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Duality of parity doublets of helicity ± 2 in $D = 2 + 1$ dimensions

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In $D = 2 + 1$ dimensions there are two dual descriptions of parity singlets of helicity ± 1, namely, the self-dual model of first order (in derivatives) and the Maxwell-Chern-Simons theory of second order. Correspondingly, for helicity ± 2 there are four models $S^{(r)}_{SD\pm}$ describing parity singlets of helicities ± 2. They are of first, second, third, and fourth order ($r = 1, 2, 3, 4$), respectively. Here we show that the generalized soldering of the opposite helicity models $S^{(4)}_{SD+}$ and $S^{(4)}_{SD-}$ leads to the linearized form of the new massive gravity suggested by Bergshoeff, Hohm, and Townsend (BHT) similarly to the soldering of $S^{(3)}_{SD+}$ and $S^{(3)}_{SD-}$. We argue why in both cases we have the same result. We also find out a triple master action which interpolates between the three dual models: linearized BHT theory, $S^{(3)}_{SD+} + S^{(3)}_{SD-}$, and $S^{(4)}_{SD+} + S^{(4)}_{SD-}$. By comparing gauge invariant correlation functions we deduce dual maps between those models. In particular, we learn how to decompose the field of the linearized BHT theory in helicity eigenstates of the dual models up to gauge transformations.

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I. INTRODUCTION

In $D = 2 + 1$ dimensions it is possible to have a local description of a massive spin-1 particle by means of one vector field without breaking gauge invariance. Such a theory is called Maxwell-Chern-Simons (MCS) and it was introduced in [1]. It is a second-order (in derivatives) model which describes a parity singlet of helicity $+1$ or -1, according to the sign in front of the Chern-Simons term. The MCS theory is invariant under the usual $U(1)$ gauge transformations $\delta_\xi A_\mu = \partial_\mu \xi$. Another model, named the self-dual (SD) model, was found later in [2]. It shares the particle content of the MCS theory but it is of first order and it has no local symmetries. Part of the SD model, namely, the Chern-Simons term, is invariant under $\delta_\xi A_\mu$. By means of a Noether embedment of this symmetry it is possible to obtain the MCS theory from the SD model (see [3]).

A similar picture applies for spin-2 particles in $D = 2 + 1$. A third-order model, the so-called topologically massive gravity, was introduced in [1] to describe a gravitational theory with a massive graviton of helicity $+2$ or -2, according to the sign in front of the gravitational Chern-Simons term, without breaking the general coordinate invariance of the Einstein-Hilbert action. The linearized version of this model about a flat background will be denoted here by $S^{(3)}_{SD\pm}$, respectively. Later [4], a self-dual model of first order $S^{(1)}_{SD\pm}$, similar to its spin-1 counterpart [2], was introduced as well as a second-order model ($S^{(2)}_{SD\pm}$) analogous to the MCS theory (see [5]). Recently, a new self-dual theory of fourth order ($S^{(4)}_{SD\pm}$) has been found [6,7]. In [6] we have shown that starting with the lowest-order model $S^{(1)}_{SD\pm}$ there is a natural sequence of Noether embedment of gauge symmetries such that $S^{(i)}_{SD\pm} \to S^{(i+1)}_{SD\pm}$ with $i = 1, 2, 3$ culminates at $S^{(4)}_{SD\pm}$. The same reasoning applied on the spin-1 case (SD $\to$ MCS) terminates at the MCS theory. Both MCS and $S^{(4)}_{SD\pm}$ consist of two terms invariant under the same set of local symmetries. Thus, there is no symmetry left for a further embedment. This indicates that those models might be the highest-order models to describe particles of helicity ± 1 and ± 2, respectively, in terms of only one fundamental field.

On the other hand, in the spin-1 case, it is well known that the Proca theory describes in $D = 2 + 1$ a parity doublet of helicities $+1$ and -1 which is the same particle content of two SD models of opposite helicities. Since both models (pair of SD and Proca) have no local symmetries, one might wonder whether they could be identified. In fact, it is easy to show [8] that the pair of SD models of opposite helicities corresponds to a first-order version of the Proca model after some trivial rotation. However, regarding its dual theory, a pair of MCS models of opposite helicities, it is not so easy to identify it with the Proca theory due to the local $U(1)$ symmetry of the MCS theory. An extra "interference term" between the opposite helicities is needed to comply with the local symmetries. This extra term can be produced by the soldering formalism [9] as shown in [10,11]. The idea of fusing two fields representing complementary aspects of some symmetry into one specific combination of fields is the core of the soldering procedure (see also [12,13]).

In the spin-2 case it is the Fierz-Pauli [14] theory which plays the role of the Proca theory. Once again it is possible to show [8] that the pair $S^{(1)}_{SD+} + S^{(1)}_{SD-}$ is equivalent, after a rotation, to a first-order version of the Fierz-Pauli (FP) theory while the dual pair $S^{(2)}_{SD\pm}$ must be soldered in order

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to furnish the FP theory. Remarkably, the soldering of a pair of third-order models $S^{(3)}_{SD\pm}$ does not reproduce the FP theory and leads to a unitary [15] fourth-order theory describing a parity doublet of helicities $+2$ and -2 just like the FP theory. This model corresponds precisely to the linearized version of the recently proposed new massive gravity theory [16], henceforth the linearized Bergshoeff, Hohm, and Townsend (LBHT) theory. It is therefore natural to try to solder also a pair of the top models $S^{(4)}_{SD\pm}$. In the next section we carry this out and end up again with the LBHT theory. This suggests the uniqueness of the LBHT model as a unitary higher-derivative model describing a parity doublet of helicities ± 2 in $D = 2 + 1$.

In previous examples of soldered second-order models for spin-1 [10,11] and spin-2 [8] it turns out that the theories before and after soldering can be shown to be equivalent at quantum level. This has been shown in [17–19] by means of the master action technique [20]. In the second part of this work (Sec. III) we define a triple master action which interpolates between the linearized BHT theory, $S^{(3)}_{SD+} + S^{(3)}_{SD-}$, and $S^{(4)}_{SD+} + S^{(4)}_{SD-}$, thus proving the quantum equivalence of all three models in agreement with the soldering predictions of [8] and Sec. II of the present work. The introduction of convenient source terms allows us to derive dual maps between gauge invariants of those theories.

II. SOLDERING $S^{(4)}_{SD+}$ AND $S^{(4)}_{SD-}$

It is necessary to fix the notation before we go on. Throughout this work indices are lowered and raised by the flat metric: $\eta_{\alpha\beta} = \mathrm{diag}(-, +, +)$. Inside integrals we use a shorthand notation similar to differential forms:

$$\int A \cdot dB \equiv \int d^3x A^{\mu\alpha} \epsilon_{\mu}{}^{\nu\lambda} \partial_\nu B_{\lambda\alpha}. \qquad (1)$$

Frequent use will be made of the rank two tensor $\Omega_\alpha{}^\gamma(h) = \epsilon^{\gamma\mu\nu}[\partial_\alpha h_{\nu\mu} - \partial_\mu(h_{\nu\alpha} + h_{\alpha\nu})]$ and of the symmetric and antisymmetric operators $\theta_{\mu\nu} = (\eta_{\mu\nu} - \partial_\mu\partial_\nu/\Box)$ and $E^{\mu\nu} = \epsilon^{\mu\nu\alpha}\partial_\alpha$, respectively.

Some of the actions here can be interpreted as quadratic truncations (linearized versions) about a flat background. In particular, with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the linearized Einstein-Hilbert action, linearized gravitational Chern-Simons term, and linearized K term [16] can be written, respectively, as

$$\int d^3x\sqrt{-g}R|_{hh} = \int d^3x h_{\mu\alpha}E^{\mu\lambda}E^{\alpha\gamma}h_{\gamma\lambda} = -\frac{1}{2}\int d^3x h \cdot d\Omega(h), \qquad (2)$$

$$\frac{1}{2}\int d^3x\Big[\epsilon^{\mu\nu\rho}\Gamma^{\epsilon}_{\mu\gamma}\Big(\partial_\nu\Gamma^{\gamma}_{\epsilon\rho} + \frac{2}{3}\Gamma^{\gamma}_{\nu\delta}\Gamma^{\delta}_{\rho\epsilon}\Big)\Big]_{hh} = -\int d^3x h_{\nu\mu}\Box\theta^{\nu\epsilon}E^{\mu\rho}h_{\epsilon\rho} = \frac{1}{4}\int d^3x\Omega(h)\cdot d\Omega(h), \qquad (3)$$

$$\int d^3x\Big[\sqrt{-g}\Big(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\Big)\Big]_{hh} = \frac{1}{2}\int d^3x h_{\sigma\mu}\Box^2(2\theta^{\sigma\gamma}\theta^{\mu\nu} - \theta^{\gamma\nu}\theta^{\mu\sigma})h_{\nu\gamma} = -\frac{1}{8}\int d^3x\Omega(h)\cdot d\Omega(\Omega(h)). \qquad (4)$$

Now we start with a couple of new self-dual models recently obtained in [6,7]. Each model $S^{(4)}_{SD\pm}$ below, though of fourth order in derivatives, is unitary [7,21] and describes one massive mode of mass $m_\pm$ and helicity ± 2 in $D = 2 + 1$ dimensions, respectively. In a convenient notation for the soldering approach we write

$$S^{(4)}_{SD+}(A) = \int d^3x\Big[\frac{1}{4}A_{\rho\sigma}\Box^2(2\theta^{\rho\nu}\theta^{\sigma\mu} - \theta^{\rho\sigma}\theta^{\mu\nu})A_{\mu\nu} + \frac{m_+}{2}A_{\lambda\mu}\Box\theta^{\lambda\alpha}E^{\mu\delta}A_{\alpha\delta}\Big], \qquad (5)$$

$$S^{(4)}_{SD-}(B) = \int d^3x\Big[\frac{1}{4}B_{\rho\sigma}\Box^2(2\theta^{\rho\nu}\theta^{\sigma\mu} - \theta^{\rho\sigma}\theta^{\mu\nu})B_{\mu\nu} - \frac{m_-}{2}B_{\lambda\mu}\Box\theta^{\lambda\alpha}E^{\mu\delta}B_{\alpha\delta}\Big]. \qquad (6)$$

The tensor fields are symmetric $A_{\alpha\beta} = A_{\beta\alpha}$, $B_{\alpha\beta} = B_{\beta\alpha}$. The first term in both actions above corresponds exactly to (4), and the second one is proportional to the quadratic truncation of the gravitational Chern-Simons term (3). As suggested in [1], the full nonlinear version of (3) together with the Einstein-Hilbert action build up the so-called topologically massive gravity. Since the Einstein-Hilbert action is substituted by the fourth-order K term in $S^{(4)}_{SD\pm}$, we may call such models a linearized higher-derivative topologically massive gravity.

Now let us recall the basic idea of the soldering procedure. The actions (5) and (6) are invariant under independent global shifts $\delta A_{\mu\nu} = \omega_{\mu\nu}$; $\delta B_{\mu\nu} = \tilde{\omega}_{\mu\nu}$. In the soldering procedure [8,10,11] one lifts the global shift symmetry to a local one and ties the fields $A_{\mu\nu}$ and $B_{\mu\nu}$ together by imposing that their local symmetry transformations are proportional to each other:

$$\delta A_{\mu\nu} = \omega_{\mu\nu}, \qquad \delta B_{\mu\nu} = \alpha\omega_{\mu\nu}, \qquad (7)$$

where α is so far an arbitrary constant. From (5)–(7) we can write down

$$\delta(S^{(4)}_{SD+}(A) + S^{(4)}_{SD-}(B)) = \int d^3x J^{\sigma}_{\alpha}\Box\theta^{\rho\alpha}\omega_{\rho\sigma}, \qquad (8)$$

with the Noether-like current J^σ_α given by

$$J^\sigma_\alpha = \frac{\Box}{2}(2\theta^{\sigma\mu}C_{\mu\alpha} - \eta^\sigma_\alpha\theta^{\mu\nu}C_{\mu\nu}) + E^{\sigma\delta}D_{\alpha\delta}, \quad (9)$$

where we have used the following field combinations:

$$C_{\mu\nu} = A_{\mu\nu} + \alpha B_{\mu\nu}; \quad D_{\alpha\delta} = m_+ A_{\alpha\delta} - \alpha m_- B_{\alpha\delta}. \quad (10)$$

At this point we may try to cancel the variation (8) by the introduction of an auxiliary field H^α_σ with a specific variation $\delta H^\alpha_\sigma = -\Box\theta^{\rho\alpha}\omega_{\rho\sigma}$ such that

$$\delta\Big(S^{(4)}_{\rm SD+}(A) + S^{(4)}_{\rm SD-}(B) + \int d^3x J^\sigma_\alpha H^\alpha_\sigma\Big) = \int d^3x \delta J^\sigma_\alpha H^\alpha_\sigma. \quad (11)$$

Since

$$\delta C_{\mu\nu} = (1+\alpha^2)\omega_{\mu\nu}, \qquad \delta D_{\mu\nu} = (m_+ - \alpha^2 m_-)\omega_{\mu\nu}, \quad (12)$$

we have

$$\begin{aligned}\delta J^\sigma_\alpha &= \frac{(1+\alpha^2)}{2}(2\Box\theta^{\sigma\mu}\omega_{\mu\alpha} - \eta^\sigma_\alpha\Box\theta^{\mu\nu}\omega_{\mu\nu}) \\ &\quad + (m_+ - \alpha^2 m_-)E^{\sigma\delta}\omega_{\alpha\delta} \\ &= -\frac{(1+\alpha^2)}{2}(2\delta H^\sigma_\alpha - \eta^\sigma_\alpha \delta H^\mu_\mu) \\ &\quad + (m_+ - \alpha^2 m_-)E^{\sigma\delta}\omega_{\alpha\delta}. \end{aligned} \quad (13)$$

In order to write the Lagrangian density on the right-hand side of (11) as a local function of the auxiliary field H^α_σ and its variation δH^α_σ, we are forced to choose

$$\alpha = \pm\sqrt{\frac{m_+}{m_-}} \quad (14)$$

which leads to the soldering action S_S invariant under the local transformations (7),

$$\begin{aligned} S_S &= S^{(4)}_{\rm SD+}(A) + S^{(4)}_{\rm SD-}(B) \\ &\quad + \int d^3x \Big[H^\alpha_\sigma J^\sigma_\alpha + \frac{(1+\alpha^2)}{4}(2H^\alpha_\sigma H^\sigma_\alpha - H^2)\Big], \end{aligned} \quad (15)$$

where $H = H^\alpha_\alpha$. Solving the algebraic equations of motion of H^β_ν we can invert them in terms of J^σ_ν and rewrite the expression (15) as

$$\begin{aligned} S_S &= S^{(4)}_{\rm SD+}(A) + S^{(4)}_{\rm SD-}(B) \\ &\quad - \frac{1}{2(1+\alpha^2)}\int d^3x (J^\sigma_\alpha J^\alpha_\sigma - J^2), \end{aligned} \quad (16)$$

where $J = J^\mu_\mu$. The quadratic term in the Noether current is interpreted [10,11] as an interference term between the opposite helicity modes necessary to patch together the actions $S^{(4)}_{\rm SD+}$ and $S^{(4)}_{\rm SD-}$ into a local theory invariant under (7). Replacing J^ν_σ from (9) in (16) we find

$$\begin{aligned} S_S &= S^{(4)}_{\rm SD+}(A) + S^{(4)}_{\rm SD-}(B) - \frac{1}{(1+\alpha^2)} \\ &\quad \times \int d^3x \Big[\frac{1}{4}C_{\mu\nu}\Box^2(2\theta^{\alpha\mu}\theta^{\beta\nu} - \theta^{\mu\nu}\theta^{\alpha\beta})C_{\alpha\beta} \\ &\quad + C_{\mu\nu}\Box\theta^{\sigma\mu}E^{\nu\gamma}D_{\sigma\gamma} - \frac{1}{2}D_{\alpha\nu}E^{\sigma\nu}E^{\alpha\gamma}D_{\sigma\gamma}\Big]. \end{aligned} \quad (17)$$

After some algebra it is possible to rewrite the soldered Lagrangian density entirely in terms of the soldered field $h_{\mu\nu} = (\alpha A_{\mu\nu} - B_{\mu\nu})/\sqrt{m_+ m_-}$ which is invariant under the local shifts (7) with α being any of the two possibilities given in (14), namely,

$$\begin{aligned} \mathcal{L}_S = \frac{1}{(1+\alpha^2)}\Big[&\frac{1}{4m_+m_-}h_{\mu\nu}\Box^2(2\theta^{\alpha\mu}\theta^{\nu\beta} - \theta^{\mu\nu}\theta^{\alpha\beta})h_{\alpha\beta} \\ &- \frac{m_+ - m_-}{2m_+m_-}h_{\mu\nu}\Box\theta^{\sigma\mu}E^{\nu\gamma}h_{\sigma\gamma} - \frac{1}{2}h_{\mu\nu}E^{\mu\alpha}E^{\nu\gamma}h_{\alpha\gamma}\Big]. \end{aligned} \quad (18)$$

By using $\alpha = \pm\sqrt{m_+/m_-}$ we can check that each of the terms in (18) is invariant under the discrete symmetry $(m_+, m_-) \to (-m_-, -m_+)$, which interchanges $S^{(4)}_{\rm SD+} \leftrightharpoons S^{(4)}_{\rm SD-}$. More importantly, up to an overall constant, the Lagrangian $\mathcal{L}_S$ corresponds precisely to the quadratic truncation of the generalized ($m_+ \neq m_-$) new massive gravity theory of [16]

$$\begin{aligned} 2(1+\alpha^2)\mathcal{L}_S = \Big[&\sqrt{-g}R - \frac{m_+ - m_-}{2m_+m_-}\epsilon^{\mu\nu\rho}\Gamma^\epsilon_{\mu\gamma} \\ &\times\Big(\partial_\nu\Gamma^\gamma_{\epsilon\rho} + \frac{2}{3}\Gamma^\gamma_{\nu\delta}\Gamma^\delta_{\rho\epsilon}\Big) - \frac{\sqrt{-g}}{m_+m_-}\Big(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\Big)\Big]_{hh}. \end{aligned} \quad (19)$$

This is a bit surprising, because we have found the same soldered theory $\mathcal{L}_S$ in [8] where we have started with two third-order self-dual models $S^{(3)}_{\rm SD+}$ and $S^{(3)}_{\rm SD-}$. This seems to indicate that the LBHT theory might be the highest-order self-consistent (unitary) theory describing a parity doublet of helicity ±2.

In order to get some clue on why the soldering of $S^{(4)}_{\rm SD+}$ and $S^{(4)}_{\rm SD-}$ leads to the same theory obtained from $S^{(3)}_{\rm SD+}$ and $S^{(3)}_{\rm SD-}$, we give below a rough argument dropping the fields' indices. The key point is some freedom in defining the Noether current due to an integration by parts. In both cases we can write

$$\delta(S^{(r)}_{\rm SD+}(A) + S^{(r)}_{\rm SD+}(B)) = \int d^3x J^{(r)}\partial^p\omega. \quad (20)$$

Where $r = 3, 4$. The symbol ∂^p stands for some differential operator of order p whose explicit form is not important and may be different in each expression. So p simply counts the order of some differential operator. Since the $S^{(r)}_{\rm SD\pm}$ model contains a term of order r plus another one of order $r-1$, the freedom to integrate by parts in (20) allows us to choose any integer value for p such that $p = 0, 1, \cdots, r-1$ and redefine the Noether current accordingly:

$$J^{(3)} = \partial^{3-p} D + \partial^{2-p} C, \tag{21}$$

$$J^{(4)} = \partial^{4-p} C + \partial^{3-p} D, \tag{22}$$

where $C = A + \alpha B$ and $D = m_+ A - \alpha m_- B$ [see (10)]. The term with an odd number of derivatives in $S^{(r)}_{\mathrm{SD}\pm}$ carries the sign of the particle's helicity and gives rise to the D combination in (21) and (22). The formula (20) suggests the auxiliary field variation $\delta H = -\partial^p \omega$ which leads to [see (11)]

$$\delta\Big(S^{(r)}_{\mathrm{SD}+}(A) + S^{(r)}_{\mathrm{SD}+}(B) + \int d^3x J^{(r)} H \Big) = \int d^3x \delta J^{(r)} H. \tag{23}$$

However, using (12) in (21) and (22) we have

$$\delta J^{(3)} = -(m_+ - \alpha^2 m_-)\partial^{3-2p} H - (1+\alpha^2)\partial^{2-2p} H, \tag{24}$$

$$\delta J^{(4)} = -(1+\alpha^2)\partial^{4-2p} H - (m_+ - \alpha^2 m_-)\partial^{3-2p} H. \tag{25}$$

Therefore [see (23)] in order to avoid any dynamics for the auxiliary field H we must choose $\alpha^2 = m_+/m_-$ in both cases $r = 3, 4$ and $p = 1$ for $r = 3$ while $p = 2$ if $r = 4$ as we have done in [8] and here, respectively. In fact, the above argument holds also for the generalized soldering of $S^{(2)}_{\mathrm{SD}+}$ and $S^{(2)}_{\mathrm{SD}-}$ carried out in [8] (see also [22]) and the generalized soldering of two MCS theories of opposite helicities ± 1 with different masses [11] (see also [10]); in such examples $p = 0$. Finally, since in both cases $r = 3, 4$ we have $\delta J^{(r)} = -(1+\alpha^2)\delta H$ and the Noether currents will be the sum of a 1st-order and a 2nd-order term, it is clear that the interference term obtained after the elimination of the auxiliary field will be quadratic in the current and can only contain terms of order 4, 3, and 2 which lead dimensionally to the generalized BHT theory $\mathcal{L}_S$.

III. MASTER ACTION AND DUAL MAPS

In the soldering procedure there is *a priori* no guarantee of quantum equivalence between the initial pair of field theories describing the opposite helicity states and the final soldered field theory. In the spin-1 case where a couple of MCS theories of opposite helicities are soldered into a Maxwell-Chern-Simons-Proca theory, even if $m_+ \neq m_-$, it is possible to prove at quantum level the equivalence of those theories before and after soldering by means of a master action [17,18]. Likewise, in the spin-2 case one can also solder [8] the opposite helicities' second-order models $S^{(2)}_{\mathrm{SD}+}$ and $S^{(2)}_{\mathrm{SD}-}$ into a kind of spin-2 Maxwell-Chern-Simons-Proca model where the role of the Maxwell-Proca terms is played by the Fierz-Pauli theory. Once again, those theories (before and after soldering) are known to be quantum equivalent [19]. On one hand, such results are not surprising since the particle content of both theories before and after soldering is the same; however, the local symmetries are in general not the same and the existence of a local dual map between gauge invariant objects is not trivial. From the above discussion and from what we have learned in the last section it is quite suggestive to think about a master action which interpolates among a couple of $S^{(4)}_{\mathrm{SD}\pm}$, a couple of $S^{(3)}_{\mathrm{SD}\pm}$, and the LBHT theory. For simplicity we assume hereafter $m_+ = m_-$ and suggest the following master action,

$$\begin{aligned} S_M[h, H, A, B] = &\frac{1}{2}\int h \cdot d\Omega(h) \\ &- \frac{1}{8m^2}\int \Omega(h)\cdot d\Omega(\Omega(h)) \\ &+ \frac{1}{2}\int \Big(H + \frac{\Omega(h)}{2m}\Big)\cdot d\Omega\Big(H + \frac{\Omega(h)}{2m}\Big) \\ &+ \frac{1}{4m}\int \Omega(a - A)\cdot d\Omega(a - A) \\ &- \frac{1}{4m}\int \Omega(b - B)\cdot d\Omega(b - B), \end{aligned} \tag{26}$$

where all fields above are second-rank symmetric tensors with $a_{\alpha\beta}$ and $b_{\alpha\beta}$ linear combinations of h and H (dropping the indices):

$$a = \frac{(h+H)}{\sqrt{2}}; \qquad b = \frac{(h-H)}{\sqrt{2}}. \tag{27}$$

The first two terms in (26) correspond to the LBHT theory. Next, there are three mixing terms. The first one is a quadratic truncation of the Einstein-Hilbert term [see (2)], while the last two are quadratic truncations of the gravitational Chern-Simons term [see (3)]. All mixing terms have no particle content and that feature plays a fundamental role in the interpolation between the different models [18,23]. In order to verify the equivalence between correlation functions of gauge invariants, we are going to add a source term to S_M. At this point we can ask what is the proper source term. The fourth-order self-dual model is invariant under linearized general coordinate transformations $\delta_\xi h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ and a linearized local Weyl symmetry $\delta_\phi h_{\mu\nu} = \phi \eta_{\mu\nu}$. On the other hand, the quadratic Einstein-Hilbert term present in the LBHT and in $S^{(3)}_{\mathrm{SD}\pm}$ breaks the local Weyl symmetry. The basic idea is to use a source term invariant under a set of symmetries common to all models to be interpolated. The lowest-order source term invariant under $\delta_\xi h_{\mu\nu}$ is given by

$$\begin{aligned} \int d^3x j^{\mu\nu} F_{\mu\nu}(h) &\equiv \int d^3x j^{\mu\nu} E^\gamma_\mu E^\lambda_\nu h_{\gamma\lambda} \\ &= -\frac{1}{2}\int j \cdot d\Omega(h). \end{aligned} \tag{28}$$

So for simplicity we first define the generating functional with only one type of source:

$$\mathcal{W}[j] = \int \mathcal{D}h_{\mu\nu}\mathcal{D}H_{\mu\nu}\mathcal{D}A_{\mu\nu}\mathcal{D}B_{\mu\nu} \times \exp i\left[S_M(h, H, A, B) - \frac{1}{2}\int j\cdot d\Omega(h)\right]. \quad (29)$$

It is easy to see that if we do the trivial shifts, dropping the indices, $A \to A + a$, $B \to B + b$, and $H \to H - \Omega(h)/2m$ in (29), the last three terms of S_M decouple completely into three terms without particle content. Integrating over $A_{\mu\nu}$, $B_{\mu\nu}$, and $H_{\mu\nu}$ we obtain up to an overall constant:

$$\mathcal{W}[j] = \int \mathcal{D}h_{\mu\nu} \exp i\left[S_{\rm LBHT}(h) - \frac{1}{2}\int j\cdot d\Omega(h)\right]. \quad (30)$$

Therefore, the spectrum of S_M coincides with the one of the quadratic truncation of the BHT theory for equal masses, i.e., a parity doublet of helicities ± 2 and mass "m." In the next two subsections we are going to derive the dual models to LBHT from (29).

A. Duality between $S^{(3)}_{\rm SD+} + S^{(3)}_{\rm SD-}$ and the linearized BHT theory

For a demonstration of equivalence of LBHT with one couple of third-order self-dual models $S^{(3)}_{\rm SD\pm}$, we rewrite the first three terms of S_M. The generating functional (29) becomes

$$\begin{aligned}\mathcal{W}[j] = &\int \mathcal{D}h_{\mu\nu}\mathcal{D}H_{\mu\nu}\mathcal{D}A_{\mu\nu}\mathcal{D}B_{\mu\nu} \\ &\times \exp i\int\Big[\frac{1}{2}h\cdot d\Omega(h) + \frac{1}{2}H\cdot d\Omega(H) \\ &+ \frac{1}{2m}\Omega(h)\cdot d\Omega(H) + \frac{1}{4m}\Omega(a-A)\cdot d\Omega(a-A) \\ &- \frac{1}{4m}\Omega(b-B)\cdot d\Omega(b-B) - \frac{1}{2}j\cdot d\Omega(h)\Big].\end{aligned} \quad (31)$$

After the shifts $A \to A + a$ and $B \to B + b$ we can integrate over A and B and get rid of the two third-order Chern-Simons mixing terms which play no role in this subsection. Then, inverting (27) we can decouple the fields in (31). Thus, the generating functional, up to an overall constant, can be rewritten as

$$\mathcal{W}[j] = \int \mathcal{D}a_{\mu\nu}\mathcal{D}b_{\mu\nu} \exp i\left[S^{(3)}_{\rm SD+}(a) + S^{(3)}_{\rm SD-}(b) - \frac{1}{2^{3/2}}\int j\cdot d\Omega(a+b)\right], \quad (32)$$

where

$$S^{(3)}_{\rm SD\pm}(a) = -\int d^3x\left[a_{\mu\alpha}E^{\mu\lambda}E^{\alpha\gamma}a_{\gamma\lambda} \pm \frac{1}{m}a_{\alpha\mu}\Box\theta^{\alpha\gamma}E^{\beta\mu}a_{\gamma\beta}\right]. \quad (33)$$

The first term represents the quadratic truncation of the Einstein-Hilbert action with a negative sign, while the second one is a similar truncation of the gravitational Chern-Simons action [see (2) and (3), respectively]. Differentiating (30) and (32) with respect to the source $j^{\mu\nu}$ we have the following relationship between the correlation functions:

$$\langle F_{\mu_1\nu_1}[h(x_1)]\cdots F_{\mu_N\nu_N}[h(x_N)]\rangle_{\rm LBHT} = \left\langle \frac{F_{\mu_1\nu_1}[(a+b)(x_1)]}{\sqrt{2}}\cdots\frac{F_{\mu_N\nu_N}[(a+b)(x_N)]}{\sqrt{2}}\right\rangle_{S^{(3)}_{\rm SD+}(a)+S^{(3)}_{\rm SD-}(b)}. \quad (34)$$

Consequently, the relevant gauge invariant quantity in the LBHT theory $F_{\mu\nu}[h(x)]$ is given in terms of a (gauge invariant) specific combination of the fields with well defined helicity: $F_{\mu_1\nu_1}[(a+b)(x_N)]/\sqrt{2}$. However, for a complete proof of equivalence between the decoupled pair $S^{(3)}_{\rm SD\pm}$ and the linearized BHT theory we should be able to compute correlation functions of $F_{\mu\nu}[a(x)]$ and $F_{\mu\nu}[b(x)]$ separately in terms of correlators of gauge invariant objects in the LBHT theory. With this purpose in mind we define a new generating function by changing the source term in (29), i.e.,

$$\begin{aligned}\mathcal{W}[j_+, j_-] = &\int \mathcal{D}h_{\mu\nu}\mathcal{D}H_{\mu\nu}\mathcal{D}A_{\mu\nu}\mathcal{D}B_{\mu\nu} \\ &\times \exp i\Big\{S_M(h, H, A, B) \\ &- \frac{1}{2}\int[j_+\cdot d\Omega(h) + j_-\cdot d\Omega(H)]\Big\}.\end{aligned} \quad (35)$$

The next steps will be totally equivalent to those we have done previously, except for the fact that the source terms are now redefined. Therefore we are going to suppress some details. Using the same sequence of shifts that we have done from (29) to (30) we can verify that (35) after some rearrangement is rewritten as

$$\begin{aligned}\mathcal{W}[j_+, j_-] = &\int \mathcal{D}h_{\mu\nu}\mathcal{D}H_{\mu\nu} \exp i\Big\{S_{\rm LBHT}(h) \\ &- \frac{1}{2}\int\Big[j_+\cdot d\Omega(h) - \frac{j_-\cdot d\Omega(\Omega(h))}{2m}\Big] \\ &+ \frac{1}{2}\int\Big(H - \frac{j_-}{2}\Big)\cdot d\Omega\Big(H - \frac{j_-}{2}\Big) \\ &- \frac{1}{8}\int j_-\cdot d\Omega(j_-)\Big\},\end{aligned} \quad (36)$$

and shifting $H \to H + j_-/2$ in (36), we can decouple $H_{\alpha\beta}$ from the sources $(j_-)_{\alpha\beta}$ and obtain an Einstein-Hilbert term for the field $H_{\alpha\beta}$ which has no particle content. Integrating over such a field we have, up to an overall constant,

$$\mathcal{W}[j_+, j_-] = \int \mathcal{D}h_{\mu\nu} \exp i\bigg\{S_{\rm LBHT}(h) - \frac{1}{2}\int\bigg[j_+\cdot d\Omega(h) - \frac{j_-\cdot\Omega(\Omega(h))}{2m}\bigg] + \mathcal{O}(j^2)\bigg\}. \quad (37)$$

On the other hand, similarly to what we have done from (31) to (32) we can write the expression for the generating functional $\mathcal{W}[j_+, j_-]$ in terms of the $S^{(3)}_{\rm SD\pm}$ models as

$$\mathcal{W}[j_+, j_-] = \int \mathcal{D}a_{\mu\nu}\mathcal{D}b_{\mu\nu} \exp i\bigg\{S^{(3)}_{\rm SD+}(a) + S^{(3)}_{\rm SD-}(b) - \frac{1}{2^{3/2}}\int[j_+\cdot d\Omega(a+b) + j_-\cdot d\Omega(a-b)]\bigg\}. \quad (38)$$

The source terms in (38) suggest the redefinition,

$$\tilde{j}_+ = \frac{j_+ + j_-}{\sqrt{2}}, \qquad \tilde{j}_- = \frac{j_+ - j_-}{\sqrt{2}}, \quad (39)$$

which gives us

$$\mathcal{W}[j_+, j_-] = \int \mathcal{D}a_{\mu\nu}\mathcal{D}b_{\mu\nu} \exp i\bigg\{S^{(3)}_{\rm SD+}(a) + S^{(3)}_{\rm SD-}(b) - \frac{1}{2}\int[\tilde{j}_+\cdot d\Omega(a) + \tilde{j}_-\cdot d\Omega(b)]\bigg\}. \quad (40)$$

Back in (37) we have

$$\mathcal{W}[j_+, j_-] = \int \mathcal{D}h_{\mu\nu} \exp i\bigg\{S_{\rm LBHT}(h) - \frac{1}{2^{3/2}m^2} \times \int\bigg[\tilde{j}_+\cdot d\Omega\bigg(h - \frac{\Omega(h)}{2m}\bigg) + \tilde{j}_-\cdot d\Omega\bigg(h + \frac{\Omega(h)}{2m}\bigg)\bigg] + \mathcal{O}(\tilde{j}^2)\bigg\}. \quad (41)$$

Differentiating (40) and (41) with respect to the sources $\tilde{j}_+$ and $\tilde{j}_-$ it is possible to map correlations functions of the gauge invariant objects $F_{\mu\nu}[a(x)]$ and $F_{\mu\nu}[b(x)]$ separately in terms of gauge invariants from the LBHT theory as follows,

$$2^{N/2}\langle F_{\mu_1\nu_1}[a(x_1)]\cdots F_{\mu_N\nu_N}[a(x_N)]\rangle_{S^{(3)}_{\rm SD+}(a)} = \langle (F_{\mu_1\nu_1} - G_{\mu_1\nu_1})[h(x_1)]\cdots(F_{\mu_N\nu_N} - G_{\mu_N\nu_N}) \times [h(x_N)]\rangle_{\rm LBHT} + \text{C.T}, \quad (42)$$

$$2^{N/2}\langle F_{\mu_1\nu_1}[b(x_1)]\cdots F_{\mu_N\nu_N}[b(x_N)]\rangle_{S^{(3)}_{\rm SD-}(b)} = \langle (F_{\mu_1\nu_1} + G_{\mu_1\nu_1})[h(x_1)]\cdots(F_{\mu_N\nu_N} + G_{\mu_N\nu_N}) \times [h(x_N)]\rangle_{\rm LBHT} + \text{C.T}, \quad (43)$$

where C. T means contact terms which are due to the quadratic terms in the sources while

$$G_{\alpha\beta}[a(x)] = -\frac{\Box}{m}(E^{\rho}_{\alpha}\theta^{\delta}_{\beta} + E^{\rho}_{\beta}\theta^{\delta}_{\alpha})a_{\rho\delta}(x) \quad (44)$$

is invariant not only under linearized general coordinate transformations $\delta_\xi a_{\rho\delta} = \partial_\rho\xi_\delta + \partial_\delta\xi_\rho$ but also under linearized Weyl symmetry $\delta_\phi a_{\rho\delta} = \phi\eta_{\rho\delta}$ (use $E^{\rho}_{\alpha}\theta_{\rho\beta} = E_{\beta\alpha}$).

From (42) and (43) the dual maps are

$$F_{\mu\nu}[a(x)]|_{S^{(3)}_{\rm SD+}(a)} \leftrightarrow \frac{1}{\sqrt{2}}(F_{\mu\nu} - G_{\mu\nu})[h(x)]|_{\rm LBHT}, \quad (45)$$

$$F_{\mu\nu}[b(x)]|_{S^{(3)}_{\rm SD-}(b)} \leftrightarrow \frac{1}{\sqrt{2}}(F_{\mu\nu} + G_{\mu\nu})[h(x)]|_{\rm LBHT}. \quad (46)$$

They are clearly consistent with (34) and the decomposition of $F_{\mu\nu}[h(x)]$ into the linear combination of gauge invariant helicity eigenstates $F_{\mu\nu}[(a+b)(x)]/\sqrt{2}$. The reader might ask what happens when we subtract (45) from (46). In this case we have $F_{\mu\nu}[(a-b)(x)]$ calculated in the $S^{(3)}_{\rm SD+}(a) + S^{(3)}_{\rm SD-}(b)$ theory in terms of $G_{\mu\nu}[h(x)]$ calculated in the linearized BHT theory. If we recall that $\Box\theta^{\mu}_{\alpha} = -E^{\mu\nu}E_{\nu\alpha}$ it is clear from (44) that $G_{\alpha\beta}[h(x)]$ can be written as a first-order differential operator applied on $F_{\mu\nu}[h(x)]$. Therefore correlation functions of $F_{\mu\nu}[(a-b)(x)]$ are given in terms of correlation functions of $F_{\mu\nu}[(a+b)(x)]$ both calculated in the $S^{(3)}_{\rm SD+}(a) + S^{(3)}_{\rm SD-}(b)$ theory, though a and b are independent helicity eigenstates. There is in fact no contradiction since we have a nontrivial first-order differential operator relating both correlation functions. This is typical of self-dual theories and it happens also when we have a pair of spin-1 MCS theories of opposite helicities [see formulas (3.9) and (3.10) of [18] for a simpler example].

In summary, we have a complete equivalence between $S^{(3)}_{\rm SD+} + S^{(3)}_{\rm SD-}$ and $S_{\rm LBHT}$. In the next subsection we show how the third-order linearized gravitational Chern-Simons mixing terms in the master action S_M allow us to interpolate also between the fourth-order models $S^{(4)}_{\rm SD+} + S^{(4)}_{\rm SD-}$ and $S_{\rm LBHT}$.

B. Duality between $S^{(4)}_{\rm SD+} + S^{(4)}_{\rm SD-}$ and the linearized BHT theory

From (27) and the intermediate expression (31) we get

$$\mathcal{W}[j] = \int \mathcal{D}a_{\mu\nu}\mathcal{D}b_{\mu\nu}\mathcal{D}A_{\mu\nu}\mathcal{D}B_{\mu\nu} \times \exp i\bigg\{S^{(3)}_{\rm SD+}(a) + S^{(3)}_{\rm SD-}(b) + \frac{1}{4m}\int\Omega(a-A)\cdot d\Omega(a-A) - \frac{1}{4m}\int\Omega(b-B)\cdot d\Omega(b-B) - \frac{1}{2^{3/2}}\int j\cdot d\Omega(a+b)\bigg\}. \quad (47)$$

The factors in front of the linearized gravitational Chern-Simons mixing terms in S_M have been fine-tuned to cancel the third-order terms of $S^{(3)}_{SD+}(a) + S^{(3)}_{SD-}(b)$. After those cancellations and some rearrangements we get

$$\mathcal{W}[j] = \int \mathcal{D}a_{\mu\nu}\mathcal{D}b_{\mu\nu}\mathcal{D}A_{\mu\nu}\mathcal{D}B_{\mu\nu} \times \exp i\Big\{S^{(4)}_{SD+}(A) + S^{(4)}_{SD-}(B) + \frac{1}{2}\int\Big(a - \frac{\Omega(A)}{2m}\Big)\cdot d\Omega\Big(a - \frac{\Omega(A)}{2m}\Big) + \frac{1}{2}\int\Big(b + \frac{\Omega(B)}{2m}\Big)\cdot d\Omega\Big(b + \frac{\Omega(B)}{2m}\Big) - \frac{1}{2^{3/2}}\int j\cdot d\Omega(a+b)\Big\}. \quad (48)$$

It is easy to see that if we make $a \to a + \Omega(A)/2m$ and $b \to b - \Omega(B)/2m$ we have

$$\mathcal{W}[j] = \int \mathcal{D}a_{\mu\nu}\mathcal{D}b_{\mu\nu}\mathcal{D}A_{\mu\nu}\mathcal{D}B_{\mu\nu} \times \exp i\Big[S^{(4)}_{+}(A) + S^{(4)}_{-}(B) - \frac{1}{2^{5/2}m}\int j\cdot d\Omega(\Omega(A-B)) + \frac{1}{2}\int(a-\sqrt{2}j)\cdot d\Omega(a-\sqrt{2}j) + \frac{1}{2}\int(b-\sqrt{2}j)\cdot d\Omega(b-\sqrt{2}j) + \mathcal{O}(j^2)\Big]. \quad (49)$$

After trivial shifts and integrating over $a_{\alpha\beta}$ and $b_{\alpha\beta}$ fields we deduce up to an overall constant:

$$\mathcal{W}[j] = \int \mathcal{D}A_{\mu\nu}\mathcal{D}B_{\mu\nu}\exp i\Big[S^{(4)}_{SD+}(A) + S^{(4)}_{SD-}(B) - \frac{1}{2^{5/2}m}\int j\cdot d\Omega(\Omega(A-B)) + \mathcal{O}(j^2)\Big]. \quad (50)$$

Differentiating (30) and (50) with respect to the source j we obtain the following relationship between correlation functions:

$$2^{N/2}\langle F_{\mu_1\nu_1}[h(x_1)]\cdots F_{\mu_N\nu_N}[h(x_N)]\rangle_{LBHT} = \langle G_{\mu_1\nu_1}[(A-B)(x_1)]\cdots G_{\mu_N\nu_N}[(A-B)\times(x_N)]\rangle_{S^{(4)}_{SD+}(A)+S^{(4)}_{SD-}(B)} + \text{C.T.} \quad (51)$$

Now we go in the reverse direction and find correlation functions mapping gauge invariant objects of $S^{(4)}_{SD+}(A)$ and $S^{(4)}_{SD-}(A)$ separately in gauge invariants of LBHT. Exactly as in the previous subsection, we replace the source term $\int j\cdot d\Omega(h)$ in (29) by $\int j_+\cdot d\Omega(h) + \int j_- d\cdot\Omega(H)$ which on one hand leads to (41) and on the other hand, following our previous steps, amounts to replacing (50) by the generating functional:

$$\mathcal{W}[j_+, j_-] = \int \mathcal{D}A_{\mu\nu}\mathcal{D}B_{\mu\nu}\exp i\Big[S^{(4)}_{SD+}(A) + S^{(4)}_{SD-}(B) - \frac{1}{4m}\int \tilde{j}_+\cdot d\Omega(\Omega(A)) + \frac{1}{4m}\int \tilde{j}_-\cdot d\Omega(\Omega(B)) + \mathcal{O}(j^2)\Big]. \quad (52)$$

Finally, differentiating (41) and (52) with respect to the sources $\tilde{j}_+$ and $\tilde{j}_-$ we find

$$2^{N/2}\langle G_{\mu_1\nu_1}[A(x_1)]\cdots G_{\mu_N\nu_N}[A(x_N)]\rangle_{S^{(4)}_{SD+}(A)} = \langle (F_{\mu_1\nu_1} + G_{\mu_1\nu_1})[h(x_1)]\cdots(F_{\mu_N\nu_N} + G_{\mu_N\nu_N})\times[h(x_N)]\rangle_{LBHT} + \text{C.T}, \quad (53)$$

$$(-2)^{N/2}\langle G_{\mu_1\nu_1}[B(x_1)]\cdots G_{\mu_N\nu_N}[B(x_N)]\rangle_{S^{(4)}_{SD-}(B)} = \langle (F_{\mu_1\nu_1} - G_{\mu_1\nu_1})[h(x_1)]\cdots(F_{\mu_N\nu_N} - G_{\mu_N\nu_N})\times[h(x_N)]\rangle_{LBHT} + \text{C.T}. \quad (54)$$

The correlation functions (53) and (54) lead to the gauge invariant maps

$$G_{\mu\nu}[A(x)]|_{S^{(4)}_{SD+}(A)} \leftrightarrow \frac{(F_{\mu\nu} + G_{\mu\nu})}{\sqrt{2}}[h(x)]|_{LBHT}, \quad (55)$$

$$G_{\mu\nu}[B(x)]|_{S^{(4)}_{SD-}(B)} \leftrightarrow -\frac{(F_{\mu\nu} - G_{\mu\nu})}{\sqrt{2}}[h(x)]|_{LBHT}, \quad (56)$$

which are consistent with (51). Analogously to the dual maps of the previous subsection, if instead of subtracting we add (55) and (56) we get a relationship between correlation functions of $G_{\mu\nu}[(A+B)(x)]$ in terms of correlation functions of a first-order differential operator acting on $G_{\mu\nu}[(A-B)(x)]$ which is again typical of self-dual models. This completes the proof of quantum equivalence between $S^{(4)}_{SD+} + S^{(4)}_{SD-}$ and the LBHT theory. In particular, we have learned how to decompose the gauge invariant sector of the LBHT theory in terms of (gauge invariant) helicity eigenstates of $S^{(4)}_{SD\pm}$; namely, $F_{\mu\nu}[h(x)]$ corresponds to $G_{\mu\nu}[(A-B)(x)]\sqrt{2}$. We remark that each $S^{(4)}_{SD\pm}$ theory is invariant under linearized general coordinate and Weyl transformations, so it is not surprising that we have the tensor $G_{\mu\nu}$ [see (44) and the comments below that formula] on the left-hand side of (53) and (54).

IV. CONCLUSION

Although previous soldering of second-order $S^{(2)}_{SD\pm}$ and third-order $S^{(3)}_{SD\pm}$ spin-2 parity singlets has led us to second-order (Fierz-Pauli theory) and fourth-order (linearized BHT theory) parity doublets, respectively, we have shown in Sec. II that the soldering of fourth-order singlets $S^{(4)}_{SD\pm}$ has brought us back to the linearized BHT model. We have technically explained why this must be so. This is an indication that the linearized BHT model [16] is the highest-order self-consistent (unitary) model which describes a parity doublet of helicities $+2$ and -2. The reader can check that, according to the argument given at the end of Sec. II, if we had a higher-derivative model $S^{(r)}_{SD\pm}$ with $r > 4$, then we could have after soldering another higher-derivative ($r > 4$) description of parity doublets of spin-2 in $D = 2 + 1$. However, the symmetry arguments given in [6] indicate that $S^{(4)}_{SD\pm}$ might be the top (highest-order) derivative model for parity singlets of spin-2. If this is really the case the linearized BHT model is in fact the highest-order description of parity doublets.

On the other hand, from the point of view of the local symmetries the soldering of $S^{(3)}_{SD+} + S^{(3)}_{SD-}$ into the linearized BHT theory is more surprising than the soldering of $S^{(4)}_{SD+} + S^{(4)}_{SD-}$ into the same theory, since in the first case the two theories (before and after soldering) are invariant under the same set of local symmetries (linearized general coordinate transformation) while in the second one the models $S^{(4)}_{SD\pm}$ are also symmetric under linearized local Weyl transformations which call for an extra term in the soldering to get rid of the Weyl symmetry. In the first case it should be possible to simply add $S^{(3)}_{SD+} + S^{(3)}_{SD-}$ in order to obtain the linearized BHT theory after eventually some trivial manipulations without adding extra terms. This is the case of the two first-order self-dual models of spin-1 and spin-2 which are known [8] to lead to its second-order counterparts (Proca and Fierz-Pauli theories, respectively, in the first-order form) after a simple addition followed by a trivial rotation. So far we have not been able to do it in the case of the models $S^{(3)}_{SD\pm}$.

In Sec. III we have written down a triple master action which interpolates between all three models, i.e., $S^{(3)}_{SD+} + S^{(3)}_{SD-}$, linearized BHT, and $S^{(4)}_{SD+} + S^{(4)}_{SD-}$. By introducing adequate source terms we have derived identities involving correlation functions in the different models allowing us to deduce a precise dual map [see (45), (46), (55), and (56)] between the relevant gauge invariants of the different dual theories. No specific gauge condition has ever been used. In particular, we have been able to decompose a gauge invariant of the linearized BHT model in terms of helicity eigenstates of both $S^{(3)}_{SD\pm}$ and $S^{(4)}_{SD\pm}$. Putting our master action (26) together with the one defined in [16] relating the Fierz-Pauli theory to the linearized BHT model, as well as using the decomposition of the Fierz-Pauli model in terms of a couple of $S^{(1)}_{SD\pm}$ models as given in [8], we can build a unifying description of all known dual versions of field theories describing parity doublets of helicities $+2$ and -2 in $D = 2 + 1$. As remarked in [23], the key ingredient in the master action approach is the use of mixing terms without particle content.

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