Uniqueness theorem for Kaluza-Klein black holes in five-dimensional minimal supergravity

Shinya Tomizawa*

Cosmophysics Group, Theory Center, Institute of Particle and Nuclear Studies, KEK, Tsukuba, Ibaraki, 305-0801, Japan
(Received 26 July 2010; published 23 November 2010)

We show a uniqueness theorem for Kaluza-Klein black holes in the bosonic sector of five-dimensional minimal supergravity. More precisely, under the assumptions of the existence of two commuting axial isometries and a nondegenerate connected event horizon of the cross-section topology S^3, or lens space, we prove that a stationary charged rotating Kaluza-Klein black hole in five-dimensional minimal supergravity is uniquely characterized by its mass, two independent angular momenta, electric charge, magnetic flux, and nut charge, provided that there exists neither a nut nor a bolt (a bubble) in the domain of outer communication. We also show that under the assumptions of the same symmetry, same asymptotics, and the horizon cross section of $S^1 \times S^2$, a black ring within the same theory—if it exists—is uniquely determined by its dipole charge and rod intervals besides the charges and magnetic flux.

I. INTRODUCTION

Higher dimensional black holes have played an important role in understanding basic properties of fundamental theories, such as string theory. A number of interesting solutions of higher dimensional black holes have been discovered recently [1–21], revealing a much richer structure of their solution space than that of four-dimensional black holes, and we are naturally led to address the question of how to classify them. There are already several papers that generalize the black hole uniqueness theorems [22–31] to higher dimensions [32–46], upon some additional assumptions concerning the horizon topology, symmetry properties, asymptotic structures, etc. In particular, five-dimensional generalizations of the uniqueness theorems have been shown, in various theories, for stationary, axisymmetric (with two rotational symmetries) black holes being noncompact, as simple higher dimensional generalizations of the well-known four-dimensional setup. However, since our real, observable world is macroscopically four dimensional, extra dimensions have to be compactified in realistic, classical spacetime models [47]. Therefore it is of great interest to consider higher dimensional Kaluza-Klein black holes, which look four-dimensional, at least at large distances. Classifying such Kaluza-Klein solutions may also help us to get some insights into the major open problem of how to compactify and stabilize extra dimensions in string theory. The purpose of this paper is to address such a classification problem, showing a uniqueness theorem for stationary Kaluza-Klein black hole solutions of five-dimensional minimal supergravity.

Perhaps the simplest example of Kaluza-Klein black holes is a black string, a direct product of a four-dimensional vacuum black hole and a circle. A more nontrivial class of Kaluza-Klein black holes is given by squashed Kaluza-Klein black holes, found recently by Ishihara-Matsuno [48], applying the squashing technique to five-dimensional black holes. The idea is that for, e.g., the simplest static vacuum case, one first views the S^3 section (or horizon manifold) of a five-dimensional Schwarzschild-type black hole spacetime as a fibre bundle of S^1 over S^2, and then considers a deformation that changes the ratio of the radii of the fibre S^1 and base S^2, so that the resultant spacetime looks, at large distances, like a twisted S^1 over a four-dimensional asymptotically flat spacetime, hence a Kaluza-Klein spacetime, while it looks like a five-dimensional black hole near the event horizon. The basic structure of squashed Kaluza-Klein black holes can in fact be seen in the much earlier works of Refs. [49,50], whose solutions asymptote to a twisted S^1 bundle over a four-dimensional spacetime as studied in [51]. A number of further generalizations of squashed Kaluza-Klein black holes have been made lately [52–60]. The recent accumulation of this new type of Kaluza-Klein black hole solutions also motivates us to address the classification problem of Kaluza-Klein black holes.

All known exact Kaluza-Klein black hole solutions in five dimensions admit the isometry group that describes the stationarity and two "axial" symmetries, one along the S^2 base space and the other along the S^1 fibre, or simply T^2. These symmetries are mutually commuting, hypersurface orthogonal, and form the isometry group $R \times U(1) \times U(1)$. In this paper, we consider Kaluza-Klein black holes that possess this symmetry property and that are purely bosonic as solutions to the minimal supergravity. The topology of horizon cross sections can be either S^3, $S^1 \times S^2$, or lens space $L(p, q)$ [61]. More precisely we shall show the following:

Theorem. Consider the bosonic sector in five-dimensional minimal supergravity, i.e., in five-dimensional Einstein-Maxwell-Chern-Simons theory with a certain

*tomizawa@post.kek.jp

special value of the Chern-Simons coupling constant [given by Eq. (1)], a stationary charged rotating black hole with finite temperature that is regular on and outside the event horizon, and asymptotically Kaluza-Klein spacetime (the precise definition is given below). Assume that the black hole spacetime admits, besides the stationary Killing vector field, two mutually commuting axial Killing vector fields so that the isometry group is $\mathbb{R} \times U(1) \times U(1)$ and that the topology of the horizon spatial cross sections is S^3, or $L(n; 1)$, or $S^1 \times S^2$. Then (i) if there exists no nut in the domain of outer communication, the black hole spacetime with S^3 or $L(n; 1)$ horizon cross section is uniquely characterized by its mass, and two independent angular momenta, electric charge, nut charge, and magnetic flux, and (ii) if the topology of the black hole exterior region is $\mathbb{R} \times \{\mathbb{R}^4 \setminus B^3 \times S^1\}$, the black hole spacetime with $S^1 \times S^2$ horizon cross section is uniquely characterized by its dipole charge and rod intervals in addition to their charges and magnetic flux.

It is known that the bosonic sector of minimal supergravity that possesses the above symmetry group can be reduced to a nonlinear sigma model [63,64], which is much more complicated than the well-known four-dimensional electrovacuum case. Nevertheless, one can derive formulas similar to those used in the four-dimensional uniqueness proof, such as the coset matrix representation of the equations of motion, the divergence (Mazur) identity, etc. [65], as we will discuss below. Apart from the difference in the sigma model, another main difference from the four-dimensional case can be seen in the boundary value analysis, in particular, along the symmetry axis and the horizon. This is because we have a larger variety of the horizon topology in five dimensions. We can specify the horizon topology in terms of the "rod structure" (or interval structure) [66].

The new part of the job that is particular for our asymptotic Kaluza-Klein case is in the boundary value analysis at infinity. We need to consider fall-off conditions of the sigma-model fields at infinity more carefully than the globally flat case. When inspecting the asymptotic fall-off behavior of the perturbations, we find that the perturbations of the metric and the gauge field decouple each other at least in the leading order. Then, imposing boundary conditions for the asymptotically Kaluza-Klein spacetime, we will identify the parameters which are used to characterize the solutions. For example, from the leading order of the fall-off behavior of the gravitational sector, we find N which describes how much the S^1 fibre (i.e., the compactified fifth dimension) is twisted with respect to the S^2 base space, and Q which may be viewed as the "angular" momentum along the fifth dimension, in addition to the usual angular momentum J along $U(1)$ of the base space. From the Maxwell part, we have, besides the electric charge q, the magnetic flux c_ϕ over the base space at infinity, and furthermore the dipole charge q_m if the

topology of the horizon cross section is $S^1 \times S^2$ (see the next section for their precise definitions).

The main interest of this paper is in the context of minimal supergravity, and we therefore restrict our attention to the Einstein-Maxwell-Chern-Simons theory with a certain value of the coupling constant. However, one can expect that a similar uniqueness theorem may also hold in other, similar theories. For example, restricting attention to some integrable sector of the five-dimensional pure Einstein-Maxwell theory, a uniqueness theorem similar to the above has recently been shown [67], based on the classification of Kaluza-Klein black holes in arbitrary, D-dimensional, vacuum Einstein gravity [68] with $D - 2$ Killing symmetries. The sigma model for the integrable sector of the five-dimensional pure Einstein-Maxwell system appears quite different, but the basic strategy for the proof is essentially the same as the one we will consider below. However, we should note that the integrable sector considered in [67] corresponds to a highly restricted class of the solutions in which the electric part of the Maxwell field and, at least, one of the two angular momenta are required to vanish. For this reason, the boundary value analysis in the asymptotic region (i.e., at large distances) in [67] seems rather simple and straightforward. Furthermore, for the solutions dealt with in [67], some of the parameters in our above theorem turn out to be identically zero.

In this paper we consider a more (perhaps the most) general class of Kaluza-Klein black hole solutions with a single horizon of the minimal supergravity that possesses the above isometry group, so that the solutions can admit two independent arbitrary "angular" momenta (one of which may be called the momentum along the compactified dimension) and a nonvanishing electric component of the Maxwell field. We find that for such a general solution, for example, the parameter c_ϕ appears, in contrast to the case of asymptotically flat cases [45,46] for which c_ϕ vanishes. Note also that for some known exact solutions [9,56–58,69,70], the parameter c_ϕ seems to be related to so-called Gödel parameter, whose square is proportional to the energy density of magnetic field. As is well known [9,69,70], the five-dimensional Gödel type universe is filled with the pressureless magnetic field, and due to the rapid rotation of the magnetic field, the spacetime admits closed timelike curves in far regions, but for Kaluza-Klein black holes [56–58] it exhibits no causal pathology outside the event horizon, thanks to the appropriate compactification. As far as we know, the most general solution with all independent parameters has not yet been found. Hence, we would like to show that such a solution is characterized by its parameters and therefore must be unique if it exists.

In the next section, we will briefly describe our strategy for the proof and write down some necessary formulas, such as the equations of motion and the definitions of

relevant sigma-model fields. In Sec. III, by solving straight-forwardly Einstein-Maxwell-Chern-Simons equations near infinity, we derive the Kaluza-Klein asymptotics—the asymptotic behaviors of the metric and gauge potential of the Maxwell-Chern-Simons field—in the Weyl-Papapetrou coordinate system. In Sec. IV we perform the boundary value analysis for black holes with a spherical horizon topology and complete our proof. In Sec. V we consider the boundary value analysis for black rings with a $S^1 \times S^2$ horizon cross section and show the uniqueness theorem. In Sec. VI we also discuss the boundary value analysis for black lenses. In Sec. VII we summarize our results.

II. EINSTEIN-MAXWELL-CHERN-SIMONS SYSTEM WITH SYMMETRIES

A. The basic strategy for the proof and mathematical formulas

First we briefly describe the basic strategy for our uniqueness proof, which roughly proceeds as follows. (i) We first reduce the (bosonic sector of) five-dimensional minimal supergravity theory with three commuting independent Killing symmetries to a nonlinear sigma model, that is, a set of equations for eight scalar fields Φ^A on two-dimensional orbits space Σ, with the target space isometry G. With the aid of G, the action of the sigma model can be described in terms of a symmetric, unimodular matrix, M, on the coset space G/H where H is an isotropy subgroup of G. Thus, the solutions of the original system can be expressed by the matrix M. Furthermore, the matrix M formally defines a conserved current, J, for the solution. (ii) Next, we introduce the deviation matrix, Ψ, which is essentially the difference between two coset matrices, say, $M_{[0]}$ and $M_{[1]}$, so that when two solutions coincide with each other, the deviation matrix vanishes, and vice versa. What we wish to show is that Ψ vanishes over the entire Σ when two solutions satisfy the same boundary conditions that specify relevant physical parameters characterizing the black hole solution of interest. For this purpose, we construct a global identity, called the Mazur identity, (the integral version of) which equates an integration along the boundary $\partial\Sigma$ of a derivative of the trace of Ψ to an integration over the whole base space Σ of the trace of the "square" of the deviation, $\mathcal{M}$, of the two conserved currents, $J_{[0]}$ and $J_{[1]}$. The latter is therefore non-negative. (iii) Then, we perform boundary value analysis of the matrix Ψ. We identify boundary conditions for M that define physical parameters characterizing black hole solutions and that guarantee the regularity of the solutions. Then we examine the behavior of Ψ near $\partial\Sigma$. For a higher dimensional case, this is the point where the topology and symmetry properties, translated into the language of the rod structure, come to play a role as additional parameters to specify solutions. Also this is the place where we have to take into consideration the nature of the asymptotic

structure of the spacetime. When the integral along the boundary $\partial\Sigma$, say, the left side of the Mazur identity, vanishes under our boundary conditions, it then follows from the right side of the identity, i.e., the non-negative integration over Σ, that $\mathcal{M}$ has to vanish; hence the two currents, $J_{[0]}$ and $J_{[1]}$, must coincide with each other over Σ, implying that the deviation matrix Ψ must be constant over Σ. Then, if Ψ is shown to be zero on some part of the boundary $\partial\Sigma$, it follows that Ψ must be identically zero over the entire Σ, thus proving the two solutions, $M_{[0]}$ and $M_{[1]}$, must be identical.

In our present case, the first two steps, (i) and (ii), completely parallel those in paper [45], and step (iii) is the new result of this paper. In the following we provide some of the formulas for steps (i) and (ii), such as the definitions of the relevant sigma-model fields, in order to establish our notation. The reader can also find them in paper [45]. Some relevant formulas, such as the coset matrix representation of the sigma-model field, are also summarized in the Appendix.

B. Einstein-Maxwell-Chern-Simons system with symmetries and the reduction to σ model

We start with the five-dimensional minimal supergravity action

$$S = \frac{1}{16\pi}\left[\int d^5x\sqrt{-g}\left(R - \frac{1}{4}F^2\right) - \frac{1}{3\sqrt{3}}\int F \wedge F \wedge A\right],$$

(1)

where we set a Newton constant to be unity and $F = dA$. Varying this action (1), we derive the Einstein equation

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = \tfrac{1}{2}(F_{\mu\lambda}F_\nu{}^\lambda - \tfrac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}),$$

(2)

and the Maxwell equation

$$d * F + \frac{1}{\sqrt{3}}F \wedge F = 0,$$

(3)

which have the extra term coming from the Chern-Simons term of (1). We are concerned with asymptotically Kaluza-Klein, stationary, charged rotating black hole solutions of this theory. We additionally impose two independent axial symmetries, so that the total isometry group is $\mathbb{R} \times U(1) \times U(1)$ with $\mathbb{R}$ being stationary symmetry, generated by mutually commuting three Killing vector fields $\xi_t = \partial/\partial t$ and $\xi_a = (\xi_\phi, \xi_w) = (\partial/\partial\phi, \partial/\partial w)$ [71]. Hereafter, we mean 1-forms by ξ_a and ξ_t, i.e., $\xi_t := g_{t\mu}dx^\mu$ and $\xi_a := g_{a\mu}dx^\mu$. Note here that as seen later, the generators of $U(1) \times U(1)$ are not ξ_a themselves but the linear combinations. Using the Einstein equation and the Maxwell equation, we can show that the generators ξ_t, ξ_a of the isometry group satisfy type of integrability conditions discussed in Refs. [66,75]. As a result, we obtain the coordinate system, $\{t, \phi, w, \rho, z\}$, in which the metric takes the Weyl-Papapetrou form

$$ds^2 = \lambda_{\phi\phi}(d\phi + a^\phi{}_t dt)^2 + \lambda_{ww}(dw + a^w{}_t dt)^2$$
$$+ 2\lambda_{\phi w}(d\phi + a^\phi{}_t dt)(dw + a^w{}_t dt)$$
$$+ |\tau|^{-1}[e^{2\sigma}(d\rho^2 + dz^2) - \rho^2 dt^2], \qquad (4)$$

and the gauge potential is written

$$A = \sqrt{3}\psi_a dx^a + A_t dt, \qquad (5)$$

where the coordinates $x^a = (\phi, w)$ denote the Killing parameters, and thus all functions λ_{ab}, $\tau := -\det(\lambda_{ab})$, $a^a{}_t$, σ, and (ψ_a, A_t) are independent of t and x^a, and where the potentials ψ_a are related to Maxwell field by Eq. (8) of paper [45] [see also Appendix A of paper [45] for the gauge choice employed in Eq. (5)]. Note that the coordinates (ρ, z) that span a two-dimensional base space, $\Sigma = \{(\rho, z)| \rho \geq 0, -\infty < z < \infty\}$, are globally well defined, harmonic, and mutually conjugate on Σ. See, e.g., [76]. Furthermore, by using the Maxwell's equation and Einstein's equation, we introduce the magnetic potential μ and twist potentials ω_a by

$$d\mu = \frac{1}{\sqrt{3}} * (\xi_\phi \wedge \xi_w \wedge F) - \epsilon^{ab}\psi_a d\psi_b, \qquad (6)$$

$$d\omega_a = *(\xi_\phi \wedge \xi_w \wedge d\xi_a) + \psi_a(3d\mu + \epsilon^{bc}\psi_b d\psi_c), \quad (7)$$

where $\epsilon^{\phi w} = -\epsilon^{w\phi} = 1$. Then, the nonlinear sigma model reduced from the theory (1) with the symmetry assumptions consists of the target space with the isometry $G = G_{2(2)}$ and the eight scalar fields $\Phi^A = (\lambda_{ab}, \omega_a, \psi_a, \mu)$ on the base space Σ. All the other fields such as σ, a^a, etc., can be determined by Φ^A through the equations of motion.

It turns out that the sigma model fields, Φ^A, can be expressed by a 7×7 symmetric unimodular coset $G_{2(2)}/SO(4)$ matrix M [See Eq. (34) of paper [45]], as shown by [63–65]. (We will provide a detailed description of the coset matrix in the Appendix.) Then we define the deviation matrix, Ψ, for two solutions, $M_{[0]}$ and $M_{[1]}$, as in Eq. (42) of paper [45], and derive the Mazur identity,

$$\int_{\partial\Sigma} \rho \partial_p \operatorname{tr}\Psi dS^p = \int_\Sigma \operatorname{tr}(\mathcal{M}^T \cdot \mathcal{M})\rho d\rho dz, \qquad (8)$$

where the *dot* denotes the inner product on Σ. As briefly mentioned above, $\mathcal{M}$, in the right side, essentially describes the difference between two matrix currents $J_{[0]}$, $J_{[1]}$, given by Eq. (47) of paper [45], of which detail is irrelevant to the discussion below. Our task is to show that the left side of Eq. (8) vanishes on the boundary, $\partial\Sigma$, and then show Ψ itself vanishes on some part of the boundary.

Now we note that the right-hand side of the identity, (8), is non-negative. Therefore, if we impose the boundary conditions at $\partial\Sigma$, under which the left-hand side of Eq. (A12) vanishes, then we must have $\overset{\odot i}{J} = 0$. In that case, it follows from Eq. (A10) that Ψ must be a constant matrix over the region Σ. Therefore, in particular, if Ψ is

shown to be zero on some part of the boundary $\partial\Sigma$, it immediately follows that Ψ must be identically zero over the base space Σ, implying that the two solutions $M_{[0]}$ and $M_{[1]}$ must coincide with each other. This is indeed the case as we will analyze in the next section.

III. KALUZA-KLEIN ASYMPTOTICS IN FIVE DIMENSIONS

Before estimating the boundary integrals in the left-hand side of the Mazur identity, Eq. (8), we must derive the asymptotic form of the gauge potential and metric at infinity for asymptotically Kaluza-Klein spacetimes including all known exact solutions in $D = 5$ minimal supergravity. Here, by the *asymptotically Kaluza-Klein spacetime*, we mean that the five-dimensional spacetime metric at large distances behaves as

$$ds^2 \simeq -dt^2 + dx^2 + dy^2 + dz^2 + dw^2, \qquad (9)$$

where the 5th coordinate w has the periodicity $\Delta w = 2\pi L$. Hence, we can see that at infinity the spacetime behaves as a four-dimensional flat spacetime with a circle. Now in order to study the asymptotics of such a spacetime, it is more convenient to use the radial coordinate r and the angular coordinate θ defined by

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad (10)$$

$$\theta = \arccos\left(\frac{z}{r}\right). \qquad (11)$$

Note that the coordinates (ρ, z) in the Weyl-Papapetrou coordinate system are related to the above defined coordinates (r, θ) by

$$\rho = r\sin\theta, \qquad (12)$$

$$z = r\cos\theta. \qquad (13)$$

A. Gauge potential

First, we determine the behavior of the gauge field, $A \simeq A^{(0)}(\theta) + A^{(1)}(\theta)/r + \mathcal{O}(r^{-2})$, near infinity. From Eq. (3), the gauge potential, A, is subject to the Maxwell-Chern-Simons equation,

$$\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}F^{\mu\nu}) + \frac{1}{4\sqrt{3}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\nu\rho}F_{\sigma\lambda} = 0. \quad (14)$$

From the t component of Eq. (14), we can derive the equation to determine the leading order of A_t,

$$\partial_\theta^2 A_t^{(0)} + \cot\theta \partial_\theta A_t^{(0)} = 0. \qquad (15)$$

Solving the above equation, we obtain

$$A_t^{(0)} = c_t + d_t \log\left|\tan\frac{\theta}{2}\right|, \qquad (16)$$

where c_t and d_t are integration constants. The regularity of the field strength $F = dA$ requires $d_t = 0$. Note that by using the gauge transformation, i.e., the gauge freedom in

adding a constant, we can also set the value of the other constant to be $c_t = 0$. After all, without loss of generality, we may put

$$A_t^{(0)} = 0. \tag{17}$$

From the leading order of the w component in Eq. (14), we derive the equation to the leading order of A_w,

$$\partial_\theta^2 A_w^{(0)} + \cot\theta \partial_\theta A_w^{(0)} = 0. \tag{18}$$

Similarly, we get

$$A_w^{(0)} = \bar{c}_w + \bar{d}_w \log \left| \tan\frac{\theta}{2} \right|, \tag{19}$$

where $\bar{c}_w$ and $\bar{d}_w$ are constants. From the same discussion, we can set the values of these two integration constants to be $\bar{c}_w = \bar{d}_w = 0$ and therefore obtain

$$A_w^{(0)} = 0. \tag{20}$$

The remaining ϕ component is written as

$$\partial_\theta^2 A_\phi^{(0)} - \cot\theta \partial_\theta A_\phi^{(0)} = 0. \tag{21}$$

The solution is written in terms of integration constants c_ϕ and d_ϕ,

$$A_\phi^{(0)} = c_\phi \cos\theta + d_\phi. \tag{22}$$

Using the degree of the gauge freedom, we may choose $d_\phi = 0$ and hence obtain

$$A_\phi^{(0)} = c_\phi \cos\theta. \tag{23}$$

Substituting Eqs. (17), (20), and (23) into Eq. (14), we can derive the equations which determine the next order of the gauge fields $A^{(1)}$. It turns out that the equations for $A_t^{(1)}$ and $A_w^{(1)}$ take exactly the same forms as for $A_t^{(0)}$ and $A_w^{(0)}$. Therefore, the next orders of A_t and A_w turn out to be, respectively,

$$A_t^{(1)} = q, \qquad A_w^{(1)} = c_w, \tag{24}$$

where q and c_w are constants. Note that by using the gauge transformation, we cannot set them to be 0. To summarize, near infinity, the gauge field behaves as

$$A \simeq \frac{q}{r}(1 + \mathcal{O}(r^{-1}))dt + c_\phi \cos\theta(1 + \mathcal{O}(r^{-1}))d\phi$$

$$+ \frac{c_w}{r}(1 + \mathcal{O}(r^{-1}))dw. \tag{25}$$

B. Metric

Next we would like to determine the next order of the metric, $g_{ij}^{(1)}$, near infinity, where $g_{ij}^{(1)}$ is defined by

$$g_{ij}(r, \theta) \simeq g_{ij}^{(0)}(\theta) \sum_{k=1}^{\infty} \left(1 + \frac{g_{ij}^{(k)}(\theta)}{r^k}\right). \tag{26}$$

From the (tt) component of Eq. (2), we can derive the equation to determine the next order of g_{tt},

$$\partial_\theta^2 g_{tt}^{(1)} + \cot\theta \partial_\theta g_{tt}^{(1)} = 0. \tag{27}$$

This can immediately be solved,

$$g_{tt}^{(1)} = c_{tt} + d_{tt} \log \left| \tan\frac{\theta}{2} \right|, \tag{28}$$

where c_{tt} and d_{tt} are integration constants. The regularity of the metric requires $d_{tt} = 0$. Hence, we obtain

$$g_{tt}^{(1)} = c_{tt}. \tag{29}$$

From the (ww) component and (tw) component of Eq. (2), we derive the equations, respectively,

$$\partial_\theta^2 g_{ww}^{(1)} + \cot\theta \partial_\theta g_{ww}^{(1)} = 0, \tag{30}$$

$$\partial_\theta^2 g_{tw}^{(1)} + \cot\theta \partial_\theta g_{tw}^{(1)} = 0. \tag{31}$$

Similarly, in terms of constants Q and c_{ww}, $g_{ww}^{(1)}$ and $g_{tw}^{(1)}$ can be written

$$g_{ww}^{(1)} = c_{ww}, \tag{32}$$

$$g_{tw}^{(1)} = Q, \tag{33}$$

respectively. From the (ϕw) component of Eq. (2), we derive the equation

$$\partial_\theta^2 g_{\phi w}^{(0)} - \cot\theta \partial_\theta g_{\phi w}^{(0)} = 0. \tag{34}$$

Solving this, we can obtain

$$g_{\phi w}^{(0)} = c_{\phi w} + N \cos\theta, \tag{35}$$

where $c_{\phi w}$ and N are constants. It turns out here that by performing the coordinate transformation, $w \rightarrow w - c_{\phi w} \phi$, the constant, $c_{\phi w}$, can be set to be 0. Therefore, $g_{\phi w}^{(0)}$ can be written as

$$g_{\phi w}^{(0)} = N \cos\theta. \tag{36}$$

From the $(\phi\phi)$ component of Eq. (2) and the above results, the equation

$$\partial_\theta^2 g_{\phi\phi}^{(1)} + \cot\theta \partial_\theta g_{\phi\phi}^{(1)} = 0 \tag{37}$$

can be derived. The regularity of the metric requires that the solution must take the form of

$$g_{\phi\phi}^{(1)} = c_{\phi\phi}, \tag{38}$$

where $c_{\phi\phi}$ is an integration constant. From the $(t\phi)$ component of Eq. (2), we derive the equation

$$\partial_\theta^2 g_{t\phi}^{(1)} - \cot\theta \partial_\theta g_{t\phi}^{(1)} + 2g_{t\phi}^{(1)} - 2QN \cos\theta = 0. \tag{39}$$

Integrating this equation, we obtain the solution

$$g_{t\phi}^{(1)} = J\sin^2\theta + d_{t\phi}\left(2\cos\theta + \sin^2\theta \log\frac{1 + \cos\theta}{1 - \cos\theta}\right)$$

$$+ QN \cos\theta, \tag{40}$$

in terms of constants J and $d_{t\phi}$. Similarly, the regularity requires $d_{t\phi} = 0$.

Here, recall that in the canonical coordinate system, the three-dimensional metric $g = (g_{ij})$ ($i, j = t, \phi, w$) is subject to the constraint

$$\det(g) = -\rho^2. \tag{41}$$

Therefore, using the constraint and the formula

$$\det(g + \delta g) = \det[g(1 + g^{-1}\delta g)]$$
$$= -\rho^2(1 + \mathrm{tr}(g^{-1}\delta g) + \det(g^{-1}\delta g))$$
$$\simeq -\rho^2(1 + \mathrm{tr}(g^{-1}\delta g)), \tag{42}$$

we can see in the next order that the metric has to satisfy the constraint

$$\sum_{i=t,\phi,w} g_{ii}^{(1)} = 0, \tag{43}$$

which is the same constraint as in the asymptotically flat case [45]. We note that though in the Weyl-Papapetrou coordinate system the asymptotic form of the metric is not diagonal, the off-diagonal component does not affect this constraint in the order of $\mathcal{O}(r^{-1})$.

Thus, to summarize, in the Weyl-Papapetrou coordinate system, the metric near infinity, $r = \sqrt{\rho^2 + z^2} \to \infty$, behaves as

$$ds^2 \simeq \left(-1 + \frac{m}{r} + \mathcal{O}(r^{-2})\right)dt^2 + r^2\sin^2\theta\left(1 + \frac{m-\eta}{2r} + \mathcal{O}(r^{-2})\right)d\phi^2 + \left(1 + \frac{m+\eta}{2r} + \mathcal{O}(r^{-2})\right)dw^2$$
$$+ \frac{2(J\sin^2\theta + QN\cos\theta)}{r}(1 + \mathcal{O}(r^{-1}))dtd\phi + \frac{2Q}{r}(1 + \mathcal{O}(r^{-1}))dtdw + 2N\cos\theta(1 + \mathcal{O}(r^{-1}))d\phi dw$$
$$+ (1 + \mathcal{O}(r^{-1}))(d\rho^2 + dz^2). \tag{44}$$

Here η is a constant that comes from gauge degrees of freedom in the choice of the coordinate z, i.e., degrees of freedom with respect to shift translation $z \to z + \alpha$. (This gauge freedom exists even after the gauge freedom of the conjugate coordinate, ρ, is fixed at infinity.) Since, for example, in the proof for the black hole case we choose the coordinate z such that the horizons are located at the interval $[-k^2, k^2]$ for two configurations $M_{[0]}$ and $M_{[1]}$, we choose the same values of η for the two solutions.

C. Asymptotic charges and flux

Now let us see the relation between the asymptotic charges and the integration constants appearing in the asymptotic form of the metric and gauge potential. We can see from Eq. (44) that for $r \to \infty$, the metric behaves as

$$ds^2 \simeq -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$
$$+ (dw + N\cos\theta d\phi)^2. \tag{45}$$

It is now clear that the metric has the structure of the S^1 bundle over the four-dimensional Minkowski spacetime and the spatial infinity is the S^1 fibre bundle over the S^2 base space. In particular, when $N = L/2$, or $N = (L/2)n$ ($|n|$: natural numbers larger than 1), the spatial infinity can be regarded as a squashed S^3, or squashed lens space $L(n; 1)$. Also note when $N = 0$, the S^1 and the Minkowski spacetime are a direct product. The asymptotic charges should be defined as boundary integrals over the spatial infinity S_∞. Since we are concerned with stationary, axisymmetric spacetimes with Killing symmetries in Einstein-Maxwell-Chern-Simons theory, the conserved charges, mass M, angular momenta J_a, and electric charge Q_e are defined as follows, and are related to the integration

constants in the asymptotic form of the metric and gauge potential by

$$M = -\frac{3}{32\pi}\int_{S_\infty} dS^{\mu\nu}\nabla_\mu(\xi_t)_\nu = \frac{3\pi mL}{4}, \tag{46}$$

$$J_\phi = \frac{1}{16\pi}\int_{S_\infty} dS^{\mu\nu}\nabla_\mu(\xi_\phi)_\nu = \frac{\pi JL}{3}, \tag{47}$$

$$J_w = \frac{1}{16\pi}\int_{S_\infty} dS^{\mu\nu}\nabla_\mu(\xi_w)_\nu = \frac{\pi QL}{2}, \tag{48}$$

$$Q_e = \frac{1}{16\pi}\int_{S_\infty}\left(*F + \frac{1}{\sqrt{3}}A \wedge F\right) = \frac{\pi qL}{2}. \tag{49}$$

As seen later, the magnetic flux Q_m is defined by

$$Q_m = \frac{1}{4\pi}\int_{S_\infty^2} F = c_\phi, \tag{50}$$

where S_∞^2 denotes the base manifold of S^2 at infinity.

IV. BOUNDARY VALUE PROBLEMS FOR BLACK HOLES

As discussed in [35,68], under the existence of two commuting axial Killing vectors, the cross-section topology of each connected component of the event horizon of stationary vacuum black hole solutions must be S^3, $S^1 \times S^2$, or a lens space. First, let us start from the boundary value analysis for black holes with a spherical horizon cross section and with Kaluza-Klein asymptotics. In terms of the Weyl-Papapetrou coordinate system and the rod structure [66], the boundary $\partial\Sigma$ of the base space

$\Sigma = \{(\rho, z)|\rho > 0, -\infty < z < \infty\}$ is described as a set of three rods and the infinity [see Fig. 1(b) about the rod diagram]:

(i) the outer axis, $\partial \Sigma_+ = \{(\rho, z)|\rho = 0, k^2 < z < \infty\}$ with the rod vector $v = (0, 1, N)$,

(ii) the horizon, $\partial \Sigma_{\mathscr{H}} = \{(\rho, z)|\rho = 0, -k^2 < z < k^2\}$,

(iii) the outer axis, $\partial \Sigma_- = \{(\rho, z)|\rho = 0, -\infty < z < -k^2\}$ with the rod vector $v = (0, 1 - N)$,

(iv) the infinity, $\partial \Sigma_\infty = \{(\rho, z)|\sqrt{\rho^2 + z^2} \to \infty$ with $z/\sqrt{\rho^2 + z^2}$ finite$\}$.

As mentioned in the previous section, the sphericity of the spatial infinity requires that the nut charge N be related to the size of the fifth dimension L by $N = L/2$. Furthermore note also that in the above rod structure there is no joint point such as a nut—the point where two spacelike rods meet with each other—outside the horizon and therefore this means that the topology of the horizon cross section is S^3 and the topology of the black hole exterior region is $\mathbb{R} \times \{\mathbb{R}^4 \setminus \mathbb{B}^4\}$. We can see this as follows. We here assume the identification $(\phi, w) \to (\phi + 2\pi, w + 2\pi N)$, $(\phi, w) \to (\phi + 2\pi, w - 2\pi N)$ and hence the periodicity of ϕ and w are 2π and $4\pi N$, respectively. Therefore, as discussed in Ref. [77], the pair of Killing vectors, $\partial/\partial\phi_\pm = \partial/\partial\phi \pm N\partial/\partial w$ [$\phi_\pm = (\phi \pm N^{-1}w)/2$], is identified as a pair of
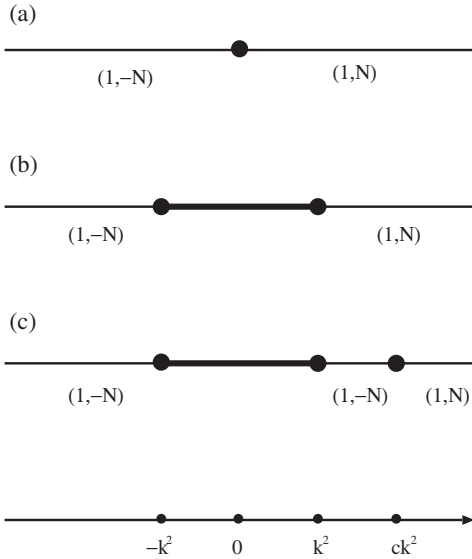
(a)

(b)

(c)

FIG. 1. The rod structures of spacetimes with Kaluza-Klein asymptotics: (a) the Gross-Perry-Sorkin monopole, (b) the black hole, and (c) the black ring. Here, the solid finite rods correspond to the horizons, the assigned vectors on the spacelike rods denote the rod vectors; i.e., the pairs of numbers $(1, \pm N)$ means that the Killing vectors, $v = (\partial/\partial\phi) \pm N(\partial/\partial w)$, have fixed points there—more precisely, the metric, $g_{ij}(0, z)$, has an eigenvalue zero for a given z. See Ref. [77] about the rod structures of well-known gravitational instantons—for example, Euclidean self-dual Taub-NUT space—with $U(1) \times U(1)$ symmetry and its classification.

2π periodic generators of the $U(1) \times U(1)$ isometry group since the identification $(\phi, w) \to (\phi + 2\pi, w + 2\pi N)$, $(\phi, w) \to (\phi + 2\pi, w - 2\pi N)$ in the coordinate (ϕ, w) can be regarded as the identification of $(\phi_+, \phi_-) \to (\phi_+ + 2\pi, \phi_-), (\phi_+, \phi_-) \to (\phi_+, \phi_- + 2\pi)$ in the coordinates (ϕ_+, ϕ_-). Accordingly, we can see that the determinant of the two rod vectors $v_\pm := \partial/\partial\phi_\pm$ is $|\det(v_+, v_-)| = 1$, which means that the horizon cross section and the spatial infinity are topologically S^3. The boundary integral in the left-hand side of the Mazur identity, Eq. (8), is decomposed into the integrals over the three rods (i)–(iii), and the integral at infinity (iv), as

$$\int_{\partial\Sigma} \rho\partial_p \,\mathrm{tr}\Psi dS^p = \int_{-\infty}^{-k^2} \rho \frac{\partial\,\mathrm{tr}\Psi}{\partial z} dz + \int_{-k^2}^{k^2} \rho \frac{\partial\,\mathrm{tr}\Psi}{\partial z} dz$$
$$+ \int_{k^2}^{\infty} \rho \frac{\partial\,\mathrm{tr}\Psi}{\partial z} dz + \int_{\partial\Sigma_\infty} \rho\partial_p \,\mathrm{tr}\Psi dS^p. \quad (51)$$

(iv) The infinity. It immediately follows from Eq. (44) that near infinity, the gravitational potentials, λ_{ab}, behave as

$$\lambda_{\phi\phi} \simeq \rho^2\left(1 + \frac{m - \eta}{2\sqrt{\rho^2 + z^2}} + \mathcal{O}((\rho^2 + z^2)^{-1})\right), \quad (52)$$

$$\lambda_{ww} \simeq 1 + \frac{m + \eta}{2\sqrt{\rho^2 + z^2}} + \mathcal{O}((\rho^2 + z^2)^{-1}), \quad (53)$$

$$\lambda_{\phi w} \simeq \frac{Nz}{\sqrt{\rho^2 + z^2}} + \mathcal{O}((\rho^2 + z^2)^{-(1/2)}). \quad (54)$$

We see directly from Eq. (25) that the electric potentials, ψ_a, behave as

$$\psi_\phi \simeq \frac{c_\phi}{\sqrt{3}} \frac{z}{\sqrt{\rho^2 + z^2}} + \mathcal{O}((\rho^2 + z^2)^{-(1/2)}), \quad (55)$$

$$\psi_w \simeq \frac{c_w}{\sqrt{3}} \frac{1}{\sqrt{\rho^2 + z^2}} + \mathcal{O}((\rho^2 + z^2)^{-1}). \quad (56)$$

From the Kaluza-Klein asymptotics (44), we see that the functions, τ and $a^a{}_t$, behave as

$$\tau = \lambda_{\phi w}^2 - \lambda_{\phi\phi}\lambda_{ww} \simeq -\rho^2, \quad (57)$$

$$a^\phi{}_t = \frac{\lambda_{\phi w} g_{tw} - \lambda_{ww} g_{t\phi}}{\tau} \simeq -\frac{J}{\sqrt{\rho^2 + z^2}}, \quad (58)$$

$$a^w{}_t = \frac{\lambda_{\phi w} g_{t\phi} - \lambda_{\phi\phi} g_{tw}}{\tau} \simeq \frac{Q}{\sqrt{\rho^2 + z^2}} \quad (59)$$

near infinity. From Eqs. (55)–(59), the derivatives of the magnetic potential behave as

$$\mu_{,\rho} = \frac{\tau}{\rho}\left(\frac{A_{t,z}}{\sqrt{3}} - a^a{}_t \psi_{a,z}\right) - \epsilon^{ab}\psi_a\psi_{b,\rho} \simeq \frac{q\rho z}{\sqrt{3}\sqrt{\rho^2 + z^2}^3}, \tag{60}$$

$$\mu_{,z} = -\frac{\tau}{\rho}\left(\frac{A_{t,\rho}}{\sqrt{3}} - a^a{}_t \psi_{a,\rho}\right) - \epsilon^{ab}\psi_a\psi_{b,z}$$

$$\simeq -\frac{q\rho^2}{\sqrt{3}\sqrt{\rho^2 + z^2}^3}. \tag{61}$$

Hence, by integrating these, we find that near infinity the magnetic potential behaves as

$$\mu \simeq -\frac{qz}{\sqrt{3}\sqrt{\rho^2 + z^2}} + \mathcal{O}((\rho^2 + z^2)^{-1}). \tag{62}$$

On the other hand, the asymptotic behaviors of the derivatives of the twist potentials can be derived

$$\omega_{\phi,\rho} = \frac{\tau}{\rho}\lambda_{\phi b}a^b{}_{t,z} + \psi_\phi(3\mu_{,\rho} + \epsilon^{bc}\psi_b\psi_{c,\rho})$$

$$\simeq \frac{-3J\rho^3 z}{\sqrt{\rho^2 + z^2}^5} + \frac{(NQ - c_\phi q)\rho z^2}{(\rho^2 + z^2)^2}, \tag{63}$$

$$\omega_{\phi,z} = -\frac{\tau}{\rho}\lambda_{\phi b}a^b{}_{t,\rho} + \psi_\phi(3\mu_{,z} + \epsilon^{bc}\psi_b\psi_{c,z})$$

$$\simeq \frac{3J\rho^4}{\sqrt{\rho^2 + z^2}^5} - \frac{(NQ - c_\phi q)\rho^2 z}{(\rho^2 + z^2)^2}, \tag{64}$$

$$\omega_{w,\rho} = \frac{\tau}{\rho}\lambda_{wb}a^b{}_{t,z} + \psi_w(3\mu_{,\rho} + \epsilon^{bc}\psi_b\psi_{c,\rho})$$

$$\simeq \frac{Q\rho z}{\sqrt{\rho^2 + z^2}^3}, \tag{65}$$

$$\omega_{w,z} = -\frac{\tau}{\rho}\lambda_{wb}a^b{}_{t,\rho} + \psi_w(3\mu_{,z} + \epsilon^{bc}\psi_b\psi_{c,z})$$

$$\simeq -\frac{Q\rho^2}{\sqrt{\rho^2 + z^2}^3} \tag{66}$$

from the definitions (7) and (55)–(59). Integrating these, we obtain

$$\omega_\phi \simeq \frac{(c_\phi q - NQ)z^2}{2(\rho^2 + z^2)} + J\left(\frac{6z}{\sqrt{\rho^2 + z^2}} - \frac{2z^3}{\sqrt{\rho^2 + z^2}^3}\right)$$

$$+ \mathcal{O}((\rho^2 + z^2)^{-(1/2)}), \tag{67}$$

$$\omega_w \simeq -\frac{Qz}{\sqrt{\rho^2 + z^2}} + \mathcal{O}((\rho^2 + z^2)^{-(1/2)}). \tag{68}$$

Then, using Eqs. (52)–(56), (62), (67), and (68), we can see that for the two configurations, $M_{[0]}$ and $M_{[1]}$, with the same constants (m, J, Q, N, q, c_ϕ), $\rho \operatorname{tr}\Psi$ near infinity behaves as

$$\operatorname{tr}\Psi \simeq \frac{6(\overset{\circ}{c}_w)^2}{\rho^2 + z^2}. \tag{69}$$

Therefore, by using the coordinates (r, θ), $\rho\partial_p \operatorname{tr}\Psi dS^p$ at infinity $r = \infty$ turns out to be

$$\rho\partial_p \operatorname{tr}\Psi dS^p \simeq 6(\overset{\circ}{c}_w)^2(r\sin\theta) \cdot (\partial_\theta r^{-2}) \cdot (rd\theta) = 0, \tag{70}$$

which does not depend on whether $\overset{\circ}{c}_w$ vanishes or not. Thus, we can show that for the two solutions with the same values of the constants (m, J, Q, N, q, c_ϕ), the boundary integral at infinity vanishes:

$$\int_{\partial\Sigma_\infty} \rho\partial_p \operatorname{tr}\Psi dS^p = 0. \tag{71}$$

(ii) The horizon: $\partial\Sigma_{\mathcal{H}} = \{(\rho, z) | \rho = 0, -k^2 < z < k^2\}$. The regularity on the horizon requires that for $\rho \to 0$,

$$\lambda_{ab} \simeq \mathcal{O}(1), \qquad \omega_a \simeq \mathcal{O}(1), \tag{72}$$

$$\psi_a \simeq \mathcal{O}(1), \qquad \mu \simeq \mathcal{O}(1). \tag{73}$$

Therefore, for $\rho \to 0$, $\rho \operatorname{tr}\Psi$ behaves as

$$\rho\partial_z \operatorname{tr}\Psi \simeq O(\rho). \tag{74}$$

(i), (iv) The outer axes: $\partial\Sigma_\pm = \{(\rho, z) | \rho = 0, k^2 < |z| < \infty\}$. We note that the rod vectors $v = (1, \pm N)$ vanish on the two outer axes. By just the same discussion as in the black lens [46], the regularity requires that for $\rho \to 0$, the potentials λ_{ab} behave as

$$\lambda_{\phi\phi} \simeq N^2 g(z) + \mathcal{O}(\rho^2), \tag{75}$$

$$\lambda_{\phi w} \simeq \mp Ng(z) + \mathcal{O}(\rho^2), \tag{76}$$

$$\lambda_{ww} \simeq g(z) + \mathcal{O}(\rho^2), \tag{77}$$

where $g(z)$ is some function of z. Note here that in the below boundary value analysis, one need not require $g(z)_{[0]} = g(z)_{[1]}$ for the two solutions with the same boundary condition. Next, let us consider the boundary conditions for the electric potentials ψ_a. It follows that for $\rho = 0$,

$$0 = -i_v F = \sqrt{3}(d\psi_\phi \pm Nd\psi_w). \tag{78}$$

Integrating this, we obtain

$$\psi_\phi \pm N\psi_w = c_0, \tag{79}$$

where c_0 is a constant. Note from Eqs. (55) and (56) that $c_0 = \psi_\phi(\rho = 0, z = \pm\infty) \pm N\psi_w(\rho = 0, z = \pm\infty) = \pm\frac{c_\phi}{\sqrt{3}}$. Therefore, we can set the electric potentials to behave as

$$\psi_\phi \simeq \pm\frac{c_\phi}{\sqrt{3}} \mp Nh(z) + \mathcal{O}(\rho^2), \tag{80}$$

$$\psi_w \simeq h(z) + \mathcal{O}(\rho^2), \tag{81}$$

with $h(z)$ being some function of z. We cannot determine how the other magnetic potential, ψ_w, behaves near the

axes and hence do not require $h(z)_{[0]} = h(z)_{[1]}$ for the two solutions.

We further consider the behavior of the magnetic potential μ defined by Eq. (6). Since the norm of the rod vector v vanishes over the outer axes, the first term in the right-hand side of Eq. (6) vanishes there. Then, it follows from Eq. (80) that the derivative of the magnetic potential, μ, is given by

$$d\mu = \mp \frac{c_\phi}{\sqrt{3}} dh(z). \qquad (82)$$

Integrating this, we obtain

$$\mu = \mp \frac{c_\phi}{\sqrt{3}} h(z) + c_1, \qquad (83)$$

where c_1 is an integration constant. Here, note from Eq. (62) that $\mu = q/\sqrt{3}$ at $z = -\infty$, $\rho = 0$ and $\mu = -q/\sqrt{3}$ at $z = \infty$, $\rho = 0$. Therefore, the constant c_1 is determined,

$$c_1 = \mp \frac{q}{\sqrt{3}}. \qquad (84)$$

Thus, we can see that the magnetic potential, μ, must behave as

$$\mu \simeq \mp \frac{c_\phi}{\sqrt{3}} h(z) \mp \frac{q}{\sqrt{3}} + \mathcal{O}(\rho^2), \qquad (85)$$

near the outer axes.

Finally, let us consider the behaviors of the twist potentials ω_a near the outer axes. From Eqs. (80) and (81), the derivatives of the twist potentials on the outer axes are give by

$$d\omega_a = \mp \frac{2c_\phi}{\sqrt{3}} \psi_a dh(z). \qquad (86)$$

Then, it follows that ω_a can be written

$$\omega_\phi = -\frac{2c_\phi^2}{3} h(z) + \frac{Nc_\phi}{\sqrt{3}} h(z)^2 + c_2,$$
$$\omega_w = \pm \frac{c_\phi}{\sqrt{3}} h(z)^2 + c_3, \qquad (87)$$

with c_2 and c_3 constants. From Eqs. (67) and (68), we easily find that

$$\omega_\phi = \pm 4J + \tfrac{1}{2}(c_\phi q - NQ), \qquad \omega_w = \mp Q \qquad (88)$$

at $\rho = 0$, $z = \pm\infty$. These boundary conditions at infinity for the twist potentials and $h(z = \pm\infty) = 0$ determine the integration constants, c_2 and c_3, as

$$c_2 = \pm 4J + \tfrac{1}{2}(c_\phi q - NQ), \qquad c_3 = \mp Q. \qquad (89)$$

Therefore, the twist potentials behave as

$$\omega_\phi \simeq -\frac{2c_\phi^2}{3} h(z) + \frac{Nc_\phi}{\sqrt{3}} h(z)^2 \pm 4J + \frac{1}{2}(c_\phi q - NQ) + \mathcal{O}(\rho^2), \qquad (90)$$

$$\omega_w \simeq \pm \frac{c_\phi}{\sqrt{3}} h(z)^2 \mp Q + \mathcal{O}(\rho^2) \qquad (91)$$

near the outer axes.

Therefore, from Eqs. (75)–(77), (80), (81), (85), (90), and (91), we can show that for $\rho \to 0$, $\rho \partial_z \operatorname{tr}\Psi$ behaves as

$$\rho \partial_z \operatorname{tr}\Psi \simeq O(\rho). \qquad (92)$$

Thus, we find from (i)–(iv) that the boundary integral, Eq. (93), vanishes on each rod and the infinity. The deviation matrix, Ψ, is constant and has the asymptotic behavior, $\Psi \to 0$. Therefore, Ψ vanishes over Σ, and the two configurations, $M_{[0]}$ and $M_{[1]}$, with the same values of constants (m, J, Q, N, q, c_ϕ), must coincide with each other. This completes our proof for the uniqueness theorem for black holes.

V. BOUNDARY VALUE PROBLEMS FOR BLACK RINGS

In this section we would like to consider the boundary value problem for asymptotically Kaluza-Klein black rings. In the Weyl-Papapetrou coordinate system, the boundaries for a black ring with the $S^1 \times S^2$ horizon topology can be given as follows [see Fig. 1(c) about the rod diagram]:

(i) the outer axis, $\partial\Sigma_+ = \{(\rho, z)|\rho = 0, ck^2 < z < \infty\}$ with the rod vector $v = (0, 1, N)$,

(ii) the inner axis, $\partial\Sigma_{in} = \{(\rho, z)|\rho = 0, k^2 < z < ck^2\}$ with the rod vector $v = (0, 1, -N)$,

(iii) the horizon, $\partial\Sigma_{\mathcal{H}} = \{(\rho, z)|\rho = 0, -k^2 < z < k^2\}$,

(iv) the outer axis, $\partial\Sigma_- = \{(\rho, z)|\rho = 0, -\infty < z < -k^2\}$ with the rod vector $v = (0, 1, -N)$,

(v) the infinity, $\partial\Sigma_\infty = \{(\rho, z)|\sqrt{\rho^2 + z^2} \to \infty$ with $z/\sqrt{\rho^2 + z^2}$ finite$\}$,

where constants c and k satisfy $c > 1$ and $0 < k^2$.

Therefore, the boundary integral in the left-hand side of the Mazur identity, Eq. (8), is decomposed into the integrals over the four rods (i)–(iv), and the integral at infinity (v), as

$$\int_{\partial\Sigma} \rho \partial_p \operatorname{tr}\Psi dS^p = \int_{-\infty}^{-k^2} \rho \frac{\partial \operatorname{tr}\Psi}{\partial z} dz + \int_{-k^2}^{k^2} \rho \frac{\partial \operatorname{tr}\Psi}{\partial z} dz$$
$$+ \int_{k^2}^{ck^2} \rho \frac{\partial \operatorname{tr}\Psi}{\partial z} dz + \int_{ck^2}^{\infty} \rho \frac{\partial \operatorname{tr}\Psi}{\partial z} dz$$
$$+ \int_{\partial\Sigma_\infty} \rho \partial_p \operatorname{tr}\Psi dS^p. \qquad (93)$$

Note that the only difference between black holes and black rings appears at the third term in the right side of Eq. (93), which corresponds to the integral over the inner axis inside the black ring. As will be seen below, because of the existence of this third integral, a dipole charge comes to appear in our boundary conditions. For the boundaries (i), (iii), (iv), and (v), the boundary conditions of the scalar fields, Φ^A, are exactly the same as those of black holes. Therefore, we consider only (ii).

Noting that the rod vector is $\nu = (1, -N)$ for the inner axis, we find that the regularity requires that the potentials, λ_{ab}, near the inner axis must behave as

$$\lambda_{\phi\phi} \simeq N^2 k(z) + \mathcal{O}(\rho^2), \qquad (94)$$

$$\lambda_{\phi w} \simeq N k(z) + \mathcal{O}(\rho^2), \qquad (95)$$

$$\lambda_{ww} \simeq k(z) + \mathcal{O}(\rho^2), \qquad (96)$$

where $k(z)$ is some function of z. The electric potentials satisfy

$$0 = -i_\nu F = \sqrt{3}(d\psi_\phi - N d\psi_w). \qquad (97)$$

Hence, integrating this, we obtain

$$\psi_\phi - N\psi_w = c_{\text{in}}, \qquad (98)$$

where c_{in} is an integration constant. Recall that the dipole charge, q_m, of a black ring is defined by

$$q_m = \frac{1}{2\pi} \int_{S^2} F = \sqrt{3}[\psi_{\phi_-}(\rho = 0, z = k^2)$$
$$- \psi_{\phi_-}(\rho = 0, z = -k^2)] = \sqrt{3}c_{\text{in}} + c_\phi. \qquad (99)$$

Therefore, we see that the constant c_{in} is related to the dipole charge q by

$$c_{\text{in}} = \frac{q_m - c_\phi}{\sqrt{3}}. \qquad (100)$$

From Eq. (98) and the requirement of regularity, we can set the electric potentials, ψ_a, to behave as

$$\psi_\phi \simeq c_{\text{in}} + N h(z) + \mathcal{O}(\rho^2), \qquad (101)$$

$$\psi_w \simeq h(z) + \mathcal{O}(\rho^2) \qquad (102)$$

in terms of some function $h(z)$ near the inner axis. Also note that in the boundary value analysis, we do not assume $h(z)_{[0]} = h(z)_{[1]}$.

Next, let us see how the magnetic potential, μ, behaves near the inner axis. From Eqs. (6), (101), and (102), the derivative of the magnetic potential on the inner axis is written as

$$d\mu = -c_{\text{in}} dh(z). \qquad (103)$$

Integrating this on the inner axis, we obtain

$$\mu = -c_{\text{in}} h(z) + \tilde{c}_1, \qquad (104)$$

where $\tilde{c}_1$ is an integration constant. On the other hand, from Eq. (85), we note that just at the joint point $(\rho, z) = (0, ck^2)$ where the outer axis $\partial\Sigma_+$ and the inner axis $\partial\Sigma_{\text{in}}$ meet with each other, the magnetic potential takes the value of

$$\mu = -\frac{c_\phi}{\sqrt{3}} h(ck^2) - \frac{q}{\sqrt{3}}. \qquad (105)$$

Hence, the continuity of the potential, μ, at the point determines the value of the integration constant $\tilde{c}_1$,

$$\tilde{c}_1 = \left(c_{\text{in}} - \frac{c_\phi}{\sqrt{3}}\right) h(ck^2) - \frac{q}{\sqrt{3}}. \qquad (106)$$

Note from Eqs. (79) and (98) and the continuity of the electric potentials that the equations,

$$\psi_\phi(\rho = 0, z = ck^2) + N\psi_w(\rho = 0, z = ck^2) = c_{\text{in}}, \qquad (107)$$

$$\psi_\phi(\rho = 0, z = ck^2) - N\psi_w(\rho = 0, z = ck^2) = \frac{c_\phi}{\sqrt{3}}, \qquad (108)$$

should hold at the point. By solving these, the value of $h(z)$ at $z = ck^2$ can be determined as

$$h(ck^2) = \psi_w(\rho = 0, z = ck^2) = -\frac{1}{2N}\left(c_{\text{in}} - \frac{c_\phi}{\sqrt{3}}\right) \qquad (109)$$

in terms of the constants N, c_ϕ, and c_{in}, i.e., N, c_ϕ, and q_m. Therefore, the magnetic potential μ behaves as

$$\mu \simeq -c_{\text{in}} h(z) - \frac{1}{2N}\left(c_{\text{in}} - \frac{c_\phi}{\sqrt{3}}\right)^2 - \frac{q}{\sqrt{3}} + \mathcal{O}(\rho^2) \qquad (110)$$

near the inner axis.

By similar computations, we can see that the twist potentials behave as

$$\omega_\phi \simeq -2c_{\text{in}}^2 h(z) + N c_{\text{in}} h(z)^2 + [4J + \tfrac{1}{2}(c_\phi q - NQ)] + \tilde{c}_2 + \mathcal{O}(\rho^2), \qquad (111)$$

$$\omega_w \simeq c_{\text{in}} h(z)^2 - Q + \mathcal{O}(\rho^2) + \tilde{c}_3, \qquad (112)$$

where the constants $\tilde{c}_2$ and $\tilde{c}_3$ are given by

$$\tilde{c}_2 = -\frac{1}{4N}\left(c_{\text{in}} - \frac{c_\phi}{\sqrt{3}}\right)^2 (5c_{\text{in}} + \sqrt{3}c_\phi), \qquad (113)$$

$$\tilde{c}_3 = -\frac{1}{4N^2}\left(c_{\text{in}} - \frac{c_\phi}{\sqrt{3}}\right)^2\left(c_{\text{in}} + \frac{c_\phi}{\sqrt{3}}\right). \qquad (114)$$

Therefore, by using Eqs. (94)–(96), (101), (102), and (110)–(112), we can show that for $\rho \to 0$, $\rho \,\mathrm{tr}\Psi$ behaves as

$$\rho \partial_z \,\mathrm{tr}\Psi \simeq O(\rho). \qquad (115)$$

Thus, we find that the boundary integral, Eq. (93), vanishes on each rod and the infinity. The deviation matrix, Ψ, is constant and has the asymptotic behavior, $\Psi \to 0$. Therefore, Ψ vanishes over Σ, and the two configurations, $M_{[0]}$ and $M_{[1]}$, with the same parameters $(m, J, Q, N, q, c_\phi, q_m)$ and same rod data c coincide with each other. This completes our proof for the uniqueness theorem for black rings.

VI. BOUNDARY VALUE PROBLEMS FOR BLACK LENSES

Finally, let us consider the boundary value analysis for black lenses. In terms of the Weyl-Papapetrou coordinate system and the rod structure [66], the boundary $\partial\Sigma$ of the base space $\Sigma = \{(\rho, z) | \rho > 0, -\infty < z < \infty\}$ is described as a set of three rods and the infinity: namely,

(i) the outer axis, $\partial\Sigma_+ = \{(\rho, z)|\rho = 0, k^2 < z < \infty\}$ with the rod vector $v = (0, 1, N)$,

(ii) the horizon, $\partial\Sigma_{\mathcal{H}} = \{(\rho, z)|\rho = 0, -k^2 < z < k^2\}$,

(iii) the outer axis, $\partial\Sigma_- = \{(\rho, z)|\rho = 0, -\infty < z < -k^2\}$ with the rod vector $v = (0, 1 - N)$,

(iv) the infinity, $\partial\Sigma_\infty = \{(\rho, z)|\sqrt{\rho^2 + z^2} \to \infty$ with $z/\sqrt{\rho^2 + z^2}$ finite$\}$.

The above rod structure is similar to that of black holes but now the relation between the nut charge N and the size of the fifth dimension L is given by $N = (L/2)n$. As mentioned in Sec. III, the spatial infinity is topologically a lens space $L(n; 1)$, and hence from the absence of nuts in the black hole exterior region, we can see that the topology of the horizon spatial cross section is $L(n; 1)$. It is clear that how to prove the uniqueness for the black lenses is entirely the same as the black hole case. Accordingly, we can conclude that the two configurations, $M_{[0]}$ and $M_{[1]}$, with the same parameters (m, J, Q, N, q, c_ϕ) coincide with each other.

VII. SUMMARY AND DISCUSSIONS

We have shown the uniqueness theorem which states that in five-dimensional minimal supergravity an asymptotically Kaluza-Klein, stationary charged black hole, or black lens, is uniquely specified by its asymptotic conserved charges and magnetic flux if (1) it admits two independent axial Killing symmetries, (2) the event horizon cross section is connected and nondegenerate, and (3) there exists neither a nut nor a bolt (a bubble) in the domain of outer communication. Furthermore, we have also shown that under the assumptions (1) and (2), a stationary charged rotating black ring with an event horizon of the cross-section topology $S^1 \times S^2$ is uniquely specified by the dipole charge and rod structure in addition to its asymptotic conserved charges and magnetic flux. Our theorem generalizes the uniqueness theorem for Kaluza-Klein black holes in five-dimensional vacuum Einstein gravity [34], or in five-dimensional

Einstein-Maxwell theory [67], to the case of the bosonic sector in five-dimensional minimal supergravity.

Finally, we comment on the assumption (3) in our proof. This assumption (3) restricts the topologies of the black hole exterior regions to the simplest cases. When there exists a nut or a bolt—joint points of two spacelike rods—outside the horizon, the rod structure can have the isolated and finite spacelike rod which cannot be connected with infinity. We here call it *inner axis*. As seen in the proof of black rings, the integration constant c' which is defined by $\psi_\phi + N'\psi_w = c'$ appears in the boundary condition on the inner axis. We have not been able to relate the integration constant to any of the other charges, except for the vacuum case ($q = c_\phi = 0$). We also see that a problem similar to that just mentioned above occurs when we consider uniqueness theorems for multirings, black Saturn, or more complicated black objects. This issue deserves further study.

APPENDIX: COSET MATRIX AND THE MAZUR IDENTITY

Here, to be self-contained, we provide the coset matrix representation and the Mazur identity for our nonlinear sigma model.

Following [65], we introduce the $G_{2(2)}/SO(4)$ coset matrix, M, defined by

$$M = \begin{pmatrix} \hat{A} & \hat{B} & \sqrt{2}\hat{U} \\ \hat{B}^T & \hat{C} & \sqrt{2}\hat{V} \\ \sqrt{2}\hat{U}^T & \sqrt{2}\hat{V}^T & \hat{S} \end{pmatrix}, \tag{A1}$$

where $\hat{A}$ and $\hat{C}$ are symmetric 3×3 matrices, $\hat{B}$ is a 3×3 matrix, $\hat{U}$ and $\hat{V}$ are 3-component column matrices, and $\hat{S}$ is a scalar, defined, respectively, by

$$\hat{A} = \begin{pmatrix} [(1-y)\lambda + (2+x)\psi\psi^T - \tau^{-1}\tilde{\omega}\tilde{\omega}^T + \mu(\psi\psi^T\lambda^{-1}\hat{J} - \hat{J}\lambda^{-1}\psi\psi^T)] & \tau^{-1}\tilde{\omega} \\ \tau^{-1}\tilde{\omega}^T & -\tau^{-1} \end{pmatrix},$$

$$\hat{B} = \begin{pmatrix} (\psi\psi^T - \mu\hat{J})\lambda^{-1} - \tau^{-1}\tilde{\omega}\psi^T\hat{J} & [(-(1+y)\lambda\hat{J} - (2+x)\mu + \psi^T\lambda^{-1}\tilde{\omega})\psi + (z - \mu\hat{J}\lambda^{-1})\omega] \\ \tau^{-1}\psi^T\hat{J} & -z \end{pmatrix},$$

$$\hat{C} = \begin{pmatrix} (1+x)\lambda^{-1} - \lambda^{-1}\psi\psi^T\lambda^{-1} & \lambda^{-1}\tilde{\omega} - \hat{J}(z - \mu\hat{J}\lambda^{-1})\psi \\ \tilde{\omega}^T\lambda^{-1} + \psi^T(z + \mu\lambda^{-1}\hat{J})\hat{J} & [\tilde{\omega}^T\lambda^{-1}\tilde{\omega} - 2\mu\psi^T\lambda^{-1}\tilde{\omega} - \tau(1 + x - 2y - xy + z^2)] \end{pmatrix},$$

$$\hat{U} = \begin{pmatrix} (1 + x - \mu\hat{J}\lambda^{-1})\psi - \mu\tau^{-1}\tilde{\omega} \\ \mu\tau^{-1} \end{pmatrix},$$

$$\hat{V} = \begin{pmatrix} (\lambda^{-1} + \mu\tau^{-1}\hat{J})\psi \\ \psi^T\lambda^{-1}\tilde{\omega} - \mu(1 + x - z) \end{pmatrix},$$

$$\hat{S} = 1 + 2(x - y),$$

with

$$\tilde{\omega} = \omega - \mu\psi, \tag{A2}$$

$$x = \psi^T \lambda^{-1} \psi, \qquad y = \tau^{-1} \mu^2, \qquad z = y - \tau^{-1} \psi^T \hat{J} \tilde{\omega}, \tag{A3}$$

and the 2×2 matrix,

$$\hat{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A4}$$

We note that this 7×7 matrix M is symmetric, $M^T = M$, and unimodular, $\det(M) = 1$. Since we choose the Killing vector fields ξ_ϕ and ξ_w to be spacelike, all the eigenvalues of M are real and positive. Therefore, there exists a $G_{2(2)}$ matrix $\hat{g}$ such that

$$M = \hat{g}\hat{g}^T. \tag{A5}$$

We define a current matrix as

$$J_i = M^{-1}\partial_i M, \tag{A6}$$

which is conserved if the scalar fields are the solutions of the equation of motion derived by the action (1). Then, the action (1) can be written in terms of J and M as follows:

$$S = \frac{1}{4} \int d\rho dz \rho \, \text{tr}(J_i J^i)$$

$$= \frac{1}{4} \int d\rho dz \rho \, \text{tr}(M^{-1}\partial_i M M^{-1}\partial^i M). \tag{A7}$$

Thus, the matrix M completely specifies the solutions to our system.

Let us now consider two sets of field configurations, $M_{[0]}$ and $M_{[1]}$, that satisfy the equations of motion derived from the action, Eq. (A7). We denote the difference between the value of the functional obtained from the field configuration $M_{[1]}$ and the value obtained from $M_{[0]}$ as a bull's eye $\odot$, e.g.,

$$\overset{\odot}{J}{}^i = J^i_{[1]} - J^i_{[0]}, \tag{A8}$$

where the subscripts [0] and [1] denote, respectively, the quantities associated with the field configurations $M_{[0]}$ and $M_{[1]}$. The deviation matrix, Ψ, is then defined by

$$\Psi = \overset{\odot}{M} M_{[0]}^{-1} = M_{[1]} M_{[0]}^{-1} - \mathbf{1}, \tag{A9}$$

where $\mathbf{1}$ is the unit matrix. Taking the derivative of this, we have the relation between the derivative of the deviation matrix and $\overset{\odot}{J}{}^i$,

$$D^i \Psi = M_{[1]} \overset{\odot}{J}{}^i M_{[0]}^{-1}, \tag{A10}$$

where D_i is a covariant derivative associated with the abstract three-metric γ. Taking, further, the divergence of the above formula and also the trace of the matrix elements, we have the following divergence identity

$$D_i D^i \, \text{tr}\Psi = \text{tr}(\overset{\odot Ti}{J} M_{[1]} \overset{\odot i}{J} M_{[0]}^{-1}), \tag{A11}$$

where we have also used the conservation equation $D_i J^i = 0$. Then, integrating this divergence identity over the region $\Sigma = \{(\rho, z) | \rho \geq 0, -\infty < z < \infty\}$, we obtain the Mazur identity,

$$\int_{\partial\Sigma} \rho \partial_p \, \text{tr}\Psi dS^p = \int_\Sigma \rho \hat{h}_{pq} \, \text{tr}(\mathcal{M}^{Tp} \mathcal{M}^q) d\rho dz, \tag{A12}$$

where $\hat{h}_{pq}$ is the two-dimensional flat metric

$$\hat{h} = d\rho^2 + dz^2, \tag{A13}$$

and the matrix $\mathcal{M}$ is defined by

$$\mathcal{M}^p = \hat{g}_{[0]}^{-1} \overset{\odot Tp}{J} \hat{g}_{[1]}. \tag{A14}$$

[1] R. C. Myers and M. J. Perry, Ann. Phys. (N.Y.) **172**, 304 (1986).

[2] R. Emparan and H. S. Reall, Phys. Rev. Lett. **88**, 101101 (2002).

[3] T. Mishima and H. Iguchi, Phys. Rev. D **73**, 044030 (2006).

[4] A. A. Pomeransky and R. A. Sen'kov, arXiv:hep-th/0612005.

[5] H. Iguchi and T. Mishima, Phys. Rev. D **75**, 064018 (2007).

[6] H. Elvang and P. Figueras, J. High Energy Phys. 05 (2007) 050.

[7] K. Izumi, Prog. Theor. Phys. **119**, 757 (2008).

[8] H. Elvang and M. J. Rodriguez, J. High Energy Phys. 04 (2008) 045.

[9] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis, and H. S. Reall, Classical Quantum Gravity **20**, 4587 (2003).

[10] J. C. Breckenridge, R. C. Myers, A. W. Peet, and C. Vafa, Phys. Lett. B **391**, 93 (1997).

[11] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, Phys. Rev. Lett. **93**, 211302 (2004).

[12] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, Phys. Rev. D **71**, 024033 (2005).

[13] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, J. High Energy Phys. 08 (2005) 042.

[14] M. Cvetič and D. Youm, Nucl. Phys. **B476**, 118 (1996).

[15] M. Cvetič, H. Lü, and C. N. Pope, Phys. Lett. B **598**, 273 (2004).

[16] Z.-W. Chong, M. Cvetič, H. Lü, and C. N. Pope, Phys. Rev. Lett. **95**, 161301 (2005).

[17] H. Elvang, R. Emparan, and P. Figueras, J. High Energy Phys. 02 (2005) 031.

[18] S. S. Yazadjiev, Phys. Rev. D **77**, 127501 (2008).

[19] S. S. Yazadjiev, Phys. Rev. D **76**, 064011 (2007).

[20] S. S. Yazadjiev, Phys. Rev. D **73**, 104007 (2006).

[21] S. S. Yazadjiev, Phys. Rev. D **78**, 064032 (2008).

[22] For review, M. Heusler, *Black Hole Uniqueness Theorems* (Cambridge University Press, Cambridge, England, 1996).

[23] W. Israel, Phys. Rev. **164**, 1776 (1967).

[24] W. Israel, Commun. Math. Phys. **8**, 245 (1968).

[25] G. L. Bunting and A. K. M. Masood-ul-Alam, Gen. Relativ. Gravit. **19**, 147 (1987).

[26] A. K. M. Masood-ul-Alam, Classical Quantum Gravity **9**, L53 (1992).

[27] B. Carter, Phys. Rev. Lett. **26**, 331 (1971).

[28] D. C. Robinson, Phys. Rev. D **10**, 458 (1974).

[29] D. C. Robinson, Phys. Rev. Lett. **34**, 905 (1975).

[30] P. O. Mazur, J. Phys. A **15**, 3173 (1982).

[31] G. L. Bunting, Ph.D. thesis, University of New England, Armidale, N. S. W., 1983.

[32] G. W. Gibbons, D. Ida, and T. Shiromizu, Prog. Theor. Phys. Suppl. **148**, 284 (2002); Phys. Rev. Lett. **89**, 041101 (2002).

[33] F. R. Tangherlini, Nuovo Cimento **27**, 636 (1963).

[34] Y. Morisawa and D. Ida, Phys. Rev. D **69**, 124005 (2004).

[35] S. Hollands and S. Yazadjiev, Commun. Math. Phys. **283**, 749 (2008).

[36] Y. Morisawa, S. Tomizawa, and Y. Yasui, Phys. Rev. D **77**, 064019 (2008).

[37] S. Hollands and S. Yazadjiev, Classical Quantum Gravity **25**, 095010 (2008).

[38] M. Rogatko, Classical Quantum Gravity **19**, 875 (2002).

[39] M. Rogatko, Classical Quantum Gravity **19**, L151 (2002).

[40] M. Rogatko, Phys. Rev. D **67**, 084025 (2003).

[41] M. Rogatko, Phys. Rev. D **70**, 044023 (2004).

[42] M. Rogatko, Phys. Rev. D **70**, 084025 (2004).

[43] M. Rogatko, Phys. Rev. D **73**, 124027 (2006).

[44] M. Rogatko, Phys. Rev. D **77**, 124037 (2008).

[45] S. Tomizawa, Y. Yasui, and A. Ishibashi, Phys. Rev. D **79**, 124023 (2009).

[46] S. Tomizawa, Y. Yasui, and A. Ishibashi, Phys. Rev. D **81**, 084037 (2010).

[47] The assumption of asymptotic global flatness becomes relevant in the context of a certain type of braneworld model, in which the size of higher dimensional black holes can be much smaller than the size of extra dimensions.

[48] H. Ishihara and K. Matsuno, Prog. Theor. Phys. **116**, 417 (2006).

[49] P. Dobiasch and D. Maison, Gen. Relativ. Gravit. **14**, 231 (1982).

[50] D. Rasheed, Nucl. Phys. **B454**, 379 (1995).

[51] G. W. Gibbons and D. L. Wiltshire, Ann. Phys. (N.Y.) **167**, 201 (1986).

[52] I. Bena, P. Kraus, and R. Warner, Phys. Rev. D **72**, 084019 (2005).

[53] D. Gaiotto, A. Strominger, and X. Yin, J. High Energy Phys. 02 (2006) 023.

[54] H. Ishihara, M. Kimura, K. Matsuno, and S. Tomizawa, Classical Quantum Gravity **23**, 6919 (2006).

[55] T. Nakagawa, H. Ishihara, K. Matsuno, and S. Tomizawa, Phys. Rev. D **77**, 044040 (2008).

[56] S. Tomizawa, H. Ishihara, K. Matsuno, and T. Nakagawa, Prog. Theor. Phys. **121**, 823 (2009).

[57] K. Matsuno, H. Ishihara, T. Nakagawa, and S. Tomizawa, Phys. Rev. D **78**, 064016 (2008).

[58] S. Tomizawa and A. Ishibashi, Classical Quantum Gravity **25**, 245 007 (2008).

[59] S. Tomizawa, Y. Yasui, and Y. Morisawa, Classical Quantum Gravity **26**, 145 006 (2009).

[60] D. V. Gal'tsov and N. G. Scherbluk, Phys. Rev. D **79**, 064020 (2009).

[61] Topology censorship in Kaluza-Klein spacetimes [62] tells that topology of T^3 is not allowed under the assumption of the null energy condition.

[62] P. T. Chruściel, G. J. Galloway, and D. Solis, Ann. Inst. Henri Poincaré, A **10**, 893 (2009).

[63] S. Mizoguchi and N. Ohta, Phys. Lett. B **441**, 123 (1998).

[64] S. Mizoguchi and G. Schröder, Classical Quantum Gravity **17**, 835 (2000).

[65] A. Bouchareb, G. Clement, C-M. Chen, D. V. Gal'tsov, N. G. Scherbluk, and T. Wolf, Phys. Rev. D **76**, 104032 (2007); **78**, 029901(E) (2008).

[66] T. Harmark, Phys. Rev. D **70**, 124002 (2004); T. Harmark and P. Olesen, Phys. Rev. D **72**, 124017 (2005).

[67] S. S. Yazadjiev, Phys. Rev. D **82**, 024015 (2010).

[68] S. Hollands and S. S. Yazadjiev, arXiv:0812.3036.

[69] E. G. Gimon and A. Hashimoto, Phys. Rev. Lett. **91**, 021601 (2003).

[70] C. A. R. Herdeiro, Classical Quantum Gravity **20**, 4891 (2003).

[71] This assumption concerning two independent axial symmetries is not fully justified, as the rigidity theorem [72–74] guarantees the existence of only a single axial symmetry for stationary black holes.

[72] S. Hollands, A. Ishibashi, and R. M. Wald, Commun. Math. Phys. **271**, 699 (2007).

[73] V. Moncrief and J. Isenberg, Classical Quantum Gravity **25**, 195 015 (2008).

[74] S. Hollands and A. Ishibashi, Commun. Math. Phys. **291**, 443 (2009).

[75] R. Emparan and H. S. Reall, Phys. Rev. D **65**, 084025 (2002).

[76] B. Carter, Phys. Rev. Lett. **26**, 331 (1971); P. T. Chruściel, J. Math. Phys. (N.Y.) **50**, 052 501 (2009).

[77] Y. Chen and E. Teo, Nucl. Phys. **B838**, 207 (2010).