Schwinger pair production in space- and time-dependent electric fields: Relating the Wigner formalism to quantum kinetic theory

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The nonperturbative electron-positron pair production (Schwinger effect) is considered for space- and time-dependent electric fields $\vec{E}(\vec{x}, t)$. Based on the Dirac-Heisenberg-Wigner formalism, we derive a system of partial differential equations of infinite order for the 16 irreducible components of the Wigner function. In the limit of spatially homogeneous fields the Vlasov equation of quantum kinetic theory is rediscovered. It is shown that the quantum kinetic formalism can be exactly solved in the case of a constant electric field $E(t) = E_0$ and the Sauter-type electric field $E(t) = E_0 \mathrm{sech}^2(t/\tau)$. These analytic solutions translate into corresponding expressions within the Dirac-Heisenberg-Wigner formalism and allow to discuss the effect of higher derivatives. We observe that spatial field variations typically exert a strong influence on the components of the Wigner function for large momenta or for late times.

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I. INTRODUCTION

Pair production in strong external electric fields is in many respects a paradigmatic phenomenon in quantum field theory [1–3]. It is nonperturbative in the coupling times the external field strength. It exemplifies the nontrivial properties of the quantum vacuum, as it manifests the instability of the vacuum against the formation of many-body states. In general, it depends strongly on the space-time structure of the external field, such that the pair-production process is expected to exhibit features of nonlocality, final state correlations and real-time dynamics. Moreover, it is a nonequilibrium process in quantum field theory and as such belonging perhaps to the least-well understood branch of modern field theory. Whereas pair proliferation is expected to occur at the critical Schwinger field strength $E_c = m^2/e \simeq 1.3 \times 10^{18}$ V/m, recent studies have suggested that pair production might become observable already at lower but dynamically modulated field strengths [4–10]. These estimates of the required field strengths indicate that pair production might already become accessible at future high-intensity laser systems such as the extreme light infrastructure (ELI) [11–13] or the European x-ray free electron laser (XFEL) [14,15].

Computing pair production in a complicated space- and time-dependent field such as a high-intensity pulse is by no means straightforward. Many different theoretical methods, such as the proper time method [3], WKB techniques [16–20], the Schrödinger-functional approach [21], functional techniques [22,23], quantum kinetic equations [24–29], being also closely related to scattering techniques [30], various instanton techniques [31–35], Borel summation [36], propagator constructions [37], and worldline numerics [38] have been developed to study pair production in external fields. Most of those approaches have only been applied to one-dimensional temporal or spatial inhomogeneities, see [39] for the only true multidimensional case. Also, finite-temperature contributions have been determined which under the assumption of local thermal equilibrium first occur at the two-loop level [40,41]. For thermal pair production from more general initial states, see [42–44].

For both, a profound understanding of the phenomenon as well as reliable quantitative predictions for realistic cases, a formalism that can deal with arbitrary space- and time-dependent fields is urgently required. This is also stressed by recent observations of characteristic and potentially easy to detect signatures of pair production in the momentum distribution of the pairs which has turned out surprisingly sensitive to the subcycle structure of high-intensity pulses [45,46], also exhibiting information about the quantum statistics of the particles involved [47,48]. Such a formalism based on suitable real-time correlation functions is indeed available and has already been studied in the context of pair production [49]. The present work is devoted to exploring this DHW formalism, putting it into the context also of other work such as quantum kinetic equations, and performing first systematic studies with the aid of both exactly soluble cases and within approximative schemes.

This paper is organized as follows: In Sec. II we briefly review the Dirac-Heisenberg-Wigner (DHW) formalism, adopting already a notation which will prove to be advantageous in the following. We describe how the Quantum Kinetic Theory (QKT) emerges as a specific limit of the DHW formalism and present some analytical solutions. In Sec. III we introduce a derivative expansion and discuss its region of validity. In Sec. IV we conclude and provide an outlook. Details about the QKT are summarized in Appendix A. The analytical results for the irreducible components of the Wigner function in the constant electric field and Sauter-type electric field are given in Appendix B.

II. THE EQUAL-TIME DHW FORMALISM

A classical statistical one-particle system is described by probability distributions $\mathcal{F}(\vec{x}, \vec{p}; t)$ in 6-dimensional phase space $\{\vec{x}, \vec{p}\}$. The generalization for a relativistic quantum field theory is obtained by choosing an appropriate density operator and performing a Wigner transformation to 8-dimensional phase space $\{x^\mu, p^\mu\}$. The corresponding Wigner operator $\hat{\mathcal{W}}(x, p)$ is manifestly Lorentz covariant but the associated Wigner function $\langle\Omega|\hat{\mathcal{W}}(x, p)|\Omega\rangle$ may not have a clear physical interpretation [50,51].

Alternatively, one may drop the manifest Lorentz covariance in favor of a canonical time evolution from the beginning and start with an equal-time density operator. The corresponding Wigner operator $\hat{\mathcal{W}}(\vec{x}, \vec{p}; t)$ is then defined in 6-dimensional phase space $\{\vec{x}, \vec{p}\}$. It is an advantage of this approach that the Wigner function $\langle\Omega|\hat{\mathcal{W}}(\vec{x}, \vec{p}; t)|\Omega\rangle$ might be interpreted as quasiprobability distribution in analogy to classical physics. It is an additional benefit that the equation of motion might be formulated as initial value problem [49,52]. Alternatively, one could also start with the Lorentz covariant formulation and switch to the equal-time formulation by performing an energy integral over p_0 [53–55].

We will adopt the equal-time formulation throughout this paper. Because of the fact that we have dropped manifest Lorentz covariance anyway, we will also fix the gauge from the beginning. We will choose the temporal gauge $A_0 = 0$ throughout, such that the electric and magnetic fields are calculable from the vector potential $\vec{A}(\vec{x}, t)$ according to

$$\vec{E}(\vec{x}, t) = -\partial_t \vec{A}(\vec{x}, t), \qquad \vec{B}(\vec{x}, t) = \vec{\nabla} \times \vec{A}(\vec{x}, t). \tag{1}$$

A. Derivation of the DHW formalism

In this section we define the equal-time Wigner operator $\hat{\mathcal{W}}(\vec{x}, \vec{p}; t)$ in the presence of an external electromagnetic field. By applying a Hartree approximation for the electromagnetic field, i.e. treating it as a C-number field instead of an operator-valued quantum field, we are able to derive the equation of motion for the corresponding Wigner function $\langle\Omega|\hat{\mathcal{W}}(\vec{x}, \vec{p}; t)|\Omega\rangle$.

For this, we consider the following equal-time density operator of two Dirac field operators in the Heisenberg picture,

$$\hat{\mathcal{C}}(\vec{x}_1, \vec{x}_2; t) \equiv e^{-ie\int_{\vec{x}_2}^{\vec{x}_1} \vec{A}(\vec{x}', t)\cdot d\vec{x}'} [\Psi(\vec{x}_1, t), \bar{\Psi}(\vec{x}_2, t)], \tag{2}$$

where we have dropped the Lorentz indices for simplicity. Here we choose the equal-time commutator, since the equal-time anticommutator is trivially fulfilled for spinor fields. Additionally, in order to preserve gauge invariance we include a Wilson line factor with an integral of the vector potential over a straight line. In fact, the choice of the integration path is not unique, but the present choice will allow for introducing a properly defined kinetic momentum variable $\vec{p}$. In terms of the center-of-mass coordinates $\vec{x} = \frac{1}{2}(\vec{x}_1 + \vec{x}_2)$ and $\vec{s} = \vec{x}_1 - \vec{x}_2$, it reads

$$\hat{\mathcal{C}}(\vec{x}; \vec{s}; t) = e^{-ie\int_{-1/2}^{1/2} \vec{A}(\vec{x}+\lambda\vec{s}, t)\cdot\vec{s}\, d\lambda} \times [\Psi(\vec{x} + \vec{s}/2, t), \bar{\Psi}(\vec{x} - \vec{s}/2, t)]. \tag{3}$$

The Wigner operator is then defined as the Fourier transform of $\hat{\mathcal{C}}(\vec{x}; \vec{s}; t)$ with respect to the relative coordinate $\vec{s}$, such that the arguments are the center-of-mass coordinate $\vec{x}$, the kinetic momentum variable $\vec{p}$ and time t:

$$\hat{\mathcal{W}}(\vec{x}, \vec{p}; t) \equiv -\frac{1}{2}\int d^3s\, e^{-i\vec{p}\cdot\vec{s}} \hat{\mathcal{C}}(\vec{x}; \vec{s}; t). \tag{4}$$

Note that if we had defined Eq. (2) with $\Psi^\dagger(\vec{x}_2, t)$ instead of $\bar{\Psi}(\vec{x}_2, t)$, the corresponding Wigner operator would have been Hermitian. With our definition, $\mathcal{W}(\vec{x}, \vec{p}; t)$ is not Hermitian but transforms like a Dirac matrix:

$$\hat{\mathcal{W}}^\dagger(\vec{x}, \vec{p}; t) = \gamma^0 \hat{\mathcal{W}}(\vec{x}, \vec{p}; t)\gamma^0. \tag{5}$$

In general, the Wigner function is then defined as the expectation value of the Wigner operator $\langle\Omega|\hat{\mathcal{W}}(\vec{x}, \vec{p}; t)|\Omega\rangle$ with respect to the full interacting vacuum. However, due to the fact that we are mainly interested in describing Schwinger pair production in the following, we restrict ourselves to the vacuum state in the Heisenberg picture $|\Omega\rangle = |0\rangle$:

$$\mathcal{W}(\vec{x}, \vec{p}; t) = -\frac{1}{2}\int d^3s\, e^{-i\vec{p}\cdot\vec{s}} \langle 0|\hat{\mathcal{C}}(\vec{x}; \vec{s}; t)|0\rangle. \tag{6}$$

In order to derive the equation of motion for the Wigner function, we take the time derivative of Eq. (6) and take the properly gauge fixed Dirac equation

$$(i\gamma^0\partial_t + i\vec{\gamma}\cdot[\vec{\nabla}_{\vec{x}} - ie\vec{A}(\vec{x}, t)] - m)\Psi(\vec{x}, t) = 0 \tag{7}$$

into account. In the course of the derivation we adopt a Hartree approximation of the electromagnetic field, which should be a good approximation for high field strengths. This means that we replace the operator-valued electromagnetic quantum field by a C-number electromagnetic field:

$$\langle 0|\hat{F}^{\mu\nu}(\vec{x}, t)\hat{\mathcal{C}}(\vec{x}; \vec{s}; t)|0\rangle \to F^{\mu\nu}(\vec{x}, t)\langle 0|\hat{\mathcal{C}}(\vec{x}; \vec{s}; t)|0\rangle. \tag{8}$$

Diagrammatically, this approximation corresponds to ignoring higher-loop radiative corrections. Physically, this implies that final state interactions as well as mass shift effects are ignored. This derivation finally yields the equation of motion for the Wigner function:

$$D_t\mathcal{W} = -\frac{1}{2}\vec{D}_{\vec{x}}[\gamma^0\vec{\gamma}, \mathcal{W}] - im[\gamma^0, \mathcal{W}] - i\vec{P}\{\gamma^0\vec{\gamma}, \mathcal{W}\}, \tag{9}$$

with D_t, $\vec{D}_{\vec{x}}$ and $\vec{P}$ denoting the following nonlocal pseudo-differential operators:

$$D_t = \partial_t + e\int_{-1/2}^{1/2} d\lambda \vec{E}(\vec{x} + i\lambda\vec{\nabla}_{\vec{p}}, t)\cdot\vec{\nabla}_{\vec{p}},$$
$$\vec{D}_{\vec{x}} = \vec{\nabla}_{\vec{x}} + e\int_{-1/2}^{1/2} d\lambda \vec{B}(\vec{x} + i\lambda\vec{\nabla}_{\vec{p}}, t)\times\vec{\nabla}_{\vec{p}}, \quad (10)$$
$$\vec{P} = \vec{p} - ie\int_{-1/2}^{1/2} d\lambda\lambda \vec{B}(\vec{x} + i\lambda\vec{\nabla}_{\vec{p}}, t)\times\vec{\nabla}_{\vec{p}}.$$

As the Wigner function $\mathcal{W}(\vec{x}, \vec{p}; t)$ is in fact a Dirac matrix, we may expand it in terms of irreducible components by choosing an appropriate complete basis set of 4×4 matrices $\{\mathbb{1}, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}$. Actually we choose 16 real functions (from now on called DHW functions) which transform under orthochronous Lorentz transformations as scalar $\mathbb{s}(\vec{x}, \vec{p}; t)$, pseudoscalar $\mathbb{p}(\vec{x}, \vec{p}; t)$, vector $\mathbb{v}_\mu(\vec{x}, \vec{p}; t)$, axialvector $\mathbb{a}_\mu(\vec{x}, \vec{p}; t)$ and tensor $\mathbb{t}_{\mu\nu}(\vec{x}, \vec{p}; t)$, respectively:

$$\mathcal{W}(\vec{x}, \vec{p}; t) = \frac{1}{4}[\mathbb{1}\mathbb{s} + i\gamma_5\mathbb{p} + \gamma^\mu\mathbb{v}_\mu + \gamma^\mu\gamma_5\mathbb{a}_\mu + \sigma^{\mu\nu}\mathbb{t}_{\mu\nu}]. \quad (11)$$

Inserting this decomposition into the equation of motion, Eq. (9), and comparing the coefficients of the basis matrices, we find a partial differential equation (PDE) system for the 16 DHW functions. Introducing the compact notation for the tensorial components,

$$(\vec{\mathbb{t}}_1)_i = \mathbb{t}_{0i} - \mathbb{t}_{i0}, \qquad (\vec{\mathbb{t}}_2)_i = \epsilon_{ijk}\mathbb{t}_{jk}, \quad (12)$$

this system reads

$$D_t\mathbb{s} - 2\vec{P}\cdot\vec{\mathbb{t}}_1 = 0 \quad (13)$$

$$D_t\mathbb{p} + 2\vec{P}\cdot\vec{\mathbb{t}}_2 = 2m\mathbb{a}_0 \quad (14)$$

$$D_t\mathbb{v}_0 + \vec{D}_{\vec{x}}\cdot\vec{\mathbb{v}} = 0 \quad (15)$$

$$D_t\mathbb{a}_0 + \vec{D}_{\vec{x}}\cdot\vec{\mathbb{a}} = 2m\mathbb{p} \quad (16)$$

$$D_t\vec{\mathbb{v}} + \vec{D}_{\vec{x}}\mathbb{v}_0 + 2\vec{P}\times\vec{a} = -2m\vec{\mathbb{t}}_1 \quad (17)$$

$$D_t\vec{\mathbb{a}} + \vec{D}_{\vec{x}}\mathbb{a}_0 + 2\vec{P}\times\vec{v} = 0 \quad (18)$$

$$D_t\vec{\mathbb{t}}_1 + \vec{D}_{\vec{x}}\times\vec{\mathbb{t}}_2 + 2\vec{P}s = 2m\vec{\mathbb{v}} \quad (19)$$

$$D_t\vec{\mathbb{t}}_2 - \vec{D}_{\vec{x}}\times\vec{\mathbb{t}}_1 - 2\vec{P}\mathbb{p} = 0. \quad (20)$$

Note that for spatially homogeneous electromagnetic fields $F^{\mu\nu}(\vec{x}, t) = F^{\mu\nu}(t)$, an enormous simplification occurs as the nonlocal operators Eq. (10) reduce to local ones:

$$D_t = \partial_t + e\vec{E}(t)\cdot\vec{\nabla}_{\vec{p}}, \qquad \vec{D}_{\vec{x}} = \vec{\nabla}_{\vec{x}} + e\vec{B}(t)\times\vec{\nabla}_{\vec{p}}, \qquad \vec{P} = \vec{p}. \quad (21)$$

It has been shown previously [49], that some of the DHW functions can be given an intuitive interpretation, whereas others do not have a classical analogue. First, the symmetrized electromagnetic current $j^\mu(\vec{x}, t) = \frac{e}{2}\times\langle 0[\bar{\Psi}(\vec{x}, t), \gamma^\mu\Psi(\vec{x}, t)]0\rangle$ is expressed as

$$j^\mu(\vec{x}, t) = e\int\frac{d^3p}{(2\pi)^3}\mathbb{v}^\mu(\vec{x}, \vec{p}; t). \quad (22)$$

Additionally, several conservation laws concerning physically observable quantities like the total charge $\mathcal{Q}$, the total energy $\mathcal{E}$, the total linear momentum $\vec{\mathcal{P}}$ and the total angular momentum $\vec{\mathcal{M}}$ are valid,

$$\frac{d}{dt}\{\mathcal{Q}; \mathcal{E}; \vec{\mathcal{P}}; \vec{\mathcal{M}}\} = 0, \quad (23)$$

with

$$\mathcal{Q} = e\int d\Gamma\mathbb{v}_0(\vec{x}, \vec{p}; t), \quad (24)$$

$$\mathcal{E} = \int d\Gamma[\vec{p}\cdot\vec{\mathbb{v}}(\vec{x}, \vec{p}; t) + m\mathbb{s}(\vec{x}, \vec{p}; t)] + \frac{1}{2}\int d^3x[|\vec{E}(\vec{x}, t)|^2 + |\vec{B}(\vec{x}, t)|^2], \quad (25)$$

$$\vec{\mathcal{P}} = \int d\Gamma\vec{p}\mathbb{v}_0(\vec{x}, \vec{p}; t) + \int d^3x\vec{E}(\vec{x}, t)\times\vec{B}(\vec{x}, t), \quad (26)$$

$$\vec{\mathcal{M}} = \int d\Gamma\left[\vec{x}\times\vec{p}\mathbb{v}_0(\vec{x}, \vec{p}; t) - \frac{1}{2}\vec{\mathbb{a}}(\vec{x}, \vec{p}; t)\right] + \int d^3x\vec{x}\times\vec{E}(\vec{x}, t)\times\vec{B}(\vec{x}, t), \quad (27)$$

with $d\Gamma = d^3xd^3p/(2\pi)^3$ denoting the phase-space volume element. According to these expressions, we may associate $\mathbb{s}(\vec{x}, \vec{p}; t)$ with a mass density, $\mathbb{v}_0(\vec{x}, \vec{p}; t)$ with a charge density and $\vec{\mathbb{v}}(\vec{x}, \vec{p}; t)$ with a current density and $\vec{\mathbb{a}}(\vec{x}, \vec{p}; t)$ with a spin density. Another important conservation law concerns the norm of the Wigner function itself:

$$\frac{d}{dt}\int d\Gamma\,\mathrm{Tr}[\mathcal{W}(\vec{x}, \vec{p}; t)\mathcal{W}^\dagger(\vec{x}, \vec{p}; t)] = 0, \quad (28)$$

which translates into a conservation law for the 16 DHW functions.

B. Quantum kinetic theory (QKT) as limit of the DHW formalism

In this subsection we show that the DHW formalism in the case of a spatially homogeneous, time-dependent electric field $\vec{E}(\vec{x}, t) = E(t)\vec{e}_3$ and vanishing magnetic field $\vec{B}(\vec{x}, t) = 0$ yields the well-known Vlasov equation of QKT for Schwinger pair production [24,25,27]. For this, we first calculate the Wigner function for pure vacuum to obtain appropriate initial conditions. In a second step, we simplify the PDE system (13)–(20) to an ODE system [49], which turns out to be equivalent to the Vlasov equation [56]. For an analysis of the relation between the Wigner function and QKT for several examples of pair production in non-Abelian fields, see [57].

In order to calculate the Wigner function for pure vacuum $\mathcal{W}_{\rm vac}(\vec{x}, \vec{p}; t)$, we consider first the general expression Eq. (6) for vanishing vector potential: $A(\vec{x}, t) = 0$. We first decompose the Dirac field operator in its Fourier basis

$$\Psi(\vec{x}, t) = \int \frac{d^3q}{(2\pi)^3} \tilde{\psi}(\vec{q}, t) e^{i\vec{q}\cdot\vec{x}}, \tag{29}$$

and introduce a decomposition in terms of anticommuting creation/annihilation operators as well as four spinors

$$\tilde{\psi}(\vec{q}, t) = \sum_s \tilde{u}_s(\vec{q}, t) a_s(\vec{q}) + \tilde{v}_s(-\vec{q}, t) b_s^\dagger(-\vec{q}). \tag{30}$$

Evaluating the vacuum expectation value and taking advantage of the four-spinor completeness relations, we finally obtain for the vacuum Wigner function

$$\mathcal{W}_{\rm vac}(\vec{x}, \vec{p}; t) = -\frac{1}{2\omega(\vec{p})}[\mathbb{1}m - \vec{\gamma}\cdot\vec{p}], \tag{31}$$

with $\omega(\vec{p}) = \sqrt{m^2 + \vec{p}^2}$. Comparing this expression with Eq. (11), we immediately see that (a) in the pure vacuum only 4 DHW functions do not vanish and (b) these vacuum functions do not depend on $\vec{x}$ and t:

$$\mathbb{s}_{\rm vac}(\vec{p}) = -\frac{2m}{\omega(\vec{p})}, \tag{32}$$

$$\vec{\mathbb{v}}_{\rm vac}(\vec{p}) = -\frac{2\vec{p}}{\omega(\vec{p})}. \tag{33}$$

After fixing the vacuum initial conditions, we consider next the PDE system Eqs. (13)–(20) for $\vec{E}(\vec{x}, t) = E(t)\vec{e}_3$ and $\vec{B}(\vec{x}, t) = 0$ in more detail: Because of spatial homogeneity, the DHW functions do not depend on the variable $\vec{x}$ and hence all spatial derivatives vanish. As an immediate consequence, $\mathbb{v}_0(\vec{p}; t)$ decouples completely. Additionally, due to the fact that the DHW functions $\{\mathbb{p}, \mathbb{a}_0, \vec{\mathbb{t}}_2\}(\vec{p}; t)$ are subject to a closed set of equations which does not couple to the nonvanishing vacuum initial conditions, these functions have to vanish as well. As a consequence, the PDE system for former 16 DHW functions reduces to a PDE system for the remaining 10 DHW functions $\vec{\mathbb{w}}(\vec{p}; t) \equiv (\mathbb{s}, \vec{\mathbb{v}}, \vec{\mathbb{a}}, \vec{\mathbb{t}}_1)(\vec{p}; t)$:

$$[\partial_t + eE(t)\partial_{p_3}]\vec{\mathbb{w}}(\vec{p}; t) = \mathcal{M}(\vec{p})\vec{\mathbb{w}}(\vec{p}; t). \tag{34}$$

Here, $\vec{\mathbb{w}}(\vec{p}; t)$ is a column vector and $\mathcal{M}(\vec{p})$ is the following 10×10 matrix:

$$\mathcal{M}(\vec{p}) = \begin{pmatrix} 0 & 0 & 0 & 2\vec{p}^{\rm T} \\ 0 & 0 & -2\vec{p}^{\times} & -2m \\ 0 & -2\vec{p}^{\times} & 0 & 0 \\ -2\vec{p} & 2m & 0 & 0 \end{pmatrix}, \tag{35}$$

with

$$\vec{p}^{\times} = \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix}. \tag{36}$$

The PDE system Eq. (34) will be simplified by applying the method of characteristics. We introduce a new parameter α and assume that the originally independent variables depend on this new parameter:

$$\vec{p} = \vec{\pi}(\alpha) \quad \text{and} \quad t = \tau(\alpha). \tag{37}$$

Imposing the following equality for any function $\mathcal{F}(\vec{p}; t)$ depending on the former independent variables $\vec{p}$ and t,

$$\left[\frac{\partial}{\partial t} + eE(t)\frac{\partial}{\partial p_3}\right]\mathcal{F}(\vec{p}; t) \stackrel{!}{=} \frac{d}{d\alpha}\mathcal{F}(\vec{\pi}(\alpha), \tau(\alpha)), \tag{38}$$

we find $\alpha = \tau = t$ and $\vec{\pi}(\vec{q}, t) = \vec{q} - eA(t)\vec{e}_3$. Note that $\vec{\pi}(\vec{q}, t)$ denotes the time-dependent kinetic momentum on a trajectory, whereas $\vec{q}$, which serves as an integration constant in the method of characteristics, corresponds to the canonical momentum. Additionally, we still have the notion of a phase-space kinetic momentum $\vec{p}$. These three types of momenta have to be clearly distinguished in the following. To be consistent throughout this paper, we always denote

$\vec{p}$ kinetic momentum in phase space,

$\vec{q}$ canonical momentum,

$\vec{\pi}(\vec{q}, t)$ kinetic momentum on a trajectory.

On the one hand, any function defined in phase space possesses only an explicit time dependence and will henceforth be denoted by $\mathcal{F}(\vec{p}; t)$. On the other hand, functions depending on the time-dependent kinetic momentum $\vec{\pi}(\vec{q}, t)$ show both an explicit and an implicit time dependence and will be denoted by $\tilde{\mathcal{F}}(\vec{q}, t)$.

Formally, the method of characteristics is applied to the PDE system Eq. (34) by replacing $\vec{p}$ by $\vec{\pi}(\vec{q}, t)$, such that the relation between the phase-space DHW functions and the DHW functions on a trajectory reads

$$\tilde{\vec{\mathbb{w}}}(\vec{q}, t) = \vec{\mathbb{w}}(\vec{p}; t)|_{\vec{p}\to\vec{q}-e\vec{A}(t)} \tag{39}$$

$$\vec{\mathbb{w}}(\vec{p}; t) = \tilde{\vec{\mathbb{w}}}(\vec{q}, t)|_{\vec{q}\to\vec{p}+e\vec{A}(t)}. \tag{40}$$

Consequently, the PDE system Eq. (34) becomes an ODE system, with the former time-independent matrix $\mathcal{M}(\vec{p})$ becoming a time-dependent quantity $\tilde{\mathcal{M}}(\vec{q}, t)$:

$$\frac{d}{dt}\tilde{\vec{\mathbb{w}}}(\vec{q}, t) = \tilde{\mathcal{M}}(\vec{q}, t)\tilde{\vec{\mathbb{w}}}(\vec{q}, t). \tag{41}$$

In order to proceed, we seek an appropriate basis to span $\tilde{\vec{\mathbb{w}}}(\vec{q}, t)$, such that Eq. (41) reduces to a simple form:

$$\vec{\mathbb{w}}(\vec{q}, t) = -2\sum_{i=1}^{10} \tilde{\chi}^i(\vec{q}, t)\vec{\tilde{\mathbb{e}}}_i(\vec{q}, t), \tag{42}$$

with the factor -2 chosen for later convenience. To this end, we exploit the vacuum initial conditions Eq. (32) and (33) and choose the first basis vector $\vec{\tilde{\mathbb{e}}}_1(\vec{q}, t)$ such that in pure vacuum the first coefficient $\tilde{\chi}^1_{\rm vac}(\vec{q}, t_{\rm vac}) = 1$, whereas all other coefficients $\tilde{\chi}^i_{\rm vac}(\vec{q}, t_{\rm vac})$ vanish. Consequently, we find a subset of basis vectors,

$$\vec{\tilde{\mathbb{e}}}_1(\vec{q}, t) = \frac{1}{\tilde{\omega}(\vec{q}, t)}\begin{pmatrix} m \\ \vec{\pi}(\vec{q}, t) \\ \vec{0} \\ \vec{0}\end{pmatrix},$$

$$\vec{\tilde{\mathbb{e}}}_2(\vec{q}, t) = \frac{1}{\epsilon_\perp \tilde{\omega}(\vec{q}, t)}\begin{pmatrix} m\pi_3(q_3, t) \\ \vec{\pi}(\vec{q}, t)\pi_3(q_3, t) - \tilde{\omega}^2(\vec{q}, t)\vec{e}_3 \\ \vec{0} \\ \vec{0}\end{pmatrix},$$

$$\vec{\tilde{\mathbb{e}}}_3(\vec{q}, t) = \frac{1}{\epsilon_\perp}\begin{pmatrix} 0 \\ \vec{0} \\ \vec{\pi}(\vec{q}, t)\times\vec{e}_3 \\ -m\vec{e}_3\end{pmatrix}, \tag{43}$$

with $\epsilon_\perp = \sqrt{m^2 + \vec{q}_\perp^2}$ and $\tilde{\omega}(\vec{q}, t) = \sqrt{\epsilon_\perp^2 + \pi_3^2(q_3, t)}$, which form an orthonormalized, complete set:

$$\tilde{M}(\vec{q}, t)\begin{Bmatrix} \vec{\tilde{\mathbb{e}}}_1 \\ \vec{\tilde{\mathbb{e}}}_2 \\ \vec{\tilde{\mathbb{e}}}_3 \end{Bmatrix}(\vec{q}, t) = 2\tilde{\omega}(\vec{q}, t)\begin{Bmatrix} \vec{0} \\ \vec{\tilde{\mathbb{e}}}_3 \\ -\vec{\tilde{\mathbb{e}}}_2 \end{Bmatrix}(\vec{q}, t), \tag{44}$$

$$\frac{d}{dt}\begin{Bmatrix} \vec{\tilde{\mathbb{e}}}_1 \\ \vec{\tilde{\mathbb{e}}}_2 \\ \vec{\tilde{\mathbb{e}}}_3 \end{Bmatrix}(\vec{q}, t) = -\frac{eE(t)\epsilon_\perp}{\tilde{\omega}^2(\vec{q}, t)}\begin{Bmatrix} \vec{\tilde{\mathbb{e}}}_2 \\ -\vec{\tilde{\mathbb{e}}}_1 \\ \vec{0} \end{Bmatrix}(\vec{q}, t). \tag{45}$$

As a consequence, only the coefficients $\tilde{\chi}^{i=\{1,2,3\}}(\vec{q}, t)$ couple to the initial vacuum state whereas all other coefficients $\tilde{\chi}^i(\vec{q}, t)$ vanish. This means that $\vec{\mathbb{w}}(\vec{q}, t)$ is fully characterized by

$$\vec{\mathbb{w}}(\vec{q}, t) = -2\sum_{i=1}^{3} \tilde{\chi}^i(\vec{q}, t)\vec{\tilde{\mathbb{e}}}_i(\vec{q}, t). \tag{46}$$

Next we introduce $\tilde{f}(\vec{q}, t) = 1 - \tilde{\chi}^1(\vec{q}, t)$ parametrizing the deviation from the vacuum state, such that in pure vacuum $\tilde{f}_{\rm vac}(\vec{q}, t_{\rm vac}) = 0$. Additionally, we define

$$\tilde{Q}(\vec{q}, t) = \frac{eE(t)\epsilon_\perp}{\tilde{\omega}^2(\vec{q}, t)}. \tag{47}$$

If we consider the ODE system Eq. (41) together with the relations Eqs. (44) and (45), we obtain

$$\frac{d}{dt}\tilde{f}(\vec{q}, t) = \tilde{Q}(\vec{q}, t)\tilde{\chi}^2(\vec{q}, t), \tag{48}$$

$$\frac{d}{dt}\tilde{\chi}^2(\vec{q}, t) = \tilde{Q}(\vec{q}, t)[1 - \tilde{f}(\vec{q}, t)] - 2\tilde{\omega}(\vec{q}, t)\tilde{\chi}^3(\vec{q}, t), \tag{49}$$

$$\frac{d}{dt}\tilde{\chi}^3(\vec{q}, t) = 2\tilde{\omega}(\vec{q}, t)\tilde{\chi}^2(\vec{q}, t), \tag{50}$$

together with vacuum initial conditions $\tilde{f}_{\rm vac}(\vec{q}, t_{\rm vac}) = \tilde{\chi}^2_{\rm vac}(\vec{q}, t_{\rm vac}) = \tilde{\chi}^3_{\rm vac}(\vec{q}, t_{\rm vac}) = 0$. This ODE system is nothing but the well-known Vlasov equation of QKT in its differential form [56] (cf. also Appendix A). Note that $\tilde{f}(\vec{q}, t)$ denotes the single-particle momentum distribution function in quantum kinetic theory. Thus, the DHW formalism in the presence of a spatially homogeneous, time-dependent electric field $\vec{E}(\vec{x}, t) = E(t)\vec{e}_3$ is completely equivalent to QKT. However, the DHW formalism is the more general approach since it allows for any time- and space-dependent electromagnetic field whereas the Vlasov equation is restricted to spatially homogeneous, time-dependent electric fields.

For some special cases, an exact solution of QKT can be found (see Sec. II C), such that we are able to calculate the DHW functions as well. To this end one uses the representation Eq. (46) and projects back to phase-space Eq. (40). For Schwinger pair production in spatially homogeneous, time-dependent electric fields; for instance, one finds that only 7 of the possible 16 DHW functions contribute:

$$\mathbb{s}(\vec{p}; t) = -\frac{2m}{\omega(\vec{p})}\chi^1(\vec{p}; t) - \frac{2mp_3}{\epsilon_\perp\omega(\vec{p})}\chi^2(\vec{p}; t), \tag{51}$$

$$\vec{\mathbb{v}}_\perp(\vec{p}; t) = -\frac{2\vec{p}_\perp}{\omega(\vec{p})}\chi^1(\vec{p}; t) - \frac{2\vec{p}_\perp p_3}{\epsilon_\perp\omega(\vec{p})}\chi^2(\vec{p}; t), \tag{52}$$

$$\mathbb{v}_3(\vec{p}; t) = -\frac{2p_3}{\omega(\vec{p})}\chi^1(\vec{p}; t) + \frac{2\epsilon_\perp}{\omega(\vec{p})}\chi^2(\vec{p}; t), \tag{53}$$

$$\mathbb{a}_1(\vec{p}; t) = -\frac{2p_2}{\epsilon_\perp}\chi^3(\vec{p}; t), \tag{54}$$

$$\mathbb{a}_2(\vec{p}; t) = \frac{2p_1}{\epsilon_\perp}\chi^3(\vec{p}; t), \tag{55}$$

$$\mathbb{t}_{1,3}(\vec{p}; t) = \frac{2m}{\epsilon_\perp}\chi^3(\vec{p}; t). \tag{56}$$

C. Exactly solvable electric fields

In this subsection we derive the analytic expressions for the single-particle momentum distribution function $\tilde{f}(\vec{q}, t)$ of QKT for both the constant electric field $E(t) = E_0$ and the Sauter-type electric field $E(t) = E_0{\rm sech}^2(t/\tau)$. The construction of the solution can be oriented along the lines of QKT, cf. Appendix A: We first seek an analytic solution for $\tilde{g}^{(+)}(\vec{q}, t)$ of Eq. (A6) and determine its normalization such that it coincides at $t_{\rm vac} \to -\infty$ with $\tilde{G}^{(+)}(\vec{q}, t)$ defined in Eq. (A17). According to Eq. (A22), we are then able to calculate the Bogoliubov coefficient $\tilde{\beta}(\vec{q}, t)$ and, consequently, the single-particle momentum distribution function $\tilde{f}(\vec{q}, t)$ according to Eq. (A20).

As soon as this solution is known, we are able to calculate the nonvanishing coefficients $\tilde{\chi}^{i=\{2,3\}}(\vec{q}, t)$ according to Eqs. (48)–(50). As an immediate consequence of Eqs. (51)–(56), all the nonvanishing DHW functions can be calculated as well.

1. Constant electric field

A constant electric field $E(t) = E_0$ might be represented by the vector potential

$$A(t) = -E_0 t, \tag{57}$$

such that Eq. (A6) reads

$$(\partial_t^2 + \epsilon_\perp^2 + (q_3 + eE_0 t)^2 + ieE_0)\tilde{g}(q_3, t) = 0. \tag{58}$$

Note that for notational simplicity we do not explicitly indicate the dependence on the orthogonal canonical momentum via $\epsilon_\perp^2 = m^2 + \vec{q}_\perp^2$. Dynamically, q_3 is the only relevant parameter such that the situation becomes effectively one-dimensional. Introducing the dimensionless parameter $\eta = \epsilon_\perp^2/eE_0$ and performing the variable transformation

$$q_3 + eE_0 t = \sqrt{\tfrac{eE_0}{2}}u, \tag{59}$$

we see that $\tilde{g}(u)$ will only depend on one dimensionless variable u due to the linear relation between q_3 and t. As a consequence, the differential equation Eq. (58) turns into the parabolic cylinder differential equation ([58], Chapter 19):

$$\left[\partial_u^2 + \frac{1}{4}u^2 + \frac{1}{2}(i + \eta)\right]\tilde{g}(u) = 0. \tag{60}$$

This second-order differential equation has two standard solutions which are given by

$$\tilde{g}^{(+)}(u) = N^{(+)} D_{-1+i\eta/2}(-ue^{-i(\pi/4)}), \tag{61}$$

$$\tilde{g}^{(-)}(u) = N^{(-)} D_{-i\eta/2}(-ue^{i(\pi/4)}) \tag{62}$$

with $D_\nu(z)$ being the parabolic cylinder function and $N^{(\pm)}$ being normalization factors. For $u \to -\infty$, these solutions behave asymptotically as

$$\tilde{g}^{(+)}(u) \overset{u\to-\infty}{\longrightarrow} \frac{1}{|u|} e^{i[(u^2/4)+(\eta/2)\log(|u|)+(\pi/4)]} e^{\pi\eta/8}, \tag{63}$$

$$\tilde{g}^{(-)}(u) \overset{u\to-\infty}{\longrightarrow} e^{-i[(u^2/4)+(\eta/2)\log(|u|)]} e^{\pi\eta/8}. \tag{64}$$

On the other hand, the adiabatic mode functions are given by, cf. Eq. (A17),

$$\tilde{G}^{(\pm)}(u) = \frac{e^{\mp i\Theta(u_0,u)}}{\sqrt{m^2\epsilon\sqrt{2\eta + u^2}(\sqrt{2\eta + u^2} \mp u)}}, \tag{65}$$

with

$$\epsilon = E_0/E_c. \tag{66}$$

The dynamical phase $\Theta(u_0, u)$ can be explicitly calculated as soon as we fix u_0. Here, we choose the symmetric point $u_0 = 0$, such that the definite integral,

$$\Theta(0, u) = \frac{1}{2}\int_0^u du' \sqrt{2\eta + u'^2}, \tag{67}$$

yields

$$\frac{1}{4}[u\sqrt{2\eta + u^2} + 2\eta\log(u + \sqrt{2\eta + u^2}) - \eta\log 2\eta]. \tag{68}$$

Consequently, we fix the normalization constants $N^{(\pm)}$ such that $\tilde{g}^{(\pm)}(u \to -\infty) = \tilde{G}^{(\pm)}(u \to -\infty)$:

$$N^{(+)} = \frac{1}{\sqrt{2m^2\epsilon}} e^{i[(\eta/4)[1+\log(2/\eta)]-(\pi/4)]} e^{-(\pi\eta/8)}, \tag{69}$$

$$N^{(-)} = \frac{1}{\sqrt{m^2\epsilon\eta}} e^{-i(\eta/4)[1+\log(2/\eta)]} e^{-(\pi\eta/8)}. \tag{70}$$

According to Eqs. (A20) and (A22), we are then able to calculate the single-particle momentum distribution function. Taking into account the general relation for parabolic cylinder functions,

$$\partial_z D_\nu(z) = \frac{1}{2}zD_\nu(z) - D_{1+\nu}(z), \tag{71}$$

we finally obtain

$$\begin{aligned}\tilde{f}(u) = \frac{1}{4}\left(1 + \frac{u}{\sqrt{2\eta + u^2}}\right)e^{-(\pi\eta/4)}|(\sqrt{2\eta + u^2} - u)\\ \times D_{-1+i\eta/2}(-ue^{-i(\pi/4)})\\ - 2e^{i(\pi/4)}D_{i\eta/2}(-ue^{-i(\pi/4)})|^2.\end{aligned} \tag{72}$$

Finally, we may show that we obtain the Schwinger result for the pair production rate in a constant electric field if we consider the limit $u \to \infty$. To this end, we take the leading term in the asymptotic expansion of the parabolic cylinder functions in Eq. (72). Neglecting terms of the order $\mathcal{O}(u^{-1})$, they are given by

$$\begin{aligned}D_{-1-i\eta/2}(-u^{i(\pi/4)})\\ \overset{u\to\infty}{\longrightarrow} \frac{\sqrt{2\pi}}{\Gamma(1 + i\eta/2)} e^{i[(u^2/4)+(\eta/2)\log(u)]} e^{-(\pi\eta/8)},\end{aligned} \tag{73}$$

$$D_{-i\eta/2}(-u^{i(\pi/4)}) \overset{u\to\infty}{\longrightarrow} e^{-i[(u^2/4)+(\eta/2)\log(u)]} e^{-(3\pi\eta/8)}, \quad (74)$$

such that the asymptotic behavior of $\tilde{f}(u)$ is given by

$$\lim_{u\to\infty} \tilde{f}(u) = 2e^{-\pi\eta}. \quad (75)$$

As a consequence, the Schwinger pair-production rate per volume and time $\dot{n}[e^+e^-]$, i.e. the first term in the Schwinger expression for the vacuum decay probability [59,60], is found

$$\dot{n}[e^+e^-] = \int \frac{d^3q}{(2\pi)^3} \partial_t \tilde{f}(\vec{q}, t) = \frac{e^2E_0^2}{4\pi^3} e^{-(m^2\pi/eE_0)}, \quad (76)$$

which completes our analytical solution for the constant electric field.

2. Sauter-type electric field

The Sauter-type electric field $E(t) = E_0 \mathrm{sech}^2(t/\tau)$ might be represented by the vector potential,

$$A(t) = -E_0\tau \tanh(\tfrac{t}{\tau}), \quad (77)$$

such that Eq. (A6) reads

$$[\partial_t^2 + \epsilon_\perp^2 + (q_3 + eE_0\tau\tanh(\tfrac{t}{\tau}))^2 + ieE_0\mathrm{sech}^2(\tfrac{t}{\tau})]\tilde{g}(q_3, t) = 0. \quad (78)$$

Again, we only indicate the dependence on q_3 whereas the dependence on the orthogonal canonical momentum will not be denoted explicitly. In the following, we introduce the dimensionless variable,

$$u = \tfrac{1}{2}[1 + \tanh(\tfrac{t}{\tau})], \quad (79)$$

such that $t \to -\infty$ corresponds to $u \to 0$ whereas $t \to \infty$ corresponds to $u \to 1$. Additionally, we introduce the Keldysh parameter $\gamma = 1/(m\epsilon\tau)$ and dimensionless momentum variables $\hat{q}_3 = q_3/m$ and $\hat{\kappa}^2 = \vec{q}_\perp^2/m^2$, such that the dimensionless kinetic momentum on the trajectory $\hat{\pi}_3(\hat{q}_3, u)$ and the dimensionless energy variable $\hat{\omega}(\hat{q}_3, u)$ read

$$\hat{\pi}_3(\hat{q}_3, u) = \frac{q_3 + eE_0\tau(2u-1)}{m} = \hat{q}_3 + \frac{2u-1}{\gamma}, \quad (80)$$

$$\hat{\omega}^2(\hat{q}_3, u) = 1 + \hat{\kappa}^2 + \hat{\pi}_3^2(\hat{q}_3, u). \quad (81)$$

Within these new variables, the differential equation Eq. (78) reads

$$[4\gamma^2\epsilon^2 u(1-u)\partial_u\{u(1-u)\}\partial_u + \hat{\omega}^2(\hat{q}_3, u) + 4i\epsilon u(1-u)]\tilde{g}(\hat{q}_3, u) = 0. \quad (82)$$

In order to solve this differential equation, we apply an ansatz for $\tilde{g}(\hat{q}_3, u)$:

$$\tilde{g}(\hat{q}_3, u) = u^{-i(\hat{\omega}(\hat{q}_3,0)/2\gamma\epsilon)}(1-u)^{i(\hat{\omega}(\hat{q}_3,0)/2\gamma\epsilon)}\tilde{h}(\hat{q}_3, u). \quad (83)$$

Plugging this ansatz into Eq. (82) yields the hypergeometric differential equation ([58], Chapter 15) for $\tilde{h}(\hat{q}_3, u)$:

$$[u(1-u)\partial_u^2 + (\tilde{c} - [\tilde{a} + \tilde{b} + 1]u)\partial_u - \tilde{a}\tilde{b}]\tilde{h}(\hat{q}_3, u) = 0, \quad (84)$$

with

$$\tilde{a}(\hat{q}_3) = -\frac{i}{\gamma\epsilon}\Big(\frac{1}{\gamma} + \frac{\hat{\omega}(\hat{q}_3, 0)}{2} - \frac{\hat{\omega}(\hat{q}_3, 1)}{2}\Big),$$
$$\tilde{b}(\hat{q}_3) = 1 + \frac{i}{\gamma\epsilon}\Big(\frac{1}{\gamma} - \frac{\hat{\omega}(\hat{q}_3, 0)}{2} + \frac{\hat{\omega}(\hat{q}_3, 1)}{2}\Big), \quad (85)$$
$$\tilde{c}(\hat{q}_3) = 1 - \frac{i}{\gamma\epsilon}\hat{\omega}(\hat{q}_3, 0).$$

Note that these parameters do not depend on u. The two linearly independent solutions $\tilde{h}^{(\pm)}(\hat{q}_3, u)$ in the neighborhood of the singular point $u = 0$ are given by

$$\tilde{h}^{(+)}(\hat{q}_3, u) = N^{(+)}F(\tilde{a}, \tilde{b}, \tilde{c}; u), \quad (86)$$

$$\tilde{h}^{(-)}(\hat{q}_3, u) = N^{(-)}u^{i(\hat{\omega}(\hat{q}_3,0)/\gamma\epsilon)}(1-u)^{-i(\hat{\omega}(\hat{q}_3,1)/\gamma\epsilon)} \times F(1-\tilde{a}, 1-\tilde{b}, 2-\tilde{c}; u), \quad (87)$$

with $F(\tilde{a}, \tilde{b}, \tilde{c}; u)$ denoting the Gauss hypergeometric function. Taking Eq. (83) into account, the asymptotic behavior of $\tilde{g}^{(\pm)}(\hat{q}_3, u)$ for $u \to 0$ is

$$\tilde{g}^{(+)}(\hat{q}_3, u) \overset{u\to 0^+}{\longrightarrow} e^{-i(\hat{\omega}(\hat{q}_3,0)/2\gamma\epsilon)\log(u)}, \quad (88)$$

$$\tilde{g}^{(-)}(\hat{q}_3, u) \overset{u\to 0^+}{\longrightarrow} e^{i(\hat{\omega}(\hat{q}_3,0)/2\gamma\epsilon)\log(u)}. \quad (89)$$

On the other hand, we are again able to give an analytic expression for the adiabatic mode functions:

$$\tilde{G}^{(\pm)}(\hat{q}_3, u) = \frac{e^{\mp i\Theta(\hat{q}_3, u_0, u)}}{\sqrt{2m^2\hat{\omega}(\hat{q}_3, u)[\hat{\omega}(\hat{q}_3, u) \mp \hat{\pi}_3(\hat{q}_3, u)]}}. \quad (90)$$

Again, the dynamical phase $\Theta(\hat{q}_3, u_0, u)$, which is given by the following integral

$$\Theta(\hat{q}_3, u_0, u) = \frac{1}{2\gamma\epsilon}\int_{u_0}^{u} du' \frac{\sqrt{1 + \hat{\kappa}^2 + \hat{\pi}_3^2(\hat{q}_3, u')}}{u'(1-u')}, \quad (91)$$

can be analytically calculated. For $u_0 \neq \{0, 1\}$ and $u \to 0$, this phase splits into a relevant divergent part and an irrelevant regular part $\Psi(\hat{q}_3, u_0, 0)$,

$$\Theta(\hat{q}_3, u_0, 0) = \Psi(\hat{q}_3, u_0, 0) + \frac{\hat{\omega}(\hat{q}_3, 0)}{2\gamma\epsilon}\log(u), \quad (92)$$

such that the normalization constants $N^{(\pm)}$ is fixed according to $\tilde{g}^{(\pm)}(\hat{q}_3, 0) = \tilde{G}^{(\pm)}(\hat{q}_3, 0)$:

$$N^{(\pm)} = \frac{e^{\mp i\Psi(\hat{q}_3, u_0, 0)}}{\sqrt{2m^2\hat{\omega}(\hat{q}_3, 0)[\hat{\omega}(\hat{q}_3, 0) \mp \hat{\pi}_3(\hat{q}_3, 0)]}}. \quad (93)$$

Again, the single-particle momentum distribution function is calculated according to Eqs. (A20) and (A22):

$$\tilde{f}(\hat{q}_3, u) = \tilde{N}_f(\hat{q}_3)\left(1 + \frac{\hat{\pi}_3(\hat{q}_3, u)}{\hat{\omega}_3(\hat{q}_3, u)}\right)|[\hat{\omega}(\hat{q}_3, u) - (1-u)\hat{\omega}(\hat{q}_3, 0) - u\hat{\omega}(\hat{q}_3, 1)]F(\tilde{a}, \tilde{b}, \tilde{c}; u) - 2i\gamma\epsilon u(1-u)\partial_u F(\tilde{a}, \tilde{b}, \tilde{c}; u)|^2 \quad (94)$$

with the normalization factor $\tilde{N}_f(\hat{q}_3)$ being given by

$$\tilde{N}_f(\hat{q}_3) = \frac{1}{2\hat{\omega}(\hat{q}_3, 0)[\hat{\omega}(\hat{q}_3, 0) - \hat{\pi}_3(\hat{q}_3, 0)]} \quad (95)$$

and

$$\partial_u F(\tilde{a}, \tilde{b}, \tilde{c}; u) = \frac{\tilde{a}\,\tilde{b}}{\tilde{c}} F(1+\tilde{a}, 1+\tilde{b}, 1+\tilde{c}; z). \quad (96)$$

Similarly to the constant electric field, we may give a simple expression for the asymptotic single-particle momentum distribution function. Applying a linear transformation formula for hypergeometric functions and considering the asymptotic limit $u \longrightarrow 1$, one first obtains

$$\tilde{f}(\hat{q}_3, 1) = \frac{2\gamma^2\epsilon^2}{\hat{\omega}(\hat{q}_3, 0)\hat{\omega}(\hat{q}_3, 1)} \frac{\hat{\omega}(\hat{q}_3, 1) + \hat{\pi}_3(\hat{q}_3, 1)}{\hat{\omega}(\hat{q}_3, 1) - \hat{\pi}_3(\hat{q}_3, 1)} \times \left|\frac{\tilde{a}\,\tilde{b}}{\tilde{c}} \frac{\Gamma(1+\tilde{c})\Gamma(1+\tilde{a}+\tilde{b}-\tilde{c})}{\Gamma(1+\tilde{a})\Gamma(1+\tilde{b})}\right|^2. \quad (97)$$

Applying the transformation formulae for gamma functions

$$\Gamma(1+a) = a\Gamma(a) \quad \text{and} \quad |\Gamma(1+ib)|^2 = \frac{\pi b}{\sinh(\pi b)}, \quad (98)$$

we obtain a relatively simple analytic expression for the asymptotic single-particle momentum distribution function $\tilde{f}(\hat{q}_3, 1)$:

$$\frac{2\sinh(\frac{\pi}{2\gamma\epsilon}[\frac{2}{\gamma} + \hat{\omega}(\hat{q}_3, 1) - \hat{\omega}(\hat{q}_3, 0)])\sinh(\frac{\pi}{2\gamma\epsilon}[\frac{2}{\gamma} - \hat{\omega}(\hat{q}_3, 1) + \hat{\omega}(\hat{q}_3, 0)])}{\sinh(\frac{\pi}{\gamma\epsilon}\hat{\omega}(\hat{q}_3, 1))\sinh(\frac{\pi}{\gamma\epsilon}\hat{\omega}(\hat{q}_3, 0))}. \quad (99)$$

III. INFLUENCE OF A SMALL SPATIAL INHOMOGENEITY

In this section, we discuss the influence of a small spatial inhomogeneity along the direction of the time-dependent electric field $\vec{E}(\vec{x}, t)$ based on the analytic results for both the constant electric field and the Sauter-type electric field. In order to estimate the effect of higher derivatives, we adopt a derivative expansion and determine the ratio between the first derivative and higher derivatives. Note that we again ignore the effect of magnetic fields for simplicity, $\vec{B}(\vec{x}, t) = 0$.

A. Derivative expansion

We consider a space- and time-dependent electric field:

$$\vec{E}(\vec{x}, t) = E(t)[1 + \Delta(x_3)]\vec{e}_3, \quad (100)$$

where $|\Delta(x_3)| \ll 1$ describes a small deviation from the spatially homogeneous electric field. The equation of motion for the 16 DHW functions $\vec{\mathbb{w}}(\vec{x}, \vec{p}; t)$ reads

$$D_t\vec{\mathbb{w}}(\vec{x}, \vec{p}; t) = \mathcal{M}(\vec{\nabla}_{\vec{x}}, \vec{p})\vec{\mathbb{w}}(\vec{x}, \vec{p}; t), \quad (101)$$

with $\vec{\mathbb{w}}(\vec{x}, \vec{p}; t)$ and $\mathcal{M}(\vec{\nabla}_{\vec{x}}, \vec{p})$ satisfying Eqs. (13)–(20). We perform a derivative expansion of the pseudodifferential operator D_t in Eq. (10), such that,

$$D_t = \partial_t + eE(t)\partial_{p_3} + eE(t)\left[\Delta(x_3)\partial_{p_3} - \frac{\Delta''(x_3)}{24}\partial^3_{p_3} + \ldots\right] \quad (102)$$

where $\Delta''(x_3)$ denotes the second derivative with respect to x_3. We may consider $\vec{\mathbb{w}}(\vec{x}, \vec{p}; t)$ in an expansion as well:

$$\vec{\mathbb{w}}(\vec{x}, \vec{p}; t) = \vec{\mathbb{w}}^{(0)}(\vec{p}; t) + \vec{\mathbb{w}}^{(1)}(\vec{x}, \vec{p}; t), \quad (103)$$

with $\vec{\mathbb{w}}^{(0)}(\vec{p}; t)$ being the exact result for the case of a spatially homogeneous electric field and $\vec{\mathbb{w}}^{(1)}(\vec{x}, \vec{p}; t)$ denoting a small deviation from the zeroth-order solution due to the spatial inhomogeneity. Neglecting terms of the order $\Delta(x_3)\vec{\mathbb{w}}^{(1)}(\vec{x}, \vec{p}; t)$, we obtain

$$[\partial_t + eE(t)\partial_{p_3} - \mathcal{M}(\vec{\nabla}_{\vec{x}}, \vec{p})]\vec{\mathbb{w}}^{(1)}(\vec{x}, \vec{p}; t) \approx -eE(t)\left[\Delta(x_3)\partial_{p_3} - \frac{\Delta''(x_3)}{24}\partial^3_{p_3}\right]\vec{\mathbb{w}}^{(0)}(\vec{p}; t), \quad (104)$$

such that the spatially homogeneous solution $\vec{\mathbb{w}}^{(0)}(\vec{p}; t)$ acts as a source term for the spatially inhomogeneous solution. In order to estimate the parameter regime for which the omission of derivatives higher than linear might be a good approximation, we consider a simple model,

$$\Delta(x_3) = \delta_0 \cos\left(\frac{x_3}{L}\right), \quad (105)$$

with L being the length scale of the spatial variation and $\delta_0 \ll 1$ being its amplitude. Introducing the dimensionless variable

$$\lambda = mL \quad (106)$$

which measures the spatial variation in units of the Compton wave length, we obtain

$$[\partial_t + eE(t)\partial_{p_3} - \mathcal{M}(\vec{\nabla}_{\vec{x}}, \vec{p})]\vec{\mathbb{w}}^{(1)}(\vec{x}, \vec{p}; t) \approx -eE(t)\Delta(x_3)\left[\partial_{p_3} + \frac{m^2}{24\lambda^2}\partial^3_{p_3}\right]\vec{\mathbb{w}}^{(0)}(\vec{p}; t). \quad (107)$$

This equation serves as the starting point for our analysis of the influence of a small spatial inhomogeneity. In order to estimate the influence of higher derivatives, we compare the terms occurring in the derivative expansion of the pseudodifferential operator D_t in Eq. (102):

$$\partial_{p_3}\vec{\mathbb{w}}^{(0)}(\vec{p}; t) \quad \text{with} \quad \frac{m^2}{24\lambda^2}\partial^3_{p_3}\vec{\mathbb{w}}^{(0)}(\vec{p}; t). \quad (108)$$

In fact, by means of this procedure we do not quantify the overall influence of the small spatial inhomogeneity $\Delta(x_3)$ on the Schwinger effect; for this we would really have to solve the PDE system Eq. (101). However, by means of the derivative expansion we might estimate a parameter region for which the higher derivatives do not play an important role such that we could restrict ourselves to the solution of the first-order PDE system:

$$[\partial_t + eE(t)\partial_{p_3}]\vec{\mathbb{w}}(\vec{x}, \vec{p}; t) = \mathcal{M}(\vec{\nabla}_{\vec{x}}, \vec{p})\vec{\mathbb{w}}(\vec{x}, \vec{p}; t). \quad (109)$$

It is known from the analysis of the Schwinger effect in spatially homogeneous electric fields that the orthogonal momentum solely acts as an additional mass term and does not change the qualitative behavior. Thus, for simplicity, we will restrict ourselves in the following to $\vec{p}_\perp = 0$, such that we deal only with the following DHW functions:

$$\mathbb{s}(\hat{p}_3; t) = -\frac{2}{\hat{\omega}(\hat{p}_3)}\chi^1(\hat{p}_3; t) - \frac{2\hat{p}_3}{\hat{\omega}(\hat{p}_3)}\chi^2(\hat{p}_3; t), \quad (110)$$

$$\mathbb{v}_3(\hat{p}_3; t) = -\frac{2\hat{p}_3}{\hat{\omega}(\hat{p}_3)}\chi^1(\hat{p}_3; t) + \frac{2}{\hat{\omega}(\hat{p}_3)}\chi^2(\hat{p}_3; t), \quad (111)$$

$$\mathbb{t}_{1,3}(\hat{p}_3; t) = 2\chi^3(\hat{p}_3; t), \quad (112)$$

where we have introduced the dimensionless phase-space kinetic momentum $\hat{p}_3 = \frac{p_3}{m}$ and the dimensionless energy variable $\hat{\omega}(\hat{p}_3) = \frac{\omega(p_3)}{m}$.

B. Example 1: Constant electric field

We explicitly calculated the time-independent coefficients $\chi^{i=\{1,2,3\}}(\vec{p})$ in Appendix. B, such that the DHW functions Eqs. (110)–(112) for $\vec{p}_\perp = 0$ and in terms of the dimensionless variables $\hat{p}_3$ and $\hat{\omega}(\hat{p}_3)$ are easily calculable. In the end, we want to compare the first with the third derivative of the source term in Eq. (107):

$$-m\epsilon\Delta(x_3)\left[\partial_{\hat{p}_3} + \frac{1}{24\lambda^2}\partial^3_{\hat{p}_3}\right]\vec{\mathbb{w}}^{(0)}(\hat{p}_3). \quad (113)$$

Therefore, we explicitly determine these derivatives according to Eq. (34), yielding

$$\partial_{\hat{p}_3}\mathbb{s}^{(0)} = \frac{2\hat{p}_3}{\epsilon}\mathbb{t}^{(0)}_{1,3},$$
$$\partial_{\hat{p}_3}\mathbb{v}^{(0)} = -\frac{2}{\epsilon}\mathbb{t}^{(0)}_{1,3}, \quad (114)$$
$$\partial_{\hat{p}_3}\mathbb{t}^{(0)}_{1,3} = -\frac{2\hat{p}_3}{\epsilon}\mathbb{s}^{(0)} + \frac{2}{\epsilon}\mathbb{v}^{(0)}_3$$

and

$$\partial^3_{\hat{p}_3}\mathbb{s}^{(0)} = \frac{4}{\epsilon^2}\left[-3\hat{p}_3\mathbb{s}^{(0)} + 2\mathbb{v}^{(0)}_3 - \frac{2\hat{p}_3\hat{\omega}^2(\hat{p}_3)}{\epsilon}\mathbb{t}^{(0)}_{1,3}\right],$$
$$\partial^3_{\hat{p}_3}\mathbb{v}^{(0)}_3 = \frac{4}{\epsilon^2}\left[\mathbb{s}^{(0)} + \frac{2\hat{\omega}^2(\hat{p}_3)}{\epsilon}\mathbb{t}^{(0)}_{1,3}\right],$$
$$\partial^3_{\hat{p}_3}\mathbb{t}^{(0)}_{1,3} = \frac{4}{\epsilon^2}\left[\frac{2\hat{p}_3\hat{\omega}^2(\hat{p}_3)}{\epsilon}\mathbb{s}^{(0)} - \frac{2\hat{\omega}^2(\hat{p}_3)}{\epsilon}\mathbb{v}^{(0)}_3 - 3\hat{p}_3\mathbb{t}^{(0)}_{1,3}\right], \quad (115)$$

where we have dropped the arguments of the DHW functions for simplicity.

According to Eq. (113), we compare $\partial_{\hat{p}_3}\vec{\mathbb{w}}^{(0)}(\hat{p}_3)$ with $\frac{1}{24}\partial^3_{\hat{p}_3}\vec{\mathbb{w}}^{(0)}(\hat{p}_3)$ in the following: due to the fact that all the DHW functions show a rather similar behavior, we restrict ourselves to $\mathbb{t}^{(0)}_{1,3}(\hat{p}_3)$ for simplicity. Figure 1 clearly shows

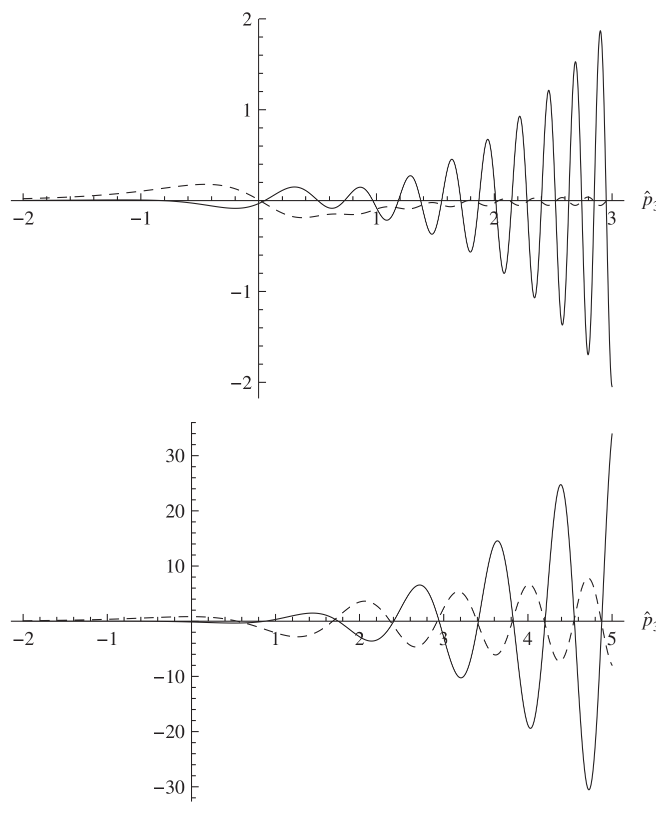

FIG. 1. Comparison of the leading-order derivative term $\partial_{\hat{p}_3}\mathbb{t}^{(0)}_{1,3}(\hat{p}_3)$ (dashed line) with the next-to-leading-order term $\frac{1}{24}\partial^3_{\hat{p}_3}\mathbb{t}^{(0)}_{1,3}(\hat{p}_3)$ (solid line) for $E(t) = E_0$ with $\epsilon = E_0/E_c = 0.2$ (upper) and $\epsilon = 1$ (lower). For higher momenta, the next-to-leading-order term eventually exceeds the leading-order term as the electric field persistently accelerates the produced pairs.

that the higher derivatives always become more important than the first derivative for large momenta $\hat{p}_3$. This might be understood in the following way: Because of the acceleration in the electric field, all length scales are ultimately probed and become important, even though the pair creation process happens on the length scale of $\mathcal{O}(\lambda)$. Note, however, that the point at which the higher derivatives are of the order of the first derivative depends on the electric field strength $\epsilon = E_0/E_c$: For higher field strengths this point is already reached for lower momenta.

In order to estimate the importance of the spatial inhomogeneity for the pair creation process itself, we should have a closer look at the pair-production rate in Fig. 2:

For $\epsilon = 0.2$, the dominant contributions to the pair-production rate stem from a region of kinetic momenta up to $\hat{p}_3 \approx \pm 2$. In this regime, the first derivatives are still of the order of the third derivatives, as shown in Fig. 1. As a consequence, the effect of higher derivatives on the pair-production process should be taken into account only for a spatial variation of the Compton wavelength $\lambda = \mathcal{O}(1)$, whereas the effect of the higher derivatives becomes suppressed for larger variation scales λ.

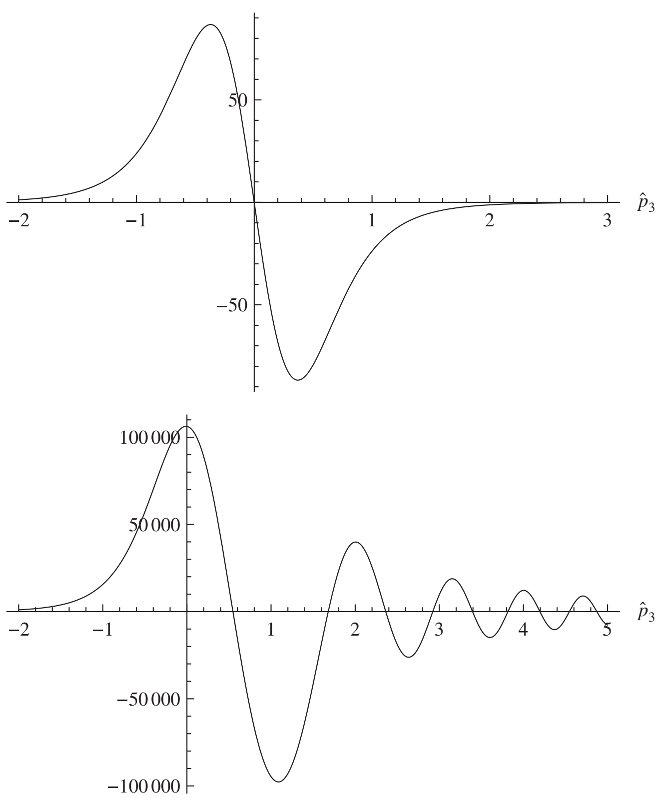

FIG. 2. Pair-production rate for $E(t) = E_0$ with $\epsilon = E_0/E_c = 0.2$ (upper) and $\epsilon = 1$ (lower). For $\epsilon = 0.2$, pair production occurs dominantly for momenta $p_3 \lesssim 2$ where next-to-leading-order derivative terms remain small, cf. Figure 1. By contrast, higher derivative terms are expected to take a quantitative influence on pair production for $\epsilon = 1$ and sizable spatial variation λ.

For $\epsilon = 1$, the situation is slightly different: Again, the dominant contributions to the pair-production rate arise from kinetic momenta up to $\hat{p}_3 \approx \pm 2$; however, there are nonvanishing contributions for higher momenta as well, as shown in Fig. 1. Because of the fact that the higher derivatives become more important than the first derivatives for large momenta, the scale of spatial variation λ has to increase as well in order to suppress the higher derivatives.

To conclude: Concerning the pair-production process, there is a strong interplay between the electric field strength ϵ and the scale of spatial variation λ. This interplay between field strength and scale of spatial variation affecting the quality of the derivative expansion has already been observed in earlier studies [38]. In order to keep the pair-production process itself unaltered by higher derivatives, the scale of spatial variation λ must get larger for higher electric field strengths ϵ. However, even if the effect of the higher derivatives on the pair-production process might be negligible, the effect on the final momentum distribution might be large: this is due to the acceleration of the pairs in the electric field which finally emphasizes higher momenta such that higher derivatives always become more important than the first derivative.

C. Example 2: Sauter-type electric field

For the Sauter-type electric field we may perform a similar analysis like for the constant electric field. Again, we only consider $\vec{p}_\perp = 0$; however, there is a huge qualitative difference since the DHW functions now depend on both the phase-space kinetic momentum $\hat{p}_3$ and the time variable u. Using the expressions given in Appendix B, we are able to analytically calculate the DHW functions Eqs. (110)–(112). Again, we consider the source term in Eq. (107), which now reads

$$-4m\epsilon u(1-u)\Delta(x_3)\left[\partial_{\hat{p}_3} + \frac{1}{24\lambda^2}\partial^3_{\hat{p}_3}\right]\vec{\mathbb{w}}^{(0)}(\hat{p}_3; u). \quad (116)$$

In what follows, we compare the first derivative $\partial_{\hat{p}_3}\vec{\mathbb{w}}^{(0)}(\hat{p}_3; u)$ with the third derivatives $\frac{1}{24}\partial^3_{\hat{p}_3}\vec{\mathbb{w}}^{(0)}(\hat{p}_3; u)$. Again, we restrict ourselves to $\mathbb{t}^{(0)}_{1,3}(\hat{p}_3; u)$. Nonetheless, there are three big differences in comparison to the constant electric field: First, we are not able to calculate the first and third derivative with respect to $\hat{p}_3$ as easily as for the constant electric field, as shown in Eqs. (114) and (115), since these derivatives now include parameter derivatives of the Gauss hypergeometric function. Nevertheless, a numerical calculation is rather simple. Second, the situation is not quasistatic anymore but it makes a difference at which moment of time the system is considered. Third, the field strength ϵ is not the only relevant parameter but we also have to consider the dependence on the pulse length τ via the Keldysh parameter $\gamma = 1/(m\epsilon\tau)$. In order to analyze the interplay between these quantities, we consider the system at three different instants of time: At $t = -\tau$ (increase of field strength), $t = 0$ (maximum of field strength) and $t = \tau$ (decrease of field strength).

Let us first concentrate on Fig. 3 with $\epsilon = 0.2$ and $\gamma = 1$, where we already observe some general features: As in the case of the constant electric field, the third derivatives show an out-of-phase behavior compared to the first derivative: At momentum values where the third derivatives show a local maximum, the first derivative shows a local minimum and vice versa. As a consequence, depending on the scale of spatial variation λ, the influence of the first derivative might be inverted due to the third derivative term. We also observe that the relative importance of the third derivatives in comparison to the first derivative becomes bigger for later times. This might be interpreted in the following way: At late times the created particles have had more time to be accelerated in the electric field and, as a consequence, have travelled over larger distances and were exposed to even large scale inhomogeneities. The effect of acceleration in the electric field might also be seen in the shift of the global maximum of the third derivatives towards higher-momentum values. If we want to suppress the higher derivatives in comparison to the first derivative, we should choose the scale of spatial variation to be at least of the order $\lambda \gtrsim \mathcal{O}(5)$.

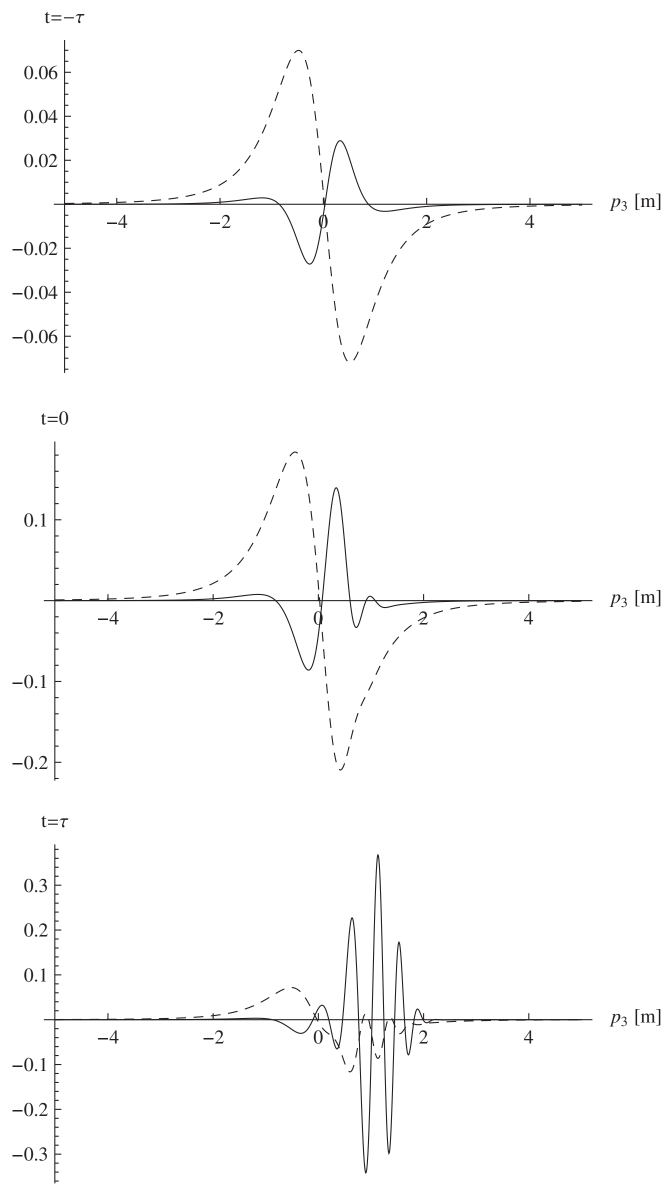

FIG. 3. Comparison of the leading-order derivative term $\partial_{\hat{p}_3}\mathbb{t}^{(0)}_{1,3}(\hat{p}_3; u)$ (dashed line) with the next-to-leading-order term $1/24\partial^3_{\hat{p}_3}\mathbb{t}^{(0)}_{1,3}(\hat{p}_3; u)$ (solid line) for $\epsilon = E_0/E_c = 0.2$ and $\gamma = 1$ for $t = -\tau$ (upper), $t = 0$ (middle) and $t = \tau$ (lower).

Next we switch to a longer pulse with $\gamma \to 0.2$ as shown in Fig. 4, which corresponds to a pulse with the same electric field strength but with a 5-times longer duration. We see that the behavior at early times $t = -\tau$ is very similar, however, the situation changes drastically for later times: First, due to the longer pulse duration the created particles are accelerated to higher momenta. Additionally, we observe a strong enhancement of the oscillatory structure and a strong increase of the magnitude of the third derivatives. Especially at $t = \tau$, the third derivatives are orders of magnitude larger than the first derivatives. This means that the scale of spatial variation λ has to be chosen much larger in order to suppress the influence of higher order derivatives.

The change in the overall magnitude is the distinctive feature when we increase the field strength $\epsilon \to 1$ as shown in Fig. 5. The general behavior is rather similar as before, however, the source term becomes more important in comparison to the left-hand side of Eq. (107). As a consequence, we expect the overall effect of spatial inhomogeneities to be more important for strong electric fields than for weak electric fields.

To summarize, Figs. 3–5 give us the following physical picture:

(i) Concerning the time variable u, we see that the role of higher derivatives is more important at late times than at early times. This suggests that even if the influence of spatial inhomogeneities on the pair-production process itself might not be very strong, the final momentum distribution can be strongly altered.

(ii) Concerning the field strength ϵ, we conclude by comparing the magnitude of the source term for the different values of ϵ that the influence of spatial inhomogeneities on the pair-production process becomes more important for higher field strengths. We interpret this observation as arising from the fact that for weak fields the created particles are not accelerated that much and, as a consequence, they do only feel inhomogeneities on large length scales but not on shorter variation scales.

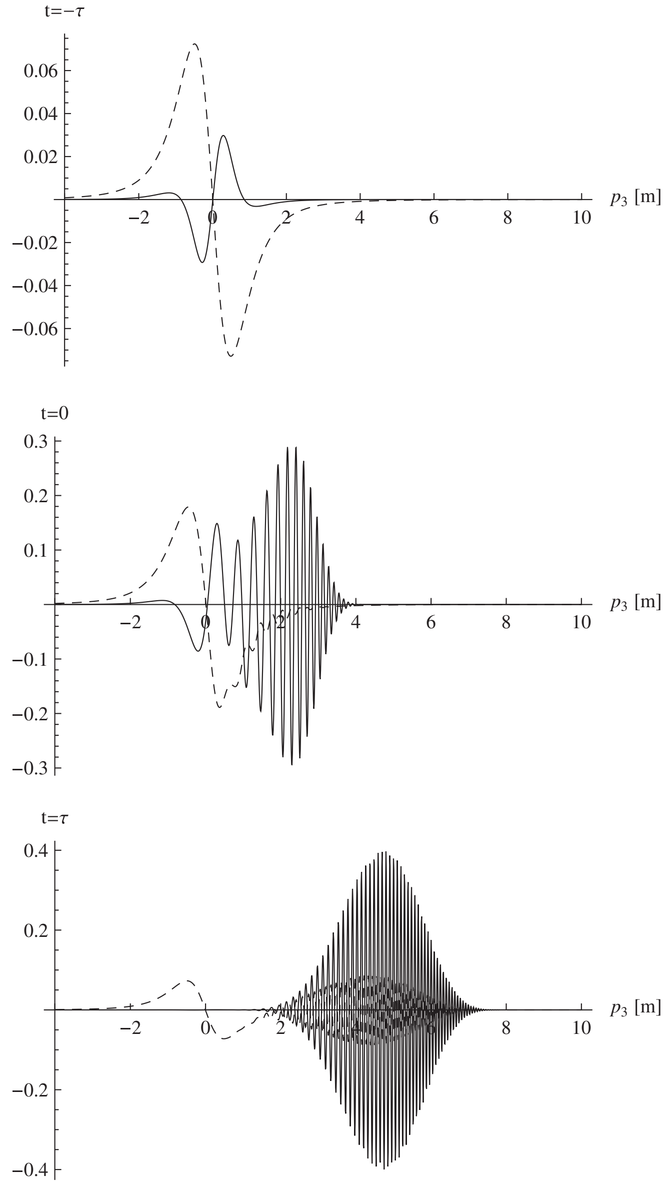

FIG. 4. Comparison of the leading-order derivative term $\partial_{\hat{p}_3}\mathbb{k}^{(0)}_{1,3}(\hat{p}_3;u)$ (dashed line) with the next-to-leading-order term $1/24\partial^3_{\hat{p}_3}\mathbb{k}^{(0)}_{1,3}(\hat{p}_3;u)$ (solid line) with $\epsilon = E_0/E_c = 0.2$ and $\gamma = 0.2$ for $t = -\tau$ (upper), $t = 0$ (middle) and $t = \tau$ (lower). Note: For $t = \tau$ the third derivative is in fact 2 magnitudes larger than the first derivative but has been scaled by a factor of 0.01.

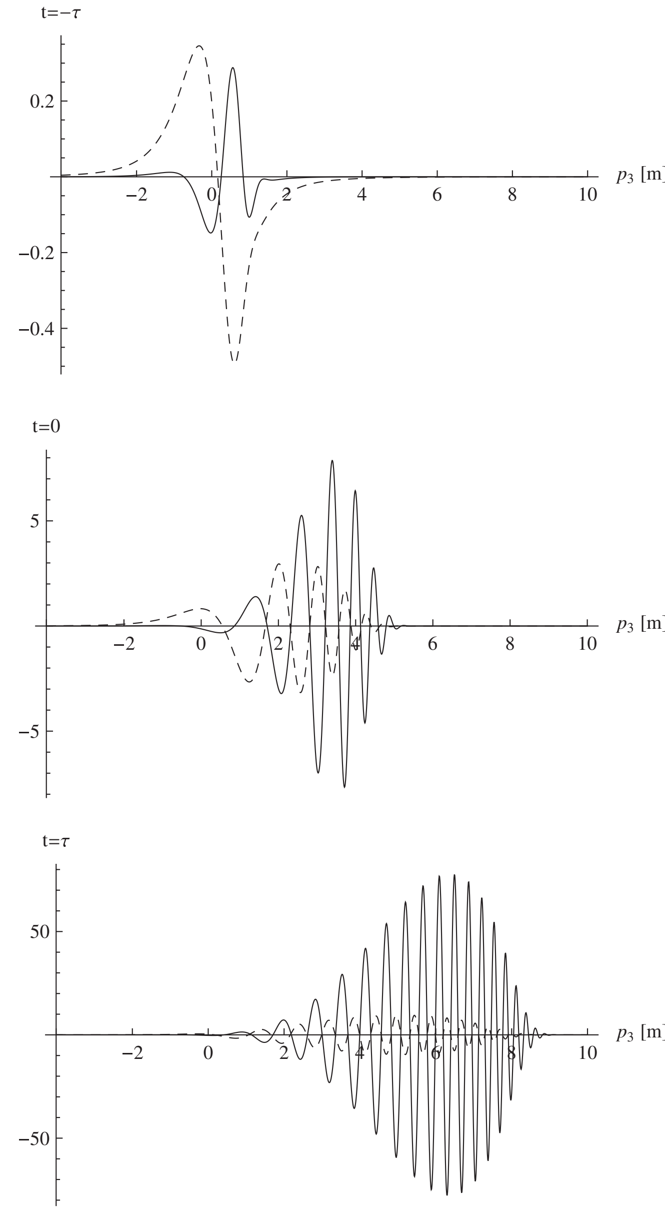

FIG. 5. Comparison of the leading-order derivative term $\partial_{\hat{p}_3}\mathbb{k}^{(0)}_{1,3}(\hat{p}_3;u)$ (dashed line) with the next-to-leading-order term $1/24\partial^3_{\hat{p}_3}\mathbb{k}^{(0)}_{1,3}(\hat{p}_3;u)$ (solid line) with $\epsilon = 1$ and $\gamma = 0.2$ for $t = -\tau$ (upper), $t = 0$ (middle) and $t = \tau$ (lower).

(iii) Concerning the Keldysh parameter γ we note that small values of γ correspond either to strong electric field strengths or to longer pulse durations τ. Therefore, particles are more substantially accelerated in the electric field and, as a consequence, spatial inhomogeneities might have a greater importance, since the particles feel the inhomogeneities even on shorter scales of variation.

As a consequence, we observe that there will generally be a complex interplay between all the relevant parameters for any type of space- and time-dependent electric field $\vec{E}(\vec{x}, t)$. Thus, it is difficult to predict *a priori* whether neglecting higher derivatives can be a good approximation or not. It is clear that in the limit of a spatially homogeneous electric fields $\lambda \to \infty$ the leading-order derivative approximation Eq. (109) becomes exact since all higher derivatives are suppressed by a factor of λ^{-2n}. However, it is not clear *a priori* for which values of λ the higher derivatives play a quantitatively important role and should be taken into account. Future investigations on that problem should help to better understand the interplay between the different scales.

IV. CONCLUSIONS AND OUTLOOK

We have investigated the Dirac-Heisenberg-Wigner (DHW) formalism for nonperturbative pair production in general electromagnetic fields. As a genuine real-time formalism, this approach provides for a comprehensive framework of addressing all aspects of pair production, most notably, the nonequilibrium character of pair production in a fully time- and space-resolved manner.

We have shown that the DHW formalism includes quantum kinetic theory (QKT) which has so far been the most successful approach to describe the real-time evolution of pair production in the limit of time-dependent but spatially homogeneous electric fields. We conclude that the DHW formalism provides for the desired generalization of QKT to the case of arbitrarily general space- and time-dependent electromagnetic fields. For a given field, the solution of the DHW formalism is parameterized in spinor QED in 4-dimensional space-time by 16 irreducible components of the Wigner function which encode the phase-space distributions of physical quantities such as mass, charge and current densities as provided by the produced pairs. From the knowledge of these quantities, although they are no semipositive definite probability distributions, physical observables such as the pair distribution function in phase-space can directly be inferred.

Whereas the DHW formalism is completely general as far as the details of the external field are concerned, we have confined ourselves in the present work to an analysis of exactly solvable cases such as the constant electric field and the Sauter potential. Of course, such exactly solvable cases always provide for a controlled starting point for more general cases, in particular, they should also serve as a benchmark for future full numerical studies. Moreover, we have used these cases in the present work to provide for a first glance at the possible use and limitations of natural approximation schemes such as the derivative expansion. The leading-order of this approximation corresponds to a *locally constant field approximation*, i.e., approximating the spatial dependence of the field locally by a constant field.

The picture arising from this investigation is rather diverse: it has already been known, for instance, from worldline instanton studies [33,34], that the locally constant field approximation underestimates the pair-production rate for fields varying in time, and overestimates the rate for fields varying in space. Whereas the exact solution, e.g. for the Sauter potential in Sec. II C 2 reflects this fact, we observe that general statements about the potential quality of the derivative or locally constant-field approximation cannot straightforwardly be made. For all concrete examples, we observe that the next-to-leading-order derivative terms in fact exceed the leading-order terms either for higher momenta or for late times. Taken at face value, this seems to imply that the derivative expansion is always bound to fail as soon as the field exhibits spatial variations. However, the reason for this strong modification of the next-to-leading-order terms lies in the fact that the persistent presence of accelerating field components, of course, exerts a strong influence on high-momentum components which eventually resolve also small spatial variations of the field. Therefore, it is only natural to expect that the derivative expansion should fail in the way it does for high-momentum components and at late times.

Nevertheless, our results also provide a guideline to a less strict view on the quality of the derivative expansion: phenomenologically, the most relevant quantity is the pair distribution function in momentum space. Quantitatively, the question needs to be addressed whether the next-to-leading-order terms of the derivative expansion exert a strong influence on this distribution function. As the higher derivative terms become dominant for higher-momentum components, we conclude that the derivative expansion can still remain a reasonable approximation as long as the dominant pair distribution is peaked at lower momenta. Whether or not this is the case, depends not only on the scale of spatial variation of the field, but also on the overall field strength and also possible further time dependencies. As a rule of thumb, we observe that the dominant low-momentum components appear to remain little affected at significantly subcritical field strengths with spatial variations being substantially larger than the Compton wavelength.

Beyond the technical question about the quality of the derivative expansion, the most interesting question is certainly as to whether a phenomenologically relevant interplay between characteristic signatures of pair production and space-time shaping of the external field exists. Based on the surprising observations that have already been made for simple time-dependencies [45,46], we expect that this question can be answered in the affirmative. From our present studies of spatial variations and the dominant effect on high-momentum components, we conclude that temporal and spatial pulse shaping can have substantial effects on the momentum structure of the pair distribution function. As to whether temporal and spatial pulse shaping can also optimize the total number of produced pairs certainly remains the most pressing question which we hope to address with the DHW formalism based on full numerical solutions in the near future.

ACKNOWLEDGMENTS

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APPENDIX A: QUANTUM KINETIC THEORY (QKT)

In this Appendix, we give a brief derivation of the quantum kinetic equation describing Schwinger pair production in spatially homogeneous, time-dependent electric fields [24,25]. As in the main part of this paper, we adopt the temporal gauge $A_0 = 0$. We choose the vector potential $\vec{A}(t) = A(t)\vec{e}_3$ such that the Dirac equation reads

$$(i\gamma^0\partial_t + i\vec{\gamma}\cdot[\vec{\nabla}_{\vec{x}} - ieA(t)\vec{e}_3] - m)\Psi(\vec{x}, t) = 0. \quad \text{(A1)}$$

Because of spatial homogeneity, we decompose the spinor field into its Fourier modes according to Eq. (29), such that the Dirac equation for the mode function $\tilde{\psi}(\vec{q}, t)$ reads

$$(i\gamma^0\partial_t - \vec{\gamma}\cdot\vec{\pi}(\vec{q}, t) - m)\tilde{\psi}(\vec{q}, t) = 0. \quad \text{(A2)}$$

Again note that $\vec{\pi}(\vec{q}, t) = \vec{q} - eA(t)\vec{e}_3$ denotes the time-dependent kinetic momentum on the trajectory whereas $\vec{q}$ denotes the canonical momentum. In order to solve this equation we apply the ansatz

$$\tilde{\psi}(\vec{q}, t) = (i\gamma^0\partial_t - \vec{\gamma}\cdot\vec{\pi}(\vec{q}, t) + m)\tilde{\phi}(\vec{q}, t), \quad \text{(A3)}$$

such that the spinor-valued function $\tilde{\phi}(\vec{q}, t)$ obeys the equation

$$(\partial_t^2 + \tilde{\omega}^2(\vec{q}, t) + ieE(t)\gamma^0\gamma^3)\tilde{\phi}(\vec{q}, t) = 0, \quad \text{(A4)}$$

with $\tilde{\omega}(\vec{q}, t)$ defined as before. It is convenient to expand $\tilde{\phi}(\vec{q}, t)$ in a basis consisting of the eigenvectors of $\gamma^0\gamma^3$, such that

$$\tilde{\phi}(\vec{q}, t) = \sum_s R_s \tilde{g}_s(\vec{q}, t) \quad \text{with} \quad \gamma^0\gamma^3 R_s = \lambda R_s. \quad \text{(A5)}$$

There are two eigenvectors $R_{s=1,2}$ with $\lambda = +1$ and two eigenvectors $R_{s=3,4}$ with $\lambda = -1$. Inserting this ansatz into Eq. (A4), each $\tilde{g}_s(\vec{q}, t)$ obeys the equation of a time-dependent oscillator:

$$(\partial_t^2 + \tilde{\omega}^2(\vec{q}, t) + ieE(t)\lambda)\tilde{g}_s(\vec{q}, t) = 0, \quad \text{(A6)}$$

which are in general not exactly solvable; exceptions are the constant electric field $E(t) = E_0$ and the Sauter-type electric field $E(t) = E_0\text{sech}^2(t/\tau)$ (see Sec. II C). Each of them is a second-order differential equation and possesses as such two linearly independent solutions $\tilde{g}_s^{\pm}(\vec{q}, t)$. Because of the fact that Eq. (A4) gives four equations but the ansatz Eq. (A5) allows for eight solutions, there is a redundancy which is removed by choosing only one set of eigenvectors, either $s = \{1, 2\}$ or $s = \{3, 4\}$. Because of the absence of magnetic fields, we impose the same initial conditions for both spin states, such that

$$\tilde{g}_1^{(\pm)}(\vec{q}, t) = \tilde{g}_2^{(\pm)}(\vec{q}, t) = \tilde{g}^{(\pm)}(\vec{q}, t). \quad \text{(A7)}$$

Consequently, we canonically quantize $\tilde{\psi}(\vec{q}, t)$ according to Eq. (30) by introducing anticommuting creation/annihilation operators as well as four spinors, with

$$\tilde{u}_s(\vec{q}, t) = (i\gamma^0\partial_t - \vec{\gamma}\cdot\vec{\pi}(\vec{q}, t) + m)\tilde{g}^{(+)}(\vec{q}, t)R_s, \quad \text{(A8)}$$

$$\tilde{v}_s(-\vec{q}, t) = (i\gamma^0\partial_t - \vec{\gamma}\cdot\vec{\pi}(\vec{q}, t) + m)\tilde{g}^{(-)}(\vec{q}, t)R_s. \quad \text{(A9)}$$

Because of the fact that we work in the Heisenberg picture, the creation/annihilation operators are time dependent in general, however, due to the choice Eqs. (A8) and (A9), the whole time-dependence can be absorbed into the four spinors. It can be shown, that in the case of vanishing electric fields, the properly normalized vacuum solutions are given by

$$\tilde{g}_{\text{vac}}^{(\pm)}(\vec{q}, t) = \frac{e^{\mp i\omega(\vec{q})t}}{\sqrt{2\omega(\vec{q})(\omega(\vec{q}) \mp q_3)}}, \quad \text{(A10)}$$

with $\omega(\vec{q}) = \sqrt{m^2 + \vec{q}^2}$. It is important to note that a particle/antiparticle interpretation of the field quanta is only possible in the case of such plane-wave solutions. However, as soon as electric fields are present, the mode functions $\tilde{g}^{(\pm)}(\vec{q}, t)$ are no plane waves anymore and an interpretation in terms of particles/antiparticles is not straightforward. It is a further consequence of the presence of electric fields that the Hamiltonian operator achieves off-diagonal elements which account for particle/antiparticle creation/annihilation.

The Hamiltonian operator might be diagonalized by performing a unitary nonequivalent change of basis to a quasiparticle representation via a time-dependent Bogoliubov transformation:

$$\tilde{a}_s(\vec{q}, t) = \tilde{\alpha}(\vec{q}, t)a_s(\vec{q}) - \tilde{\beta}^*(\vec{q}, t)b_s^\dagger(-\vec{q}), \quad \text{(A11)}$$

$$\tilde{b}_s^\dagger(-\vec{q}, t) = \tilde{\beta}(\vec{q}, t)a_s(\vec{q}) + \tilde{\alpha}^*(\vec{q}, t)b_s^\dagger(-\vec{q}), \quad \text{(A12)}$$

with the creation/annihilation operators becoming time dependent but still fulfilling the equal-time anticommutation relations. In order to be a canonical transformation, the Bogoliubov coefficients $\tilde{\alpha}(\vec{q}, t)$ and $\tilde{\beta}(\vec{q}, t)$ have to fulfill

$$|\tilde{\alpha}(\vec{q}, t)|^2 + |\tilde{\beta}(\vec{q}, t)|^2 = 1. \quad \text{(A13)}$$

In pure vacuum when no electric fields are present, the two different operator bases coincide such that $\tilde{\alpha}_{\text{vac}}(\vec{q}, t) = 1$ and $\tilde{\beta}_{\text{vac}}(\vec{q}, t) = 0$. Note, that this relation also holds in the presence of electric fields at asymptotic times $t \to -\infty$. Within this so-called adiabatic basis, the Fourier modes $\tilde{\psi}(\vec{q}, t)$ read

$$\tilde{\psi}(\vec{q}, t) = \sum_s \tilde{U}_s(\vec{q}, t)\tilde{a}_s(\vec{q}, t) + \tilde{V}_s(-\vec{q}, t)\tilde{b}_s^\dagger(-\vec{q}, t). \quad \text{(A14)}$$

The adiabatic four spinors are chosen in close analogy to Eqs. (A8) and (A9) such that they coincide with the vacuum solutions in the case of vanishing electric fields:

$$\tilde{U}_s(\vec{q}, t) = (\gamma^0\tilde{\omega}(\vec{q}, t) - \vec{\gamma}\cdot\vec{\pi}(\vec{q}, t) + m)\tilde{G}^{(+)}(\vec{q}, t)R_s, \quad \text{(A15)}$$

$$\tilde{V}_s(-\vec{q}, t) = (-\gamma^0\tilde{\omega}(\vec{q}, t) - \vec{\gamma}\cdot\vec{\pi}(\vec{q}, t) + m)\tilde{G}^{(-)}(\vec{q}, t)R_s, \quad \text{(A16)}$$

with the adiabatic mode function $\tilde{G}^{(\pm)}(\vec{q}, t)$ given by

$$\tilde{G}^{(\pm)}(\vec{q},t) = \frac{e^{\mp i\Theta(\vec{q},t_0,t)}}{\sqrt{2\tilde{\omega}(\vec{q},t)(\tilde{\omega}(\vec{q},t)\mp\pi_3(q_3,t))}}, \quad (A17)$$

and the dynamical phase $\Theta(\vec{q}, t_0, t)$ being defined as

$$\Theta(\vec{q},t_0,t) = \int_{t_0}^{t} \tilde{\omega}(\vec{q},t')dt'. \quad (A18)$$

The lower bound t_0 is not determined since it only fixes an arbitrary phase at a given instant of time. Note that an interpretation in terms of particles/antiparticles is only straightforward at asymptotic times when the external electric field vanishes and the solutions Eq. (A17) behave like plane waves.

In order to define the single-particle momentum distribution function $\tilde{f}(\vec{q},t)$, we assume that we start with vacuum initial conditions at $t\to-\infty$:

$$\langle 0|a_s^\dagger(\vec{q})a_s(\vec{q})|0\rangle = \langle 0|b_s^\dagger(\vec{q})b_s(\vec{q})|0\rangle = 0. \quad (A19)$$

We then define $\tilde{f}(\vec{q},t)$ as the instantaneous quasiparticle number density for a given canonical momentum $\vec{q}$. Because of the absence of magnetic fields, we take the sum over both spin states, such that

$$\tilde{f}(\vec{q},t) = \lim_{V\to\infty}\sum_{s=1,2}\frac{\langle 0|\tilde{a}_s^\dagger(\vec{q},t)\tilde{a}_s(\vec{q},t)|0\rangle}{V} = 2|\tilde{\beta}(\vec{q},t)|^2 \quad (A20)$$

with V being the (infinite) configuration space volume. As a consequence, the knowledge of $\tilde{\beta}(\vec{q},t)$ allows for the calculation of $\tilde{f}(\vec{q},t)$. In fact, the different representations of $\tilde{\psi}(\vec{q},t)$ Eqs. (30) and (A14) translate into an expression for the Bogoliubov coefficients:

$$\tilde{\alpha}(\vec{q},t) = i\epsilon_\perp \tilde{G}^{(-)}(\vec{q},t)[\partial_t - i\tilde{\omega}(\vec{q},t)]\tilde{g}^{(+)}(\vec{q},t), \quad (A21)$$

$$\tilde{\beta}(\vec{q},t) = -i\epsilon_\perp \tilde{G}^{(+)}(\vec{q},t)[\partial_t + i\tilde{\omega}(\vec{q},t)]\tilde{g}^{(+)}(\vec{q},t), \quad (A22)$$

such that their time derivatives form an ODE system:

$$\frac{d}{dt}\tilde{\alpha}(\vec{q},t) = \frac{1}{2}\tilde{Q}(\vec{q},t)\tilde{\beta}(\vec{q},t)e^{2i\Theta(\vec{q},t_0,t)}, \quad (A23)$$

$$\frac{d}{dt}\tilde{\beta}(\vec{q},t) = -\frac{1}{2}\tilde{Q}(\vec{q},t)\tilde{\alpha}(\vec{q},t)e^{-2i\Theta(\vec{q},t_0,t)}, \quad (A24)$$

with $\tilde{Q}(\vec{q},t)$ defined as in Eq. (47). Introducing $\tilde{\mathcal{C}}(\vec{q},t) = 2\tilde{\alpha}(\vec{q},t)\tilde{\beta}^*(\vec{q},t)$, this ODE system might be rewritten as

$$\frac{d}{dt}\tilde{\mathcal{C}}(\vec{q},t) = -\tilde{Q}(\vec{q},t)[1-\tilde{f}(\vec{q},t)]e^{-2i\Theta(\vec{q},t_0,t)}, \quad (A25)$$

$$\frac{d}{dt}\tilde{f}(\vec{q},t) = -\tilde{Q}(\vec{q},t)\,\mathrm{Re}[\tilde{\mathcal{C}}(\vec{q},t)e^{2i\Theta(\vec{q},t_0,t)}]. \quad (A26)$$

Formally integrating the first equation from a time of pure vacuum $t_{\mathrm{vac}}\to-\infty$ to t, yields the Vlasov equation for the single-particle momentum distribution function $\tilde{f}(\vec{q},t)$ in its integro-differential form:

$$\frac{d}{dt}\tilde{f}(\vec{q},t) = \tilde{Q}(\vec{q},t)\int_{t_{\mathrm{vac}}}^{t} dt'\tilde{Q}(\vec{q},t')[1-\tilde{f}(\vec{q},t')] \times \cos[2\Theta(\vec{q},t',t)], \quad (A27)$$

with $\tilde{f}(\vec{q},t_{\mathrm{vac}})=0$. It is possible to rewrite this integro-differential equation in terms of an equivalent ODE system by introducing auxiliary functions $\tilde{\rho}(\vec{q},t)$ and $\tilde{\sigma}(\vec{q},t)$:

$$\frac{d}{dt}\tilde{f}(\vec{q},t) = \tilde{Q}(\vec{q},t)\tilde{\rho}(\vec{q},t), \quad (A28)$$

$$\frac{d}{dt}\tilde{\rho}(\vec{q},t) = \tilde{Q}(\vec{q},t)[1-\tilde{f}(\vec{q},t)] - 2\tilde{\omega}(\vec{q},t)\tilde{\sigma}(\vec{q},t), \quad (A29)$$

$$\frac{d}{dt}\tilde{\sigma}(\vec{q},t) = 2\tilde{\omega}(\vec{q},t)\tilde{\rho}(\vec{q},t), \quad (A30)$$

with appropriate vacuum initial conditions $\tilde{\rho}(\vec{q},t_{\mathrm{vac}}) = \tilde{\sigma}(\vec{q},t_{\mathrm{vac}}) = 0$.

APPENDIX B: DHW FUNCTIONS FOR EXACTLY SOLVABLE ELECTRIC FIELDS

In this Appendix we give the analytic expressions for the DHW functions for the exactly solvable cases of the constant electric field and the Sauter-type electric field. The DHW functions are obtained as follows: We already derived the analytic expressions for the single-particle momentum distribution function $\tilde{f}(\vec{q},t)$ in Sec. II C. As a consequence, according to Eqs. (48)–(50) we are able to calculate the nonvanishing coefficients $\tilde{\chi}^{i=\{2,3\}}(\vec{q},t)$, cf. Eq. (46), as well. Finally, we obtain the DHW functions according to Eqs. (51)–(56) after performing the phase-space projection Eq. (40).

1. Constant electric field

In order to simplify the expression for the single-particle momentum distribution function $\tilde{f}(u)$ derived in Eq. (72), we introduce the following abbreviations:

$$\tilde{d}_1(u) = |D_{-1+i\eta/2}(-ue^{-i(\pi/4)})|^2 \quad (B1)$$

$$\tilde{d}_2(u) = |D_{i\eta/2}(-ue^{-i(\pi/4)})|^2 \quad (B2)$$

$$\tilde{d}_3(u) = e^{i(\pi/4)}D_{-1-i\eta/2}(-ue^{-i(\pi/4)}) \times D_{i\eta/2}(-ue^{-i(\pi/4)}) + c.c. \quad (B3)$$

$$\tilde{d}_4(u) = e^{-i(\pi/4)}D_{-1-i\eta/2}(-ue^{-i(\pi/4)}) \times D_{i\eta/2}(-ue^{-i(\pi/4)}) + c.c. \quad (B4)$$

which fulfill

$$\partial_u \tilde{d}_1(u) = \tilde{d}_4(u), \tag{B5}$$

$$\partial_u \tilde{d}_2(u) = -\frac{\eta}{2}\tilde{d}_4(u), \tag{B6}$$

$$\partial_u \tilde{d}_3(u) = -u\tilde{d}_4(u). \tag{B7}$$

We may then express $\tilde{f}(u)$ in terms of $\tilde{d}_i(u)$, such that $\tilde{\chi}^{i=\{1,2,3\}}(u)$ are given by

$$\tilde{\chi}^1(u) = 1 - e^{-(\pi\eta/4)}\bigg\{\frac{\eta}{2}\bigg(1 - \frac{u}{\sqrt{2\eta + u^2}}\bigg)\tilde{d}_1(u) + \bigg(1 + \frac{u}{\sqrt{2\eta + u^2}}\bigg)\tilde{d}_2(u) - \frac{\eta}{\sqrt{2\eta + u^2}}\tilde{d}_3(u)\bigg\}, \tag{B8}$$

$$\tilde{\chi}^2(u) = \sqrt{\frac{\eta}{2}}\frac{1}{\sqrt{2\eta + u^2}}e^{-(\pi\eta/4)}\{-\eta\tilde{d}_1(u) + 2\tilde{d}_2(u) + u\tilde{d}_3(u)\}, \tag{B9}$$

$$\tilde{\chi}^3(u) = \frac{\sqrt{2\eta}}{(2\eta + u^2)^{3/2}} + \sqrt{\frac{\eta}{2}}\frac{1}{(2\eta + u^2)^{3/2}} \times e^{-(\pi\eta/4)}\{-\eta\tilde{d}_1(u) - 2\tilde{d}_2(u) + (2\eta + u^2)^{3/2}\tilde{d}_4(u)\}, \tag{B10}$$

with $\tilde{\chi}^1(u) = 1 - \tilde{f}(u)$. In order to obtain the DHW functions we perform the variable transformation $\vec{q} \to \vec{p} + e\vec{A}(t)$. Because of the linear relation between q_3 and t, this phase-space projection is trivial and reads

$$u \to \sqrt{\tfrac{2}{\epsilon}}\hat{p}_3 \quad \text{and} \quad d_i(\vec{p}) = \tilde{d}_i\bigg(\sqrt{\tfrac{2}{\epsilon}}\hat{p}_3\bigg), \tag{B11}$$

where we introduced the dimensionless phase-space kinetic momentum $\hat{p}_3 = \frac{p_3}{m}$. Note that the $d_i(\vec{p})$ implicitly depend on the orthogonal kinetic momentum $\vec{p}_\perp$ by means of $\eta = \epsilon_\perp^2/eE_0$ with $\epsilon_\perp^2 = m^2 + \vec{p}_\perp^2$. Obviously, $d_i(\vec{p})$ do not depend on the time variable t but only on the kinetic momentum $\vec{p}$ such that the Schwinger effect in a constant electric field might be regarded as a quasistatic problem. The phase-space coefficients $\chi^{i=\{1,2,3\}}(\vec{p})$ which allow for the calculation of the DHW functions Eqs. (51)–(56) thus read

$$\chi^1(\vec{p}) = 1 - e^{-((\pi\epsilon_\perp^2)/(4eE_0))}\bigg\{\frac{\epsilon_\perp^2}{2eE_0}\bigg(1 - \frac{p_3}{\omega(\vec{p})}\bigg)d_1(\vec{p}) + \bigg(1 + \frac{p_3}{\omega(\vec{p})}\bigg)d_2(\vec{p}) - \frac{\epsilon_\perp^2}{\sqrt{2eE_0}\omega(\vec{p})}d_3(\vec{p})\bigg\}, \tag{B12}$$

$$\chi^2(\vec{p}) = \frac{\epsilon_\perp}{2\omega(\vec{p})}e^{-((\pi\epsilon_\perp^2)/(4eE_0))}\bigg\{-\frac{\epsilon_\perp^2}{eE_0}d_1(\vec{p}) + 2d_2(\vec{p}) + \sqrt{\frac{2}{eE_0}}p_3 d_3(\vec{p})\bigg\}, \tag{B13}$$

$$\chi^3(\vec{p}) = \frac{eE_0\epsilon_\perp}{2\omega^3(\vec{p})}\bigg(1 + \frac{1}{2}e^{-((\pi\epsilon_\perp^2)/(4eE_0))}\bigg\{-\frac{\epsilon_\perp^2}{eE_0}d_1(\vec{p}) - 2d_2(\vec{p}) + \bigg(\frac{2}{eE_0}\bigg)^{3/2}\omega^3(\vec{p})d_4(\vec{p})\bigg\}\bigg). \tag{B14}$$

2. Sauter-type electric field

We start from the expression for the single-particle momentum distribution function $\tilde{f}(\hat{q}_3, u)$ given in Eq. (94) and introduce the following abbreviations:

$$\tilde{h}_1(\hat{q}_3, u) = |F(\tilde{a}, \tilde{b}, \tilde{c}; u)|^2 \tag{B15}$$

$$\tilde{h}_2(\hat{q}_3, u) = \bigg|\frac{\tilde{a}\,\tilde{b}}{\tilde{c}}F(1 + \tilde{a}, 1 + \tilde{b}, 1 + \tilde{c}; u)\bigg|^2 \tag{B16}$$

$$\tilde{h}_3(\hat{q}_3, u) = -i\frac{\tilde{a}\,\tilde{b}}{\tilde{c}}F(1 + \tilde{a}, 1 + \tilde{b}, 1 + \tilde{c}; u) \times F(\tilde{a}^*, \tilde{b}^*, \tilde{c}^*; u) + \text{c.c.} \tag{B17}$$

and

$$\tilde{\varrho}_1(\hat{q}_3, u) = [\hat{\omega}(\hat{q}_3, u) - (1 - u)\hat{\omega}(\hat{q}_3, 0) - u\hat{\omega}(\hat{q}_3, 1)]^2 \tag{B18}$$

$$\tilde{\varrho}_2(\hat{q}_3, u) = 4\gamma^2\epsilon^2 u^2(1 - u)^2 \tag{B19}$$

$$\tilde{\varrho}_3(\hat{q}_3, u) = 2\gamma\epsilon u(1 - u) \times [\hat{\omega}(\hat{q}_3, u) - (1 - u)\hat{\omega}(\hat{q}_3, 0) - u\hat{\omega}(\hat{q}_3, 1)] \tag{B20}$$

such that $\tilde{f}(\hat{q}_3, u)$ can be written as

$$\tilde{f}(\hat{q}_3, u) = \tilde{N}_f(\hat{q}_3)\bigg(1 + \frac{\hat{\pi}_3(\hat{q}_3, u)}{\hat{\omega}(\hat{q}_3, u)}\bigg)\sum_{i=1}^{3}\tilde{\varrho}_i(\hat{q}_3, u)\tilde{h}_i(\hat{q}_3, u). \tag{B21}$$

Again note that $\{\tilde{a}, \tilde{b}, \tilde{c}\}$, which have been defined in Eq. (85), only depend on $\hat{q}_3$ but not on u. Taking into account the general derivation formula for the Gauss hypergeometric function Eq. (96), we can explicitly calculate the first and second derivative of Eq. (B21). After calculating $\partial_u\tilde{f}(\hat{q}_3, u)$ and $\partial_u^2\tilde{f}(\hat{q}_3, u)$, we are able to determine the coefficients $\tilde{\chi}^{i=\{1,2,3\}}(\hat{q}_3, u)$ according to

$$\tilde{\chi}^1(\hat{q}_3, u) = 1 - \tilde{f}(\hat{q}_3, u) \tag{B22}$$

$$\tilde{\chi}^2(\hat{q}_3, u) = \frac{\gamma\hat{\omega}^2(\hat{q}_3, u)}{2\sqrt{1 + \hat{\kappa}^2}}\partial_u\tilde{f}(\hat{q}_3, u) \tag{B23}$$

$$\tilde{\chi}^3(\hat{q}_3, u) = \frac{2\epsilon\sqrt{1+\hat{\kappa}^2}u(1-u)}{\hat{\omega}^3(\hat{q}_3, u)}\bigg\{1 - \tilde{f}(\hat{q}_3, u) - \frac{\gamma\hat{\omega}^2(\hat{q}_3, u)\hat{\pi}_3(\hat{q}_3, u)}{1+\hat{\kappa}^2}\partial_u\tilde{f}(\hat{q}_3, u) - \frac{\gamma^2\hat{\omega}^4(\hat{q}_3, u)}{4(1+\hat{\kappa}^2)}\partial_u^2\tilde{f}(\hat{q}_3, u)\bigg\}. \quad \text{(B24)}$$

In order to obtain the coefficients in phase-space, we have to perform the variable transformation $\vec{q} \rightarrow \vec{p} + e\vec{A}(t)$, which reads

$$\hat{q}_3 \rightarrow \hat{p}_3 - \frac{2u-1}{\gamma}. \quad \text{(B25)}$$

As a consequence, the quantities $\hat{\omega}(\hat{q}_3, u)$ and $\hat{\pi}_3(\hat{q}_3, u)$ only depend on $\hat{p}_3$ but not on u after performing this variable transformation:

$$\hat{\pi}_3(\hat{q}_3, u) \rightarrow \hat{p}_3, \quad \text{(B26)}$$

$$\hat{\omega}(\hat{q}_3, u) \rightarrow \hat{\omega}(\hat{p}_3) = \sqrt{1+\hat{\kappa}^2+\hat{p}_3^2}. \quad \text{(B27)}$$

On the other hand, any function of the canonical momentum $\hat{q}_3$ only, e.g. $\hat{\omega}(\hat{q}_3, 0)$ or $\hat{\omega}(\hat{q}_3, 1)$, acquires a dependence on both the phase-space kinetic momentum $\hat{p}_3$ and the time variable u:

$$\hat{\pi}_3(\hat{q}_3, 0) = \hat{q}_3 - \frac{1}{\gamma} \rightarrow \hat{p}_3 - \frac{2u}{\gamma}, \quad \text{(B28)}$$

$$\hat{\pi}_3(\hat{q}_3, 1) = \hat{q}_3 + \frac{1}{\gamma} \rightarrow \hat{p}_3 - \frac{2u-2}{\gamma}, \quad \text{(B29)}$$

and

$$\hat{\omega}(\hat{q}_3, 0) \rightarrow \sqrt{1+\hat{\kappa}^2+\left(\hat{p}_3 - \frac{2u}{\gamma}\right)^2}, \quad \text{(B30)}$$

$$\hat{\omega}(\hat{q}_3, 1) \rightarrow \sqrt{1+\hat{\kappa}^2+\left(\hat{p}_3 - \frac{2u-2}{\gamma}\right)^2}. \quad \text{(B31)}$$

Therefore, whereas the functions $\tilde{h}_i(\hat{q}_3, u)$ depend on u solely through the last argument of the Gauss hypergeometric function, the transformed functions $h_i(\hat{p}_3; u)$ have a twofold u dependence: On the one hand, there is still the u dependence due to the last argument. On the other hand, due to the fact that the parameters Eq. (85) were function of $\hat{q}_3$ only, they will depend on both $\hat{p}_3$ and u after the transformation to phase space.

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