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Static solutions for fourth order gravity

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The Lichnerowicz and Israel theorems are extended to higher order theories of gravity. In particular it is shown that Schwarzschild is the *unique* spherically symmetric, static, asymptotically flat, black-hole solution, provided the spatial curvature is less than the quantum gravity scale *outside* the horizon. It is then shown that in the presence of matter (satisfying certain positivity requirements), the only static and asymptotically flat solutions of general relativity that are also solutions of higher order gravity are the vacuum solutions.

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I. INTRODUCTION

While general relativity (GR) remains the most successful classical theory we have, one may question whether it is possible for there to be corrections to it. Indeed we expect GR to break down as we approach the Planck scale where the quantum theory of gravity (whatever it is) will become dominant. In this sense GR is not a full physical theory but only an excellent approximation to some (presumably) complicated underlying theory, with the approximation getting better and better as the scales we consider are further and further from the Planck scale.[1]

Over the past century there have been many important theorems proved about the solutions and structure of GR and an important question is whether these theorems are also valid in theories that deviate from GR at some scale. In this paper we will look, in particular, at two theorems, first the Lichnerowicz theorem [2], which tells us that the only static, asymptotically flat, geodesically complete, vacuum, solution to Einstein's equations is flat space-time. We then consider the Israel theorem [3], which demonstrates that the only static, asymptotically flat, vacuum space-time, which contains past and future event horizons (that intersect on a surface that is topologically S^2) is given by the Schwarzschild metric.[2] This "no-hair" theorem is a striking result in classical GR, and extensions of it to more general gravity theories, in particular, as we approach the scales on which GR is expected to be violated, would provide us with insight into the transition from GR to full quantum gravity.

The action for GR is the well-known Einstein-Hilbert action which is linear in the Ricci scalar R (here for simplicity we neglect the cosmological constant). A natural extension of this is to include terms of higher powers of curvature invariants to find

$$\mathcal{S}_{\text{Grav.}} = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{\gamma}{\kappa^2}R - \alpha_0\hbar R^{abcd}R_{abcd} - \alpha\hbar R^{ab}R_{ab} + \beta\hbar R^2 + \mathcal{O}(R^3)\right), \tag{1.1}$$

where we have written only terms that are only second order in the curvature. Throughout we will use Latin indices to label space-time components $a, b, \ldots = 0, \ldots, 3$, our signature is $(-, +, +, +)$, and our conventions for curvature are $R_{abc}{}^{d} = \partial_b\Gamma^d{}_{ac} + \cdots$ and $R_{ab} = R^c{}_{acb}$. In Eq. (1.1) the coefficients α_0, α, β, and γ are dimensionless numbers and $\kappa^2 = 32\pi G$, so GR is recovered simply by setting the coefficients of the higher order terms to zero ($\alpha_0 = \alpha = \beta = 0$) and taking $\gamma = 2$. Also note that the presence of the $\hbar$ in Eq. (1.1) is only due to dimensions; i.e. this remains a classical theory. If these correction terms are motivated from some quantum gravity theory, then we may expect the coefficients α_0, α, and β to be of the order of unity; however, in general they can take any value.

In fact, in 4 dimensions, the Gauss-Bonnet combination of second order curvature invariants integrates to a topological invariant, i.e.

$$R^{abcd}R_{abcd} - 4R^{ab}R_{ab} + R^2 = \text{total divergence}, \tag{1.2}$$

which we can use to eliminate the $R^{abcd}R_{abcd}$ term from Eq. (1.1). Thus, up to second order in the curvature, the most general correction to GR comes from the action

$$\mathcal{S}_{\text{Grav.}} = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{\gamma}{\kappa^2}R - \alpha\hbar R^{ab}R_{ab} + \beta\hbar R^2\right). \tag{1.3}$$

This is the theory that we will work with, and in the following we refer to it as fourth order gravity, since the equations of motion involve, at most, fourth derivatives of

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[1]It is important to note that here the Planck scale can include both very small (UV) scales *and* very large (IR) scales (see for example [1]).

[2]The Israel theorem also shows that the Reissner-Nordström metric must be the solution if the vacuum is replaced by an electromagnetic field.

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the metric. Historically, theories of this type with $\gamma = 0$ were considered for many reasons, not least because of their improved behavior under renormalization; however, it was shown that without a term linear in R, the theory (for $\alpha = 0$) does not couple to matter correctly [4]. The theory with general γ was first considered in [5] and later in [6], where a perturbative analysis around the Schwarzschild solution was performed, while the initial value formulation of the theory was given in [7]. The consequence for black holes in higher dimensions have been considered in [8]. The cosmological implications of fourth order gravity were investigated in [9–12], with the extension to include (tensor) perturbations appearing only recently [13]. The special case in which the correction terms are conformal (when $\alpha = 3\beta$, which gives the Bach-Einstein equations) has received considerable attention (see for example [14]) and has recently appeared in the context of noncommutative geometry [15], with much work going into restricting the parameters of this form of the theory via cosmological and astrophysical observations [16–20]. More generally, corrections to GR of this type are expected from string theory (see for example [21]), renormalization group techniques [22], and loop quantum gravity [23].

In addition there has been a great deal of phenomenological work on the cosmological consequences of higher order gravity theories, usually in the form of $f(R)$ theories (see, for example, [24–26]). Despite this it is important to note that theories defined by Eq. (1.3) cannot represent true physical theories for arbitrary coefficients due to the presence of ghosts [6,27,28]. Not only are ghosts a problem for the quantization of the theory, they also lead to a breakdown of causality at the classical level. Note however that with certain restrictions on the coefficients (e.g. $\alpha = 0$ and $\beta < 0$), all the difficulties associated with these ghosts can be eliminated. Here we will take the (conservative) view that, just as for GR, fourth order gravity should be considered only as an approximation to the underlying, full theory (which is presumable free of ghosts, etc.).

In this paper we shall demonstrate three key results, the first two being an extension of the Lichnerowicz (Sec. III) and Israel (Sec. IV) theorems to fourth order gravity (for space-times satisfying some restriction on the magnitude of the spatial curvature) suffice to demonstrate that Schwarzschild is the *unique*, spherically symmetric, static, asymptotically flat, vacuum solution to the theory.[3] The final result (Sec. V) considers nonvacuum solutions to the theory and demonstrates that for static, asymptotically flat space-times with matter satisfying certain positivity requirements, it is *only* the vacuum solutions of GR and fourth order gravity that agree; all other solutions of GR fail to be solutions of fourth order gravity.

II. STATIC SOLUTIONS OF THE EQUATIONS OF MOTION

Varying the action given in Eq. (1.3) with respect to the metric, one finds [6]

$$\begin{aligned} H_{ab} &= (\alpha - 2\beta)R_{;a;b} - \alpha\Box R_{ab} - \frac{1}{2}(\alpha - 4\beta)g_{ab}\Box R \\ &\quad + 2\alpha R^{cd}R_{acbd} - 2\beta R R_{ab} \\ &\quad - \frac{1}{2}g_{ab}(\alpha R^{cd}R_{cd} - \beta R^2) - \frac{\gamma}{\hbar\kappa^2}\left(R_{ab} - \frac{1}{2}g_{ab}R\right) \\ &= -\frac{1}{2\hbar}T_{ab}. \end{aligned} \tag{2.1}$$

Considering the vacuum, the trace part gives the Klein-Gordon equation for the Ricci scalar:

$$(\alpha - 3\beta)\left(\Box - \frac{\gamma\kappa^{-2}}{2(\alpha - 3\beta)}\right)R = 0. \tag{2.2}$$

It is well known that there are no static, asymptotically constant solutions of the Klein-Gordon equation for a scalar field (see for example [29]); however, since the method used to prove this result will be the one that is later extended to the full equation of motion, Eq. (2.1), we give it explicitly here.

Theorem II.1.—If the space-time is static ($\mathcal{L}_t g_{ab} = 0$) and $D_a R \to 0$ sufficiently fast at infinity, then $(\Box - m^2)R = 0$, for $m \neq 0 \in \mathbb{R}$ implies $R = 0$. If the space-time is static and $R \to 0$ sufficiently fast at infinity, the equation $\Box R = 0$ implies $R = 0$.

Proof.—We decompose the metric into components parallel and perpendicular to the constant time spacelike hypersurfaces $\mathcal{S}$ via $g_{ab} = h_{ab} - \frac{1}{\lambda}t_a t_b$, where t_a is the timelike Killing field, which has norm $t^a t_a = -\lambda$ and h_{ab} is spatial metric on $\mathcal{S}$. Note the following useful identities:

$$\mathcal{L}_t R = t^b \nabla_b R = 0, \tag{2.3}$$

$$\nabla_a t_b = \frac{1}{2\lambda}(t_b \nabla_a \lambda - t_a \nabla_b \lambda), \tag{2.4}$$

and

$$t^a \nabla_a \lambda = 0. \tag{2.5}$$

Using these one can readily show that

$$\begin{aligned} g^{ab}\nabla_a\nabla_b R - m^2 R &= D^a D_a R + \frac{1}{2\lambda}(D^a\lambda)(D_a R) \\ &\quad - m^2 R = 0, \end{aligned} \tag{2.6}$$

where D_a is the spatial covariant derivative compatible with h_{ab}. Multiplying this equation by $\lambda^{1/2}R$ and integrating it over the entire spatial slice, one finds

[3]This result holds for black holes in which the spatial curvature outside the horizon is small relative to the scale set between $\gamma\kappa^{-2}$ and $\hbar\alpha$. If these corrections come from some quantum gravity theory, this restriction is essentially that the spatial curvature be small compared to the quantum gravity scale.

$$\int_{\mathcal{S}} \sqrt{h}\mathrm{d}^2x\bigg[\lambda^{1/2}RD^aD_aR + \frac{1}{2}\lambda^{-1/2}R(D^a\lambda)(D_aR) - m^2\lambda^{1/2}R^2\bigg] = 0. \quad (2.7)$$

Integrating the first term by parts this becomes

$$\int_{\mathcal{S}} \sqrt{h}\mathrm{d}^3x[D^a(\lambda^{1/2}RD_aR) - \lambda^{1/2}(D^aR)(D_aR) - m^2\lambda^{1/2}R^2] = 0. \quad (2.8)$$

If the spatial slice is asymptotically constant, i.e. $D_aR \to 0$ sufficiently fast at infinity, then the boundary term vanishes. The second and third terms in the integrand of Eq. (2.8) are manifestly negative definite and hence each term must vanish at every spatial point. If $m \neq 0$, this implies $R = 0$, for the case of $m = 0$ this gives $D_aR = 0$, and hence we additionally require that $R \to 0$ at infinity in order to find $R = 0$. ■

Note that here we require only that $D_aR \to 0$ (or $R \to 0$) sufficiently fast at infinity, which is satisfied by asymptotically constant (or flat) space-times but is in general a weaker condition. In the following we will consider asymptotically constant (or flat) space-times since they are of more interest; however, all the results hold also for the weaker condition given above.

The basic method of this proof is the following. Decompose the metric onto (and perpendicular to) constant time spacelike hypersurfaces and use this to write the four-dimensional covariant derivatives as spatial, three-dimensional covariant derivatives plus correction terms. Multiply the resulting equation by a suitable factor and integrate over the spacelike hypersurface. Prove that, up to boundary terms, the integrand has a definite sign and hence vanishes at each spatial point. This is precisely the same method that we will use to prove that, with some restrictions on the magnitude of the spatial curvature on $\mathcal{S}$, there are no static, asymptotically constant solutions to the trace-free part of Eq. (2.1), which, using Theorem II.1 can be written as

$$\hbar\alpha\bigg(\Box R_{ab} - \frac{\gamma}{\hbar\alpha\kappa^2}R_{ab} + \frac{1}{2}g_{ab}R^{cd}R_{cd} - 2R^{cd}R_{acbd}\bigg) = 0, \quad (2.9)$$

for $\alpha - 3\beta \geq 0$. Since Eq. (2.9) is rather more complicated than Eq. (2.2), we proceed by first proving a series of lemmas.[4]

Lemma II.2.—If the space-time is static and asymptotically constant, then $(\Box - m^2)R_{ab} = 0$, for $m \neq 0 \in \mathbb{R}$ implies $R_{ab} = 0$. If the space-time is static and asymptotically flat, then $\Box R_{ab} = 0$ implies $R_{ab} = 0$.

Proof.—We begin by noting several identities, which are proved in the appendix:

$$t^a\nabla_a t^b\nabla_b R_{cd} = 2[(R_{a(c}\nabla_{d)}t^b)(\nabla_b t^a) + R_{ab}(\nabla_c t^b)(\nabla_d t^a)], \quad (2.10)$$

$$(\nabla_d t^c)(\nabla_a t^d) = \frac{-1}{4\lambda^2}t^c t_a(\nabla_d\lambda)(\nabla^d\lambda) + \frac{1}{4\lambda}(\nabla^c\lambda)(\nabla_a\lambda), \quad (2.11)$$

$$\frac{1}{2\lambda^2}h^{ab}(\nabla_a\lambda)t^c t^d\nabla_b R_{cd} = \frac{1}{2\lambda}(\nabla^a\lambda)(\nabla_a^{(3)}R), \quad (2.12)$$

and

$$\frac{1}{\lambda}t^c t^d h^{ab}\nabla_a h_b^e\nabla_e R_{cd} = D^a D_a^{(3)}R, \quad (2.13)$$

where ${}^{(3)}R$ is the three-dimensional Ricci scalar formed from the spatial metric h_{ab}. Now, decomposing $g_{ab} = h_{ab} - \frac{1}{\lambda}t_a t_b$ one finds

$$(\Box - m^2)R_{ab} = h^{ab}\nabla_a h_b^e\nabla_e R_{cd} - \frac{1}{\lambda}t^a t^b\nabla_a\nabla_b R_{cd} - m^2R_{cd} = 0. \quad (2.14)$$

Noting the staticity of R_{ab},

$$\mathcal{L}_t R_{ab} \equiv t^a\nabla_a R_{cd} + R_{ad}\nabla_c t^a + R_{ca}\nabla_d t^a = 0, \quad (2.15)$$

and using Eq. (2.10) and (2.11), one can show (see the appendix) that Eq. (2.14) becomes

$$\begin{aligned} &h^{ab}\nabla_a h_b^e\nabla_e R_{cd} + \frac{1}{2\lambda}h^{ab}(\nabla_a\lambda)(\nabla_b R_{cd}) \\ &+ \frac{1}{4\lambda^3}[t_c t^a(\nabla_b\lambda)(\nabla^b\lambda) - \lambda(\nabla_c\lambda)(\nabla^a\lambda)]R_{ad} \\ &- \frac{1}{2\lambda^2}[t^a\nabla_d\lambda - t_d\nabla^a\lambda][t^b\nabla_c\lambda - t_c\nabla^d\lambda]R_{ba} \\ &+ \frac{1}{4\lambda^3}[t_d t^a(\nabla_b\lambda)(\nabla^b\lambda) - \lambda(\nabla_d\lambda)(\nabla^a\lambda)]R_{ac} \\ &- m^2R_{cd} = 0. \end{aligned} \quad (2.16)$$

We now project this equation onto the spatial slice $\mathcal{S}$ with $h_e^c h_f^d$ and define the projection of R_{ab} onto $\mathcal{S}$ as $\bar{R}_{ab} \equiv h_a^c h_b^d R_{cd}$ to find

$$\begin{aligned} &D^aD_a\bar{R}_{ef} + \frac{1}{2\lambda}(D_a\lambda)(D^a\bar{R}_{ef}) \\ &- \frac{1}{4\lambda^2}(D_e\lambda)(D^a\lambda)\bar{R}_{af} - \frac{1}{2\lambda^3}(D_f\lambda)(D_e\lambda)t^a t^b R_{ab} \\ &- \frac{1}{4\lambda^2}(D^a\lambda)(D_f\lambda)\bar{R}_{ea} - m^2\bar{R}_{ef} = 0. \end{aligned} \quad (2.17)$$

Noting that the extrinsic curvature of the spatial slice vanishes (as can be shown by direct calculation) the Codacci and Gauss equations reduce to

$$h_a^b t^c R_{bc} = 0, \quad (2.18)$$

[4] A similar line of argument has used in [29].

$$\frac{1}{\lambda}t^a t^b R_{ab} = {}^{(3)}R, \tag{2.19}$$

where we have use Theorem II.1 to set $R = 0$. Using these relations in Eq. (2.17) we find

$$D^a D_a \bar{R}_{cd} + \frac{1}{a\lambda}(D_a\lambda)(D^a\bar{R}_{cd}) - \frac{1}{4\lambda^2}(D_c\lambda)(D^a\lambda)\bar{R}_{ad} - \frac{1}{2\lambda^2}(D_d\lambda)(D_c\lambda)^{(3)}R - \frac{1}{4\lambda^2}(D^a\lambda)(D_d\lambda)\bar{R}_{ca} - m^2\bar{R}_{cd} = 0. \tag{2.20}$$

Multiplying this by $\lambda^{1/2}\bar{R}^{cd}$ and integrating over the three-dimensional spatial slice gives

$$\int_S \sqrt{h}\mathrm{d}^3x\Big[D^a(\lambda^{1/2}\bar{R}^{cd}D_a\bar{R}_{cd}) - \lambda^{1/2}(D^a\bar{R}^{cd})(D_a\bar{R}_{cd}) - \frac{1}{2}\lambda^{-3/2}\bar{R}^{cd}\bar{R}_{ad}(D_c\lambda)(D^a\lambda) - \frac{1}{2}\lambda^{-3/2(3)}R\bar{R}^{cd}(D_d\lambda)(D_c\lambda) - \lambda^{1/2}m^2\bar{R}_{cd}\bar{R}^{cd}\Big] = 0. \tag{2.21}$$

We now project Eq. (2.16) perpendicular to the spacelike hypersurface with $t^b t^c$ and again use the Codacci and Gauss equations [Eq. (2.18) and (2.19), respectively] to find

$$t^c t^d h^{ab}\nabla_a h_b^e \nabla_e R_{cd} + \frac{1}{2\lambda}t^c t^d h^{ab}(\nabla_a\lambda)\nabla_b R_{cd} - \frac{1}{2\lambda}(\nabla_b\lambda)(\nabla^b\lambda)^{(3)}R - \frac{1}{2\lambda}(\nabla^a\lambda)(\nabla^b\lambda)R_{ab} - \lambda m^{2(3)}R = 0. \tag{2.22}$$

Multiplying this by $\lambda^{-1/2(3)}R$, integrating over the three-dimensional hypersurface, and using the identities given in Eqs. (2.12) and (2.13) gives

$$\int_S \sqrt{h}\mathrm{d}^3x\Big[D^a(\lambda^{1/2(3)}RD_a^{(3)}R) - \lambda^{1/2}(D^{a(3)}R)(D_a{}^{(3)}R) - \frac{1}{2}\lambda^{-3/2}(D_a\lambda)(D^a\lambda)({}^{(3)}R)^2 - \frac{1}{2}\lambda^{-3/2}(D^a\lambda)(D^b\lambda)\bar{R}_{ab}{}^{(3)}R - \lambda^{1/2}m^2({}^{(3)}R)^2\Big] = 0. \tag{2.23}$$

Finally, adding Eqs. (2.21) and (2.23) one finds

$$\int_S \sqrt{h}\mathrm{d}^3x\Big[D^a(\lambda^{1/2}\bar{R}^{cd}D_a\bar{R}_{cd} + \lambda^{1/2(3)}RD_a{}^{(3)}R) - \lambda^{1/2}(D^a\bar{R}^{cd})(D_a\bar{R}_{cd}) - \lambda^{1/2}(D^{a(3)}R)(D_a{}^{(3)}R) - \frac{1}{2}\lambda^{-3/2}(\bar{R}^{cd}D_c\lambda + {}^{(3)}RD^d\lambda)(\bar{R}_{ad}D^a\lambda + {}^{(3)}RD_d\lambda) - \lambda^{1/2}m^2(\bar{R}^{cd}\bar{R}_{cd} + ({}^{(3)}R)^2)\Big] = 0. \tag{2.24}$$

This is sufficient to prove the desired result; however, there is a simplification that can be made by noting that the contracted Bianchi identity $\nabla^a(R_{ab} - \frac{1}{2}g_{ab}R) = 0$ implies that

$$\bar{R}_{dc}D^d\lambda + {}^{(3)}RD_c\lambda = -2\lambda D^a\bar{R}_{ac}. \tag{2.25}$$

Substituting this into Eq. (2.24) explicitly removes the derivatives of λ from the integrand and ensures that λ only appears with positive powers. This will be crucial when we consider space-times with (null) horizons.

In any case, asymptotic constancy, i.e. $D_a R_{cd} \to 0$ at infinity, ensures that the boundary terms in Eq. (2.24) vanish. All the remaining terms are negative definite and hence each term must independently vanish at every spatial point. For $m \neq 0$ this implies ${}^{(3)}R = 0$ and $\bar{R}_{cd} = 0$, which together with Theorem II.1 and Eq. (2.18) implies $R_{ab} = 0$. For the $m = 0$ case we find that $D_a^{(3)}R = 0$ and $D_a\bar{R}_{cd} = 0$ and hence additionally require asymptotic flatness in order to find $R_{ab} = 0$. ■

This result is easily extended to include the third term in Eq. (2.9) by using the fact that by Theorem II.1 we can set $R = 0$.

Lemma II.3.—If the space-time is static and asymptotically constant, then the pair of equations $R = 0$ and $(\Box - m_1^2)R_{ab} + m_2 g_{ab}R^{cd}R_{cd} = 0$, for $m_1 \neq 0 \in \mathbb{R}$ and $m_2 \in \mathbb{R}$, imply $R_{ab} = 0$, while for $m_1 = 0$, additionally requiring the space-time to be asymptotically flat implies $R_{ab} = 0$.

Proof.—From the equation $(\Box - m_1^2)R_{ab} + m_2^2 g_{ab}R^{cd}R_{cd} = 0$ it follows that

$$[\lambda^{1/2}\bar{R}^{cd}h_c^a h_d^b + \lambda^{-1/2(3)}Rt^a t^b][(\Box - m_1^2)R_{ab} + m_2^2 g_{ab}R^{cd}R_{cd}] = 0. \tag{2.26}$$

Recalling that $t_a t^a = -\lambda$ and noting that $h^{ab}\bar{R}_{ab} = {}^{(3)}R$ we see that Eq. (2.26) reduces to

$$[\lambda^{1/2}\bar{R}^{cd}h_c^a h_d^b + \lambda^{-1/2(3)}Rt^a t^b](\Box - m_1^2)R_{ab} = 0, \tag{2.27}$$

which is independent of the value of m_2. By Lemma II.2, for $m_1 \neq 0$, this implies $R_{ab} = 0$, provided the space-time is static and asymptotically constant, while for $m_1 = 0$ and the space-time being static and asymptotically flat implies $R_{ab} = 0$. Indeed integrating Eq. (2.27) over the spatial slice exactly gives Eq. (2.24). ■

The final step that is required in order to allow us to consider Eq. (2.9) is to extend the above lemmas to include a term of the form $R^{ab}R_{acbd}$. To do this note that by using the definition of the Riemann tensor

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)R_{cd} = R_{abc}{}^e R_{ed} + R_{abd}{}^e R_{ce} \tag{2.28}$$

and the fact that the contracted Bianchi identity, for $R = 0$, gives

$$\nabla_c R_b{}^c = \tfrac{1}{2}\nabla_b R = 0, \tag{2.29}$$

we can write (for $R = 0$)

$$R^{cd}R_{acbd} = -\nabla_c\nabla_a R_b{}^c + R_{ac}R_b{}^c. \tag{2.30}$$

Thus we see that this term is in fact closely related to $\Box R_{ab}$ and can be dealt with in a similar manner.

Lemma II.4.—For a static space-time the following relation holds:

$$\begin{aligned}\int \sqrt{h}\mathrm{d}^3x[(\lambda^{1/2}\bar{R}^{ef}h^a_e h^b_f + \lambda^{-1/2(3)}Rt^a t^b)R_{acbd}R^{cd}] \\ = \int \sqrt{h}\mathrm{d}^3x[D^a(-\lambda^{1/2}\bar{R}^{ef}D_e\bar{R}af + \lambda^{1/2(3)}RD^b\bar{R}_{ab}) \\ + \lambda^{1/2}(D_a\bar{R}_{ef})(D^e\bar{R}^{af}) + \lambda^{1/2}(D^a\bar{R}_{ac})(D_b\bar{R}^{bc}) \\ - 2\lambda^{1/2}(D^c\bar{R}_{ca})(D^{a(3)}R) + \lambda^{1/2}[\bar{R}^a{}_b\bar{R}^b{}_c\bar{R}^c{}_a - ({}^{(3)}R)^3]].\end{aligned} \tag{2.31}$$

Proof.—To show this we project Eq. (2.30) onto the spatial surface $\mathcal{S}$ with $h^a{}_e h^b{}_f$ and use the identity

$$\begin{aligned}&h^a{}_e h^b{}_f\nabla_c\nabla_a R_b{}^c \\ &= h^a{}_e h^b{}_f\left(h^{cd} - \frac{1}{\lambda}t^c t^d\right)\nabla_c\nabla_a R_{bd} = D_a D_e \bar{R}^a_f \\ &\quad - \frac{1}{4\lambda^2}(D_e\lambda)(D_f\lambda)^{(3)}R - \frac{1}{4\lambda^2}(D_e\lambda)(D^d\lambda)\bar{R}_{fd} \\ &\quad + \frac{1}{2\lambda}(D_f\lambda)(D^{(3)}_e R) + \frac{1}{2\lambda}(D^a\lambda)(D_e\bar{R}_{fa}),\end{aligned} \tag{2.32}$$

where the second equality comes from the tedious but straightforward algebra coming from bringing the $h^b{}_f t^d$ inside both covariant derivatives and repeatedly using Eqs. (2.4), (2.5), (2.18), and (2.19).

Substituting this expression into Eq. (2.30), multiplying by $\lambda^{1/2}\bar{R}^{ef}$, and integrating over $\mathcal{S}$, one finds

$$\begin{aligned}\int_{\mathcal{S}}\sqrt{h}\mathrm{d}^3x[\lambda^{1/2}\bar{R}^{ef}h^a_e h^b_f R_{acbd}R^{cd}] = \int_{\mathcal{S}}\sqrt{h}\mathrm{d}^3x\Big[D^a(-\lambda^{1/2}\bar{R}^{ef}D_e\bar{R}_{af}) + \lambda^{1/2}(D_a\bar{R}_{ef})(D^e\bar{R}^{af}) \\ + \frac{1}{4}\lambda^{-3/2(3)}R\bar{R}^{ef}(D_e\lambda)(D_f\lambda) + \frac{1}{4}\lambda^{-3/2}\bar{R}^{ef}\bar{R}_{fd}(D_e\lambda)(D^d\lambda) - \frac{1}{2}\lambda^{-1/2}\bar{R}^{ef}(D_f\lambda)(D^{(3)}_e R) + \lambda^{1/2}\bar{R}^a{}_b\bar{R}^b{}_c\bar{R}^c{}_a\Big].\end{aligned} \tag{2.33}$$

Similarly we project Eq. (2.30) along $t^a t^b$ to find

$$\begin{aligned}t^a t^b R^{cd}R_{acbd} &= -t^a t^b\nabla_c\nabla_a R_b{}^c + R_{ac}R_b{}^c t^a t^b \\ &= \tfrac{1}{2}[(D_a\bar{R}^a{}_b)(D^b\lambda) + \bar{R}^{ab}D_a D_b\lambda \\ &\quad + {}^{(3)}RD_a D^a\lambda] - \lambda[{}^{(3)}R]^2,\end{aligned} \tag{2.34}$$

where the second equality follows from the staticity of the space-time and, again, repeated use of Eqs. (2.4), (2.5), (2.18), and (2.19).

Multiplying Eq. (2.34) by $\lambda^{-1/2(3)}R$ and integrating over $\mathcal{S}$, one finds

$$\begin{aligned}&\int_{\mathcal{S}}\sqrt{h}\mathrm{d}^3x[\lambda^{-1/2(3)}Rt^a t^b R^c dR_{acbd}] \\ &= \int_{\mathcal{S}}\sqrt{h}\mathrm{d}^3x\Big[D^a\Big(\frac{1}{2}\lambda^{-1/2(3)}R\bar{R}_{ab}D^b\lambda \\ &\quad + \frac{1}{2}\lambda^{-1/2(3)}R^{(3)}RD^a\lambda\Big) + \frac{1}{4}\lambda^{-3/2(3)}\bar{R}^{ab}(D_a\lambda)(D_b\lambda) \\ &\quad + \frac{1}{4}\lambda^{-3/2(3)}R^{(3)}R(D^a\lambda)(D_a\lambda) - \frac{1}{2}\lambda^{-1/2}\bar{R}^{ab}(D^{(3)}_a R) \\ &\quad \times(D_b\lambda) - \lambda^{1/2(3)}R(D^{(3)}_a R)(D^a\lambda) - \lambda^{1/2}[{}^{(3)}R]^3\Big].\end{aligned} \tag{2.35}$$

Finally, adding Eqs. (2.33) and (2.35) and using the contracted Bianchi identity given in Eq. (2.25) demonstrates the desired result, Eq. (2.31). ■

III. GENERALIZED LICHNEROWICZ THEOREM FOR FOURTH ORDER GRAVITY

Using the lemmas proved in the previous section, we will now extend the Lichnerowicz theorem to fourth order gravity (for $\alpha \neq 0$ and $\alpha \geq 3\beta$), provided the spatial scalar curvature satisfies certain bounds.

Theorem III.1.—Consider a static space-time, with a spatial slice that is topologically $\mathbb{R}^3$, in which the spatial curvature everywhere satisfies the following two conditions:

$$m^2 - {}^{(3)}R \geq 0, \qquad \bar{R}^a{}_b\bar{R}^b{}_a(m^2 + \mathcal{R}) \geq 0, \tag{3.1}$$

where $\mathcal{R}$ is defined as

$$\mathcal{R} \equiv \frac{\bar{R}^a{}_b\bar{R}^b{}_c\bar{R}^c{}_a}{\bar{R}^a{}_b\bar{R}^b{}_a}. \tag{3.2}$$

Let the space-time be asymptotically constant, unless both of the inequalities in Eq. (3.1) are saturated, in which case let the space-time be asymptotically flat. Then for such a space-time the equations $R = 0$ and

$$\Box R_{ab} - m^2 R_{ab} + \tfrac{1}{2}g_{ab}R^{cd}R_{cd} - 2R^{cd}R_{acbd} = 0 \tag{3.3}$$

imply $R_{ab} = 0$.

Proof.—The equation

$$\Box R_{ab} - m^2 R_{ab} + \tfrac{1}{2}g_{ab}R^{cd}R_{cd} - 2R^{cd}R_{acbd} = 0 \tag{3.4}$$

implies,

$$I \equiv (\lambda^{1/2}\bar{R}^{ef}h^a{}_e h^b{}_f + \lambda^{-1/2(3)}Rt^a t^b)(\Box R_{ab} - m^2 R_{ab} + \tfrac{1}{2}g_{ab}R^{cd}R_{cd} - 2R^{cd}R_{acbd}) = 0. \quad (3.5)$$

By Lemmas II.3 and II.4 and Eqs. (2.24) and (2.25) of Lemma II.2 the integral of Eq. (3.5) over the constant time slice $\mathcal{S}$ is

$$\int_{\mathcal{S}}\sqrt{h}\mathrm{d}^3x(I) = \int_{\mathcal{S}}\sqrt{h}\mathrm{d}^3x\{D^a[\lambda^{1/2}({}^{(3)}RD_a^{(3)}R + \bar{R}^{cd}D_a\bar{R}_{cd} - \bar{R}^{ef}D_e\bar{R}_{af} - {}^{(3)}RD^b\bar{R}_{ba})] - \lambda^{1/2}(D^a\bar{R}^{cd})(D_a\bar{R}_{cd}) - 2\lambda^{1/2}(D_a\bar{R}_{bc}) \times (D^b\bar{R}^{ac}) - \lambda^{1/2}[D^{a(3)}R - 2D^c\bar{R}^a{}_c] \times [D_a^{(3)} - 2D^b\bar{R}_{ab}] - \lambda^{1/2}[\bar{R}_{cd}\bar{R}^{cd}(m^2 + \mathcal{R}) + ({}^{(3)}R)^2(m^2 - {}^{(3)}R)]\} = 0. \quad (3.6)$$

Asymptotic constancy ensures that the boundary terms vanish, while the final two terms are negative definite if the spatial curvature satisfies the properties

$$m^2 - {}^{(3)}R \geq 0, \qquad \bar{R}^a{}_b\bar{R}^b{}_a(m^2 + \mathcal{R}) \geq 0. \quad (3.7)$$

Finally, as shown in the appendix, the combination

$$(D^a\bar{R}^{cd})(D_a\bar{R}_{cd}) + 2(D_a\bar{R}_{bc})(D^b\bar{R}^{ac}) \geq 0 \quad (3.8)$$

for all $\bar{R}_{ab}$. Thus each term in the integrand of Eq. (3.6) is negative definite and hence required to vanish. Thus $D_cR_{ab} = 0$ and if either of the inequalities in the expressions given in Eq. (3.7) fail to be saturated in any open region, Eq. (3.6) implies $R_{ab} = 0$. If the expressions in Eq. (3.7) are saturated everywhere, then asymptotic flatness is required to imply $R_{ab} = 0$ everywhere. ■

The equations of motion of fourth order gravity (with $\alpha - 3\beta \geq 0$) imply $R = 0$ (by Theorem II.1) and [from Eq. (2.9)]

$$\hbar\,\alpha(\Box R_{ab} - m^2R_{ab} + \tfrac{1}{2}g_{ab}R^{cd}R_{cd} - 2R^{cd}R_{acbd}) = 0, \quad (3.9)$$

with $m^2 = \gamma\kappa^{-2}\hbar^{-1}\alpha^{-1}$. Thus, provided the spatial curvature everywhere obeys the bounds

$${}^{(3)}R \leq \frac{\gamma}{\hbar\alpha\kappa^2}, \qquad \mathcal{R} \geq -\frac{\gamma}{\hbar\alpha\kappa^2}, \quad (3.10)$$

there are no nontrivial, static, asymptotically constant vacuum solutions to fourth order gravity (with $\alpha - 3\beta \geq 0$ and $\alpha \neq 0$).

IV. GENERALIZED ISRAEL THEOREM FOR FOURTH ORDER GRAVITY

The result proved in Sec. III demonstrates that there are no static, vacuum solutions to fourth order gravity for space-times which have a spatial section that is topologically $\mathbb{R}^3$, but one may ask whether a similar result holds for space-times in which the spatial slice contains a boundary. In particular, in order to consider space-times with a (static) black hole, we need to allow for a null, interior boundary to our space-time, i.e. the equal time hypersurfaces become null. At first sight the presence of boundary terms in Eq. (3.6) would appear to be a significant difficulty for such a space-time; however, on the horizon, the normal to the spatial hypersurface, n_a, becomes null, i.e. $n^an_a = \lambda = 0$, and hence the boundary terms vanish. Thus, if the space-time is asymptotically constant (which ensures the boundary terms at infinity also vanish) and all interior boundaries are null, the result of Sec. III will still apply.

Theorem IV.1.—Consider a static space-time, in which the spatial curvature everywhere satisfies the following two conditions:

$${}^{(3)}R \leq \frac{\gamma}{\hbar\alpha\kappa^2}, \qquad \mathcal{R} \geq -\frac{\gamma}{\hbar\alpha\kappa^2}. \quad (4.1)$$

Let the space-time be asymptotically constant, unless both of the inequalities in Eq. (4.1) are saturated, in which case let the space-time be asymptotically flat. Let the spatial slice be bounded by a null surface that is topologically $\mathbb{S}^2$. Then the only solution to fourth order gravity (with $\alpha \neq 0$ and $\alpha - 3\beta \geq 0$), in the region exterior to the null surface, is $R_{ab} = 0$.

Proof.—As in Theorem III.1, the equations of motion for fourth order gravity imply [Eq. (3.6)]

$$\int\sqrt{h}\mathrm{d}^3x(I) = \int\sqrt{h}\mathrm{d}^3x\{D^a[\lambda^{1/2}({}^{(3)}RD_a^{(3)}R + \bar{R}^{cd}D_a\bar{R}_{cd} - \bar{R}^{ef}D_e\bar{R}_{af} - {}^{(3)}RD^b\bar{R}_{ba})] - \lambda^{1/2}(D^a\bar{R}^{cd})(D_a\bar{R}_{cd}) - 2\lambda^{1/2}(D_a\bar{R}_{bc})(D^b\bar{R}^{ac}) - \lambda^{1/2}[D^{a(3)}R - 2D^c\bar{R}^a{}_c][D_a^{(3)} - 2D^b\bar{R}_{ab}] - \lambda^{1/2}[\bar{R}_{cd}\bar{R}^{cd}(\gamma\kappa^{-2}\alpha^{-1} + \mathcal{R}) + ({}^{(3)}R)^2(\gamma\kappa^{-2}\alpha^{-1} - {}^{(3)}R)]\} = 0. \quad (4.2)$$

At infinity the boundary terms vanish because the spatial section is asymptotically constant, while on the interior boundary they vanish because $\lambda = 0$. Thus, just as in Theorem III.1, provided the three-dimensional curvature satisfies the conditions

$$\frac{\gamma}{\hbar\alpha\kappa^2} - {}^{(3)}R \geq 0, \qquad \frac{\gamma}{\hbar\alpha\kappa^2} + \mathcal{R} \geq 0,$$

each term in the integrand of Eq. (4.2) is negative definite and hence vanishes. If either of these inequalities is not saturated everywhere, asymptotic constancy is sufficient to imply $R_{ab} = 0$; otherwise, asymptotic flatness is required. ■

Since fourth order gravity implies $R_{ab} = 0$ (for asymptotically constant and static space-times), provided

$$^{(3)}R \leq \frac{\gamma}{\hbar\alpha\kappa^2}, \qquad \mathcal{R} \geq -\frac{\gamma}{\hbar\alpha\kappa^2}, \tag{4.3}$$

even in the presence of null boundaries, it follows that the only spherically symmetric solution is the (exterior of the) Schwarzschild metric.

It is important to note here that this result relies on the existence of a null boundary to the equal time hypersurfaces and the existence of a vacuum ($T_{ab} = 0$) everywhere between this horizon and infinity. This is true for a (static, vacuum) space-time containing an event horizon; however, the result cannot be applied to the exterior of a general spherical object that does not contain such a horizon. In particular this result does not imply that the metric outside a spherically symmetric distribution of matter will be Schwarzschild, unless that matter is entirely contained within the event horizon. This agrees with the (perturbative) results of (for example) [6].

V. NONVACUUM SOLUTIONS

In Secs. III and IV we showed that for $T_{ab} = 0$, the static, asymptotically constant solutions of GR are also solutions to fourth order gravity (with $\alpha \neq 0$ and $\alpha - 3\beta \geq 0$), at least in the case of space-times that have a spatial topology of $\mathbb{R}^3$ (Theorem III.1) or space-times that have internal null boundaries (Theorem IV.1), and whose spatial curvature everywhere satisfies Eq. (4.3). A natural question that one may ask is whether this connection between the solutions of GR and fourth order gravity extends to the nonvacuum case, i.e. $T_{ab} \neq 0$. We will now demonstrate that it does not. In fact below we show that for static, asymptotically flat space-times containing a barotropic fluid with equation of state $\omega \geq -3^{-1/3} \approx -0.6933$, *only* the vacuum solutions of the two theories coincide, with all other solutions being distinct.

Theorem V.1.—Consider an asymptotically flat solution to GR, ${}^{\mathrm{GR}}g_{ab}$, with matter satisfying

$$\int \sqrt{h}\mathrm{d}^3 x T^a{}_b T^b{}_c T^c{}_a \geq 0. \tag{5.1}$$

Then ${}^{\mathrm{GR}}g_{ab}$ is a solution to fourth order gravity (with $\alpha \neq 0$ and $\alpha \neq 3\beta$) iff $R_{ab} = 0$.

Proof.—Consider a solution to Einstein's equations:

$$\gamma\kappa^{-2}(R_{ab} - \tfrac{1}{2}g_{ab}R) = \tfrac{1}{2}T_{ab} \tag{5.2}$$

that is static and asymptotically flat. Substituting this into the equations of motion for fourth order gravity [Eq. (2.1)], we find

$$\begin{aligned}(\alpha - 2\beta)R_{;a;b} - \alpha\Box R_{ab} - \tfrac{1}{2}(\alpha - 4\beta)g_{ab}\Box R \\ + 2\alpha R^{cd}R_{acbd} - 2\beta R R_{ab} \\ - \tfrac{1}{2}g_{ab}(\alpha R^{cd}R_{cd} - \beta R^2) = 0.\end{aligned} \tag{5.3}$$

The trace of Eq. (5.3) is

$$(\alpha - 3\beta)\Box R = 0. \tag{5.4}$$

Then by Theorem II.1, for $\alpha - 3\beta \neq 0$, this implies $R = 0$. Thus Eq. (5.3) becomes

$$-\alpha[\Box R_{ab} + \tfrac{1}{2}g_{ab}R^{cd}R_{cd} - 2R^{cd}R_{acbd}] = 0, \tag{5.5}$$

which by Theorem III.1 implies [see Eq. (3.6)]

$$\begin{aligned}\int \sqrt{h}\mathrm{d}^3 x(I) = \int \sqrt{h}\mathrm{d}^3 x\{D^a[\lambda^{1/2}({}^{(3)}R D^{(3)}_a R + \bar{R}^{cd}D_a\bar{R}_{cd} - \bar{R}^{ef}D_e\bar{R}_{af} - {}^{(3)}R D^b\bar{R}_{ba})] - \lambda^{1/2}(D^a\bar{R}^{cd})(D_a\bar{R}_{cd}) \\ - 2\lambda^{1/2}(D_a\bar{R}_{bc})(D^b\bar{R}^{ac}) - \lambda^{1/2}[D^{a(3)}R - 2D^c\bar{R}^a{}_c][D^{(3)}_a - 2D^b\bar{R}_{ab}] - \lambda^{1/2}[\bar{R}^a{}_b\bar{R}^b{}_c\bar{R}^c{}_a - ({}^{(3)}R)^3]\} = 0.\end{aligned} \tag{5.6}$$

Thus, if

$$\int \sqrt{h}\mathrm{d}^3 x[\bar{R}^a{}_b\bar{R}^b{}_c\bar{R}^c{}_a - ({}^{(3)}R)^3] \geq 0, \tag{5.7}$$

then each term in the integrand is negative definite and hence vanishes.

However it is easy to show that

$$\begin{aligned}R^a{}_b R^b{}_c R^c{}_a &= g^{ad}g^{be}g^{cf}R_{bd}R_{ce}R_{fa} \\ &= \bar{R}^a{}_b\bar{R}^b{}_c\bar{R}^c{}_a - ({}^{(3)}R)^3,\end{aligned} \tag{5.8}$$

where the second equality follows from decomposing $g^{ab} = h^{ab} - \frac{1}{\lambda}t^a t^b$, using the Codacci and Gauss equations [Eqs. (2.19) and (2.18)] and noting that $R = 0$. Thus each term in the integrand of Eq. (5.6) is negative definite, and hence vanishes, provided

$$\int \sqrt{h}\mathrm{d}^3 R^a{}_b R^b{}_c R^c{}_a \geq 0. \tag{5.9}$$

Using Eq. (5.2) and the fact that $R = 0$ for these solutions, this condition can be written as

$$\int \sqrt{h}\mathrm{d}^3 x T^a{}_b T^b{}_c T^c{}_a \geq 0. \tag{5.10}$$

Thus, for matter satisfying Eq. (5.10), a static, asymptotically flat solution to GR is also a solution to fourth order gravity *iff* $R_{ab} = 0$, which, by Theorems III.1 and IV.1 corresponds to the (static, asymptotically flat) vacuum solutions in both GR and fourth order gravity. Thus, other than the vacuum, there are no static, asymptotically flat solutions to GR, with matter satisfying Eq. (5.10), that are also solutions to fourth order gravity. ■

If we consider a perfect fluid, the energy-momentum tensor can be written as $T^a{}_b = \rho u^a u_b + P(\delta^a{}_b + u^a u_b)$, with u_a a unit timelike vector tangent to the observer's worldline, P the fluid's pressure, and ρ its energy density. Equation (5.10) then becomes

$$\int \sqrt{h} \mathrm{d}^3 x (\rho^3 + 3P^3) \geq 0. \tag{5.11}$$

Assuming the strong energy condition holds, $\rho \geq 0$, then this condition is trivial for all matter with positive pressure. If we consider a barotropic fluid with equation of state parameter ω, i.e. $P = \omega\rho$, then Eq. (5.10) reduces to

$$\int \sqrt{h} \mathrm{d}^3 x (1 + 3\omega^3) \geq 0. \tag{5.12}$$

This is, in particular, satisfied for $\omega \geq -3^{-1/3} \approx -0.6933$ everywhere, and hence for dust ($\omega = 0$) and radiation ($\omega = 1/3$) fluids.

Thus for a static, asymptotically flat, space-time containing a single, barotropic fluid for which $\omega \geq -3^{-1/3}$ everywhere, no solutions to GR remain (static and asymptotically flat) solutions to fourth order gravity.

One can also consider, for example, the energy-momentum tensor of a Maxwell field:

$$T^a{}_b = \frac{1}{4\pi}\left(F^a{}_c F^c{}_b - \frac{1}{4} g^a{}_b F^d{}_e F^e{}_d\right), \tag{5.13}$$

with F_{ab} the electromagnetic field tensor. For such a field, one can directly check that

$$T^a{}_b T^b{}_c T^c{}_a = \frac{1}{64\pi^3}\left[\frac{1}{8}(\mathrm{Tr}\mathbf{F}^2)^3 - \frac{3}{4}(\mathrm{Tr}\mathbf{F}^4)(\mathrm{Tr}\mathbf{F}^2) - \mathrm{Tr}\mathbf{F}^6\right] \geq 0, \tag{5.14}$$

where the matrix $\mathbf{F}$ is given by $F^a{}_b$ and hence is antisymmetric and the final inequality follows from direct computation, for a general antisymmetric matrix.

VI. CONCLUSIONS

In Secs. (III and IV) we demonstrated that there are no nontrivial, static, asymptotically constant solutions to fourth order gravity, *provided* the three-dimensional curvature satisfies the following two conditions:

$$^{(3)}R \leq \frac{\gamma}{\hbar\alpha\kappa^2} \quad \text{and} \quad \mathcal{R} \geq -\frac{\gamma}{\hbar\alpha\kappa^2}, \tag{6.1}$$

everywhere outside a null horizon. The meaning of these conditions is the following. We require the three-dimensional scalar curvature, measured by both $^{(3)}R$ and $\mathcal{R}$, to be everywhere smaller in magnitude than the scale set by the ratio between $\gamma\kappa^{-2}$ and $\hbar\alpha$. This is exactly the scale at which corrections to general relativity will become significant. If these corrections are motivated by quantum corrections to general relativity, then this will be the quantum gravity scale. Thus our results can interpreted as follows: There are no nontrivial, static, asymptotically constant, vacuum solutions to fourth order gravity, so long as the three-dimensional scalar curvature is everywhere less than the quantum gravity scale. In particular for black holes, this implies that the Schwarzschild solution is the *unique*, spherically symmetric, static, asymptotically constant vacuum solution to fourth order gravity, unless the three-dimensional scalar curvature exceeds the quantum gravity scale *outside the horizon*, a condition that is easily achieved for macroscopic black holes.

This also demonstrates the fact that the no-hair theorem applies to fourth order gravity provided that the spatial curvature (outside the horizon) is everywhere less than the quantum gravity scale. Note however that the converse does not follow; i.e. if the spatial curvature outside the horizon is greater than the quantum gravity scale, Theorem IV.1 *does not* imply that there are additional (static, asymptotically flat) solutions.

Finally, in Sec. V, we demonstrated that for matter satisfying the rather mild condition

$$\int \sqrt{h} \mathrm{d}^3 x T^a{}_b T^b{}_c T^c{}_a \geq 0, \tag{6.2}$$

except for the vacuum, none of the static, asymptotically flat solutions to GR are solutions to fourth order gravity. In particular, this includes space-times containing single, barotropic fluids with equations of state everywhere satisfying $\omega \geq -3^{-1/3}$, such as relativistic and nonrelativistic baryons and fermions. A specific application of this is that the GR solutions for (static, asymptotically flat) stars will fail to be solutions in fourth order gravity (for the same matter distribution). This agrees with the results of [6], which gives the solutions to fourth order gravity for a particular, extended, spherically symmetric, system (calculated perturbatively and close to the origin) which is shown not to be Schwarzschild.

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APPENDIX

In this appendix we prove various useful (if rather laborious) identities. First consider $t^a\nabla_a t^b \nabla_b R_{cd}$, for a static space-time. Using the staticity condition

$$\mathcal{L}_t R_{ab} = 0 \Rightarrow t^a \nabla_a R_{cd} = -2R_{a(d}\nabla_{c)} t^a, \tag{A1}$$

we can write

$$t^a\nabla_a t^b\nabla_b R_{cd} = 2[(\nabla_b t^a)R_{a(d}(\nabla_{c)}t^b) + R_{ab}(\nabla_{(d}t^a)(\nabla_{c)}t^b)] - R_{bd}t^a\nabla_a\nabla_c t^b - R_{cb}t^a\nabla_a\nabla_d t^b. \tag{A2}$$

Using $t^c\nabla_c\lambda = 0$ and hence that $\nabla_a(t^c\nabla_c\lambda) = 0$, we find

$$t^c\nabla_c\nabla_a\lambda = -(\nabla_a t^c)(\nabla_c\lambda). \tag{A3}$$

Thus,

$$t^c\nabla_c\nabla_a t^d = \frac{1}{2\lambda}(t^c t^d\nabla_c\nabla_a\lambda - t_a t^c\nabla_c\nabla^d\lambda) = \frac{1}{2\lambda}(-t^d(\nabla_a t^c)(\nabla_c\lambda) + t_a(\nabla^d t^c)(\nabla_c\lambda)) = 0, \tag{A4}$$

where the final equality follows from expanding $\nabla_a t_c = \frac{1}{\lambda}(t_{[c}\nabla_{a]}\lambda)$. Using this in Eq. (A2) gives

$$t^a\nabla_a t^b\nabla_b R_{cd} = 2[(\nabla_b t^a)(R_{a(d}\nabla_{c)}t^b) + R_{ab}(\nabla_c t^b)(\nabla_d t^a)], \tag{A5}$$

which is Eq. (2.10).

By using $\nabla_a t_c = \frac{1}{\lambda}(t_{[c}\nabla_{a]}\lambda)$ and recalling that $t^a\nabla_a\lambda = 0$, we find

$$(\nabla_b t^c)(\nabla_a t^d) = \frac{1}{4\lambda^2}(t^c\nabla_d\lambda - t_d\nabla^c\lambda)(t^d\nabla_a\lambda - t_a\nabla^d\lambda) = \frac{-1}{4\lambda^2}t^c t_a(\nabla_d\lambda)(\nabla^d\lambda) + \frac{1}{4\lambda}(\nabla^c\lambda)(\nabla_a\lambda), \tag{A6}$$

which is just Eq. (2.11).

In order to show Eq. (2.14) note that

$$\frac{1}{\lambda}t^c t^d\nabla_b R_{cd} = \nabla_b\left(\frac{1}{\lambda}t^c t^d R_{cd}\right) - R_{cd}\nabla\left(\frac{1}{\lambda}t^c t^d\right). \tag{A7}$$

Now we use the fact that for a static space-time, the extrinsic curvature of the constant time hypersurfaces vanishes (this can be checked by direct computation). Thus the Gauss equation [Eq. (2.19)] gives

$$\frac{1}{\lambda}t^a t^b R_{ab} = {}^{(3)}R - R, \tag{A8}$$

while the Codacci equation [Eq. (2.18)] is simply $t^a h^b{}_c R_{ab} = 0$. A consequence of the Codacci equation and $t^c\nabla_c\lambda = 0$ is

$$t^a(\nabla^b\lambda)R_{ab} = 0. \tag{A9}$$

Using this and the Gauss equation in Eq. (A7) gives

$$\frac{1}{\lambda}t^c t^d\nabla_b R_{cd} = \nabla_b({}^{(3)}R - R). \tag{A10}$$

With this we directly find Eq. (2.14):

$$\frac{1}{2\lambda^2}h^{ab}(\nabla_a\lambda)t^c t^d\nabla_b R_{cd} = \frac{1}{2\lambda}(\nabla^a\lambda)(\nabla_a({}^{(3)}R - R)). \tag{A11}$$

Similarly, by noting

$$\frac{1}{\lambda}t^c t^d h^{ab}\nabla_a h^e{}_b\nabla_e R_{cd} = h^a b\nabla_a\left(\frac{1}{\lambda}t^c t^d h^e{}_b\nabla_e R_{cd}\right) - h^{ab}\left[\nabla_a\left(\frac{1}{\lambda}t^c t^d\right)\right]\nabla_b R_{cd} \tag{A12}$$

and using Eq. (A10) we find

$$\frac{1}{\lambda}t^c t^d h^{ab}\nabla_a h^e{}_b\nabla_e R_{cd} = D_a D^a({}^{(3)}R - R), \tag{A13}$$

which is Eq. (2.17).

A key equation of Lemma II.2 is Eq. (2.16), which we derive here. By decomposing $g_{ab} = h_{ab} - \frac{1}{\lambda}t_a t_b$ we have

$$\Box R_{ab} - m^2 R_{ab} = h^{ab}\nabla_a\nabla_b R_{cd} - \frac{1}{\lambda}t^a t^b\nabla_a\nabla_b R_{cd} - m^2 R_{cd} = 0. \tag{A14}$$

Splitting the projection $h^{ab} = h^{ac}h^b{}_c$ and bringing one of the 3-metrics, inside the covariant derivative the first term becomes

$$h^{ab}\nabla_a\nabla_b R_{cd} = h^{ab}\nabla_a h^e{}_b\nabla_e R_{cd}, \tag{A15}$$

where we used the fact that, because $\nabla_a g_{bc} = 0$, we have

$$\nabla_a h_{bc} = \nabla_a\left(\frac{1}{\lambda}t_b t_c\right), \tag{A16}$$

and $h^{ab}\nabla_a t_b = 0$ by symmetry. For the second term in Eq. (A14), we again use the fact that $\nabla_c g_{ab} = 0$ to find

$$\nabla_c h_{ab} = \nabla_c\left(\frac{1}{\lambda}t_a t_b\right) \Rightarrow t^a\nabla_c t_a = -t^a\nabla_a t_c = -\frac{1}{2}\nabla_c\lambda, \tag{A17}$$

where the second line follows by expanding the right-hand side, taking the trace over (a, b) and noting that $\nabla_{(a}t_{b)} = 0$. Using Eqs. (A5) and (A15) in Eq. (A14) gives

$$h^{ab}\nabla_a h^e{}_b\nabla_e R_{cd} + \frac{1}{2\lambda}h^{ab}(\nabla_a\lambda)(\nabla_b R_{cd}) + \frac{1}{4\lambda^3}[t_c t^a(\nabla_b\lambda)(\nabla^b\lambda) - \lambda(\nabla_c\lambda)(\nabla^a\lambda)]R_{ad} - \frac{1}{2\lambda^2}[t^a(\nabla_d\lambda) - t_d(\nabla^a\lambda)][t^b(\nabla_c\lambda) - t_c(\nabla^d\lambda)]R_{ba} + \frac{1}{4\lambda^3}[t^a t_d(\nabla_b\lambda)(\nabla^b\lambda) - \lambda(\nabla^a\lambda)(\nabla_d\lambda)]R_{ca} - m^2 R_{cd} = 0, \tag{A18}$$

which is Eq. (2.16).

Finally, we need to show that the terms given in Eq. (3.8) are positive, i.e. that

$$(D_a\bar{R}_{bc})(D^a\bar{R}^{bc}) + 2(D_a\bar{R}_{bc})(D^b\bar{R}^{ac}) \geq 0. \tag{A19}$$

To do this, consider an arbitrary tensor T_{abc} that is symmetric in the final two indices, i.e. $T_{abc} = T_{acb}$. Now we use a tetrad basis, e^i_a, that is orthonormal, i.e. $e^i_a e^a_j = \delta^i_j$, to decompose the (positive definite) metric as

$$h_{ab} = e^i_a e^j_b \delta_{ij}. \tag{A20}$$

With this one can write

$$T^{abc}T_{abc} = h^{ad}h^{de}h^{cf}T_{abc}T_{def} = T_{ijk}T_{mnp}\delta^{im}\delta^{jn}\delta^{kp},$$
$$T^{abc}T_{bac} = h^{ae}h^{bd}h^{cf}T_{abc}T_{def} = T_{ijk}T_{mnp}\delta^{in}\delta^{jm}\delta^{kp}. \tag{A21}$$

We now form the combination

$$T^{abc}T_{abc} + 2T^{abc}T_{bac} \tag{A22}$$

and expand out the Einstein summations. Since T_{abc} is a three index tensor, the three possibilities we need to consider are the indices are the same, one is distinct, and all are distinct. The first possibility clearly involves only positive terms. In the orthonormal tetrad basis given by Eq. (A20) these are

$$T^{abc}T_{abc} + 2T^{abc}T_{bac} = 3(T_{111}T_{111} + T_{222}T_{222} + T_{333}T_{333}) + \cdots, \tag{A23}$$

where the dots remind us that there are other terms we need to account for. The terms in Eq. (A22) that arise from one index being distinct, with the other two being the same, can, by using Eq. (A20), be grouped to give

$$T^{abc}T_{abc} + 2T^{abc}T_{bac} = \sum_{i\neq j}(2T_{iij} + T_{ijj})(2T_{iij} + T_{ijj}) + \cdots, \tag{A24}$$

where there is no implicit summation over repeated indices and again the dots indicate that there we are considering only certain terms. Finally we need to account for the terms in Eq. (A22) for which all the indices are distinct, which are given by

$$T^{abc}T_{abc} + 2T^{abc}T_{bac} = \sum_{i\neq j\neq k} 2(T_{ijk} + T_{jik} + T_{kij})(T_{ijk} + T_{jik} + T_{kij}) + \cdots, \tag{A25}$$

where again there is no implicit summation over repeated indices and the dots indicate that there are additional terms. In Eqs. (A24) and (A25), crucial use has been made of the fact that $T_{ijk} = T_{ikj}$. Finally, putting these together we find that, in this orthonormal tetrad basis,

$$T^{abc}T_{abc} + 2T^{abc}T_{bac} = 3\sum_i T_{iii}T_{iii} + \sum_{i\neq j}(2T_{iij} + T_{ijj})(2T_{iij} + T_{ijj}) + \sum_{i\neq j\neq k} 2(T_{ijk} + T_{jik} + T_{kij})(T_{ijk} + T_{jik} + T_{kij}), \tag{A26}$$

where once again there is no implicit summation over repeated indices. Clearly each term in Eq. (A26) is positive. It is worth noting here that this result holds in any number of dimensions, provided the metric h_{ab} is positive definite.

One may be concerned by the fact that if T_{abc} were also antisymmetric in the first two indices, the combination given on the left-hand side of Eq. (A26) would appear to be negative. However there are no three index tensors that are antisymmetric in the first pair of indices and symmetric in the second pair, as can easily be checked:

$$T_{abc} = -T_{bac} = -T_{bca} = T_{cba} = T_{cab} = -T_{acb} = -T_{abc}. \tag{A27}$$

Equation (A26) holds for any three index tensor that is symmetric in its final pair of indices and, in particular, it holds for $T_{abc} = D_a\bar{R}_{bc}$; thus,

$$(D_a\bar{R}_{bc})(D^a\bar{R}^{bc}) + 2(D_a\bar{R}_{bc})(D^b\bar{R}^{ac}) \geq 0, \tag{A28}$$

as required.

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