^{d-1}(\boldsymbol{P}' - \boldsymbol{P}) \cdot \beta \cdot 2E_{\boldsymbol{p}}. \tag{14}$$

The operator T_{zz}, which measures the stress in the z direction, is a scalar operator from the point of view of physics within an $z =$ constant slice. Therefore, its matrix elements can be parametrized as

$$\langle \psi', P' | T_{zz}(0) | \psi, P \rangle = 2MM' f_3(\psi', \psi; \boldsymbol{q}^2). \tag{15}$$

We use the standard notation $q = P' - P$, $\bar{P} = \frac{1}{2}(P + P')$, and have accounted for the possibility that the mass of the final state M' differs from the mass of the initial state M.

Second, we note that (T_{0z}, T_{kz}) is a conserved vector from the point of view of a $z =$ constant slice, if one restricts oneself to matrix elements between states that are translationally invariant in the z direction:

$$\partial_0 T_{0z} + \partial_k T_{kz} + \underbrace{\partial_z T_{zz}}_{=0} = 0. \tag{16}$$

We are thus dealing with the vector form factor of a scalar object, hence (by analogy with the pion electromagnetic form factor),

$$\langle \psi', P' | T_{\mu z}(0) | \psi, P \rangle = M \bar{P}_\mu f(\psi', \psi; \boldsymbol{q}^2), \quad \mu \neq z. \tag{17}$$

However, $T_{\mu z}$ is odd under the reflection $z \to -z$. For matrix elements with $P_z = P'_z = 0$, this implies that f must vanish identically [18].

Finally, the components of $T_{\mu\nu}$ not containing the index 'z' form a tensor with respect to the SO(d) group. Taking again into account the fact that these components form a conserved tensor in the subspace of states invariant under translations along the y direction, one finds that the general form of the matrix elements of $T_{\mu\nu}$ is

$$\langle \psi', P' | T_{\mu\nu}(0) | \psi, P \rangle = -2\bar{P}_\mu \bar{P}_\nu f_1(\psi', \psi; \boldsymbol{q}^2) + 2(q_\mu q_\nu - q^2 \delta_{\mu\nu}) f_2(\psi', \psi; \boldsymbol{q}^2), \quad \mu, \nu \neq z. \tag{18}$$

Thus the transverse structure of the ground state of the string is characterized by a total of three form factors $\{f_i\}_{i=1}^3$. The matrix elements

$$\langle \psi, P | T_{\mu\nu}(0) | \psi, P \rangle = -2P_\mu P_\nu, \tag{19}$$

$$\langle \psi, P | T_{zz}(0) | \psi, P \rangle = \beta \frac{\partial E_{\boldsymbol{p}}^2}{\partial \beta}, \tag{20}$$

determine the forward, diagonal matrix elements of f_1 and f_3,

$$f_1(\psi, \psi; \boldsymbol{0}) = 1, \qquad f_3(\psi; \psi; \boldsymbol{0}) = \frac{\beta}{2M^2} \frac{\partial E_{\boldsymbol{p}}^2(\beta)}{\partial \beta}. \tag{21}$$

The interpretation of these form factors is that f_3 measures the transverse distribution of longitudinal stress in the string, while f_1 measures the transverse distribution of energy. The form factor f_2 is somewhat less obvious to interpret. For two states with momenta equal and opposite aligned along the direction $\hat{1}$ (Breit frame), it describes the ability of T_{22} to induce a transition between these states per unit (momemtum transfer)^2 (this interpretation requires $d \geq 3$). Indeed, in this kinematic configuration, f_1 does not contribute to the matrix element (18).

IV. TRANSVERSE STRUCTURE OF OPEN AND CLOSED STRINGS

The goal of this section is to derive the spectral representation of a three-point function where a local operator is used to probe the structure of the confining string. We begin by recalling the spectral representation of the Polyakov loop two-point function. The geometry of the Polyakov correlator is illustrated in Fig. 1.

The open-string representation of the Polyakov loop two-point function reads, setting $r^2 \equiv x_1^2 + \boldsymbol{x}^2$ and with w_n integer weights,

$$\langle P_0(x_1, \boldsymbol{x}) P_0^*(0, \boldsymbol{0}) \rangle = \sum_n w_n e^{-V_n(r)L}. \tag{22}$$

Upon introducing the matrix elements [19]

$$b_n \equiv \langle \mathrm{vac} | P_0(0, \boldsymbol{0}) | n, \boldsymbol{p} \rangle, \tag{23}$$

the closed-string representation of the same correlation function reads

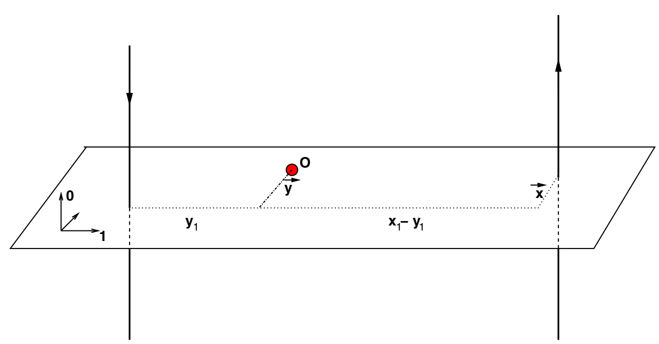

FIG. 1 (color online). The geometry of the three-point function.

$$\langle P_0(x_1, \boldsymbol{x}) P_0^*(0, \boldsymbol{0})\rangle = \frac{1}{\beta}\sum_n |b_n|^2 \int \frac{\mathrm{d}^{d-1}\boldsymbol{p}}{(2\pi)^{d-1}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \frac{e^{-E_n(\boldsymbol{p})x_1}}{2E_n(\boldsymbol{p})}, \tag{24}$$

$$= \sum_n |b_n|^2 \frac{r}{\beta M_n}\left(\frac{M_n}{2\pi r}\right)^{(d/2)} K_{((1/2)(d-2))}(M_n r), \tag{25}$$

$$\sim \sum_n \frac{|b_n|^2}{2\beta M_n}\left(\frac{M_n}{2\pi r}\right)^{((d-1)/2)} e^{-M_n r}. \tag{26}$$

In the last line we have used the asymptotic form of the modified Bessel function, $K_\nu(x) \sim e^{-x}\sqrt{\frac{\pi}{2x}}$; the result is equivalent to using nonrelativistic kinematics to begin with. This expression dictates the functional dependence on r of the Polyakov loop correlator. As usual in deriving relations between open and closed strings, the correlation function cannot be simultaneously dominated by a single open-string state *and* a single closed-string state. Let β be the length of the closed strings. For $\beta \gg r$, a single open-string state dominates, but $\mathrm{O}(\beta/r)$ closed-string states contribute in Eq. (26).

Consider now the connected correlation function of a pair of Polyakov loops in the direction $\hat{0}$ and a local operator $\mathcal{O}$. Figure 1 illustrates the geometry of the correlator. Its spectral interpretation in terms of open-string states reads, for $\beta \gg \sigma^{-(1/2)}$,

$$\langle P_0(x_1, \boldsymbol{x})\mathcal{O}(y_0, y_1, \boldsymbol{y}) P_0^*(0, \boldsymbol{0})\rangle = \sum_n e^{-V_n(r)\beta}\langle \mathcal{O}(y_1, \boldsymbol{y})\rangle_n. \tag{27}$$

In terms of closed-string states it can also be written as

$$\langle P_0(x_1, \boldsymbol{x})\mathcal{O}(y_0, y_1, \boldsymbol{y}) P_0^*(0, \boldsymbol{0})\rangle = \int \frac{\mathrm{d}^{d-1}\boldsymbol{p}'}{(2\pi)^{d-1}} e^{-i\boldsymbol{p}'\cdot\boldsymbol{x}} \int \frac{\mathrm{d}^{d-1}\boldsymbol{q}}{(2\pi)^{d-1}} e^{-i\boldsymbol{q}\cdot\boldsymbol{y}} f(\boldsymbol{p}', \boldsymbol{q}, x_1, y_1), \tag{28}$$

where

$$f(\boldsymbol{p}', \boldsymbol{q}, x_1, y_1) = \int \mathrm{d}^{d-1}\boldsymbol{x} e^{i\boldsymbol{p}'\cdot\boldsymbol{x}} \times \int \mathrm{d}^{d-1}\boldsymbol{y} e^{i\boldsymbol{q}\cdot\boldsymbol{y}} \langle P_0(x_1, \boldsymbol{x})\mathcal{O}(y_0, y_1, \boldsymbol{y}) P_0^*(0, \boldsymbol{0})\rangle \tag{29}$$

is the correlation function in momentum space, which has a more natural interpretation from the closed-string point of view. Here $\boldsymbol{p}$ and $\boldsymbol{q}$ have $d-1$ components. Because of the translation invariance of the Polyakov loops along the $\hat{0}$ direction, f has no dependence on y_0, which we therefore choose to be zero.

With the normalization of states given by Eq. (14), we parametrize the matrix elements by

$$\langle m, \boldsymbol{p}'|\mathcal{O}|n, \boldsymbol{p}\rangle = 2M_m M_n F^{m,n}(\bar{\boldsymbol{p}}, \boldsymbol{q}), \tag{30}$$

$$\boldsymbol{p} = \boldsymbol{p}' - \boldsymbol{q}, \qquad \bar{\boldsymbol{p}} = \boldsymbol{p}' - \tfrac{1}{2}\boldsymbol{q} = \tfrac{1}{2}(\boldsymbol{p} + \boldsymbol{p}'). \tag{31}$$

This parametrization is designed for dimension $(d+1)$ operators, for which $F^{m,n}$ is dimensionless. We can now write the spectral representation of f,

$$f(\boldsymbol{p}', \boldsymbol{q}, x_1, y_1) = \sum_{n,m} b_m \frac{e^{-E_m(\boldsymbol{p}')(x_1-y_1)}}{2E_m(\boldsymbol{p}')\beta} 2M_m M_n F^{m,n}(\bar{\boldsymbol{p}}, \boldsymbol{q}) \times \frac{e^{-E_n(\boldsymbol{p})y_1}}{2E_n(\boldsymbol{p})\beta} b_n^*, (\boldsymbol{p} = \boldsymbol{p}' - \boldsymbol{q}). \tag{32}$$

Next we specialize to the case of a scalar operator with respect to the symmetry group $\mathrm{SO}(d)$ of a time slice. Examples thereof are T_{00} or $T_{\mu\mu}$. We will return to the case of an operator with a more general tensor structure in Sec. IV D. Thus, F is a function of $\boldsymbol{q}^2$ alone, hence

$$\langle P_0(x_1, \boldsymbol{x})\mathcal{O}(y_0, y_1, \boldsymbol{y}) P_0^*(0, \boldsymbol{0})\rangle = \sum_{m,n}\frac{b_m b_n^*}{\beta^2} \times \int \frac{\mathrm{d}^{d-1}\boldsymbol{q}}{(2\pi)^{d-1}} e^{-i\boldsymbol{q}\cdot\boldsymbol{y}} 2M_m M_n F^{m,n}(\boldsymbol{q}^2)\mathcal{I}_{mn}(y_1, x_1 - y_1, \boldsymbol{x}, \boldsymbol{q}),$$

$$\mathcal{I}_{mn}(y_1, y_2, \boldsymbol{x}, \boldsymbol{q}) = \int \frac{\mathrm{d}^{d-1}\boldsymbol{p}'}{(2\pi)^{d-1}} \frac{e^{-i\boldsymbol{p}'\cdot\boldsymbol{x} - E_m(\boldsymbol{p}')y_2 - E_n(\boldsymbol{p}'-\boldsymbol{q})y_1}}{2E_m(\boldsymbol{p}')2E_n(\boldsymbol{p}'-\boldsymbol{q})}. \tag{33}$$

We choose without loss of generality $\boldsymbol{x} = 0$. The quantity $\mathcal{I}$ is a massive one-loop integral,

$$\mathcal{I}(y_1, y_2, 0, \boldsymbol{q}) = \int \frac{\mathrm{d}\omega}{2\pi} e^{i\omega y_1} \int \frac{\mathrm{d}\omega'}{2\pi} e^{i\omega' y_2} \int \frac{\mathrm{d}^{d-1}\boldsymbol{p}}{(2\pi)^{d-1}} \times \frac{1}{\omega^2 + (\boldsymbol{p}-\boldsymbol{q})^2 + M_n^2}\frac{1}{\omega'^2 + \boldsymbol{p}^2 + M_m^2}. \tag{34}$$

This integral can be treated by standard techniques of quantum field theory, see for instance [20] p. 327. However we anticipate that nonrelativistic kinematics is sufficient to study the long-distance behavior of the correlators (in the effective string theory, this will be guaranteed as long as $\sigma\beta y_1 \gg 1$),

$$\mathcal{I}_{mn}(y_1, y_2, \boldsymbol{0}, \boldsymbol{q}) \sim \left(\frac{M_m M_n}{y_2 M_n + y_1 M_m}\right)^{((1/2)(d-1))} \times \frac{\exp-(M_m y_2 + M_n y_1 + \frac{\boldsymbol{q}^2}{2}\frac{y_1 y_2}{M_m y_1 + M_n y_2})}{(2\pi)^{((1/2)(d-1))} 2M_m \cdot 2M_n}. \tag{35}$$

Therefore, with $y_2 \doteq x_1 - y_1$,

$$\langle P_0(x_1, \boldsymbol{0})\mathcal{O}(y_0, y_1, \boldsymbol{y}) P_0^*(0, \boldsymbol{0})\rangle \sim \sum_{m,n}\frac{b_m b_n^*}{\beta^2}\left[\frac{M_m M_n}{y_2 M_n + y_1 M_m}\right]^{((d-1)/2)} \frac{e^{-(y_2 M_m + y_1 M_n)}}{2(2\pi)^{((1/2)(d-1))}} \cdot \times \int \frac{\mathrm{d}^{d-1}\boldsymbol{q}}{(2\pi)^{d-1}} e^{-i\boldsymbol{q}\cdot\boldsymbol{y}} F^{m,n}(\boldsymbol{q}^2) e^{-(\boldsymbol{q}^2/2)((y_1 y_2)/(M_m y_1 + M_n y_2))}. \tag{36}$$

This expression dictates the leading-order functional dependence on x_1 and y_1 of the three-point function that the effective string theory must respect.

Expression (36) can be viewed as a distribution in $\boldsymbol{y}$. The quantity we will confront with a prediction from the effective string theory is its second moment at $x_1 \doteq r$,

$$w^2(r,\beta,y_1) \equiv \frac{\int d^{d-1}\boldsymbol{y}\boldsymbol{y}^2\langle P_0(r,\mathbf{0})\mathcal{O}(y_0,y_1,\boldsymbol{y})P_0^*(0,\mathbf{0})\rangle}{\int d^{d-1}\boldsymbol{y}\langle P_0(r,\mathbf{0})\mathcal{O}(y_0,y_1,\boldsymbol{y})P_0^*(0,\mathbf{0})\rangle}. \tag{37}$$

Based on (36), we obtain

$$\begin{aligned} w^2(r,\beta,y_1) &= -2(d-1)\frac{d}{d\boldsymbol{q}^2} \\ &\times \log\Big\{\sum_{m,n} b_m b_n^*\Big[\frac{M_m M_n}{y_2 M_n + y_1 M_m}\Big]^{((d-1)/2)} \\ &\times \exp-\Big(y_2 M_m + y_1 M_n + \frac{\boldsymbol{q}^2}{2}\frac{y_1 y_2}{M_m y_1 + M_n y_2}\Big) \\ &\times F^{m,n}(\boldsymbol{q}^2)\Big\}_{\boldsymbol{q}=0}, \end{aligned} \tag{38}$$

where $y_2 \doteq r - y_1$. At $y_1 = y_2 = \frac{r}{2}$, the expression simplifies slightly,

$$\begin{aligned} w^2\Big(r,\beta,\frac{r}{2}\Big) &= -2(d-1)\frac{d}{d\boldsymbol{q}^2} \\ &\times \log\Big\{\sum_{m,n} b_m b_n^* \mu_{mn}^{(1/2)(d-1)} \\ &\times \exp-\frac{r}{2}\Big(M_m + M_n + \frac{1}{2}\frac{\boldsymbol{q}^2}{M_m + M_n}\Big) \\ &\times F^{m,n}(\boldsymbol{q}^2)\Big\}_{\boldsymbol{q}=0}, \end{aligned} \tag{39}$$

where μ_{mn} is the reduced mass of M_m and M_n defined by $\mu_{mn}^{-1} = M_m^{-1} + M_n^{-1}$.

A. Effective string theory prediction

On the other hand, Allais and Caselle [21] (see also the recent two-loop result [9]) obtained within the effective bosonic string theory the leading-order result, for $x_1 = 2y_1 = r$,

$$w_{lo}^2\Big(r,\beta,\frac{r}{2}\Big) = \frac{d-1}{2\pi\sigma}\log\frac{r}{r_0} + \frac{1}{\pi\sigma}\log\frac{Z_0^2(\beta,r)}{Z_0(2\beta,r)}. \tag{40}$$

Written in this form, it is clear that the second term can be interpreted as a difference of free energies.

B. Transverse structure of the ground state of the closed string

The limit $y_1 \gg \beta$ is most transparent from the closed-string point of view, since the correlation function is then dominated by the closed-string ground state. Equation (38) yields in that limit

$$\begin{aligned} w^2(r,\beta,y_1) &= -2(d-1)\Big[-\frac{y_1(r-y_1)}{2\sigma\beta r} + \frac{(F^{0,0})'}{F^{0,0}}\Big] \\ &\times \overset{y_1=r/2}{=} -2(d-1)\Big[-\frac{r}{8\sigma\beta} + \frac{(F^{0,0})'}{F^{0,0}}\Big], \end{aligned} \tag{41}$$

where we have used the leading-order relation $M_n = \sigma L$. The form factors are now evaluated at $\boldsymbol{q}^2 = 0$, and the prime denotes differentiation with respect to $\boldsymbol{q}^2$. In the regime $y_1 \gg \beta$, the effective string expression (40) behaves as

$$w_{lo}^2\Big(r,\beta,\frac{r}{2}\Big) = \frac{d-1}{2\pi\sigma}\log\frac{\beta}{4r_0} + \frac{d-1}{4\beta\sigma}r + \mathrm{O}(e^{-2\pi r/\beta}). \tag{42}$$

It is consistent with the general expression (41) derived from the spectral representation of the correlator. The linear term turns out to agree automatically between the two expressions. From the closed-string point of view, this term is essentially a kinematic effect; we will return to its significance in the open-string interpretation of the three-point function.

The rms radius of the closed string, defined in the standard way from the derivative of the form factor at the origin, can be identified with the r-independent term,

$$\langle r^2\rangle_{\rm closed} \equiv -\frac{2(d-1)}{F^{0,0}(\mathbf{0})}\frac{dF^{0,0}(\boldsymbol{q}^2)}{d\boldsymbol{q}^2}\Big|_{\boldsymbol{q}=0} = \frac{d-1}{2\pi\sigma}\log\frac{\beta}{4r_0}. \tag{43}$$

This term thus measures the logarithmic broadening of the *closed* string with its length β. The prefactor is the same as for the open string, but the UV length scale appearing inside the logarithm is 4 times larger than in the open-string case.

When $r \gg \beta$, the open-string ensemble is at finite temperature $1/\beta$. The local operator then probes the profile of the open-string states, averaged over with the Boltzmann weight. Equation (41) shows that the profile at mid string grows linearly with the length of the open string [21]. This linear rise is likely due to the fact that $\mathrm{O}(r/\beta)$ open-string states contribute to the correlation function when $r \gg \beta$, and the width results from a stochastic superposition of these contributions. A linear increase is in fact nothing exotic, since for a screened potential $V(r) \sim e^{-mr}$, the profile goes like $e^{-m\sqrt{(r/2)^2+\boldsymbol{y}^2}}$, and hence the mean square radius is given by $(d-1)\frac{r}{2m}$ for large r.

C. Interpretation of excited closed-string contributions

Both the general expression (39) and the bosonic string formula (40) can be expanded in a series of exponentials that fall off increasingly fast. We require that the coefficients of these exponentials match.

In the following, we use the leading-order relation between the matrix elements b_n and the multiplicity (integer) factors w_n,

$$\left|\frac{b_n}{b_0}\right|^2 \stackrel{l.o.}{=} \frac{w_n}{w_0}, \tag{44}$$

we choose them to be real and use the fact that $w_0 = 1$. We recall the values $w_1 = d - 1$ and $w_2 = 1 + (d-1) + \frac{1}{2} \times (d+1)(d-2)$ [11], and also define $\Delta M_n \equiv M_n - M_0$. We start by analyzing the leading correction to (42), which comes solely from $Z_0(2\beta)$,

$$w_{lo}^2\left(r, \beta, \frac{r}{2}\right) \supset -\frac{w_1}{\pi\sigma} e^{-((\Delta M_1 r)/2)}. \tag{45}$$

Expanding (36), one finds that the O(r) term cancels out automatically. From the O(r^0) term, we obtain the consistency condition

$$2(d-1)\frac{d}{d\boldsymbol{q}^2}\left[\frac{\mathrm{Re}F^{1,0}(\boldsymbol{q}^2)}{F^{0,0}(\boldsymbol{q}^2)}\right]_{\boldsymbol{q}=0} = \frac{\sqrt{w_1}}{2\pi\sigma}, \tag{46}$$

which dictates the strength of the off-diagonal matrix element between the lightest two string states at small momentum transfer.

We now turn to the term of order $e^{-\Delta M_1 r}$, which is of precisely the same order as $e^{-(1/2)\Delta M_2 r}$ for the leading-order spectrum. This time, both $Z_0(2\beta)$ and $Z_0^2(\beta)$ contribute, and we find

$$w_{lo}^2\left(r, \beta, \frac{r}{2}\right) \supset \frac{2w_1 - w_2 + \frac{1}{2}w_1^2}{\pi\sigma} e^{-\Delta M_1 r}, \tag{47}$$

while from the general expression, we extract

$$w^2\left(\beta, r, \frac{r}{2}\right) \supset -2(d-1)e^{-\Delta M_1 r}\frac{d}{d\boldsymbol{q}^2}\left[w_1\frac{F^{1,1}}{F^{0,0}} + 2\sqrt{w_2}\frac{\mathrm{Re}F^{2,0}}{F^{0,0}} - 2w_1\left(\frac{\mathrm{Re}F^{1,0}}{F^{0,0}}\right)^2\right]_{\boldsymbol{q}=0}. \tag{48}$$

The comparison of Eqs. (47) and (48) yields predictions for the form factor at small momentum transfer. By generalizing w^2 to values of $y_1 \neq y_2$, one could disentangle $F^{2,0}$ from $F^{1,1}$ and obtain separate predictions for these form factors. In this way, a sequence of predictions are obtained for the form factors between low-lying states.

D. Three-point function with a nonscalar probe operator

We now come back to (32) in the case of an operator with a more complicated tensor structure. Consider the case of $\mathcal{O} = T_{11}$. Recall that direction "1" plays the role of time from the point of view of the closed strings. Then we replace Eq. (30) by [see Eq. (18)]

$$\langle m, \boldsymbol{p}'|\mathcal{O}|n, \boldsymbol{p}\rangle = \tfrac{1}{2}[E_m(\boldsymbol{p}') + E_n(\boldsymbol{p})]^2 f_1^{m,n}(\boldsymbol{q}^2) - 2\boldsymbol{q}^2 f_2^{m,n}(\boldsymbol{q}^2). \tag{49}$$

In this case we can write

$$\langle P_0(x_1,\boldsymbol{x})\mathcal{O}(y_0,y_1,\boldsymbol{y})P_0^*(0,\boldsymbol{0})\rangle = \sum_{m,n}\frac{b_m b_n^*}{\beta^2}\int\frac{\mathrm{d}^{d-1}\boldsymbol{q}}{(2\pi)^{d-1}}e^{-i\boldsymbol{q}\cdot\boldsymbol{y}}\cdot \times\left[\frac{1}{2}f_1^{m,n}(\boldsymbol{q}^2)\left(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}\right)^2 - 2\boldsymbol{q}^2 f_2^{m,n}(\boldsymbol{q}^2)\right]\mathcal{I}_{mn}(y_1,y_2,\boldsymbol{x},\boldsymbol{q}), \tag{50}$$

where y_2 is set to $x_1 - y_1$ at the end. We now note that at leading order for large $r = 2y_1 = 2y_2$,

$$\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\right)^2 \mathcal{I}_{mn}(y_1, y_2, \boldsymbol{0}, \boldsymbol{q}) \sim (M_m + M_n)^2 \mathcal{I}_{mn}\left(\frac{r}{2}, \frac{r}{2}, \boldsymbol{0}, \boldsymbol{q}\right). \tag{51}$$

Hence to leading order the preceding analysis still applies, with the substitution

$$F^{m,n}(\boldsymbol{q}^2) \to f_1^{m,n}(\boldsymbol{q}^2) - \frac{\boldsymbol{q}^2}{M_m M_n} f_2^{m,n}(\boldsymbol{q}^2). \tag{52}$$

A special feature of the operator T_{11} is that

$$f_1^{m,n}(\boldsymbol{0}) = \delta_{mn}, \tag{53}$$

since the states $|n, \boldsymbol{p}\rangle$ are energy eigenstates. In particular, Eq. (46) simplifies slightly to

$$-2(d-1)\mathrm{Re}(f_1^{1,0})'(\boldsymbol{0}) = \frac{\sqrt{w_1}}{2\pi\sigma}. \tag{54}$$

It is interesting that, with our normalization of states (14), the transition form factor is independent of β.

V. LATTICE DEFINITION OF THE ENERGY-MOMENTUM TENSOR IN $(d + 1)$ DIMENSIONS

In this section, we derive the lattice form of the energy-momentum tensor in $(d+1)$-dimensional $SU(N)$ gauge theory. Our main motivation is that these operators have been mostly studied in the $d = 3$ case, but recently there has been extensive work on strings in $d = 2$ $SU(N)$ gauge theories [6,22]. This preparatory work will help us interpret those results.

We will follow the treatment [23] and generalize it to d dimensions. The idea is to identify the operators whose expectation value yield the thermodynamic energy density and pressure. We start from the Wilson action [1] on an anisotropic lattice [24],

$$S_{\mathrm{g}} = \sum_x \beta_\sigma S_\sigma(x) + \beta_\tau S_\tau(x). \tag{55}$$

The action has two bare parameters, β_σ and β_τ, and there are two "renormalized" parameters, the spatial lattice spacing a_σ and the renormalized anisotropy $\xi = a_\sigma/a_\tau$.

At the isotropic point, $\beta_\sigma = \beta_\tau = \beta$ (not to be confused with the symbol used for the closed-string length in the previous sections). The function S_σ and S_τ of the link variables $U_\mu(x)$ contain exclusively spatial and temporal Wilson loops, respectively. The partition function Z depends on β_σ, β_τ and the lattice dimensions, $N_\tau \cdot N_\sigma^d$. The latter are related to its physical size by $L = N_\sigma a_\sigma$, $L_0 = 1/T = N_\tau a_\tau$. We define the renormalized quantity $\bar{Z}$ by

$$\log\bar{Z}(\beta_\sigma, \beta_\tau, N_\sigma, N_\tau) = \log Z(\beta_\sigma, \beta_\tau, N_\sigma, N_\tau) - \frac{N_\tau}{N_\tau^{\rm ref}} \log Z(\beta_\sigma, \beta_\tau, N_\sigma, N_\tau^{\rm ref}). \quad (56)$$

The conditions that $\bar{Z}$ does not depend on a_σ or on the anisotropy ξ translate, respectively, into

$$\frac{L\partial \log\bar{Z}}{\partial L} + \frac{L_0\partial \log\bar{Z}}{\partial L_0} = -\sum_x \frac{\partial\beta_\sigma}{\partial \log a_\sigma}\langle S_\sigma\rangle + \frac{\partial\beta_\tau}{\partial \log a_\sigma}\langle S_\tau\rangle, \quad (57)$$

$$\frac{L_0\partial \log\bar{Z}}{\partial L_0} = \sum_x \frac{\beta_\sigma}{\partial \log\xi}\langle S_\sigma\rangle + \frac{\partial\beta_\tau}{\partial \log\xi}\langle S_\tau\rangle, \quad (58)$$

where it is understood that the expectation values of S_σ and S_τ on the $N_\tau^{\rm ref} \cdot N_\sigma^d$ lattice are subtracted. We then recall the thermodynamic definitions of energy density and pressure,

$$e = -\frac{1}{L_0 L^d}\frac{L_0\partial \log\bar{Z}}{\partial L_0}, \qquad p = \frac{1}{dL_0L^d}\frac{L\partial \log\bar{Z}}{\partial L}. \quad (59)$$

With these definitions, we obtain at the isotropic point $\xi = 1$,

$$a^{d+1}(e - dp) = \frac{\partial\beta_\sigma}{\partial \log a_\sigma}\langle S_\sigma\rangle + \frac{\partial\beta_\tau}{\partial \log a_\sigma}\langle S_\tau\rangle, \quad (60)$$

$$\frac{d}{d+1}a^{d+1}(e+p) = -\left(\frac{\partial\beta_\sigma}{\partial \log\xi} + \frac{1}{d+1}\frac{\partial\beta_\sigma}{\partial \log a_\sigma}\right)\langle S_\sigma\rangle - \left(\frac{\partial\beta_\tau}{\partial \log\xi} + \frac{1}{d+1}\frac{\partial\beta_\tau}{\partial \log a_\sigma}\right)\langle S_\tau\rangle. \quad (61)$$

On the other hand, from the definition of the stress-energy tensor, we expect that

$$\langle\theta\rangle \equiv \langle T_{\mu\mu}\rangle = e - dp, \qquad \langle T_{00}\rangle = e. \quad (62)$$

We also define

$$\theta_{\mu\nu} = T_{\mu\nu} - \frac{1}{d+1}\delta_{\mu\nu}\theta, \quad (63)$$

so that in particular

$$\langle\theta_{00}\rangle = \frac{d}{d+1}(e+p). \quad (64)$$

Since Eqs. (60) and (61) hold at every temperature, we infer that

$$a^{d+1}\theta = \frac{\partial\beta_\sigma}{\partial \log a_\sigma}S_\sigma + \frac{\partial\beta_\tau}{\partial \log a_\sigma}S_\tau, \quad (65)$$

$$a^{d+1}\theta_{00} = -\left(\frac{\partial\beta_\sigma}{\partial \log\xi} + \frac{1}{d+1}\frac{\partial\beta_\sigma}{\partial \log a_\sigma}\right)S_\sigma - \left(\frac{\partial\beta_\tau}{\partial \log\xi} + \frac{1}{d+1}\frac{\partial\beta_\tau}{\partial \log a_\sigma}\right)S_\tau. \quad (66)$$

Recall that the magnetic field has $\frac{d(d-1)}{2}$ components, while the electric field has d components. The lattice action can be expressed in terms of these fields,

$$S_\sigma = \frac{a_\sigma^4}{N_c}{\rm Tr}\{\boldsymbol{B}^2\}, \qquad S_\tau = \frac{a_\sigma^2 a_\tau^2}{N_c}{\rm Tr}\{\boldsymbol{E}^2\}. \quad (67)$$

An important observation is now that at the isotropic point $\xi = 1$, the operators

$$S_\tau - \frac{2}{d-1}S_\sigma \quad \text{and} \quad S_\tau + S_\sigma \quad (68)$$

belong to irreducible representations of the cubic group in $(d+1)$ dimensions [25]. Since in both cases there is no other gauge-invariant operator of dimension $(d+1)$ in the same representation, both of them renormalize multiplicatively.

A. The case $d = 2$

Since the $d = 3$ case is well known [23,24], we focus here on the $d = 2$ case. The $d = 2$ theory is super-renormalizable, which leads to considerable simplifications. At tree level on the anisotropic lattice, we have the following expressions for the bare parameters in terms of the renormalized ones,

$$\beta_\sigma = \frac{2N_c}{g^2 a_\sigma}\frac{1}{\xi}, \qquad \beta_\tau = \frac{2N_c}{g^2 a_\sigma}\xi \qquad \text{(tree level)}. \quad (69)$$

Hence,

$$\frac{\partial\beta_\sigma}{\partial \log a_\sigma} \simeq -\beta_\sigma, \qquad \frac{\partial\beta_\tau}{\partial \log a_\sigma} \simeq -\beta_\tau, \quad (70)$$

$$\frac{\partial\beta_\sigma}{\partial \log\xi} \simeq -\beta_\sigma, \qquad \frac{\partial\beta_\tau}{\partial \log\xi} \simeq \beta_\tau. \quad (71)$$

Inserting these expressions into Eq. (66), we get the following tree level expressions at the isotropic point,

$$a^3\theta = -\beta(S_\sigma + S_\tau), \quad (72)$$

$$a^3\theta_{00} = \tfrac{2}{3}\beta(2S_\sigma - S_\tau). \quad (73)$$

Since we already know that these linear combinations renormalize multiplicatively (see the remarks at the end of the last section), the full expressions for θ and θ_{00} read

$$\theta = \frac{d\beta}{d\log a}(S_\sigma + S_\tau), \quad (74)$$

$$\theta_{00} = \tfrac{2}{3}\beta Z(\beta)(2S_\sigma - S_\tau), \tag{75}$$

with Z of the form $Z(\beta) = 1 + O(\beta^{-1})$ and $\frac{d\beta}{d\log a} = -\beta(1 + O(\beta^{-1}))$. Now comparing these expressions with Eqs. (65) and (66), we obtain at $\xi = 1$ the relations

$$-\frac{\partial(\beta_\sigma + 2\beta_\tau)}{\partial \log\xi} = \frac{d\beta}{d\log a}, \tag{76}$$

$$\frac{\partial(\beta_\tau - \beta_\sigma)}{\partial \log\xi} = 2\beta Z(\beta). \tag{77}$$

Combining (74) and (75),

$$a^3 T_{00} = S_\sigma\left(\frac{4}{3}\beta Z(\beta) + \frac{1}{3}\frac{\partial\beta}{\partial \log a}\right) + S_\tau\left(-\frac{2}{3}\beta Z(\beta) + \frac{1}{3}\frac{\partial\beta}{\partial\log a}\right). \tag{78}$$

By Euclidean symmetry, one then obtains also the expression for the diagonal stress operator,

$$T_{xx} = S_{0y}\left(\frac{4}{3}\beta Z(\beta) + \frac{1}{3}\frac{\partial\beta}{\partial\log a}\right) + (S_{0x} + S_{xy}) \times\left(-\frac{2}{3}\beta Z(\beta) + \frac{1}{3}\frac{\partial\beta}{\partial\log a}\right), \tag{79}$$

and similarly for T_{yy}.

In summary, we have derived the lattice expressions for the renormalized diagonal components of the energy-momentum tensor. A simplification of the $d = 2$ case over the usual $d = 3$ case is that the one-loop quantum corrections to Z and $\frac{d\beta}{d\log a}$ amount to $O(a)$ effects, and the two-loop effects would amount to $O(a^2)$ corrections. The latter are parametrically of the same order as the usual $O(a^2)$ cutoff effects that are expected to occur in lattice gauge theory. For that reason, a one-loop computation is sufficient to yield a fully renormalized energy-momentum tensor.

1. Application: Width of the confining string

In [6], the width of the string, stretched between two static charges separated by a distance r along the x direction, was extracted from the measurement of the $P_{0x} = -S_{0x} + \mathrm{cst}$ plaquette expectation value at the midpoint $x = r/2$ (we now specialize to the case of the Wilson action; the additive constant drops out when subtracting the vacuum expectation value of the plaquette). We now interpret this result in terms of the energy-momentum tensor derived above.

Working at tree level,

$$\begin{pmatrix} T_{00} \\ T_{xx} \\ T_{yy} \end{pmatrix} = \frac{\beta}{a^3}\begin{pmatrix} +1 & -1 & -1 \\ -1 & -1 & +1 \\ -1 & +1 & -1 \end{pmatrix}\begin{pmatrix} S_{xy} \\ S_{0x} \\ S_{0y} \end{pmatrix}. \tag{80}$$

Inverting the matrix, one finds that

$$S_{xy} = -\frac{a^3}{2\beta}(T_{xx} + T_{yy}), \tag{81}$$

$$S_{0x} = -\frac{a^3}{2\beta}(T_{00} + T_{xx}), \tag{82}$$

$$S_{0y} = -\frac{a^3}{2\beta}(T_{00} + T_{yy}). \tag{83}$$

Now we use the general sum rules

$$\frac{\langle\psi|\int d^d\boldsymbol{x} T_{00}(x)|\psi\rangle}{\langle\psi|\psi\rangle} = E, \tag{84}$$

$$\frac{\langle\psi|\int d^d\boldsymbol{x} T_{xx}(x)|\psi\rangle}{\langle\psi|\psi\rangle} = L_x\frac{\partial E}{\partial L_x}. \tag{85}$$

Here L_x represents an external parameter that E depends on. Thus, for a string of length r along the x direction,

$$-\frac{\langle\psi|\beta\sum_{\boldsymbol{x}} S_{xy}(x)|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{1}{2}ar\frac{\partial E}{\partial r}, \tag{86}$$

$$-\frac{\langle\psi|\beta\sum_{\boldsymbol{x}} S_{0x}(x)|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{1}{2}a\left(E + r\frac{\partial E}{\partial r}\right), \tag{87}$$

$$-\frac{\langle\psi|\beta\sum_{\boldsymbol{x}} S_{0y}(x)|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{1}{2}aE. \tag{88}$$

These can be viewed as the $d = 2$ version of the Michael sum rules [26]. For a long string, where $E \propto r$, we expect the various plaquettes (summed over a time slice) to come in the fractions

$$\langle S_{xy}\rangle : \langle S_{0x}\rangle : \langle S_{0y}\rangle = \tfrac{1}{2} : 1 : \tfrac{1}{2}. \tag{89}$$

We finish with a numerical application of Eq. (87) based on the data of [6]. For the rest of this section, we set the lattice spacing to unity. The profile obtained in [6] from the S_{0x} operator is to a good approximation Gaussian, with

$$\int_{-\infty}^{\infty} dy A \exp\left(-\frac{1}{2}y^2/R^2\right) = \sqrt{2\pi}\cdot A\cdot R. \tag{90}$$

From Fig. (2) of [6], one reads off $R \approx \sqrt{12.1}$, and from Fig. 1, $A \approx 0.000\,38$. Thus, the left-hand side (LHS) of Eq. (87) roughly amounts to

$$\sqrt{2\pi}\beta A R r. \tag{91}$$

If we neglect the quark self-energies and the string corrections, $E \approx \sigma r$ and the right-hand side (RHS) amount to

$$\sigma r. \tag{92}$$

Numerically [6], after simplifying the common factor r, we have LHS ≈ 0.030 and RHS ≈ 0.026. Given the approximations we have made, in particular, the neglect of the quark self-energies and the use of the tree level renormalization factors for the plaquette, the agreement is satisfactory.

Based on the remarks of Sec. II, we expect all three plaquettes to yield the same string profile, to leading order in the string fluctuations: each of them contains a piece of either the energy density T_{00} or the longitudinal stress T_{xx}. This is indeed what the authors of [6] observed. We would however expect T_{yy}, whose expressions in terms of plaquettes can be read off Eq. (80), to yield a different profile.

B. The case $d = 3$

It was numerically observed a long time ago [27] in $d = 3$ dimensions and for the gauge group $SU(2)$ that the trace anomaly operator $T_{\mu\mu}$ yields a large string width that grows with r in the observable (37). The linear combination $3T_{00} - T_{11} - (T_{22} + T_{33})$ was found to yield a smaller value, and no clear evidence for a growing width could be seen. According to our conjecture for the coupling of the stress-energy tensor to the string, this operator should measure the same width at leading order for a long enough string, but in the range $r\sqrt{\sigma} < 2$ reached in the study the corrections could be significant.

A few years later, a new numerical study was carried out [28,29] in the same theory. The authors considered the operators $T_{\mu\mu}$ and $(T_{00} + T_{11} - (T_{22} + T_{33}))$, which yield similar profiles, as expected from the leading terms in Eq. (3) and (8). The operator that the authors call the "transverse energy" is proportional to $T_{00} - T_{11}$, and there is some evidence that the profile measured with this linear combination is indeed different, as we would expect based on the arguments of Sec. II. The reader is reminded that we are using Euclidean conventions here, see the comment at the end of Sec. II.

VI. CONCLUSION

In this paper we have analyzed the transverse structure of the confining string in non-Abelian gauge theories. We argued that the stress-energy tensor n-point functions in the presence of the confining string are mapped onto the corresponding n-point functions of the world sheet stress-energy tensor, appropriately shifted by a multiple of the unit operator. The latter term accounts for the stress-energy of the string at rest. We then derived the closed-string representation of the three-point function from which the string profile can be extracted. For this purpose we first enumerated the gravitational form factors that characterize the string profile. The functional form of the leading-order prediction for the string's square width is then found to be in agreement with the closed-string spectral representation. Most importantly, we showed that the square radius of the ground-state closed string, defined from the slope of its form factor, grows logarithmically with the length of the string, just as the square radius of the open string does. We also obtained a prediction for the transition form factor between the ground and the first excited state.

More generally, one can ask how a generic local gauge-invariant operator is represented in the effective string theory. We expect that it is mapped onto a linear combination of world sheet operators sharing its symmetries, with unknown low-energy coefficients. The operator T_{22}, for instance, is mapped onto $\delta(\boldsymbol{h} - \boldsymbol{y})(\alpha\partial_a\boldsymbol{h}\cdot\partial_a\boldsymbol{h} + \ldots)$, where α must be determined by a matching procedure. Some open questions remain, e.g. it is not quite clear yet what role the ambiguity in the form of the world sheet energy-momentum tensor (canonical vs improved) plays beyond the quadratic order.

Finally, we wish to comment on the prospects of fully characterizing the QCD string's structure. In the analysis of hadron structure, Generalized Parton Distributions have provided a powerful way to characterize the structure of a relativistic bound state such as the proton (see [30] for a review of the subject). Their moments in the longitudinal momentum fraction are given by the form factors of twist-two operators and are thus computable in the Euclidean theory [31]. These moments can be interpreted as Fourier transforms of the transverse distribution of partons [17]. The higher the dimension of the operator, the higher the longitudinal momentum fraction of the partons that it is measuring the transverse distribution of.

It is a fascinating program to think about an analogous comprehensive way of characterizing the structure of the confining string. Here there is no need to go to the infinite-momentum frame, which leads to kinematic simplifications in the proton case, because the string is parametrically heavy compared to its transverse width. This warrants the interpretation of form factors as the Fourier transforms of "parton" densities. By analogy with the analysis of proton structure [32], a rationale for which tower of operators to concentrate on may be provided by a "deeply virtual graviton scattering" gedanken experiment. Higher-dimensional operators will presumably correspond to probing the transverse distribution of "gravipartons" which carry a higher fraction of the string's energy. By the well-known arguments, we would expect to find smaller transverse radii for these operators.

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