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Multipole moments for black objects in five dimensions

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In higher dimensions than four, conventional uniqueness theorem in asymptotically flat space-times does not hold, i.e., black objects cannot be classified only by the mass, angular momentum, and charge. In this paper, we define multipole moments for black objects and show that Myers-Perry black hole and black ring can be distinguished by quadrupole moments. This consideration gives us a new insight for the uniqueness theorem for black objects in higher dimensions.

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I. INTRODUCTION

In four dimensions, stationary and asymptotically flat black hole solutions can be classified by their mass, angular momentum, and charge completely. This is the famous uniqueness theorem [1]. On the other hand, this uniqueness property of black objects does not hold in higher dimensions. As presented by Emparan and Reall [2–4] (see also Ref. [5]), in five dimensions, there is the black ring solution, which can have the same mass and angular momentum as the Myers-Perry black hole [6]. If we do not restrict our consideration to cases with a single horizon, there are many, probably infinite, regular solutions with the same mass and angular momentum [7–11]. This shows that there are much richer properties of black object solutions in higher dimensions compared to four dimensions. At the same time, however, it is unlikely that the complete classifications of these black objects are possible. Now, note that there are a sort of uniqueness theorems in some restricted cases. For example, in static and vacuum space-times, the Schwarzschild-Tangherlini solution [12] is only regular black hole solution [13]. For stationary solutions which have a single horizon and two axial commuting Killing vectors, if one specifies the topology of the horizon (S^3 or $S^1 \times S^2$), the solution can be uniquely determined (the Myers-Perry solutions or black ring solutions, respectively) [14–16]. Furthermore, in more general situations of five-dimensional, stationary, and two rotational symmetric asymptotically flat space-times, if one specifies its mass, angular momentum, and so-called rod structure [2,17], which represents the positions of event horizons and rotational axis, the regular solution is determined uniquely [18]. Theorems like these can be extended to nonvacuum cases and so on [19–25]. However, these uniqueness theorem for five-dimensional stationary black objects are not satisfactory in physical point of view. This is because the relation between the rod structure and global charge is unclear and we want to classify black object space-times in terms of global charges or quantities observed at infinity. Unfortunately, the rod structure is quasilocal concept. The purpose of this paper is the classification of stationary black objects by multipole moments, which are defined at spatial infinity. The asymptotic quantities like multipole moments might be useful to study the properties of solutions(black ring solutions in $d > 5$ dimensions [26] or non-Myers-Perry black hole with spherical topology of event horizon [27,28]) which are conjectured to be exist.

Geroch [29,30] and Hansen [31] defined the multipole moments by using the conformal completion to obtain the property of space-times at spatial infinity Λ in four dimensions. In Ref. [30], Geroch conjectured that (A) two solutions of the four-dimensional Einstein equations having the same multipole moments coincide each others at least in a neighborhood of Λ, and (B) given any sets of multipole moments, subject to the appropriate convergence condition, there exist a solution of Einstein equations having precisely those moments. About conjecture (A), Beig and Simon and Kundu showed the validity for static [32] and stationary space-times [33,34]. For conjecture (B), there is no rigorous proof and it is an open issue even in four dimensions until now. There is also the coordinate based definition of multipole moments by Thorne [35]. It was shown that Thorne's multipole moments are same as Geroch and Hansen's multipole moments under certain conditions [36]. Following Geroch's idea on four dimensional static cases, Tomizawa and one of the present authors proposed the definition of multipole moments in higher dimensional *static* space-times [37]. In this paper, we discuss the definition of multipole moments in five-dimensional *stationary* space-times and then show that asymptotically flat, stationary, and two rotational symmetric solutions with a *single* horizon are completely classified by the mass monopole, quadrupole moments, and angular dipole moments. This successful result will encourage us to study the classification for general cases including multiple horizon cases.

The rest of this paper is organized as follows. In the next section, we define mass multipole moments and angular multipole moments in five-dimensional stationary space-times. We will emphasize that the definition of the angular multipole moments is a rather nontrivial task. Some details related to the definitions are shown in Appendix A. In Sec. III, as an exercise, we shall consider the multipole moments of the static black objects and discuss the classification of them. In Sec. IV, we will compute the multipole moments for stationary black objects. Then we show that black ring can be distinguished from the Myers-Perry solutions by the ''reduced'' quadrupole moments which are well defined in center-of-mass gauge. In Sec. V, we summarize our result and discuss the possibility of uniqueness theorem using the multipole moments. In Appendix A, we write down the field equations and see why our definition of multipole moments is appropriate. In Appendix B, we compute the multipole moments for the black ring solutions with two angular momenta. In Appendix C, for comparison, we compute the multipole moments for black objects with multiple horizons solutions like black Saturn [7] and orthogonal black di-ring solutions [10].

II. DEFINITION OF MULTIPOLE MOMENTS

In this section, following Refs. [30,37], we shall define the multipole moments in five-dimensional stationary space-times. At first, we describe the definition of the asymptotically flatness based on the conformal completion method briefly. Then we will give a definition of the multipole moments. The definition of the multipole moments associated with angular momentum is not given by a simple extension from four to five dimensions. We also address the gauge dependence which comes from the gauge freedom of the conformal transformation.

A. Asymptotic flatness

For stationary space-times, we introduce the notion of asymptotic flatness at spatial infinity based on conformal completion method [38,39]. The metric of stationary space-times can be written as

$$\hat{g}_{ab} = \frac{1}{\lambda}\xi_a\xi_b + \hat{h}_{ab}, \tag{1}$$

where $\xi = \partial/\partial t$ is the timelike Killing vector, $\lambda = \hat{g}_{ab}\xi^a\xi^b$, and $\hat{h}_{ab}$ is the metric on $t = \text{const}$ hypersurfaces. Since the multipole moments will be defined on $t = \text{const}$ hypersurfaces, we can focus on only the metric on $t = \text{const}$ hypersurfaces.

Let us consider the conformal transformation as

$$h_{ab} = \Omega^2\hat{h}_{ab}. \tag{2}$$

If there is a function Ω which satisfies the conditions

$$\Omega \hat{=} 0, \qquad D_a\Omega \hat{=} 0, \qquad D_aD_b\Omega \hat{=} 2h_{ab}, \tag{3}$$

$t = \text{const}$ hypersurface is called asymptotically flat space and the point $\Omega = 0$ is identified as the spatial infinity Λ. Here, D_a is the covariant derivative with respect to the metric h_{ab}, and $\hat{=}$ stands for the evaluation on Λ. At the spatial infinity, h_{ab} becomes the flat metric and we use the coordinate as follows:

$$ds^2 = h_{ab}dx^adx^b \hat{=} d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2). \tag{4}$$

The above conditions on Ω imply the asymptotic behavior of Ω as $\Omega \sim 1/r^2 \sim \rho^2$ near Λ. In this paper, we consider only the solutions of the vacuum Einstein equations $\hat{R}^{(5)}_{ab} = 0$ for simplicity.

B. Multipole moments

At first, we define mass 2^s-pole moments $P_{a_1a_2\cdots a_s}$ as

$$P \equiv \Omega^{-1}(1 - \sqrt{\lambda}), \tag{5}$$

$$P_a = D_aP, \tag{6}$$

$$P_{a_1a_2\cdots a_s} = \mathcal{O}\left[D_{a_1}P_{a_2\cdots a_s} - \frac{(s-1)^2}{2}R_{a_1a_2}P_{a_3\cdots a_s}\right], \tag{7}$$

where $\mathcal{O}[T_{ab\cdots}]$ denotes the totally symmetric trace free part of the tensor $T_{ab\cdots}$. In four dimensions, the mass multipole moments in stationary space-times are defined from the scalar potential P which consists of the lapse function λ and the twist potential σ satisfying a conformal invariant equation [31]. As we will show soon, the corresponding twist potential is represented by a vector σ_a, not a scalar, in five dimensions. Hence, we use only the lapse functions λ for the definition in five dimensions, and this does not matter. In fact, we can check that the twist potential does not contribute to the multipole moments in four and five dimensions. The role of σ will be important only in the proof of the smoothness of the multipole moments in four dimensions [31,33,34]. Note that the relation between the Arnowitt-Deser-Misner (ADM) mass and the mass monopole is shown in Ref. [17],

$$M_{\rm ADM} = \frac{3\pi}{4}P. \tag{8}$$

For the definition of angular multipole moments, we assume the presence of the two commuting axisymmetric Killing vectors $m = \partial/\partial\phi$ and $l = \partial/\partial\psi$. In this case, there are many known exact solutions. However, note that this assumption might be rather strong in some senses. This is because the existence of only a single axisymmetric Killing vector is guaranteed from the stationarity [40,41]. Although it is interesting to consider the multipole moments of single rotational symmetric cases too [42], this is beyond the scope of our current paper.

Now, we introduce the tensor $\hat{\sigma}_{ab}$ as

$$\hat{\sigma}_{ab} = \hat{\varepsilon}_{abcde}\xi^c\hat{\nabla}^d\xi^e. \qquad (9)$$

Since $\hat{\sigma}_{ab}$ satisfies $\hat{D}_{[a}\hat{\sigma}_{bc]} = 0$ by the vacuum Einstein equation ${}^{(5)}\hat{R}_{ab} = 0$, $\hat{\sigma}_{ab}$ can be written by the twist potential $\hat{\sigma}_a$ as

$$\hat{\sigma}_{ab} = \hat{D}_{[a}\hat{\sigma}_{b]}. \qquad (10)$$

$\hat{\sigma}_a$ satisfies the four-dimensional Maxwell-type equation (see Appendix A). Therefore, under the conformal transformation of Eq. (2), we can take $\hat{\sigma}_{ab}$ as conformal invariant quantity, i.e., $\hat{\sigma}_{ab} = \sigma_{ab}$ and $\hat{\sigma}_a = \sigma_a$. From this twist potential σ_a, we can construct two scalar potentials as

$$J^\phi = \frac{\sigma_a l^a}{(l_a l^a)^{1/2}} \quad \text{and} \quad J^\psi = \frac{\sigma_a m^a}{(m_a m^a)^{1/2}}. \qquad (11)$$

Then we can define the angular multipole moments in the same way as the mass multipole moments discussed above. That is, the angular 2^s-pole moments $J^\phi_{a_1a_2\cdots a_s}$ and $J^\psi_{a_1a_2\cdots a_s}$ are defined recursively as

$$J^\phi_{a_1a_2\cdots a_s} = \mathcal{O}\left[D_{a_1}J^\phi_{a_2\cdots a_s} - \frac{(s-1)^2}{2}R_{a_1a_2}J^\phi_{a_3\cdots a_s}\right], \qquad (12)$$

and

$$J^\psi_{a_1a_2\cdots a_s} = \mathcal{O}\left[D_{a_1}J^\psi_{a_2\ldots a_s} - \frac{(s-1)^2}{2}R_{a_1a_2}J^\psi_{a_3\cdots a_s}\right]. \qquad (13)$$

Here we have a remark: if one considers the cases having single rotational symmetry, σ_a cannot be written only by scalar potential in a natural way. This means that it is not easy to construct the angular multipole moments in single rotational symmetric cases. The resolution to this difficulty is left for future study.

C. Unphysical gauge dependence

Before computing the multipole moments for known black objects, we should comment on the gauge dependence of the multipole moments defined above. There are gauge freedoms in the conformal completion of Eq. (2) as

$$\Omega \to \omega\Omega, \qquad (14)$$

where

$$\omega \simeq 1 + \frac{f(\theta, \phi, \ldots)}{r} + O(1/r^2). \qquad (15)$$

Under this gauge transformation, multipole moments are transformed as

$$P_{a_1\cdots a_s} \to P_{a_1\cdots a_s} - s^2\mathcal{O}[P_{a_1\cdots a_{s-1}}D_{a_s}\omega] \qquad (16)$$

in the linear order of ω.[1] $D_a\omega$ represents the $1/r$-order part in ω [see Eq. (15)] and corresponds to the choice of the origin of the coordinate in the physical coordinate $\hat{x}^a$ [38]. In the definition of Thorne's multipole moments [35], there is such gauge freedom, which is just a translation. Thus, the freedom of the order of $D_a\omega$ can be fixed by gauge conditions just like a center-of-mass gauge. Higher order parts $O(1/r^2)$ in Eq. (15), which can be written as $D_aD_b\omega, D_aD_bD_c\omega, \cdots$, do not contribute to the transformation of multipole moments in the linear order. On the other hand, in nonlinear order of ω, the changes of multipole moments under transformations of Eq. (14) depend on not only $D_a\omega$, but also higher order terms. For example, the octupole moments are transformed as

$$P_{abc} \to P_{abc} - 9\mathcal{O}[P_{ab}D_c\omega] + g\mathcal{O}[D_a\omega D_b D_c\omega]P + \cdots, \qquad (17)$$

where g is a numerical constant. Hence the value of the octupole or higher-pole moments depend on the choice of $O(1/r^2)$ parts in ω, while monopole, dipole, and quadrupole moments depend only on $D_a\omega$. Higher multipole moments than quadrupole, that is, octupole and 2^4-pole moments, have gauge ambiguities from the conformal transformation even in center-of-mass gauge. Since there is no tractable way to fix the gauge freedom of $O(1/r^2)$ parts in ω, we will focus on the computation of only monopole, dipole and quadrupole moments for known black objects solutions in the center-of-mass gauge.

D. Modes

In the practical calculations of the multipole moments, we must specify the concrete coordinate and modes. In this paper, we use the coordinate in Eq. (4). Then it is easy to see that the quadrupole moments have nine modes as

$$\begin{gathered}\cos 2\theta, \quad \sin^2\theta\sin 2\phi, \quad \sin^2\theta\cos 2\phi, \\ \cos^2\theta\sin 2\psi, \quad \cos^2\theta\cos 2\psi, \quad \cos\theta\sin\theta\sin\phi\sin\psi, \\ \cos\theta\sin\theta\sin\phi\cos\psi, \quad \cos\theta\sin\theta\cos\phi\sin\psi, \\ \cos\theta\sin\theta\cos\phi\cos\psi. \end{gathered} \qquad (18)$$

In two rotational symmetric cases, nonvanishing quadrupole moment mode is only one mode of $\cos 2\theta$ in center-of-mass gauge. In this paper, then, we define the coefficients of this mode in $P_{\rho\rho}$ as mass quadrupole moments Q.[2]

III. STATIC CASES

In this section, as a first step, we compute the multipole moments for known static solutions. Then we will discuss the classification of space-times with single horizon. Following the uniqueness theorem [13], regular static black object solutions of the vacuum Einstein equation

[1] Transformations of $J^\phi_{a_1\cdots a_s}$ and $J^\psi_{a_1\cdots a_s}$ are same as Eq. (16).

[2] If we define the quadrupole moment as the coefficient in other components of P_{ab}, the difference will be only sign.

are completely classified by its mass, that is, the mass monopole. Other solutions like the static black ring have conical singularities and they are not regular solutions. However, by computing the multipole moments of these nonregular solutions, we can study the dependence of the multipole moments on the topology of horizon. Since all angular multipole moments vanish in static cases, we will consider only mass multipole moments. This section will be helpful to study the multipole moments for stationary cases in which we are interested more.

The metric of static and two rotational symmetric spacetimes in five dimensions can be written in the Weyl coordinate [2],

$$ds^2 = -e^{2U_t}dt^2 + e^{2U_\phi}d\phi^2 + e^{2U_\psi}d\psi^2 + e^{2\nu}(dR^2 + dz^2), \tag{19}$$

where $U_t + U_\phi + U_\psi = \log R$. The solutions are represented by the rod structure which is composed of the zero points of g_{tt}, $g_{\phi\phi}$, and $g_{\psi\psi}$ and stand for the positions of event horizons and rotational axis. For computation of the multipole moments, it is better to use new coordinate given by

$$R = \tfrac{1}{2}r^2 \sin 2\theta, \tag{20}$$

and

$$z = \tfrac{1}{2}r^2 \cos 2\theta, \tag{21}$$

and take the conformal factor as $\Omega = 1/r^2 = \rho^2$. The coordinate t, r, θ, ϕ, and ψ here are same as those in Eq. (4). The Weyl coordinate has the gauge freedom $z \to z +$ constant and this gauge freedom corresponds to $r \to r(1 + O(1/r^2))$ in the coordinate of Eq. (4). As mentioned in the previous section, the monopole, dipole, and quadrupole moments we compute in the following are independent on this $O(1/r^2)$ order gauge transformations. This implies that those moments should be written by the difference $a_i - a_j$ as seen soon later.

A. Schwarzschild black holes

At first, we will compute the multipole moments of the Schwarzschild black hole as a trivial example. The rod structure of the Schwarzschild black holes are shown in Fig. 1. The Schwarzschild black hole is described by two parameters $a_{\rm Sch}$ and $b_{\rm Sch}$ in the Weyl coordinate. As there is a gauge freedom $z \to z +$ constant, however, the independent parameter is only $b_{\rm Sch} - a_{\rm Sch}$. After all, the multipole moments of the Schwarzschild black hole are computed as

$$P = b_{\rm Sch} - a_{\rm Sch}, \qquad P_a = 0, \tag{22}$$

$$Q = 0. \tag{23}$$

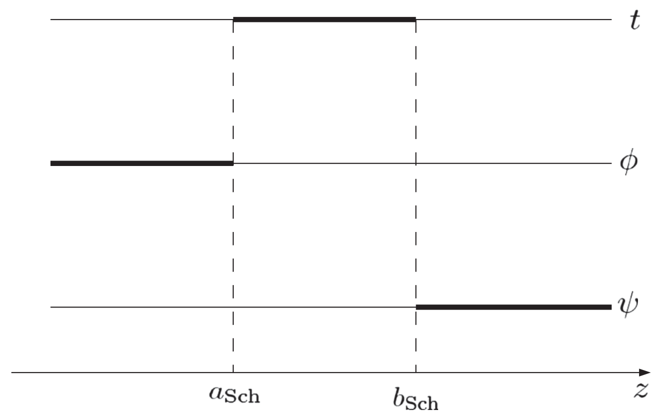

FIG. 1. Rod structure of Schwarzschild black hole.

Hence, the Schwarzschild black hole has only monopole as nontrivial multipole moments, which is proportional to the ADM mass. Here note that the 2^4-pole moment L is given by $L \propto (a_{\rm Sch}^3 - b_{\rm Sch}^3)$. As we stressed before, however, it has the unphysical gauge dependence and then the physical meaning of this L is unclear.

B. Static black ring

Next, we compute the multipole moments of static black ring solution (the rod structure is shown in Fig. 2). Static black ring solutions have two independent parameters $b_{\rm BR} - a_{\rm BR}$ and $c_{\rm BR} - b_{\rm BR}$. Then, after short calculation, we can see that the multipole moments are

$$P = b_{\rm BR} - a_{\rm BR}, \qquad P_a = 0, \tag{24}$$

$$Q = -4(b_{\rm BR} - a_{\rm BR})(c_{\rm BR} - b_{\rm BR}). \tag{25}$$

In our definition of quadrupole moments, Q is always nonpositive. In addition, only if we take the Schwarzschild limit of $b_{\rm BR} = c_{\rm BR}$ or flat limit of $a_{\rm BR} = b_{\rm BR}$, the quadrupole moment vanishes.

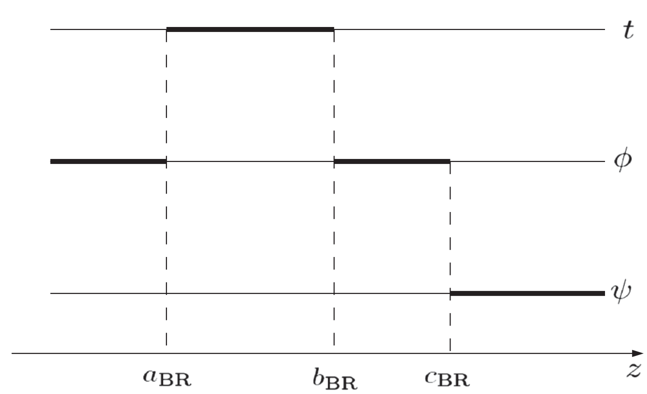

FIG. 2. Rod structure of static black ring (BR).

C. Classification issue

We can distinguish the Schwarzschild black hole from black ring solutions by quadrupole moments. Then, if the single horizon is assumed, black objects are completely classified by the mass monopole and quadrupole moments, which are well defined in center-of-mass gauge.

Although the cases with multiple horizons are beyond of our current consideration, we have some comments. In Appendix C, we computed the multipole moments for black Saturn and orthogonal black di-ring cases as examples. Static black Saturn and orthogonal di-ring solutions have three and four independent parameters, respectively. On the other hand, the mass monopole and quadrupole moments can determine only two independent parameters. Then the multipole moments up to the quadrupole moments are not enough parameters to specify the space-times uniquely. That is, higher multipole moments are needed for the classification of these solutions. As mentioned in the previous section, however, the higher multipole moments have the unphysical gauge ambiguities of ω. Thus, we should fix the gauge about the conformal factor Ω completely or improve the definition of the higher multipole moments.

IV. STATIONARY CASES

In static cases, we have shown that black objects with single horizon can be classified by mass monopole and quadrupole moments completely. In this section, we consider the black objects with angular momentum and single horizon. As we will see later soon, in stationary cases, rotating black objects can be classified by mass monopole, quadrupole, and angular dipole moments.

Using the fact of $\sigma_\phi \sim -\tan^2\theta g_{t\psi}$ and $\sigma_\psi \sim \cot^2\theta g_{t\phi}$ near the spatial infinity, we define the coefficient of $\cos\theta$ in J^ϕ_ρ and $\sin\theta$ in J^ψ_ρ as angular dipole moments J_ϕ and J_ψ. Here note that $\cos\theta$ is $l = 1$ mode of scalar harmonics in ϕ-rotational plane with the metric $d\theta^2 + \sin^2\theta d\phi^2$ and $\sin\theta$ is one in the ψ-rotational plane with the metric $d\theta^2 + \cos^2\theta d\psi^2$. Note that the relations between the angular dipole moments and the ADM angular momentum are given by [17]

$$J^\phi_{\rm ADM} = \frac{\pi}{4} J_\phi, \qquad J^\psi_{\rm ADM} = -\frac{\pi}{4} J_\psi. \tag{26}$$

A. Myers-Perry black holes

Let us examine the Myers-Perry solutions. The metric of the Myers-Perry black holes is given by

$$\begin{aligned} ds^2 = &-dt^2 + \frac{M}{\Sigma}(dt - j_\phi \sin^2\theta d\phi - j_\psi \cos^2\theta \psi)^2 \\ &+ (r^2 + j_\phi^2)\sin^2\theta d\phi^2 + (r^2 + j_\psi^2)\cos^2\theta d\psi^2 \\ &+ \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2, \end{aligned} \tag{27}$$

where

$$\Sigma = r^2 + j_\phi^2 \cos^2\theta + j_\psi^2 \sin^2\theta, \tag{28}$$

and

$$\Delta = r^2\left(1 + \frac{j_\phi^2}{r^2}\right)\left(1 + \frac{j_\psi^2}{r^2}\right) - M. \tag{29}$$

Introducing the new coordinate defined by $\rho = 1/r$ and taking the conformal factor as $\Omega = \rho^2$, we can compute the multipole moments of the Myers-Perry black holes. The results are

$$P = \frac{M}{2}, \qquad P_a = 0, \qquad Q = -(j_\phi^2 - j_\psi^2)M, \tag{30}$$

$$J = 0, \qquad J_\phi = j_\phi M, \qquad J_\psi = -j_\psi M, \tag{31}$$

$$J^\phi_{ab} = J^\psi_{ab} = 0. \tag{32}$$

Contrasted with the Schwarzschild black hole case, the rotating black holes have nonzero mass quadrupole moments, which are contributions from the rotations. To measure the deviation of other black object solutions from the Myers-Perry black holes, it is better to define the reduced mass quadrupole moments as

$$Q^{\rm red} = Q + \frac{J_\phi^2 - J_\psi^2}{2P}. \tag{33}$$

It is chosen so that the reduced mass quadrupole moments of the Myers-Perry black holes vanishes, that is,

$$Q^{\rm red}_{\rm MP} = 0. \tag{34}$$

B. Black ring with single angular momentum

Next, we compute the multipole moments of black ring with single angular momentum. For the case of black ring with two angular momenta, see Appendix. B. The metric is given by [4]

$$\begin{aligned} ds^2 = &-\frac{F(y)}{F(x)}\left(dt - CR\frac{1+y}{F(y)}d\phi\right)^2 + \frac{R^2}{(x-y)^2}F(x) \\ &\times\left[-\frac{G(y)}{F(y)}d\phi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\psi^2\right], \end{aligned} \tag{35}$$

where

$$F(\xi) = 1 + \lambda\xi, \qquad G(\xi) = (1 - \xi^2)(1 + \nu\xi), \tag{36}$$

$$C = \sqrt{\lambda(\lambda - \nu)\frac{1+\lambda}{1-\lambda}}, \tag{37}$$

and the parameter range is $0 < \nu \le \lambda < 1$. For the regular black ring solution which has no conical singularities, the parameters λ and ν must satisfy the relation as

$\lambda = 2\nu/(1+\nu^2)$. In the following, we will consider this regular black ring solution.

For the computation of the multipole moments, it is better to introduce the coordinate (ρ, θ) defined by

$$x = -1 + \frac{2R^2(1-\lambda)}{1-\nu}\rho^2\cos^2\theta, \qquad y = -1 - \frac{2R^2(1-\lambda)}{1-\nu}\rho^2\sin^2\theta, \tag{38}$$

and take conformal factor as $\Omega = \rho^2$. Then, the multipole moments of black ring with single angular momentum are evaluated as

$$P = \frac{R^2\lambda}{1-\nu}, \qquad P_a = 0, \tag{39}$$

$$Q = -2R^4\lambda\frac{(1+\lambda-3\nu+\lambda\nu)}{(1-\nu)^3}, \tag{40}$$

$$J = 0, \qquad J_\phi = \frac{2R^3\sqrt{\lambda(1+\lambda)(\lambda-\nu)}}{(1-\nu)^2}, \qquad J_{ab} = 0, \tag{41}$$

and the reduced quadrupole moment becomes

$$Q_{\rm BR}^{\rm red} = -\frac{2R^4\nu(1-\lambda)^2}{(1-\nu)^3} \le 0. \tag{42}$$

As in static cases, the reduced quadrupole moments of black ring solutions have always nonpositive value. In the appearance of naked singularity with $\lambda = 0$, 1 or in the Myers-Perry limit or flat metric limit, the reduced quadrupole moment becomes to be zero.

C. Classification issue

As shown above, the Myers-Perry black hole and black ring solutions with single angular momentum are classified by (reduced) mass quadrupole moments completely. When one specifies the ADM mass and angular momentum of black ring solutions, there are two different solutions: thin ring and fat ring solutions in a certain of parameter region. To see this, it is useful to introduce the quantity j^2 as

$$j^2 \equiv \frac{27}{32}\frac{J_{\rm ADM}^2}{M_{\rm ADM}^3} = \frac{(1+\nu)^3}{8\nu}. \tag{43}$$

Regular black ring solution has the two independent parameters R and ν satisfying $0 < R$ and $0 < \nu < 1$ as in Eq. (35). If $M_{\rm ADM}$ and $J_{\rm ADM}$ of black ring are given, we can compute the value of j^2 and determine the parameter ν. However, in the range $27/32 < j^2 < 1$, it is known that there are two different solutions, that is, thin ring $(1 > \nu > 1/2)$ and fat ring solutions $(\nu < 1/2)$ shown in Fig. 3. Then, even if we assume the horizon topology of $S^1 \times S^2$, we cannot specify the solution only by $M_{\rm ADM}$ and $J_{\rm ADM}$. This is a well-known fact.

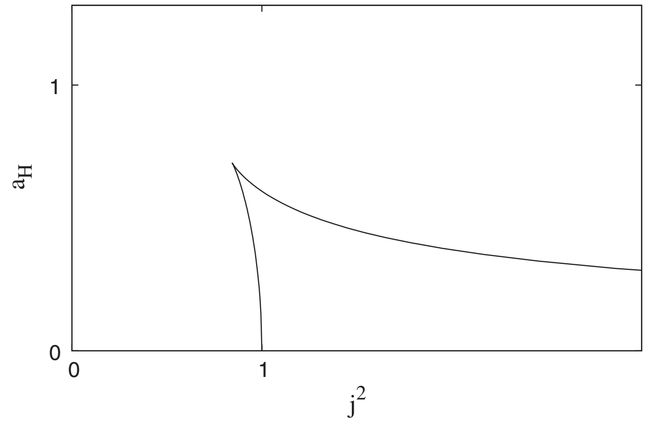

FIG. 3. Normalized horizon area $a_H = \sqrt{2\nu(1-\nu)}$ vs spin. Even if given the spin $j \sim \nu$, the fat ring $(\nu < 1/2)$ and thin ring $(\nu > 1/2)$ are not distinguished.

Then, one wonders if one can distinguish these two solutions (thin or fat) by the reduced quadrupole moment. The answer is yes. To see this, we define q as

$$q = \frac{-4Q_{\rm BR}^{\rm red}}{P^2} = \frac{2(1-\nu)^3}{\nu}. \tag{44}$$

The relation between the normalized area of the event horizon a_H [4] and q is shown in Fig. 4. From Fig. 4, one can see that we can determine the parameter R and ν if the reduced quadrupole moments $Q^{\rm red}$ and mass monopole P of black ring solutions are given. Thus, black ring solutions are completely specified by the mass monopole and the reduced quadrupole moments.

Here we comment on the multipole moments for black objects with *multiple* horizons, e.g., black Saturn [7] and orthogonal black di-ring solutions [10,11], although this is beyond the scope of our current paper. As shown in Appendix C, these solutions all have nonvanishing reduced

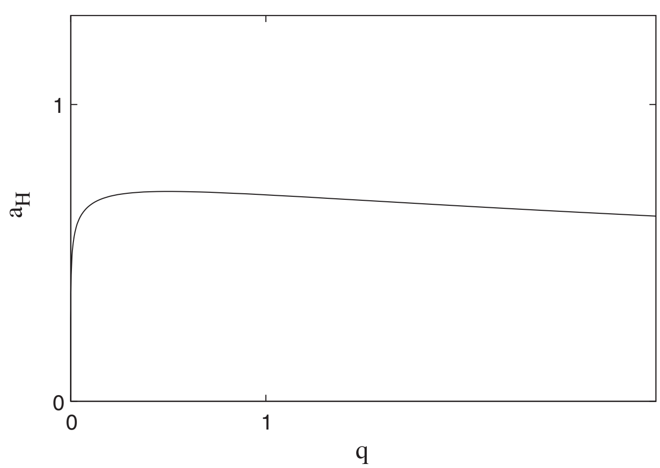

FIG. 4. Normalized horizon area vs quadrupole moment. If given the quadrupole q, ν is completely determined, that is, thin or fat is specified.

quadrupole moment. Regular black Saturn solution which has no conical singularity has four independent parameters. Hence, by tuning these parameters, black Saturn has the same (reduced) quadrupole and angular dipole moments of black ring solution. This means that these multiple horizon solutions cannot be classified only by monopole and quadrupole moments. Hence, we need the information about the higher multipole moments for the complete classification as we pointed out in static cases. For the details, see Appendix C.

V. SUMMARY AND DISCUSSION

In this paper, we have defined mass and angular multipole moments in five dimensional stationary space-times. It is known that black holes and the black ring with or without rotations cannot be distinguish by mass monopole (ADM mass) and angular dipole moment (ADM angular momentum). But, we could show that these black objects with the single horizon can be classified by introducing the (reduced) quadrupole moment. These moments are well-defined in the center-of-mass gauge. In static cases, we could see that the mass quadrupole moments capture the existence of finite spacelike rods and the quadrupole moments detects the deviation of the topology of the event horizon from sphere. As seen in the previous section, this interpretation is valid for the reduced quadrupole moment in stationary cases.

Let us discuss the remaining works. When one wants to classify black objects with multiple horizons, we need the gauge independent definition of higher multipole moments. However, our current definition of multipole moments higher than quadrupole moments are not gauge invariant even in center-of-mass gauge. Therefore, we have to define the multipole moment carefully. Since the computation itself of the multipole moment defined here is a hard task, we would guess that the improvement is also a hard one. This might be done by adding some extra terms in our current definition of the multipole moments. It is also interesting to extend the definition of multipole moments to nonvacuum cases like Einstein-Maxwell system or so.

It is known that the metric is determined completely if we specify the all mass and angular multipole moments in four-dimensional space-times [32–34]. The method of the proof of this theorem does not hold in five dimensions, because the fact that the Weyl tensor for h_{ab} trivially vanishes plays a key role of the proof in four dimensions. Therefore, it is rather nontrivial if the metric can be determined only by mass and angular multipole moments. To investigate this, it may be useful to use the rod structure. If we can show that the parameters of the rod structure can be constructed only by mass and angular multipole moments, the five-dimensional black objects with single or multiple horizons in stationary and two rotational symmetric space-times will be classified only by mass and angular multipole moments. This is also our future work.

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APPENDIX A: FIELD EQUATIONS

In this appendix, we describe the key ingredient behind the definition of the mass multipole moments. We first write down the vacuum Einstein equations $R^{(5)}_{ab} = 0$ for the metric of Eq. (1) as

$$\hat{D}^2\lambda = \frac{1}{2\lambda}\hat{D}_a\lambda\hat{D}^a\lambda - \frac{1}{2\lambda}\hat{\sigma}_{ab}\hat{\sigma}^{ab}, \tag{A1}$$

$$\hat{D}^a\hat{\sigma}_{ab} = -\frac{3}{2\lambda^2}\hat{\sigma}_{ab}\hat{D}^a\lambda. \tag{A2}$$

$$\hat{R}_{ab} = \frac{1}{2\lambda}\hat{D}_a\hat{D}_b\lambda - \frac{1}{4\lambda}\hat{D}_a\lambda\hat{D}_b\lambda - \frac{1}{4\lambda^2}(\hat{h}_{ab}\hat{\sigma}_{mn}\hat{\sigma}^{mn} - \hat{\sigma}_{am}\hat{\sigma}_b^{\ m}), \tag{A3}$$

where $\hat{D}$ and $\hat{R}_{ab}$ are the covariant derivative and Ricci tensor for $\hat{h}_{ab}$ respectively, and $\hat{\sigma}_{ab} = \hat{\epsilon}_{abcde}\xi^c\hat{\nabla}^d\xi^e$. Using the function $\hat{P} = 1 - \sqrt{\lambda}$, we rewrite Eq. (A1) as

$$\left(\hat{D}^2 - \frac{\hat{R}}{6}\right)\hat{P} = -\frac{\hat{\sigma}_{ab}\hat{\sigma}^{ab}}{8}(2 - \hat{P}), \tag{A4}$$

where $\hat{R}$ is the Ricci scalar of $\hat{h}_{ab}$. We can regard Eq. (A2) as the Maxwell equations on the $t =$ const hypersurfaces and $\hat{\sigma}_{ab}$ as the fields strength. From the conformal invariance of the Maxwell equation in four dimensions, we can suppose that the Maxwell field $\hat{\sigma}_{ab}$ is conformal invariant $\hat{\sigma}_{ab} = \sigma_{ab}$ under the conformal completion of Eq. (2). Then, the conformal transformation transforms Eq. (A4) into

$$\left(D^2 - \frac{R}{6}\right)P = \Omega^2\frac{\sigma_{ab}\sigma^{ab}P}{8} - \frac{\Omega}{4}\sigma_{ab}\sigma^{ab}, \tag{A5}$$

where $P = \Omega^{-1}\hat{P}$. We can regard Eq. (A5) as a Poisson-like equation with a certain of regular source. At spatial infinity Λ, it becomes

$$D^2P \hat{=} 0. \tag{A6}$$

Therefore, it is natural to define multipole moment using P and σ_{ab}.

APPENDIX B: BLACK RING SOLUTIONS WITH TWO ANGULAR MOMENTA

In the main text, we focused on the black objects with single angular momentum mainly. This is because we wanted the argument to be as compact as possible. In this appendix, we compute the multipole moments for black ring with two angular momenta. The metric of black ring with two angular momenta is given by [5]

$$\begin{aligned} ds^2 = & \frac{H(y,x)}{H(x,y)}(dt+\Omega)^2 + \frac{F(x,y)}{H(y,x)}d\phi^2 \\ & + 2\frac{J(x,y)}{H(y,x)}d\phi d\psi - \frac{F(y,x)}{H(y,x)}d\psi^2 \\ & - \frac{2k^2 H(x,y)}{(x-y)^2(1-\nu)^2}\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right), \end{aligned} \tag{B1}$$

where

$$\begin{aligned} \Omega = & -\frac{2k\lambda\sqrt{(1+\nu)^2-\lambda^2}}{H(y,x)}\Bigg\{(1-x^2)y\sqrt{\nu}d\psi \\ & + \frac{(1+y)}{1-\lambda+\nu}(1+\lambda-\nu+x^2y\nu(1-\lambda-\nu) \\ & + 2\nu x(1-y))d\phi\Bigg\}, \end{aligned} \tag{B2}$$

and

$$G(x) = (1-x^2)(1+\lambda x+\nu x^2), \tag{B3}$$

$$\begin{aligned} H(x,y) = & 1+\lambda^2-\nu^2+2\lambda\nu(1-x^2)y+2\lambda x(1-\nu^2y^2) \\ & + \nu x^2y^2(1-\lambda^2-\nu^2), \end{aligned} \tag{B4}$$

$$\begin{aligned} J(x,y) = & \frac{2k^2(1-x^2)(1-y^2)\lambda\sqrt{\nu}}{(x-y)(1-\nu)^2}\{1+\lambda^2-\nu^2 \\ & + 2(x+y)\lambda\nu - xy\nu(1-\lambda^2-\nu^2)\}, \end{aligned} \tag{B5}$$

$$\begin{aligned} F(x,y) = & \frac{2k^2}{(x-y)^2(1-\nu)^2}[G(x)(1-y^2)\{((1-\nu)^2-\lambda^2) \\ & \times (1+\nu) + y\lambda(1-\lambda^2+2\nu-3\nu^2)\} \\ & + G(y)(2\lambda^2 + x\lambda\{(1-\nu)^2+\lambda^2\} + x^2\{(1-\nu)^2 \\ & - \lambda^2\}(1+\nu) + x^3\lambda(1-\lambda^2-3\nu^2+2\nu^3) \\ & - x^4\nu(1-\nu)(-1+\lambda^2+\nu^2))]. \end{aligned} \tag{B6}$$

The parameter ranges are $0<\nu<1$, $2\sqrt{\nu}<\lambda<1+\nu$. Regular black ring solutions with two angular momenta have three independent parameters ν, λ, and k.

To compute the multipole moments, we introduce the new coordinate (ρ, θ) defined by

$$\begin{aligned} x &= -1 + \frac{4k^2(1-\lambda+\nu)}{1-\nu}\rho^2\cos^2\theta, \\ y &= -1 - \frac{4k^2(1-\lambda+\nu)}{1-\nu}\rho^2\sin^2\theta, \end{aligned} \tag{B7}$$

and use the conformal factor of $\Omega = \rho^2$. Then, the multipole moments are computed as

$$P = \frac{4k^2\lambda}{(1-\lambda+\nu)}, \qquad P_a = 0, \tag{B8}$$

$$Q = -\frac{16\lambda k^4(1-5\nu-5\nu^2+\nu^3-8\lambda\nu+3\lambda^2(1+\nu))}{(1-\nu)^2(1-\lambda+\nu)^2}, \tag{B9}$$

$$\begin{aligned} J_\psi &= \frac{16k^3\lambda\sqrt{\nu}\sqrt{(1+\nu)^2-\lambda^2}}{(1-\nu)^2(1-\lambda+\nu)}, \\ J_\phi &= \frac{8k^3\lambda(1+\lambda-6\nu+\lambda\nu+\nu^2)\sqrt{(1+\nu)^2-\lambda^2}}{(1-\nu)^2(1-\lambda+\nu)^2}. \end{aligned} \tag{B10}$$

Then the reduced quadrupole moment becomes

$$Q^{\rm red} = -\frac{8\lambda k^4(1-\lambda+\nu)}{(1-\nu)^2}. \tag{B11}$$

As in one rotational case, the black ring has a negative value for the mass quadrupole moment. Thus, in two rotational case, the Myers-Perry black hole and black ring with two angular momenta can be classified by the mass quadrupole moment.

APPENDIX C: CASES WITH MULTIPLE HORIZONS

In this appendix, we consider the cases with multiple horizons. In the main text, we focused on single horizon cases and we could show that space-times are uniquely specified by the multipole moments up to the quadrupole components. We can show that it is not true for the cases with multiple horizons. This section will be useful for future study or comparison with single horizon cases.

1. Static cases

Here we compute multipole moments for static black Saturn solution and static orthogonal black di-ring solution (these rod structures are shown in Figs. 5 and 6). The multipole moments are

$$P = (b_{\rm BS} - a_{\rm BS}) + (d_{\rm BS} - c_{\rm BS}), \qquad P_a = 0, \tag{C1}$$

$$Q = -4(b_{\rm BS} - a_{\rm BS})(c_{\rm BS} - b_{\rm BS}) \tag{C2}$$

for static black Saturn, and

$$P = (b_{\rm OBD} - a_{\rm OBD}) + (d_{\rm OBD} - c_{\rm OBD}), \quad P_a = 0, \tag{C3}$$

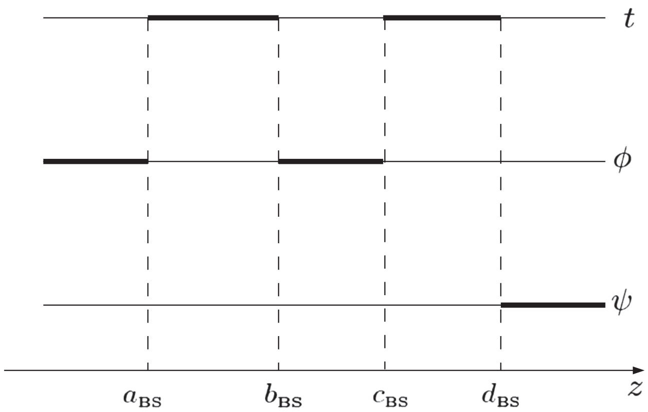

FIG. 5. Rod structure of static black Saturn (BS).

$$Q = Q_1 + Q_2, \tag{C4}$$

$$Q_1 = -4(b_{\rm OBD} - a_{\rm OBD})(c_{\rm OBD} - b_{\rm OBD}),$$
$$Q_2 = 4(e_{\rm OBD} - d_{\rm OBD})(d_{\rm OBD} - c_{\rm OBD}) \tag{C5}$$

for static orthogonal black di-ring.

We can see that in the quadrupole moments of the static orthogonal black di-ring Q_1 are the quadrupole moments of the black ring in the ϕ rotational plane, and Q_2 is the quadrupole moment of the black ring in ψ rotational plane. Hence, monopole and quadrupole moments are "linear" moments. Only by monopole and quadrupole moments, we cannot distinguish between the static black ring, black Saturn, and black di-ring. Static black Saturn solutions have three independent parameters, for example, $b_{\rm BS} - a_{\rm BS}$, $c_{\rm BS} - b_{\rm BS}$, and $d_{\rm BS} - c_{\rm BS}$. By tuning these parameters, static black Saturn can have the same mass monopole and quadrupole moments as the static black ring's. As in static orthogonal black di-ring solutions, we can do the same thing because its solution has four independent parameters. That is, there are several different solutions with the same P and Q. This result suggests that higher multipole moments are needed to classify all of these solutions.

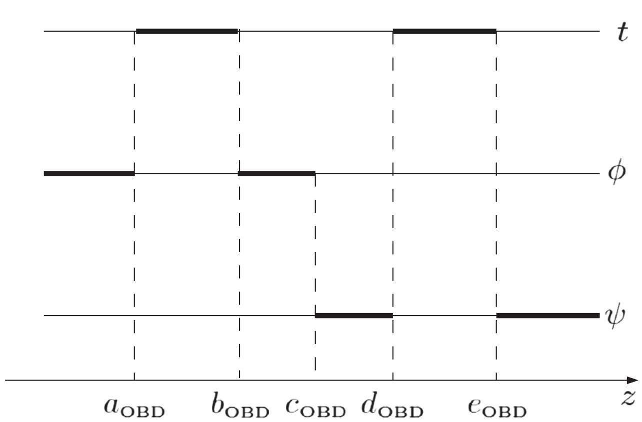

FIG. 6. Rod structure of static orthogonal black di-ring (OBD).

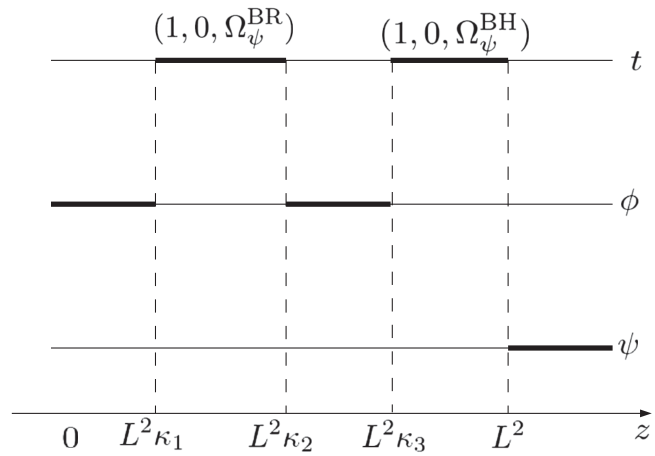

FIG. 7. Rod structure of black Saturn.

2. Stationary cases

Here, we compute the multipole moments for the stationary solutions with multiple horizons, black Saturn [7], and orthogonal black di-ring [10]. Since the explicit form of the metric is complicated, we show only the rod structure of the solutions (see Figs. 7 and 8). The metric of stationary and two rotational symmetric solutions in five dimensions can be written by Weyl form as

$$ds^2 = G_{AB}dx^A dx^B + e^{2\nu}(dR^2 + dz^2), \tag{C6}$$

where $x^A = (t, \phi, \psi)$. The rod structure is described by the zero point of $\det G_{AB}$ and their direction ξ^A determined from $G_{AB}\xi^A = 0$ at $R = 0$. For computing the multipole moments, it is better to introduce the new coordinate (ρ, θ) defined through the relation

$$R = \frac{1}{2\rho^2}\sin 2\theta, \qquad z = \frac{1}{2\rho^2}\cos 2\theta, \tag{C7}$$

and we choose the conformal factor as $\Omega = \rho^2$.

The rod structure of black Saturn solution is shown in Fig. 7. The angular velocities are given by

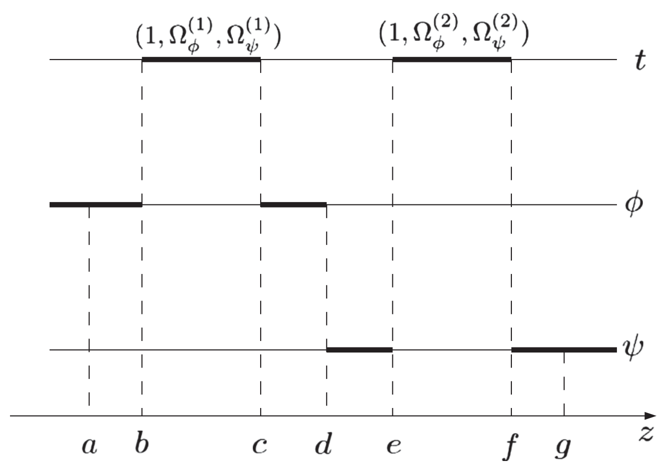

FIG. 8. Rod structure of orthogonal black di-ring.

$$\Omega_\psi^{\rm BH} = \frac{1}{L}(1+\kappa_2\bar{c}_2) \times \sqrt{\frac{\kappa_2\kappa_3}{2\kappa_1}} \frac{\kappa_3(1-\kappa_1) - \kappa_1(1-\kappa_2)(1-\kappa_3)\bar{c}_2}{\kappa_3(1-\kappa_1) + \kappa_1\kappa_2(1-\kappa_2)(1-\kappa_3)\bar{c}_2^2}, \quad \text{(C8)}$$

$$\Omega_\psi^{\rm BR} = \frac{1}{L}(1+\kappa_2\bar{c}_2) \times \sqrt{\frac{\kappa_1\kappa_3}{2\kappa_2}} \frac{\kappa_3 - \kappa_2(1-\kappa_3)\bar{c}_2}{\kappa_3 - \kappa_3(\kappa_1-\kappa_2)\bar{c}_2 + \kappa_1\kappa_2(1-\kappa_3)\bar{c}_2^2}, \quad \text{(C9)}$$

for each event horizons, that is, for the central black hole and outer black ring. Note that the following regularity condition is imposed:

$$\bar{c}_2 = \frac{1}{\kappa_2}\left[\epsilon \frac{\kappa_1-\kappa_2}{\sqrt{\kappa_1(1-\kappa_2)(1-\kappa_3)(\kappa_1-\kappa_3)}}\right], \quad \text{(C10)}$$

where $\epsilon = 1(-1)$ for $\bar{c}_2 > -\kappa_2^{-1}(\bar{c}_2 < -\kappa_2^{-1})$. Thus, black Saturn solutions has four independent parameters L, κ_1, κ_2, and κ_3.

After some length calculations, the mass multipole moments for black Saturn are computed as

$$P = L^2 \frac{\kappa_3(1-\kappa_1+\kappa_2) - 2\kappa_2\kappa_3(\kappa_1-\kappa_2)\bar{c}_2 + \kappa_2[\kappa_1 - \kappa_2\kappa_3(1+\kappa_1-\kappa_2)]\bar{c}_2^2}{\kappa_3(1+\kappa_2\bar{c}_2)^2}, \quad \text{(C11)}$$

$$P_a = 0, \quad \text{(C12)}$$

$$\begin{aligned} Q = \frac{4L^4}{(\kappa_3^2(1+\kappa_2\bar{c}_2)^4)}(&\kappa_3^2[(\kappa_2-\kappa_1)(\kappa_2-\kappa_3)-\kappa_3] + 2\kappa_2\kappa_3^2[1+2(\kappa_2-\kappa_1)(\kappa_2-\kappa_3)-\kappa_3]\bar{c}_2 \\ &+ \kappa_2\kappa_3[3\kappa_2\kappa_3\{1+2(\kappa_2-\kappa_1)(\kappa_2-\kappa_3)\} - \kappa_1(1-\kappa_1+\kappa_2+2\kappa_3)]\bar{c}_2^2 + 2\kappa_2^2\kappa_3[\kappa_2\kappa_3\{2(\kappa_2-\kappa_3)(\kappa_2-\kappa_1)+\kappa_3\} \\ &+ \kappa_1(1+\kappa_1-\kappa_2-2\kappa_3)]\bar{c}_2^3 + \kappa_2^2[\kappa_2^2\kappa_3^2\{(\kappa_2-\kappa_1)(\kappa_2-\kappa_3)-(1-\kappa_3)\} - \kappa_1\{\kappa_1-\kappa_2\kappa_3(3+\kappa_1-\kappa_2-2\kappa_3)\}]\bar{c}_2^4). \end{aligned} \quad \text{(C13)}$$

And the angular multipole moments are also computed as

$$J^\phi = 0, \quad \text{(C14)}$$

$$\begin{aligned} J_\phi = \frac{4L^3}{\kappa_3(1+\kappa_2\bar{c}_2)}\sqrt{\frac{\kappa_2}{\kappa_1\kappa_3}}[&\kappa_3^2 - \kappa_3\bar{c}_2[(\kappa_1-\kappa_2) \\ &\times(1-\kappa_1+\kappa_3) + \kappa_2(1-\kappa_3)] + \kappa_2\kappa_3\bar{c}_2^2[(\kappa_1-\kappa_2) \\ &\times(\kappa_1-\kappa_3) + \kappa_1(1+\kappa_1-\kappa_2-\kappa_3)] \\ &- \kappa_1\kappa_2\bar{c}_2^3[\kappa_1 - \kappa_2\kappa_3(2+\kappa_1-\kappa_2-\kappa_3)]], \end{aligned} \quad \text{(C15)}$$

$$J^\phi_{ab} = 0. \quad \text{(C16)}$$

Next we consider the orthogonal black di-ring solution. The rod structure is shown in Fig. 8. The angular velocities are given by

$$\Omega_\phi^{(1)} = -j_1\frac{(g-e)(g-f)}{2(g-c)^2(g-a)}, \quad \Omega_\psi^{(1)} = -j_2\frac{(e-a)^2}{2(c-a)(f-a)(g-a)}, \quad \text{(C17)}$$

$$\Omega_\phi^{(2)} = -j_1\frac{(g-f)}{2(g-a)(g-d)}, \quad \Omega_\psi^{(2)} = -j_2\frac{(d-a)^2}{2(f-a)(g-a)}. \quad \text{(C18)}$$

The labeling of (1) and (2) specifies which event horizons we consider. The regularity conditions is also imposed as

$$j_1^2 = 2\frac{(g-a)(g-c)^2(g-d)}{(g-b)(g-e)(g-f)}, \quad j_2^2 = 2\frac{(b-a)(c-a)(f-a)(g-a)}{(d-a)(e-a)^2}, \quad \text{(C19)}$$

$$(a-d)(a-f)(a-g)(b-e)(b-g)(c-d)(c-e)(c-f) = (a-e)^2(b-d)^2(b-f)^2(c-g)^2, \quad \text{(C20)}$$

$$(g-d)(g-b)(g-a)(f-c)(f-a)(e-d)(e-c)(e-b) = (g-c)^2(f-d)^2(f-b)^2(e-a)^2. \quad \text{(C21)}$$

Note that there is the gauge freedom of $z \to z + \text{const}$. Thus, the orthogonal black di-ring has the four free parameters.

The mass multipole moments for orthogonal black di-ring are computed as

$$P = (c-a) + (g-e), \quad \text{(C22)}$$

$$P_a = 0, \quad \text{(C23)}$$

$$Q = Q_\phi - Q_\psi, \quad \text{(C24)}$$

where

$$Q_\phi = -4(c-a)[(b-a)+(d-c)+(f-e)],$$
$$Q_\psi = -4(g-e)[(e-d)+(c-b)+(g-f)]. \quad (C25)$$

We can interpret the Q_ϕ and Q_ψ as the quadrupole moment of the black ring in the ϕ- and ψ-rotational planes, respectively.

The angular momentum multipole moment are

$$J^{\phi,\psi} = 0, \quad (C26)$$

$$J_\phi = 2j_1(d-a), \qquad J_\psi = -2j_2(g-b)(g-e)(g-f), \quad (C27)$$

$$J^{\phi,\psi}_{ab} = 0. \quad (C28)$$

As in static cases, regular black objects with multiple horizons have nontrivial quadrupole moments. These solutions with multiple horizons have independent parameters more than three. Hence, to classify these solutions, it is necessary to evaluate higher multipole moments such as mass 2^4-pole moments.

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