PHYSICAL REVIEW D **82,** 104025 (2010)

Ultraspinning instability of rotating black holes

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(Received 12 July 2010; published 10 November 2010)

Rapidly rotating Myers-Perry black holes in $d \geq 6$ dimensions were conjectured to be unstable by Emparan and Myers. In a previous publication, we found numerically the onset of the axisymmetric ultraspinning instability in the singly spinning Myers-Perry black hole in $d = 7, 8, 9$. This threshold also signals a bifurcation to new branches of axisymmetric solutions with pinched horizons that are conjectured to connect to the black ring, black Saturn and other families in the phase diagram of stationary solutions. We firmly establish that this instability is also present in $d = 6$ and in $d = 10, 11$. The boundary conditions of the perturbations are discussed in detail for the first time, and we prove that they preserve the angular velocity and temperature of the original Myers-Perry black hole. This property is fundamental to establishing a thermodynamic necessary condition for the existence of this instability in general rotating backgrounds. We also prove a previous claim that the ultraspinning modes cannot be pure gauge modes. Finally we find new ultraspinning Gregory-Laflamme instabilities of rotating black strings and branes that appear exactly at the critical rotation predicted by the aforementioned thermodynamic criterium. The latter is a refinement of the Gubser-Mitra conjecture.

DOI: 10.1103/PhysRevD.82.104025 PACS numbers: 04.50.Gh

I. INTRODUCTION

The turn of the century witnessed a renewed interest in Einstein's gravity. The research focus extends nowadays to numerical simulations of the full-time evolution (inspiral, merger and ring-down phases) of black hole binary systems [1], numerical simulations of high-energy collisions of black holes [2], and the study of higher-dimensional black holes [3], mainly in vacuum and in asymptotically anti–de Sitter (AdS) spacetimes. The former two endeavors are of the utmost interest for gravitational wave experiments. On the other hand, the latter program was triggered by the quest for a microscopic description of black holes in a theory of quantum gravity (string theory), by the emergence of TeV-scale gravity scenarios relevant for the LHC, and by the realization that black holes describe thermal phases of gauge theories in holographic gauge/gravity dualities. The properties of higher-dimensional black holes, namely, their stability, will be the main focus of the present work, and we review its motivations in more detail next.

Black hole thermodynamics and the thermal spectrum of Hawking radiation strongly suggest the existence of a statistical mechanical description of black holes in terms of some underlying microscopic degrees of freedom. This microscopic description necessarily requires quantum gravity. One of the most compelling candidates for such a theory is string theory, which inevitably requires gravity in extra dimensions. Furthermore, the observation that the black hole entropy is proportional to the horizon area strongly suggests that the quantum degrees of freedom of the black hole are distributed over a surface [4]. This led to the formulation of the holographic principle according to which quantum gravity in a given volume should have a description in terms of a quantum field theory on its boundary. The last decade witnessed concrete realizations of this holographic idea, namely, the AdS/conformal field theory (CFT) duality and more generically the gauge/gravity correspondence [5]. The application of these ideas to certain classes of higher-dimensional black holes has already provided a statistical description of some of their thermodynamic properties. Successful examples include: i) a statistical counting of the Bekenstein-Hawking entropy [6], ii) a microscopic description of Hawking emission in near-extremal black holes [7], iii) a microscopic description of superradiant emission [8], iv) the map between the Hawking-Page phase transition in asymptotically AdS black holes and the confinement/deconfinement phase transition in gauge theories that share some common properties with QCD [9], and v) the identification of the quasinormal modes of oscillation of a black hole with the thermalization frequencies of the perturbed holographic field theory [10], to mention but a few. Following these early successes, the gauge/gravity correspondence has matured and is nowadays a field of research on its own.

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Mainly due to the fact that this is a weak/strong coupling duality, it is recognized as a powerful technical tool to understanding not only gravity but also strongly coupled gauge theories. In recent years this correspondence has been extended to other systems of interest, namely, the fluid/gravity correspondence [11,12] and more recently the condensed matter/gravity correspondence [13]. These developments indicate that gravitational physics is becoming a valuable technical tool to understanding several other branches of physics. Given the holographic nature of these correspondences, and since we are often interested in gauge theories in four dimensions, it is therefore important to develop our understanding of gravity in five and higher dimensions.

The LHC will soon be operating at the TeV energy frontier. Although it was designed to find the Higgs boson and study particle physics beyond the standard model, there is the possibility that the LHC will discover extra dimensions. Indeed, in recent years various TeV-scale gravity or braneworld scenarios have been proposed according to which extra dimensions might be detected at the energy scales probed by the LHC (see [14] for a review). The motivation for these proposals is the solution of the hierarchy problem, i.e., the huge difference between the Planck scale and the electroweak scale. By proposing the existence of sufficiently "large" extra dimensions, the fundamental (d-dimensional) Planck mass can be of the order of the electroweak energy (~ 1 TeV). The most fascinating outcome of these scenarios is the possibility of producing microscopic black holes at the LHC. This could allow for the experimental determination of the fundamental scale of gravity, the number of extra dimensions, and the decay product of higher-dimensional black holes, including the first evidence of Hawking evaporation [14]. This fascinating possibility provides an extra motivation to undertake the study of higher-dimensional black holes.

Stimulated by the advances in higher-dimensional gravitational physics, there is an intensive ongoing program to extend analytical tools and numerical relativity to higher dimensions. The Newman-Penrose and Geroch-Held-Penrose formalisms, and the resulting Petrov classification, which are important to classify solutions and study their stability, have been extended to $d > 4$ [15]. The separability of some class of perturbation equations and the existence of completely integrable geodesic equations in some higher-dimensional black holes are possible due to the fact that there are hidden and explicit symmetries associated with the principal conformal Killing-Yano tensor [16]. The study of higher-dimensional gravity is also motivating the development of completely new analytical tools, like the blackfold approach [17,18], to construct perturbative solutions based on the fact that for higher-dimensional black holes the horizons can have widely separated length scales. Simultaneously, the first nonlinear numerical codes in higher dimensions are starting to be developed, mainly to study the time evolution of instabilities in black holes [19–21]. This is expected to be an area of active research in the near future. In addition, the challenges of higher-dimensional black hole physics have motivated the introduction of new numerical tools, like the spectral methods [22–24].

Last but not least, another motivation to study higher-dimensional black holes is the understanding of these objects *per se.* Indeed, to uncover the mathematical structure of Einstein's gravity and its solutions one should treat the number of spacetime dimensions d as a free parameter in the theory. We should be able to distinguish between the universal properties of the theory and the dimension-dependent ones.

It has been known since the 1970s that $4d$ black holes are the simplest gravitational objects in Nature. Hawking's black hole topology theorem states that a $4d$ black hole must have an event horizon with spherical topology [25]. Together with Hawking's rigidity theorem [25], this led to the proof of the $4d$ uniqueness theorems [26]. As a result, a $4d$ vacuum black hole is fully specified by its conserved charges: mass M and angular momentum J. Therefore, there is a unique black hole in $4d$ Einstein gravity—the Kerr solution—which has an upper bound on its angular momentum, $J \leq GM^2$ (G being Newton's constant). However, are these universal properties of gravity? Or are some of them dimension dependent?

The last decade has provided a cascade of fascinating answers to these questions (see [3] for a review). The higher-dimensional arena is populated by new solutions besides the Myers-Perry (MP) black hole with S^{d-2} topology [27]—the (nontrivial) counterpart of the $4d$ Kerr solution. The simplest black object in d dimensions is a black string with horizon topology $S^{d-3} \times \mathbb{R}$ or a black brane with $S^{d-2-n} \times \mathbb{R}^n$, constructed by adding trivial flat direction(s) to the metric of the Schwarzchild geometry. The most surprising analytical solution however is the $5d$ black ring found by Emparan and Reall [28]. This solution is asymptotically flat, has horizon topology $S^2 \times S^1$ and in some regions of the parameter space can carry the same conserved charges as the singly spinning Myers-Perry black hole. Therefore, this solution explicitly demonstrates that the topology and uniqueness theorems do not generalize straightforwardly to higher dimensions when rotation is considered.[1] Indeed, the generalization of the black hole topology theorem imposes much weaker restrictions on the horizon topology [31]. In addition, the $5d$ uniqueness theorems, restricted to geometries with two $U(1)$ Killing isometries, explicitly show that, in order to uniquely specify a black hole solution, one has to fix the conserved

[1]The situation is considerably different for static solutions. Indeed, it is proven that the Schwarzschild-Tangherlini black hole is the unique static solution [29], and moreover this black hole is stable at the linear mode level [30].

charges *and* other parameters not related to them [32] (see also [33] for some earlier work). Using the complete integrability of the five-dimensional Einstein's vacuum equations restricted to solutions with $\mathbb{R} \times U(1)^2$ Killing isometries, it has also been possible to construct explicitly the black rings with rotation only along the S^2 (which possess conical singularities) [34],[2] the regular doubly spinning black ring [36], and regular asymptotically flat, multiblack-hole objects like black Saturns [37], di-rings [38], and bi-cycling rings [39]. These solutions provide further examples of the richness of higher d gravity. In $d > 5$, the integrability methods are not available, but the perturbative blackfold approach of [17] has provided evidence for the existence of black objects with other (nonspherical) topologies [18,40]. All these solutions have multiple $U(1)$ rotational symmetries and need to be organized in a classification scheme. To provide such a classification we must address the question of to what extent the rigidity theorem extends to higher dimensions. In higher dimensions, stationarity only implies the existence of one rigid rotational symmetry [41]. However, all higher-dimensional solutions known exactly have more than one such symmetry. A fundamental question is whether there are stationary black holes with less symmetry than those solutions. This possibility was first raised in [42], where the existence of stationary black holes with a single $U(1)$ was conjectured. We shall return to this fundamental issue below.

Having these explicit solutions, the next natural question is whether they are dynamically stable against linear perturbations. Start with the Kerr solution in vacuum. Smarr observed that, as the rotation increases, the horizon Gaussian curvature at the poles starts positive, becomes zero at a critical rotation, and then goes negative before the Kerr bound $J = GM^2$ is reached [43]. He further noticed that a similar behavior occurs in rotating fluid droplets held by surface tension, where this behavior signals an instability. However, soon after Smarr made this remark, Teukolsky used the Newman-Penrose formalism to find the master equation that governs the gravitational perturbations of the Kerr black hole, which was used by Whiting to show the mode-by-mode stability of the Kerr black hole [44].

What about MP black holes? For $d \geq 6$, the rotation of a singly spinning MP can grow unbounded, as already noticed in [27]. This led Emparan and Myers to propose that, for sufficiently high rotation, these black holes should become unstable against what was called the *ultraspinning* limit [45]. The natural way to check the presence of this instability would be to apply a linear perturbation analysis using the Newman-Penrose formalism, this time for $d \geq 6$. Unfortunately, although this formalism and the associated Geroch-Held-Penrose formalism have been extended to higher dimensions, it is extremely difficult if not impossible to manipulate the equations to get decoupled master equations for the perturbations [15]. So, in practice, we cannot use it yet to treat the perturbation problem analytically.[3] Given these limitations, Emparan-Myers provided solid heuristic arguments, that we highlight next, to conjecture the existence of an instability for a sufficiently large rotation [45]. Take an MP black hole rotating along a single plane for simplicity. In the ultraspinning regime, $a \gg r_+$ (where $a = \frac{d-2}{2} \frac{J}{M}$, and r_+ is the horizon radius), the black hole horizon flattens out along the plane of rotation, and its shape can be approximated by that of a black disk of radius a and thickness r_+ [45]. In fact, one can take a precise limit in which the MP black hole metric near the rotation axis reduces to the metric of a static black membrane with horizon topology $S^{d-4} \times \mathbb{R}^2$, and where the world volume directions of the membrane correspond to the directions along the original rotation plane [45]. This is a far-reaching observation since black membranes are afflicted by the Gregory-Laflamme (GL) instability [48]. This led the authors of [45] to conjecture that ultraspinning MP black holes should be unstable under a Gregory–Laflamme-type of instability (see Sec. II A for more details). According to the arguments of [45], this instability should become active when $a \sim r_+$, and it provides an effective dynamical bound on the angular momentum. As often occurs in physical systems, and, in particular, in the context of black branes [49], the stationary threshold of the ultraspinning instability is expected to signal also a bifurcation point in a phase diagram of solutions into a new branch of pinched black objects. New thresholds are expected at higher rotations, from which additional branches of pinched solutions should bifurcate. These conjectured stationary solutions preserve the original MP symmetries, but they have spherical horizons distorted by ripples along the polar direction and are conjectured to connect to the black ring and black Saturn families in a phase diagram of stationary solutions [40].

The main purpose of the present study is to firmly establish the properties of the threshold of this ultraspinning instability. A practical use of the Newman-Penrose formalism in higher dimensions is not yet available, so we will resort to a numerical approach. In a previous publication, we have already addressed this problem in $d = 7, 8, 9$ [23]. This paper, however, extends the former study in several directions. We present for the first time the properties of the instability in the six-dimensional case. This is an important case since $d = 6$ is the lowest dimension where the instability is present, and it was not addressed

[2]Reference [35] was able to construct the same solution in "ringlike" coordinates without using integrability methods.

[3]Some subsectors of the perturbations of some classes of MP black holes have been decoupled [46,47], but none of them shows signs of any instability, and indeed they do not contain the kind of perturbations relevant for the ultraspinning instability.

in [23] due to numerical difficulties. Moreover, we discuss the instability in $d = 10$ and $d = 11$ dimensions that are relevant for string theory. We also address the $d = 5$ case, in which the black hole does not have an ultraspinning regime (in the sense that is defined in Sec. II B). Consequently, no instability that preserves the spatial isometries of the background can be present, as we confirm numerically. At a more technical level, we will provide the first detailed discussion of the boundary conditions of the problem. While doing so we will prove that the ultraspinning modes preserve the temperature and angular velocity of the background, an aspect that was not discussed in [23]. This is a key point in formally establishing the thermodynamic criterium (which is a necessary condition, not a sufficient one) for the critical rotation above which black holes can be afflicted by the ultraspinning instability. It can be seen as a refinement of the Gubser-Mitra conjecture [50,51]. This thermodynamic criterium was first proposed in [23] and further developed in [24]. We will also prove the claim of [23] that the ultraspinning threshold modes that we find cannot be pure gauge modes.

The ultraspinning instability is not unique to singly spinning MP black holes. Indeed, [24] found that it is also present in the case of MP black holes with equal angular momenta along the $\lfloor (d-1)/2 \rfloor$ rotation planes that are allowed in d dimensions (here $\lfloor\rfloor$ stands for the smallest integer part). This occurs in spite of the fact that, for these black holes, contrary to the singly spinning case, the angular momenta have an upper bound. The key point is that the aforementioned thermodynamic criterium (to be discussed in Sec. II B) still allows for an ultraspinning regime. In the equal spins case and in odd spacetime dimensions, the geometry is codimension-1, i.e., it only depends on the radial coordinate. This simplifies considerably the analysis of the perturbed Einstein equations since they reduce to a coupled system of ordinary differential equations. For this reason, the detailed study of the time dependence of the instability is much simpler, and [24] found, as expected, that the modes responsible for the ultraspinning instability grow exponentially with time. This fact provides strong confidence for the expectation that a similar situation occurs in the singly spinning case.

The analysis of the ultraspinning instability in the equal-spins case [24] gave the first evidence for the existence of a new family of topologically spherical black hole solutions with only a *single* rotational symmetry, i.e., solutions that saturate Hawking's rigidity theorem in higher dimensions [41]. There is a previous example of a black object with a single rotational symmetry but different horizon topology [18]: the "helical" black rings that can be perturbatively constructed using the blackfold approach [17]. These developments confirm the conjecture put forward in [42] and can fairly be seen as one of the most important legacies of the ultraspinning instability studies and of the blackfold approach. More recently [53], near-horizon geometries of extremal black holes have been found with a single rotational symmetry.

The plan of the paper is as follows. In Sec. II, we will review the ultraspinning instability conjecture as formulated in [45], together with the refined thermodynamic criterium of [23,24], and we will formulate the problem of finding the modes responsible for the ultraspinning instability as an eigenvalue problem. In Sec. III, we will discuss the boundary conditions appropriate to the problem at hand, and we will prove that the threshold modes searched numerically cannot be pure gauge. In Sec. IV, we will present our results, and finally we will close with a discussion (Sec. V). The Appendix contains the technical details of the horizon embeddings.

II. THE ULTRASPINNING INSTABILITY

In Sec. II A, we will review the work of , where the conjecture of the ultraspinning instability of singly spinning MP black holes was first formulated. In Sec. II B, we will review the thermodynamic arguments that led the authors of [23,24] to extend the ultraspinning conjecture to other classes of black holes which need not admit an arbitrarily large angular momentum (per unit mass). Finally, in Sec. II C, we will formulate our perturbation problem as an eigenvalue problem.

A. Motivations

The metric of the d-dimensional asymptotically flat MP black hole spinning in a single plane is given by [27]:

$$\begin{aligned} ds^2 = &-\frac{\Delta(r)}{\Sigma(r,\theta)}(dt - a\sin^2\theta d\phi)^2 \\ &+ \frac{\sin^2\theta}{\Sigma(r,\theta)}[(r^2+a^2)d\phi - a dt]^2 + \frac{\Sigma(r,\theta)}{\Delta(r)}dr^2 \\ &+ \Sigma(r,\theta)d\theta^2 + r^2\cos^2\theta d\Omega^2_{(d-4)}, \end{aligned} \tag{2.1}$$

where $d\Omega^2_{(d-4)}$ is the line element of a unit $(d-4)$-sphere and

$$\Sigma(r,\theta) = r^2 + a^2\cos^2\theta, \quad \Delta(r) = r^2 + a^2 - \frac{r_m^{d-3}}{r^{d-5}}. \tag{2.2}$$

This solution of the Einstein vacuum equations is characterized by two parameters, namely, the mass-radius r_m and the rotation parameter a,

$$r_m^{d-3} = \frac{16\pi GM}{(d-2)\mathcal{A}_{d-2}}, \qquad a = \frac{d-2}{2}\frac{J}{M}, \tag{2.3}$$

where $\mathcal{A}_{d-2} = 2\pi^{(d-1)/2}/\Gamma[(d-1)/2]$ is the volume of a unit-radius $(d-2)$-sphere, G denotes Newton's constant, and M and J are the Arnowitt-Deser-Misner mass and angular momentum, respectively. The (outer) event horizon lies at the largest real root $r = r_+$ of $\Delta(r) = 0$, that is,

$$r_+^2 + a^2 - \frac{r_m^{d-3}}{r_+^{d-5}} = 0. \tag{2.4}$$

For $d = 4$ a regular horizon exists for all values of a up to the Kerr bound, $a = r_m/2$ ($a = GM$), which corresponds to an extremal (i.e., zero temperature) black hole with finite size horizon area. In $d = 5$ the situation is somewhat similar: a regular solution exists for all values of a strictly smaller than r_m.[4] The solution with $a = r_m$ corresponds to a naked singularity. On the other hand, for $d \geq 6$, $\Delta(r)$ always has a (single) positive real root for *all* values of a, and therefore there exist black holes (which are always nonextremal) with arbitrarily large angular momentum per unit mass. These black holes were dubbed ''ultraspinning'' in [45] and they will be the object of our study in this paper.

In the limit of large angular momentum, these black holes can be characterized by two widely separated length scales on the horizon [45]. Let $\ell_\parallel$ denote the characteristic length scale in the directions parallel to the rotation plane, and $\ell_\perp$ in the directions perpendicular to it. For $a \gg r_+$, we have $\ell_\parallel \sim a$ and $\ell_\perp \sim r_+$, that is, $\ell_\parallel \gg \ell_\perp$. Therefore, the MP black hole spreads out along the plane of rotation and it resembles a black membrane.[5] This observation has far-reaching consequences because black branes are known to be unstable [48]. This led the authors of [45] to conjecture that rapidly rotating MP black holes should be unstable under Gregory–Laflamme-type perturbations that preserve the symmetries of the $(d-4)$-sphere. More precisely, choosing a cylindrical basis in polar coordinates ($\sigma = a\sin\theta$, ϕ) on the plane of the membrane, in the $a \to \infty$ limit, the unstable modes should be of the form [45]

$$h_{\mu\nu} \sim e^{\Gamma t} J_m(\kappa\sigma) e^{im\phi} \tilde{h}_{\mu\nu}(r), \tag{2.5}$$

where κ is the wave number along the direction parallel to the rotation. In this paper we will only address the axisymmetric case ($m = 0$), for which the radial profile is given by a cylindrical wave $J_0(\kappa\sigma)$. By extrapolating these observations to finite (but sufficiently large) a, one concludes that there should exist unstable modes that depend only on r and θ and do not break any of the symmetries of the background black hole [45].

By considering the thermodynamics of singly spinning MP black holes, the authors of [45] observed that, for $d \geq 6$, these objects exhibit two markedly different behaviors depending on the value of a. For instance, consider the temperature of these black holes:

$$T = \frac{1}{4\pi}\left(\frac{2r_+^{d-3}}{r_m^{d-2}} + \frac{d-5}{r_+}\right). \tag{2.6}$$

For fixed mass, r_+ is a monotonically decreasing function of a. Therefore, it follows from (2.6) that, starting from $a = 0$, the temperature decreases as a increases, as in the Kerr black hole case. However, at

$$\left(\frac{a}{r_+}\right)_{\rm mem} = \sqrt{\frac{d-3}{d-5}} \Rightarrow \left(\frac{a}{r_m}\right)^{d-3}_{\rm mem} = \frac{d-3}{2(d-4)}\left(\frac{d-3}{d-5}\right)^{(d-5)/2}, \tag{2.7}$$

the temperature reaches a minimum and then it starts growing like $\sim r_+^{-1}$ as a increases. This is the kind of behavior for the temperature of a black membrane as the rotation increases. This point was proposed to give an order of magnitude for the appearance of the instability [45]. Notice from (2.7) that the membranelike behavior occurs for $a \gtrsim r_+$ in all dimensions (see the first column in Table I). The main result of this paper, which builds on [23], will be to show that this picture is indeed correct.

Furthermore, it is known [49] that the threshold zero-mode ($|k| = k_c$, $\Gamma = 0$) of the GL instability gives rise to a new branch of static nonuniform strings. This led the authors of [45] to conjecture that the zero-modes of the ultraspinning instability should give rise to a sequence of new branches of stationary and axisymmetric black holes that preserve all the $\mathbb{R} \times U(1) \times SO(d-3)$ isometries of the singly spinning MP black hole and that have ripples along the polar direction θ. This conjecture was further refined in [40], which argued that this structure of an infinite sequence of lumpy black holes is in fact needed in order to connect singly spinning MP black holes to black rings, black Saturns and other black objects which include multiple concentric rings that should exist in $d \geq 6$ (see Fig. 1).

B. Black hole thermodynamics and the ultraspinning instability conjecture

In this subsection, we will refine the arguments of [45] that we have reviewed above, and we will present the closely related conjecture made in [23]. The basic statement of this conjecture is that classical instabilities associated with stationary zero-modes can only appear in a regime which we call ultraspinning. We analyze this regime for the MP family of solutions. We will also argue

TABLE I. Values of the rotation a/r_m for the first four harmonics of stationary perturbation modes ($k_c = 0$). The estimated numerical error is $\pm 3 \times 10^{-3}$ in $d = 6$, 7 and $\pm 5 \times 10^{-3}$ in $d = 8$, 9, 10, 11.

d	$(a/r_m)\|_{\ell=1}$	$(a/r_m)\|_{\ell=2}$	$(a/r_m)\|_{\ell=3}$	$(a/r_m)\|_{\ell=4}$
6	1.097	1.572	1.849	2.036
7	1.075	1.714	2.141	2.487
8	1.061	1.770	2.275	2.725
9	1.051	1.792	2.337	2.807
10	1.042	1.795	2.361	2.855
11	1.035	1.798	2.373	2.879

[4]Notice that for the black ring of [28] the angular momentum is bounded from below, but otherwise it can be arbitrarily large.

[5]Reference [45] showed that by taking $a \to \infty$, $r_m \to \infty$ and $\theta \to 0$ while keeping $\hat{r}_m^{d-5} = r_m^{d-3}/a^2$ and $\sigma = a\sin\theta$ fixed, the singly spinning MP black hole metric (2.1) near the rotation axis $\theta = 0$ reduces to the metric of a black membrane. In particular, the spatial directions along the world volume of the membrane correspond to the directions along the original plane of rotation in (2.1).

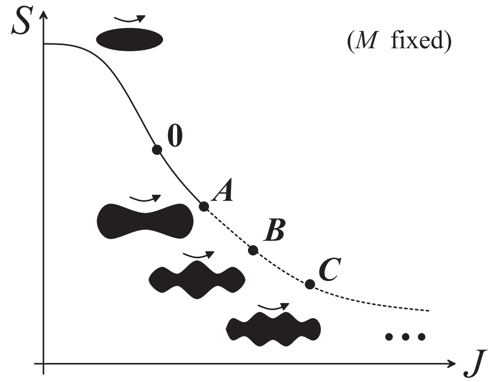

FIG. 1. Phase diagram of entropy vs angular momentum, at fixed mass, for MP black holes in $d \geq 6$ illustrating the conjecture of [45] (see also [40]): At sufficiently large spin the MP solution becomes unstable, and at the threshold of the instability a new branch of black holes with a central pinch appears (A). As the spin grows, new branches of black holes with further axisymmetric pinches ($B, C, \ldots$) appear. We determine the points where the new branches appear, but it is not yet known in which directions they run. We also indicate that, at the inflection point (0), where $\partial^2 S/\partial J^2 = 0$, there is a stationary perturbation that should correspond to neither an instability nor a new branch but rather to a zero-mode that moves the solution along the curve of MP black holes, as discussed in Sec. II B.

that a stationary zero-mode of the black hole can correspond (i) to a change in the parameters of the solution, if the zero-mode can be predicted by the condition of local thermodynamic stability, or (ii) to the threshold of a classical instability of the black hole, which in our case will correspond to the prediction of [45].

1. The ultraspinning regime

As discussed before, it was conjectured in [45] that rapidly rotating MP black holes with a single spin in $d \geq 6$ were unstable for Gregory–Laflamme-type modes. An order of magnitude estimate for the critical rotation, Eq. (2.7), was given by considering the thermodynamic behavior.

The estimate for branelike behavior is actually a zero-mode of the thermodynamic Hessian

$$-S_{\alpha\beta} \equiv -\frac{\partial^2 S(x_\gamma)}{\partial x_\alpha \partial x_\beta}, \qquad x_\alpha = (M, J_i), \tag{2.8}$$

which we readily generalized to accommodate several spins. The condition of local thermodynamic stability is the positivity of this Hessian. It was shown in [24] that $-S_{\alpha\beta}$ possesses at least one negative eigenvalue for any asymptotically flat vacuum black hole (only the Smarr relation and the first law are required). Hence all such black holes are locally thermodynamically unstable. In the MP family, this is the only negative eigenvalue for small rotations. For $d \geq 6$, as the rotation increases, the black hole acquires an additional thermodynamic instability, and a corresponding Gregory–Laflamme-type instability of the associated black branes. (This is a refinement of the Gubser-Mitra conjecture [50].)

Reference [51] (see [53,24] for the rotating case) showed that a zero-mode of the thermodynamic Hessian is also a zero-mode of the action, i.e., an on-shell stationary perturbation of the black hole. However, this type of zero-mode consists of an infinitesimal change in the asymptotic charges of the black hole solution, within the MP family. In the singly spinning case, Eq. (2.7) signals a degenerate point for which an infinitesimal change in the angular momentum at fixed mass does not change the temperature or the angular velocity of the MP black hole (indeed, our metric perturbations preserve these quantities, as we discuss later). This corresponds to the inflection point 0 in Fig. 1. Notice that we are using the MP equation of state $S(M, J_i)$ to identify the zero-mode. Hence it does not have to be related to the bifurcation to a new family of solutions, and to the instability commonly associated with a bifurcation. Only zero-modes of the black hole which are not associated with the degeneracy of the Hessian (2.8) can represent bifurcations and instabilities.

Reference [23] and the present work confirm numerically the conjecture of [45], showing that a zero-mode appears at (2.7), i.e., when the Hessian (2.8) is degenerate. More important, it is confirmed that additional zero-modes which are not thermodynamic in origin occur for higher rotations. The latter zero-modes are thresholds for *classical* instabilities of the black hole, as explicitly verified for MP codimension-1 solutions in [24].

Zero-modes which are thermodynamic in origin can be identified as the zero-modes of the simpler reduced Hessian

$$H_{ij} \equiv -\left(\frac{\partial^2 S}{\partial J_i \partial J_j}\right)_M = -S_{ij}, \tag{2.9}$$

as shown in [24]. In the MP family, the eigenvalues of H_{ij} are all positive for small enough angular momenta, at fixed M. However, as some or all of the angular momenta are increased, some eigenvalues of H_{ij} may become negative. We define the boundary of the region in which H_{ij} is positive definite to be the *ultraspinning surface*. Following [23], we shall say that a given black hole is ultraspinning if it lies outside the ultraspinning surface. Hence, as one crosses the ultraspinning surface, the black hole will develop a new thermodynamic instability, and the associated black branes will develop a new classical instability. This is in addition to the usual Gregory-Laflamme instability already present at small angular momenta.

Reference [23] conjectured that classical instabilities whose threshold is a stationary and axisymmetric zero-mode occur only for rotations higher than a thermodynamic zero-mode, i.e., in the ultraspinning regime. We emphasize that the conjecture gives a necessary condition for this instability, not a sufficient one. Notice that by "axisymmetric" we mean that the ∂_ϕ Killing isometry (or $\Omega_i \partial_{\phi_i}$ for more spins) is preserved.

The intuition leading to the conjecture is that modes of lower symmetry are usually the most unstable ones. For instance, the original Gregory-Laflamme instability occurs for the s-wave of the transversal black hole. An additional classical instability will arise after a critical value of the rotation, and it will correspond to a p-wave of the transversal black hole. As the rotation is increased, higher-order waves may become unstable. Now, if we consider a black hole instead of a black brane, the s-wave and the p-wave are associated with the asymptotic charges, mass and angular momenta. Therefore they are associated with purely thermodynamic instabilities. Higher-order waves, on the other hand, may become classically unstable as the rotation is increased, starting with the d-wave. Notice that these waves do not affect the asymptotic charges.[6]

The fact that the thresholds of classical instabilities should be associated with bifurcations to different black hole families highlights the connection between stability and uniqueness. See, e.g., [40].

2. Zero-modes of the Myers-Perry family

Let us now examine the particular form of H_{ij} in the case of general MP solutions, with $N = \lfloor (d-1)/2 \rfloor$ spins. We use the expressions in [27]. The horizon area, related to the entropy by $S = A/4$, is

$$A = \frac{\mathcal{A}_{d-2}}{r_+^{1-\epsilon}} \prod_{i=1}^{N} (r_+^2 + a_i^2), \tag{2.10}$$

where $\epsilon = 0, 1$ for odd and even d, respectively. The surface gravity κ, related to the temperature by $T = \kappa/2\pi$, and the angular velocities on the horizon Ω_i are given by

$$\kappa = r_+ \sum_{i=1}^{N} \frac{1}{r_+^2 + a_i^2} - \frac{2-\epsilon}{2r_+}, \qquad \Omega_i = \frac{a_i}{r_+^2 + a_i^2}. \tag{2.11}$$

The asymptotic charges, the mass M and the angular momenta J_i, which uniquely specify a solution, are given by

$$M = \frac{(d-2)\mathcal{A}_{d-2}}{16\pi} r_m^{d-3}, \qquad J_i = \frac{2}{d-2} a_i M. \tag{2.12}$$

The reduced thermodynamic Hessian can be explicitly derived,

$$H_{ij} = \frac{(d-2)\pi}{M\kappa}\left\{\frac{r_+^2 - a_i^2}{(r_+^2 + a_i^2)^2}\delta_{ij} + 2\frac{\Omega_i\Omega_j}{\kappa}\left[\frac{r_+}{(r_+^2 + a_i^2)^2} + \frac{r_+}{(r_+^2 + a_j^2)^2} - \frac{1}{2r_+} + \frac{\tilde{\Omega}^2}{\kappa}\right]\right\}, \tag{2.13}$$

where there is no sum over i, j, and we have defined $\tilde{\Omega}^2 \equiv \sum_i \Omega_i^2$. The matrix is positive definite in the static case, $a_i = 0\ \forall\ i$, and also in (nonextremal) $d = 4, 5$.

In the singly spinning case, say, $a \equiv a_1 \neq 0$, we have

$$H_{11} = \frac{2(d-2)(d-3)\pi r_+ (r_+^2 + a^2)}{M[(d-3)r_+^2 + (d-5)a^2]^3} \times [(d-3)r_+^2 - (d-5)a^2],$$
$$H_{ij} = \frac{(d-2)\pi}{M\kappa r_+^2}\delta_{ij} \quad \text{for } (i,j) \neq (1,1). \tag{2.14}$$

There is a single zero-mode in $d \geq 6$ which occurs precisely for (2.7). The associated eigenvector is δ_{i1}, so that the angular momenta which vanish in the background solution are not excited by the perturbation, i.e., the zero-mode keeps the black hole within the singly spinning MP family. The ultraspinning regime occurs for rotations higher than Eq. (2.7). Reference [23] and the present work find classical instabilities whose thresholds occur in that regime only.

In the equal spins case, whose analysis is also simple, we have

$$H_{ij} = \frac{(d-2)\pi}{M\kappa}\left\{\frac{r_+^2 - a^2}{(r_+^2 + a^2)^2}\delta_{ij} + 2\frac{\Omega^2}{\kappa}\left[\frac{2r_+}{(r_+^2 + a^2)^2} - \frac{1}{2r_+} + \frac{n\Omega^2}{\kappa}\right]Q_{ij}\right\}, \tag{2.15}$$

where $Q_{ij} = 1\ \forall\ i, j$. An eigenvector V_i of H_{ij} must then be an eigenvector of Q_{ij}, which leaves only two options: the eigenvector is such that $V_i = V\ \forall\ i$, or it is such that $\sum_i V_i = 0$. In the former case, there can be no zero-mode, since this would require

$$\frac{r_+^2 - a^2}{(r_+^2 + a^2)^2} + 2\frac{\Omega^2}{\kappa}\left[\frac{2r_+}{(r_+^2 + a^2)^2} - \frac{1}{2r_+} + \frac{N\Omega^2}{\kappa}\right]N = 0, \tag{2.16}$$

which can be simplified to $a^2 = -(d-3)r_+^2/(2-\epsilon)$, and thus has no solution for a. Hence no instability occurs for modes that preserve the equality between the spins, which is consistent with the results of [24,54]. However, the $N-1$ eigenvalues associated with $\sum_i V_i = 0$ do change sign once at $|a| = r_+$.[7] Therefore, an ultraspinning region exists even though the angular momenta are bounded by extremality, and indeed a classical instability was found in $d = 9$ [24].

[6]In the codimension-1 case (odd-dimensional equal spins) studied in [24], where there is a precise harmonic decomposition of perturbations in terms of scalar harmonics of CP^N, it was explicitly shown that the first zero-mode changed only the angular momenta, breaking their equality, and that the next zero-modes (associated with classical instabilities/bifurcations) did not affect the asymptotic charges.

[7]Notice that [24] used a different radial variable, related to the variable used in this paper by $\tilde{r}^2 = r^2 + a^2$, so that the ultraspinning surface is at $|a| = \tilde{r}_+/\sqrt{2}$.

C. The eigenvalue problem

According to the preceding discussion, we are interested in modes that preserve the $\mathbb{R}\times U(1)\times SO(d-3)$ symmetries of the background MP black hole (2.1), depending only on the radial and polar coordinates, r and θ. Thus we consider the following general ansatz for the perturbed metric $h_{\mu\nu}$[8]:

$$\begin{aligned} ds^2 = &-\frac{\Delta(r)}{\Sigma(r,\theta)}e^{\delta\nu_0}[dt - a\sin^2\theta e^{\delta\omega}d\phi]^2 \\ &+\frac{\sin^2\theta}{\Sigma(r,\theta)}e^{\delta\nu_1}[(r^2+a^2)d\phi - ae^{-\delta\omega}dt]^2 \\ &+\frac{\Sigma(r,\theta)}{\Delta(r)}e^{\delta\mu_0}[dr + \delta\chi\sin\theta d\theta]^2 \\ &+\Sigma(r,\theta)e^{\delta\mu_1}d\theta^2 + r^2\cos^2\theta e^{\delta\Phi}d\Omega^2_{(d-4)}, \end{aligned} \quad (2.17)$$

where $\{\delta\nu_0, \delta\nu_1, \delta\mu_0, \delta\mu_1, \delta\omega, \delta\chi, \delta\Phi\}$ are small quantities that describe our perturbations and they are functions of (r,θ) only. Unfortunately, a decoupled master equation that governs the gravitational perturbations, analogous to the Teukolsky equation [44] in the case of the Kerr black hole, is not known for (2.1). Therefore, in this paper we will solve numerically the coupled partial differential equations that govern the class of perturbations we are interested in. Choosing the traceless-transverse (TT) gauge,

$$h^\mu{}_\mu = 0 \quad \text{and} \quad \nabla^\mu h_{\mu\nu} = 0, \quad (2.18)$$

the variation of the Ricci tensor in vacuum gives the following equations of motion:

$$(\triangle_L h)_{\mu\nu} \equiv -\nabla_\rho\nabla^\rho h_{\mu\nu} - 2R_\mu{}^\rho{}_\nu{}^\sigma h_{\rho\sigma} = 0, \quad (2.19)$$

where $\triangle_L$ is the Lichnerowicz operator. Following [23] (see also [24]), we will consider a more general eigenvalue problem,

$$(\triangle_L h)_{\mu\nu} = -k_c^2 h_{\mu\nu}. \quad (2.20)$$

This problem arises when one considers the stability of a uniform rotating black string,

$$ds_{\text{string}} = g_{\mu\nu}dx^\mu dx^\nu + dz^2, \quad (2.21)$$

under perturbations of the form

$$ds_{\text{string}} \to ds_{\text{string}} + e^{ik_c z}h_{\mu\nu}(x)dx^\mu dx^\nu, \quad (2.22)$$

where $g_{\mu\nu}$ is the metric of the singly spinning MP black hole (2.1). The same problem arises when one considers the quadratic quantum corrections to the gravitational partition function in the saddle-point approximation [55] (see [22] for the application to the Kerr-AdS black hole).

Our reason for considering (2.20) instead of trying to solve (2.19) directly is that MATHEMATICA has very powerful built-in routines to solve generalized eigenvalue problems of this form. Thus, our approach will be to look for solutions of (2.20) (which will generically have $k_c \neq 0$), and vary the rotation parameter a until we find a zero-mode, i.e., a mode with $k_c = 0$. Therefore, the solutions with $k_c \neq 0$ will correspond to new kinds of GL instabilities and inhomogeneous phases of ultraspinning black strings (see also [54]). On the other hand, the $k_c = 0$ modes will correspond to asymptotically flat vacuum black holes with deformed horizons of the kind conjectured in [40,45], as shown in Fig. 1.

Finally, notice that the ansatz in (2.17) is the most general one that respects the isometries of the background MP black hole and is preserved under diffeomorphisms that depend only on (r,θ). Ultimately, this is necessary and sufficient to guarantee that (2.20) forms a closed system of equations.[9]

[8]This ansatz is equivalent to the one presented in [23], but we found that this new form was better suited for the numerics (see Sec. III E).

III. BOUNDARY CONDITIONS AND GAUGE FIXING

In the following subsections we will discuss in detail the boundary conditions that we need to impose on the metric perturbations in order to solve (2.20). In the present situation, we have to specify boundary conditions at the horizon, $r = r_+$, at asymptotic infinity, $r \to \infty$, at the rotation axis $\theta = 0$, and at the equator $\theta = \pi/2$. In the following, we shall discuss the appropriate boundary conditions at the boundaries of the integration domain. In the final part of this section we show that the threshold stationary axisymmetric zero-modes obeying these boundary conditions cannot be pure gauge.

A. Boundary conditions at the horizon

We shall demand regularity of the metric perturbations on the (future event) horizon $\mathcal{H}^+$ by imposing the regularity of the Euclideanized perturbed geometry on $\mathcal{H}^+$.[10] This approach allows us to discuss more straightforwardly the perturbations in the temperature and in the angular velocity.

For $r \approx r_+$, we can write $\Delta(r) = \Delta'(r_+)\times(r-r_+) + O[(r-r_+)^2]$ with $\Delta'(r_+) > 0$,[11] so that the near-horizon geometry of (2.1) reads

[9]For the same reasoning, if we were considering time-dependent nonaxisymmetric perturbations [that preserve the transverse $(d-4)$-sphere], we would also need to excite metric components of the type h_{tr}, $h_{t\theta}$, $h_{\phi r}$, $h_{\phi\theta}$, *etc*. We leave this interesting problem for future work.

[10]This is equivalent to transforming the metric into regular coordinates on $\mathcal{H}^+$ (e.g., in-going Eddington-Finkelstein coordinates) and requiring that metric perturbations are finite on $\mathcal{H}^+$ in the new coordinate system.

[11]Recall that for $d \geq 6$, the singly spinning MP black hole cannot be extremal and therefore $\Delta'(r_+) > 0$ holds for all values of the rotation parameter a. The $d = 5$ case is special because there is a bound on the rotation whose saturation gives a nakedly singular solution. Therefore we shall only consider nonextremal solutions with $\Delta'(r_+) > 0$.

$$\begin{aligned} ds^2 \approx &-\frac{\Sigma(r_+,\theta)\Delta'(r_+)(r-r_+)}{(r_+^2+a^2)^2}dt^2 \\ &+\frac{\Sigma(r_+,\theta)}{\Delta'(r_+)(r-r_+)}dr^2+\Sigma(r_+,\theta)d\theta^2 \\ &+\frac{(r_+^2+a^2)^2\sin^2\theta}{\Sigma(r_+,\theta)}\left(d\phi-\frac{a}{r_+^2+a^2}dt\right)^2 \\ &+r_+^2\cos^2\theta d\Omega^2_{(d-4)}. \end{aligned} \tag{3.1}$$

This suggests the introduction of a new azimuthal coordinate

$$\tilde{\phi}=\phi-\Omega_H t, \qquad \Omega_H=\frac{a}{r_+^2+a^2}, \tag{3.2}$$

with $\tilde{\phi}\sim\tilde{\phi}+2\pi$. Wick-rotating the time coordinate,

$$\begin{aligned} &t=-\mathrm{i}\tilde{\tau}, \qquad \tilde{\tau}=\frac{\tau}{2\pi T_H} \quad \text{with} \\ &T_H=\frac{\Delta'(r_+)}{4\pi(r_+^2+a^2)}, \quad \text{and} \quad \tilde{\tau}\sim\tilde{\tau}+\frac{1}{T_H}, \end{aligned} \tag{3.3}$$

and defining a new radial coordinate ρ as

$$r=r_+ +\frac{\Delta'(r_+)}{4}\rho^2, \tag{3.4}$$

the Euclidean near-horizon ($r\approx r_+$) geometry of the background solution (2.1) can be written in a manifestly regular form:

$$\begin{aligned} ds^2 \approx &\,\Sigma(r_+,\theta)[\rho^2 d\tau^2+d\rho^2]+\Sigma(r_+,\theta)d\theta^2 \\ &+\frac{(r_+^2+a^2)^2\sin^2\theta}{\Sigma(r_+,\theta)}d\tilde{\phi}^2+r_+^2\cos^2\theta d\Omega^2_{(d-4)}. \end{aligned} \tag{3.5}$$

Indeed, at the ∂_ϕ axis of rotation ($\theta=0$) we have an explicitly regular S^2 with no conical singularity. Moreover, the polar coordinate singularity at $\rho=0$ can be removed by a standard coordinate transformation into Cartesian coordinates (x, y). Note that a conical singularity at $\rho=0$ is avoided because we have chosen the period of the original Euclidean time coordinate $\tilde{\tau}$ to be $\beta=1/T_H$. To sum up, regularity at the horizon of the background solution requires that we identify $(\tilde{\tau},\phi)\sim(\tilde{\tau},\phi+2\pi)\sim(\tilde{\tau}+\beta,\phi-\mathrm{i}\Omega_H\beta)$. Furthermore, this procedure identifies Ω_H with the angular velocity of the black hole and T_H with its temperature.

The boundary conditions for the metric perturbations can now be determined, demanding that $h_{\mu\nu}dx^\mu dx^\nu$ is a regular symmetric 2-tensor when expressed in coordinates where the background metric is regular. To do this, introduce manifestly regular 1-forms,

$$E^\tau=\rho^2 d\tau=xdy-ydx, \quad E^\rho=\rho d\rho=xdx+ydy. \tag{3.6}$$

In terms of these 1-forms, the metric perturbation near the horizon reads

$$\begin{aligned} h_{\mu\nu}dx^\mu dx^\nu \approx &\,\Sigma(r_+,\theta)\left[\delta\nu_0-2a^2\sin^2\theta\left(\frac{\Sigma(r_+,\theta)\Delta'(r_+)+2r_+(r_+^2+a^2)}{\Sigma^2(r_+,\theta)\Delta'(r_+)}\right)\delta\omega\right]\rho^2 d\tau^2+\Sigma(r_+,\theta)\delta\mu_0 d\rho^2 \\ &+\frac{4\Sigma(r_+,\theta)\sin\theta}{\Delta'(r_+)}\frac{\delta\chi}{\rho^2}E^\rho d\theta-\frac{4\mathrm{i}a(r_+^2+a^2)^2\sin^2\theta}{\Sigma(r_+,\theta)\Delta'(r_+)}\frac{\delta\omega}{\rho^2}E^\tau d\tilde{\phi}+\Sigma(r_+,\theta)\delta\mu_1 d\theta^2 \\ &+\frac{(r_+^2+a^2)^2\sin^2\theta}{\Sigma(r_+,\theta)}\delta\nu_1 d\tilde{\phi}^2+r_+^2\cos^2\theta^2\delta\Phi d\Omega^2_{d-4}. \end{aligned} \tag{3.7}$$

Regularity of the perturbed geometry then requires that

$$\begin{aligned} &\delta\chi, \delta\omega=O(\rho^2), \qquad \delta\nu_0-\delta\mu_0=O(\rho^2), \\ &\partial_\rho\delta\mu_1, \partial_\rho\delta\nu_1, \partial_\rho\delta\Phi=O(\rho), \end{aligned} \tag{3.8}$$

near $\rho=0$.

It is important to note that we have imposed regularity of both, and *separately*, the background metric $\bar{g}_{\mu\nu}$ and the perturbed metric $h_{\mu\nu}$. This implies that perturbations obeying (3.8) preserve the angular velocity and temperature of the background black hole.[12]

[12]Notice that we could have chosen different boundary conditions by requiring regularity of the full metric $g_{\mu\nu}=\bar{g}_{\mu\nu}+h_{\mu\nu}$. Perturbations obeying such boundary conditions would generically change the temperature and angular velocity of the background geometry, and we shall not consider them here, but they would necessarily lead to a singular $h_{\mu\nu}$ seen as a 2-tensor on the background $g_{\mu\nu}$.

The regularity analysis of the boundary conditions is not complete without checking that the boundary conditions (3.8) are consistent both with the eigenvalue Lichnerowicz equations (2.19) and with the TT gauge conditions (2.18). We have explicitly confirmed this consistency. That is, the first term in the series expansion of the eigenvalue Lichnerowicz equations vanishes after we impose (3.8). On the other hand, we can use the TT gauge conditions to express, e.g., $\{\delta\nu_0, \delta\nu_1, \delta\Phi\}$ as functions of $\{\delta\mu_0, \delta\mu_1, \delta\omega, \delta\chi\}$ and their first derivatives. Again, the first term of a series expansion of these TT gauge conditions is consistent with (3.8).

B. Boundary conditions at the axis of rotation

We follow the same strategy as in the previous subsection, and we first discuss the regularity of the unperturbed background geometry (2.1) at the axis of rotation ($\theta=0$), where ∂_ϕ vanishes. In the region near $\theta = 0$, the background geometry (2.1) reads

$$ds^2 \approx -\frac{\Delta(r)}{r^2+a^2}dt^2 + 2a\left(1-\frac{\Delta(r)}{r^2+a^2}\right)dtd\phi + (r^2+a^2) \times (d\theta^2 + \theta^2 d\phi^2) + \frac{r^2+a^2}{\Delta(r)}dr^2 + r^2 d\Omega^2_{(d-4)}. \qquad (3.9)$$

This metric can then be cast in manifestly regular form by changing to standard Cartesian coordinates (x, y) in the $\theta - \phi$ plane.

To find the boundary conditions that the metric perturbations must satisfy at $\theta = 0$ we require that $h_{\mu\nu}dx^\mu dx^\nu$ is a regular symmetric 2-tensor when expressed in coordinates where the background metric is regular. Introducing the manifestly regular 1-forms,

$$E^\theta = \theta d\theta = xdx + ydy, \quad E^\phi = \theta^2 d\phi = xdy - ydx, \qquad (3.10)$$

the metric perturbation reads

$$h_{\mu\nu}dx^\mu dx^\nu \approx -\frac{\Delta(r)}{r^2+a^2}\delta\nu_0 dt^2 + \frac{r^2+a^2}{\Delta(r)}\delta\mu_0 dr^2 + \frac{r^2+a^2}{\Delta(r)}\delta\chi E^\theta dr + \frac{2a}{r^2+a^2}[(r^2+a^2 + \Delta(r))\delta\omega - (r^2+a^2)\delta\nu_1 + \Delta(r)\delta\nu_0]E^\phi dt + (r^2+a^2)[\delta\mu_1 d\theta^2 + \delta\nu_1\theta^2 d\phi^2] + r^2\delta\Phi d\Omega^2_{(d-4)}. \qquad (3.11)$$

Regularity of the perturbed geometry then implies

$$\delta\nu_1 - \delta\mu_1 = O(\theta^2), \quad \partial_\theta\delta\chi, \partial_\theta\delta\omega, \partial_\theta\delta\mu_0, \partial_\theta\delta\nu_0, \partial_\theta\delta\Phi = O(\theta). \qquad (3.12)$$

Again we have explicitly checked that the boundary conditions (3.12) are consistent both with the eigenvalue Lichnerowicz equations (2.19) and with the TT gauge conditions (2.18).

C. Boundary conditions at the equator

Introducing a new polar coordinate $x = \cos\theta$, we find that the metric (3.13) near $x = 0$ $(\theta = \pi/2)$ is given by

$$ds^2 \approx -\frac{\Delta(r) - a^2}{r^2}dt^2 - \frac{2a}{r^2}[r^2 + a^2 - \Delta(r)]dtd\phi + \frac{(r^2+a^2)^2 - a^2\Delta(r)}{r^2}d\phi^2 + \frac{r^2}{\Delta(r)}dr^2 + r^2(dx^2 + x^2 d\Omega^2_{(d-4)}). \qquad (3.13)$$

Once more, this geometry can be put in a manifestly regular form by changing to Cartesian coordinates.

As in the previous subsections, we now demand that the metric perturbation $h_{\mu\nu}dx^\mu dx^\nu$ is a regular symmetric 2-tensor when expressed in coordinates where the background metric is regular at $x = 0$. Introducing the manifestly regular 1-forms,

$$E^x = xdx, \qquad E^\Omega = x^2 d\Omega_{(d-4)}, \qquad (3.14)$$

the metric perturbation near $x = 0$ reads

$$h_{\mu\nu}dx^\mu dx^\nu \approx -\frac{[\Delta(r) - a^2]\delta\nu_0 + 2a^2\delta\omega}{r^2}dt^2 + \frac{(r^2+a^2)^2\delta\nu_1 - a^2\Delta(r)(\delta\nu_0 + 2\delta\omega)}{r^2}d\phi^2 + \frac{2a}{r^2}[(r^2+a^2+\Delta(r))\delta\omega - (r^2+a^2)\delta\nu_1 + \Delta(r)\delta\nu_0]dtd\phi + \frac{r^2}{\Delta(r)}\delta\mu_0 dr^2 - \frac{r^2}{\Delta(r)}\left(\frac{\delta\chi}{x} - \partial_x\delta\chi\right)E^x dr + r^2(\delta\mu_1 dx^2 + x^2\delta\Phi d\Omega^2_{(d-4)}). \qquad (3.15)$$

Regularity of the perturbed geometry for $x \to 0$ then requires that

$$\delta\chi = O(x), \qquad \delta\Phi - \delta\mu_1 = O(x^2), \quad \partial_x\delta\omega, \partial_x\delta\mu_0, \partial_x\delta\nu_0, \partial_x\delta\nu_1 = O(x). \qquad (3.16)$$

Again we have explicitly checked that the boundary conditions (3.16) are consistent both with the eigenvalue Lichnerowicz equations (2.19) and with the TT gauge conditions (2.18).

D. Boundary conditions at the asymptotic infinity

At spatial infinity, $r \to \infty$, we will require that the perturbations preserve the asymptotic flatness of the spacetime. This means that they must decay strictly faster than the background asymptotic solution which, near spatial infinity, reduces to Minkowski spacetime.[13]

In the asymptotic region, the eigenvalue Lichnerowicz equations (2.19) reduce simply to $\Box h_{\mu\nu} \simeq -k_c^2 h_{\mu\nu}$. The regular solutions (there are also irregular solutions that grow exponentially) of these equations decay as

$$h_{\mu\nu}|_{r\to\infty} \sim \frac{1}{r^\alpha}e^{-k_c r} \to 0, \qquad (3.19)$$

[13]More precisely, consider a spacetime $(\mathcal{M}, g)$ which contains a spacelike hypersurface $M_{\rm ext}$ diffeomorphic to $\mathbb{R}^n \setminus B(R)$, where $B(R)$ is a coordinate ball of radius R. The spacetime is said to be asymptotically flat if the induced metric h on $M_{\rm ext}$ and the extrinsic curvature K satisfy

$$h_{ij} - \delta_{ij} = O_k(r^{-\alpha}), \qquad K_{ij} = O_{k-1}(r^{-1-\alpha}), \qquad (3.17)$$

where r is the radial coordinate in $\mathbb{R}^n$, and we write $f = O_k(r^\beta)$ if f satisfies

$$\partial_{k_1}\ldots\partial_{k_\ell}f = O(r^{\beta-\ell}), \qquad 0 \le \ell \le k. \qquad (3.18)$$

for some constant $\alpha \geq 0$ that depends on the particular metric component and the number of spacetime dimensions d. Therefore, for $k_c \neq 0$, our perturbations decay exponentially to zero at the spatial infinity and hence asymptotic flatteness is guaranteed.

Ultimately one would be interested in studying modes with $k_c = 0$, since these correspond to exact perturbative solutions to Einstein *vacuum* equations in d spacetime dimensions. In this case, one would have to worry about the fall off of the metric perturbations. However, in our numerical method we will never be able to find modes for which $k_c = 0$ *exactly*, and therefore the fall-off conditions at infinity for these modes are not an issue for us.

E. Imposing the TT gauge conditions and the boundary conditions

We have to solve the Lichnerowicz eigenvalue problem (2.20) for the seven metric perturbations described in (2.17), namely, $\{\delta\mu_0, \delta\mu_1, \delta\chi, \delta\omega, \delta\nu_0, \delta\nu_1, \delta\Phi\}$, subject to the TT gauge conditions (2.18). The gauge conditions allow us to eliminate three functions in terms of the other four and their first derivatives. In this paper we choose to express $\{\delta\nu_0, \delta\nu_1, \delta\Phi\}$ as functions of $\{\delta\mu_0, \delta\mu_1, \delta\chi, \delta\omega\}$ and their first derivatives. Notice that this choice differs from that in [23], where the independent variables were chosen to be $\{\delta\mu_0, \delta\mu_1, \delta\chi, \delta\Phi\}$. The reason is that the present choice allowed us to obtain good numerical results in $d = 5, 6$. As discussed in Sec. III D, for $k_c \neq 0$, our (regular) perturbations vanish exponentially for $r \to \infty$. However, in addition to the exponential decay, one also has a power law behavior as in (3.19). This apparently irrelevant extra power law decay seems to make all the difference for the stability and/or accuracy of the numerical code in $d = 5, 6$. For $d = 7, 8, 9$, we obtain exactly the same results as in [23].

To summarize, we solve the gauge conditions (2.18) for $\{\delta\nu_0, \delta\nu_1, \delta\Phi\}$ in terms of $\{\delta\mu_0, \delta\mu_1, \delta\chi, \delta\omega\}$ and their first derivatives. Making these substitutions in the full set of the perturbation equations (2.20), we find that only four equations remain of second order in $\{\delta\mu_0, \delta\mu_1, \delta\chi, \delta\omega\}$. Explicitly, these equations are

$$
\begin{aligned}
&(\triangle_L h)_{rr} = -k_c^2 h_{rr}, \qquad (\triangle_L h)_{r\theta} = -k_c^2 h_{r\theta},\\
&\quad(\triangle_L h)_{\theta\theta} = -k_c^2 h_{\theta\theta},\\
&\quad a(\triangle_L h)_{tt} + \frac{r^2 + a^2 + a^2\sin^2\theta}{(r^2 + a^2)\sin^2\theta}(\triangle_L h)_{t\phi}\\
&\quad + \frac{a}{(r^2 + a^2)\sin^2\theta}(\triangle_L h)_{\phi\phi}\\
&\quad = -k_c^2\bigg[a h_{tt} + \frac{r^2 + a^2 + a^2\sin^2\theta}{(r^2 + a^2)\sin^2\theta} h_{t\phi}\\
&\quad + \frac{a}{(r^2 + a^2)\sin^2\theta} h_{\phi\phi}\bigg],
\end{aligned}
\tag{3.20}
$$

and they constitute our final set of equations to be solved.

A nontrivial consistency check of our procedure is to verify that the final equations (3.20) imply the remaining equations, which are of third order in the independent variables. We have verified that this is indeed the case.

We will solve numerically the final eigenvalue problem (3.20) using spectral methods (see, e.g., [56]). In order to do so, we find it convenient to introduce new radial and polar coordinates,

$$
y = 1 - \frac{r_+}{r}, \qquad x = \cos\theta, \tag{3.21}
$$

so that $0 \leq y \leq 1$ and $0 \leq x \leq 1$. The implementation of the method requires less computational power if all functions obey Dirichlet boundary conditions on all boundaries. Therefore, we redefine our independent functions according to

$$
\begin{aligned}
q_1(y, x) &= \left(1 - \frac{r_+}{r}\right)x(1 - x)\delta\mu_0(y, x),\\
q_2(y, x) &= r_m^{-1}(1 - x)\delta\chi(y, x),\\
q_3(y, x) &= \left(1 - \frac{r_+}{r}\right)x(1 - x)\delta\mu_1(y, x),\\
q_4(y, x) &= x\delta\omega(y, x),
\end{aligned}
\tag{3.22}
$$

so that the q_i's vanish at all boundaries. This guarantees that the boundary conditions discussed in the previous subsections, equations (3.8), (3.19), (3.16), and (3.12), are correctly imposed.

F. Zero-modes are not pure gauge

In this subsection, we will argue that the TT gauge conditions (2.18) plus our boundary conditions (3.8), (3.19), (3.16), and (3.12) ensure that our modes are physical and not a gauge artifact.

For $k_c > 0$, the TT conditions completely fix the gauge since the action of the Lichnerowicz operator on a pure gauge mode is trivial, $\Delta_L \nabla_{(\mu}\xi_{\nu)} = 0$. However, TT perturbations with $k_c = 0$ can be pure gauge [48], and therefore it could be that the stationary perturbation $k_c \to 0$ marking the onset of a new ultraspinning instability is not physical. In the rest of this subsection, we will show that there is no regular pure gauge perturbation that obeys our boundary conditions, (3.8), (3.19), (3.16), and (3.12).

Under a gauge transformation with gauge parameter ξ^μ, the metric perturbation transforms as

$$
h_{\mu\nu} \to h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}. \tag{3.23}
$$

The most general gauge parameter that preserves our ansatz (2.17) is of the form

$$
\xi = \xi_r(r, x)dr + \xi_x(r, x)dx, \tag{3.24}
$$

where $x = \cos\theta$. We will now prove that such a gauge parameter ξ^μ cannot generate a pure gauge metric perturbation that is regular on all boundaries.

A TT gauge perturbation generated by ξ^μ must satisfy $\nabla_\mu \xi^\mu = 0$ and $\Box \xi^\nu = 0$. If we introduce the antisymmetric tensor $F_{\mu\nu} = \nabla_{[\mu}\xi_{\nu]}$ and we consider Ricci flat backgrounds, these conditions reduce to

$$\partial_\mu(\sqrt{-g}\xi^\mu) = 0, \qquad \partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0. \tag{3.25}$$

Assuming that the solutions of these equations are separable,

$$\xi_r(r,x) = R_r(r)X_r(x), \qquad \xi_x(r,x) = R_x(r)X_x(x), \tag{3.26}$$

the first equation in (3.25) reduces to

$$\begin{aligned}&\partial_r(r^{d-4}\Delta R_r) = \lambda r^{d-4}R_x,\\ &\partial_x(x^{d-4}(1-x^2)X_x) = -\lambda x^{d-4}X_r,\end{aligned} \tag{3.27}$$

where λ is a separation constant. Similarly, a combination of the r and x components of the second equation in (3.25) yields

$$\begin{aligned}&\partial_r(r^{d-4}\Delta \partial_r R_x) = \kappa r^{d-4}R_x,\\ &\partial_x(x^{d-4}(1-x^2)\partial_x X_r) = -\kappa x^{d-4}X_r,\end{aligned} \tag{3.28}$$

where κ is a second separation constant. Combining the equations (3.27) and (3.28), we further find that

$$R_r = \frac{\lambda}{\kappa}R_x', \qquad X_x = \frac{\lambda}{\kappa}X_r'. \tag{3.29}$$

We will take Eqs. (3.27) and (3.29) as our independent equations.

The second equation in (3.28) has the solution

$$\begin{aligned}X_r(x) = {}& C_1 F\left(\frac{d-3-K}{4}, \frac{d-3-K}{4}, \frac{d-3}{2}, x^2\right)\\ &+ C_2 x^{5-d} F\left(\frac{7-d-K}{4}, \frac{7-d-K}{4}, \frac{7-d}{2}, x^2\right),\end{aligned} \tag{3.30}$$

where $C_{1,2}$ are integration constants, and we have defined $K \equiv \sqrt{(d-3)^2 + 4\kappa}$. For $d \geq 6$ (where we find the unstable modes), this solution diverges as $C_2 x^{5-d}$ at $x = 0$ unless we set $C_2 = 0$. In addition, a $\log(1-x)$ divergence at $x = 1$ can only be avoided if κ satisfies the following quantization condition:

$$\kappa = 2\ell(d-3+2\ell), \quad \text{for } \ell = 0, 1, 2, \ldots, \tag{3.31}$$

which follows from the property $\Gamma(-\ell) = \infty$ for nonnegative integer ℓ.

At this point we just have to solve the first equation in (3.28) with κ given by (3.31). For our purposes it is sufficient to obtain the behavior of the solution near the horizon and near infinity. Near the horizon, $r \approx r_+$, the most general solution behaves like

$$\begin{aligned}R_x(r) &\approx a_0 + a_1 \log\left(1 - \frac{r_+}{r}\right) \Rightarrow R_r(r)\\ &\approx \frac{\lambda a_1}{2\ell(d-3+2\ell)}\frac{1}{r-r_+},\end{aligned} \tag{3.32}$$

where we used (3.29). However, (3.23) implies that such a diffeomorphism would generate metric perturbations of the form

$$h_{rr} \approx \frac{\lambda a_1}{2\ell(d-3+2\ell)}\frac{1}{(r-r_+)^2}, \quad h_{rx} \approx \frac{\lambda a_1 P(x)}{2(r-r_+)}, \tag{3.33}$$

which diverge at the horizon. Therefore, our regularity requirements at the horizon force us to set $a_1 = 0$. Similarly, near infinity a Frobenius analysis gives the asymptotic solution

$$R_x(r) \approx b_0 r^{-(d-3+2\ell)} + b_1 r^{2\ell}. \tag{3.34}$$

The term in this equation proportional to b_1 generates, through (3.23), a metric perturbation of the form $h_{rr} \propto b_1 r^{2(\ell-1)}$ near infinity. This perturbation must decay strictly faster than the unperturbed background solution (which at infinity is Minkowski), which implies that we have to impose $b_1 = 0$ for any $\ell \geq 0$.

Summarizing, the gauge parameter (3.24) that could potentially generate a metric perturbation that is regular both at the horizon and at infinity must have the asymptotic behavior

$$R_x(r)|_{r\to r_+} \approx a_0, \qquad R_x(r)|_{r\to\infty} \approx b_0 r^{-(d-3+2\ell)}. \tag{3.35}$$

We can now complete our proof. Notice that

$$\begin{aligned}0 &\leq \int_{r_+}^{\infty} dr r^{d-4}\Delta(r)[\partial_r R_x(r)]^2\\ &= r^{d-4}\Delta(r)R_x(r)\partial_r R_x(r)|_{r_+}^{\infty}\\ &\quad - \int_{r_+}^{\infty} dr \partial_r[r^{d-4}\Delta(r)\partial_r R_x(r)]R_x(r)\\ &= -2\ell(d-3+2\ell)\int_{r_+}^{\infty} r^{d-4}R_x(r)^2 \leq 0,\end{aligned} \tag{3.36}$$

where we used (3.35) and (3.28). But these relations can be satisfied only for $R_x(r) = 0$, which in turn implies that $R_r(r) = 0$ and hence $\xi^\mu = 0$.

Therefore, we have proved that there is no regular gauge parameter (3.24) that could potentially generate the metric perturbations that we consider, Eq. (2.17). Thus, we conclude that our regular zero-mode perturbations cannot be pure gauge.

IV. RESULTS

In this section, we present our results for the spectrum of negative modes of the Lichnerowicz operator. The actual spectrum in $d = 5, 6, 7$ is displayed in Fig. 2; for the other values of d up to $d = 11$, the results are qualitatively similar to $d = 6, 7$ and we expect the same to be true for all values of d with $d > 11$. Following [23], we plot the

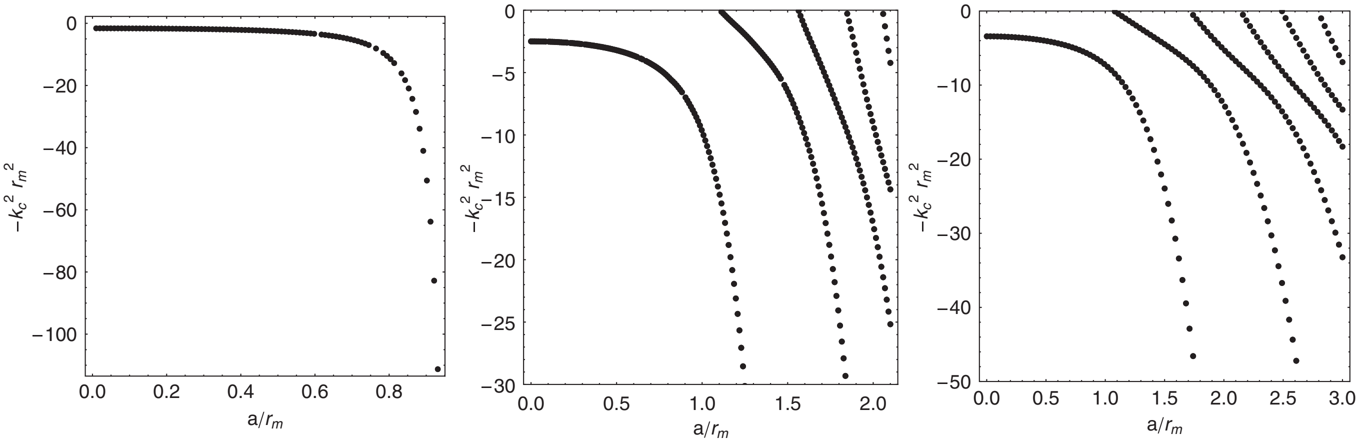

FIG. 2. Negative modes of the singly spinning MP black hole in $d = 5$ *(left)*, $d = 6$ *(center)*, and $d = 7$ *(right)*.

dimensionless negative eigenvalue $-k_c^2 r_m^2$ as a function of the (dimensionless) rotation parameter a/r_m. In Figs. 3 and 4 we show the actual metric perturbations $\{\delta\mu_0, \delta\mu_1, \delta\chi, \delta\omega\}$ for $d = 7$, and in Fig. 6 we display the embeddings of the unperturbed and the perturbed horizons for the same number of spacetime dimensions.

A. Results for $d = 5$

Reference [24] proved that MP black holes (not necessarily singly spinning) are always locally thermodynamically unstable. Therefore, it follows from the arguments in Sec. II B that the spectrum of the Lichnerowicz operator should admit at least one negative eigenvalue. Since in $d = 5$ the Hessian (2.14) is positive definite, the expectation is that the Lichnerowicz operator should admit one and only one negative mode. Our results for the spectrum of the Lichnerowicz operator are depicted in Fig. 2 (left), and indeed they confirm this expectation. As the rotation is increased, k_c increases, in agreement with the results of [22,24,54]. If we interpret this result from the black string perspective, then more modes $|k| < k_c$ are GL-unstable, which is expected since the centrifugal force should make the string "more unstable." The negative mode diverges as $k_c \propto 1/(r_m - a)$, in the singular limit $a \to r_m$.

B. Results for $6 \leq d \leq 11$

The situation in $d \geq 6$ is more interesting. The spectrum of the Lichnerowicz operator in $d = 6$, 7 is displayed in Fig. 2 (center) and (right). We label the different branches that intersect $k_c = 0$ at finite a/r_m by successive integers $\ell = 1, 2, 3, \ldots$, and we refer to the corresponding modes as harmonics, although the equations we are solving do not seem to separate. The values of a/r_m at which the first few branches intersect $k_c = 0$ are summarized in Table I. We note that for each branch the corresponding integer ℓ coincides with the number of zeros that the metric perturbations $\delta\mu_0(x, y)$ and $\delta\mu_1(x, y)$ have on the horizon $y = 0$ (see Fig. 3).

First, we notice that the Lichnerowicz operator has a negative eigenvalue for all values of a/r_m. This is in agreement with the thermodynamic argument in Sec. II B. However, for $d \geq 6$, singly spinning MP black holes admit an ultraspinning regime. The ultraspinning surface, which determines the boundary of this region (in parameter space), is given by the onset of the membrane-like behavior (see (2.7)):

$$\left(\frac{a}{r_m}\right)^{d-3}_{\text{mem}} = \frac{d-3}{2(d-4)}\left(\frac{d-3}{d-5}\right)^{(d-5)/2}. \tag{4.1}$$

According to the thermodynamic argument in Sec. II B, precisely at this value of a/r_m the Lichnerowicz operator should develop a new negative eigenvalue, and we find that this is indeed the case. In the first column of Table I, we display our numerical results for the critical values of a/r_m for the $\ell = 1$ mode in various dimensions. We find that the numerical results agree very well with the values of a/r_m computed from (4.1). We emphasize that the $\ell = 1$ mode should correspond to a variation of the parameters within the MP family of solutions such that it preserves the temperature and angular velocity of the background. (This mode is in correspondence with point 0 in Fig. 1.) Therefore this mode should not correspond to an instability of the black hole. However, it should give rise to a new type of classical instability of the associated black string, in which the horizon is deformed along both the direction of the string and the polar direction of the sphere.

For $a/r_m > (a/r_m)_{\text{mem}}$, i.e., in the ultraspinning regime, there appears an infinite sequence of new negative modes. The values of a/r_m at which the $\ell = 2$, 3, 4 branches intersect $k_c = 0$ are displayed in Table I. These modes do not admit a thermodynamic interpretation and therefore they should correspond to new perturbative black holes with deformed horizons (see points A, B, and C in Fig. 1). In particular, the $\ell = 2$ mode should signal the onset of the ultraspinning instability conjectured in [45]. For $d = 7$ the actual metric perturbations $\delta\mu_0$, $\delta\mu_1$, $\delta\chi$, $\delta\Phi$ for the $\ell = 1, 2$ modes are displayed in Figs. 3 and 4. Figures 3(a)–3(d)

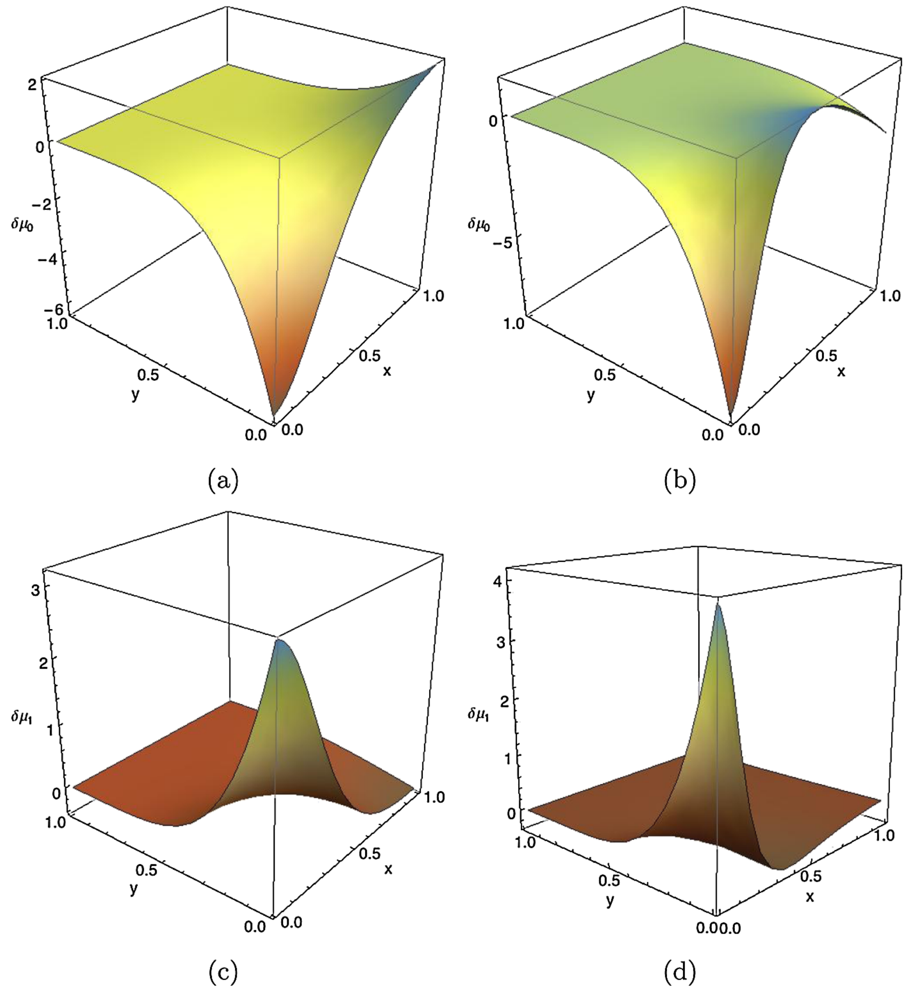

FIG. 3 (color online). Functions $\delta\mu_0(x, y)$ and $\delta\mu_1(x, y)$ for the $\ell = 1$ mode [Figs. 3(a) and 3(c)] and $\ell = 2$ mode [Figs. 3(b) and 3(d)], respectively. The number of zeros at $y = 0$ (the horizon) coincides with the integer ℓ.

show that, for each of the $\ell = 1, 2$ modes, the number of zeros that $\delta\mu_0(x, y)$ and $\delta\mu_1(x, y)$ have on the horizon ($y = 0$) coincides with the integer ℓ. We have checked that the same is also true for the higher harmonics with $\ell > 2$.

According to the discussion in Sec. II A, an important prediction of [45] is that, in the $a \to \infty$ limit *and* in a region close to the axis $\theta \approx 0$, the threshold axisymmetric modes at the horizon should be well approximated by a Bessel function $J_0(\kappa\sigma)$. In order to check if, for large a and close to the horizon and the axis, our perturbations reduce to a Bessel function (at least qualitatively), we have fitted our numerical results in $d = 7$ for $\Delta(r)h_{rr}$ with $a/r_m = 4.38$ and $k_c^2 r_m^2 = 0.114$,[14] with a Bessel function:

$$\alpha J_0(\kappa\sigma), \quad \text{with} \quad \alpha = -1.868, \qquad \kappa = 5.564, \tag{4.2}$$

and $\sigma = a \sin\theta$. The results are depicted in Fig. 5, and they show that the agreement is quite remarkable. We should emphasize though that the argument of [45] only applies in the strict limit mentioned above. For our data, a/r_m is relatively large (compared to the onset of the ultraspinning instability) but nevertheless finite, and therefore only a qualitative agreement with the prediction of [45] should be expected. This is what Fig. 5 shows.

To confirm our interpretation and visualize the effect of these perturbations on the horizon of the background MP black hole, in Fig. 6 we compare the embedding diagrams of the unperturbed and the perturbed horizons in $d = 7$. (See the Appendix for the technical details of the embedding diagram construction.) For other values of d the picture is qualitatively similar. From the embeddings we conclude that the effect of the $\ell = 2$ modes is to create a pinch centered on the axis of rotation; the $\ell = 3$ modes create a pinch at a finite latitude and the $\ell = 4$ modes create two pinches, one centered on the rotation axis and the other at a finite latitude. These are precisely the kind of deformations depicted in Fig. 1.

Finally we notice that the value of $(a/r_m)_{\rm crit}$ for every $\ell > 1$ increases with the number of spacetime dimensions d. It would be interesting to explore if the perturbative

[14]For these results, $y = 0.0034$.

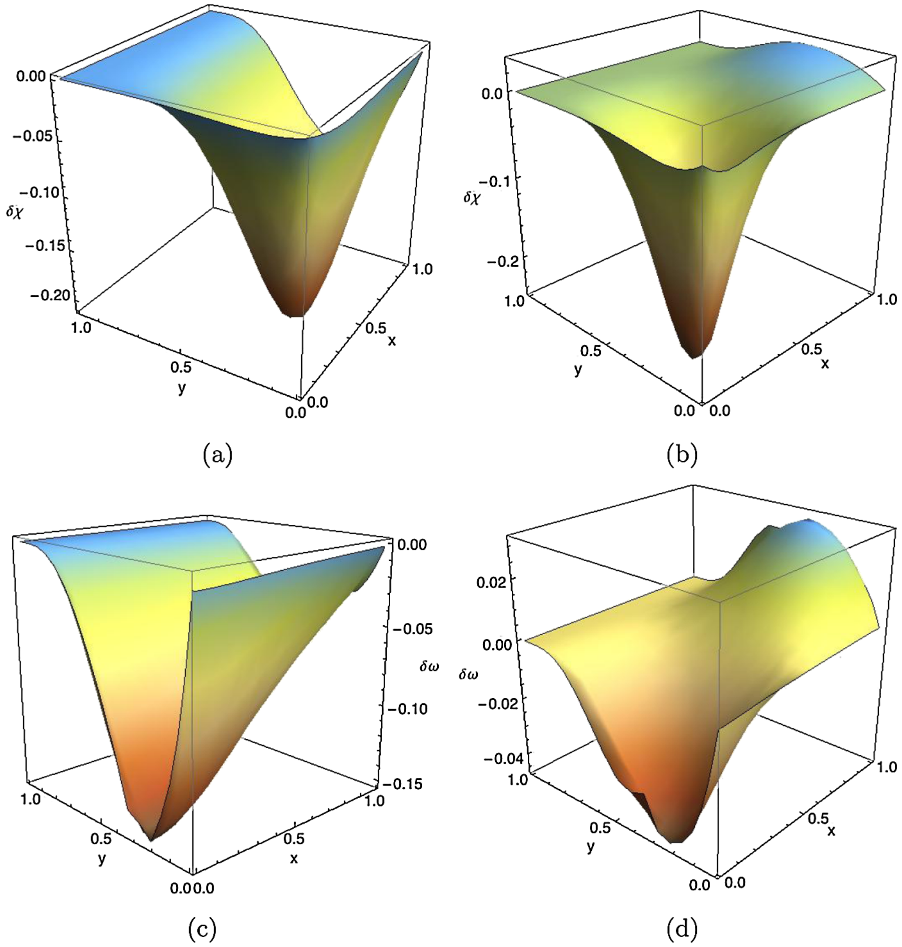

FIG. 4 (color online). Functions $\delta\chi(x, y)$ and $\delta\omega(x, y)$ for the $\ell = 1$ mode [Figs. 4(a) and 4(c)] and $\ell = 2$ mode [Figs. 4(b) and 4(d)], respectively.

approach of [17] can capture the dynamics of these instabilities in the $a \longrightarrow \infty$ limit.

V. DISCUSSION

In this paper we have studied in detail the properties of the onset of the ultraspinning instability in singly spinning MP black holes in $d \geq 6$. This instability is captured by a class of perturbations that preserve the $\mathbb{R} \times U(1) \times SO(d-3)$ symmetries of the background, as well as the angular velocity and temperature of the original MP black hole. These perturbations (in the TT gauge) must satisfy the Lichnerowicz system of equations (2.19). Our strategy however was to solve the more general *eingenvalue* problem (2.20) because: i) there are powerful numerical routines to solve generalized eigenvalue problems of this kind; and ii) this strategy also allows us to study Gregory–Laflamme-like instabilities of the associated black string. In short, varying the rotation parameter a of the background black hole (2.1), we searched for the negative modes of the problem (2.20), i.e., for solutions that have generically $k_c \neq 0$. We found several families of negative modes that exhibit an underlying harmonic structure, although the equations that we solve do not seem to separate. We used this to suggestively label the several branches by an integer ℓ. This integer coincides with the number of zeros of the metric perturbations on the horizon. The family $\ell = 0$ always has $k_c \neq 0$ and coincides with the well-known negative mode of the Schwarzchild-Tangherlini solution when the rotation parameter a vanishes [55,57]. This family therefore describes the critical mode $|k| = k_c$ of the original Gregory-Laflamme instability of the black string [48] when $a = 0$, and our results show how it evolves as the rotation increases: the value of the threshold wave number k_c increases with the rotation.

The branches with $\ell \geq 1$ are more interesting since they intersect the $k_c = 0$ axis at a critical $a > 0$ (see Table I). Thus they not only describe *new* types of Gregory-Laflamme instabilities of rotating black strings, but they also represent, for $\ell \geq 2$, true instabilities of the MP *black hole*. For black strings, the onset of these instabilities is conjectured to signal a bifurcation to new branches of nonuniform black strings in which the horizon is deformed along both the direction of the string *and* the polar

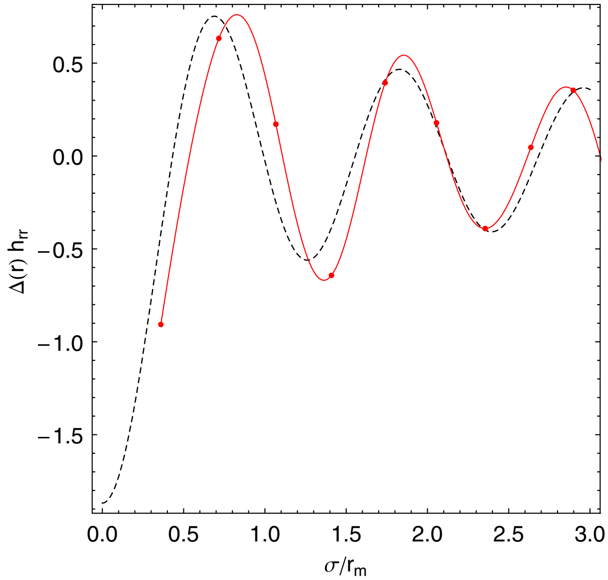

FIG. 5 (color online). Comparison between our $\Delta(r)h_{rr}$ for $a/r_m = 4.38$ and $k_c^2 r_m^2 = 0.114$ in $d = 7$ (dots) and the fitting function $\alpha J_0(\kappa\sigma)$ (dashed line). The solid (red) line corresponds to the interpolating polynomial to our data and should only serve to guide the eye. To obtain the plots, the minimum value of x was taken to be $x_{\min} = 0.69804$, since the comparison should be relevant near the rotation axis at $x = 1$.

direction of the transverse sphere. At a given rotation, the $\ell = 1$ mode has the shortest wavelength k_c^{-1}, and hence it should dominate the instability of the black string.

From now on, we focus our attention only on the consequences of our findings for the stability of the MP black holes. The critical rotation where the $\ell = 1$ mode appears can be predicted by the thermodynamic argument of [23,24] (see Sec. II B), which can be seen as a refinement of the Gubser-Mitra conjecture. Accordingly, this mode does *not* correspond to a true instability of the MP black hole. Instead, we have interpreted it as a thermodynamic mode that corresponds to a variation of the parameters within the MP family of solutions such that it preserves the temperature and angular velocity of the background (Sec. IV, and point 0 in Fig. 1).

The modes with $\ell \geq 2$ describe the onset of true ultraspinning instabilities of the MP black hole. In Sec. IV, we have studied the deformations that these modes produce on the horizon of the black hole for $d = 7$ (just for concreteness) and found that they give rise to the kind of deformations predicted in [40]. In addition, we have shown that, for large a and near $\theta = 0$, our numerical results are well approximated by a Bessel function, which is in agreement with the heuristic arguments of [45]. Altogether, these results provide solid evidence in favor of the ultraspinning instability conjectured in [40,45]. The thresholds of the ultraspinning instabilities are expected to signal bifurcations to new branches of axisymmetric solutions with pinched horizons in a phase diagram of stationary solutions (see points $A, B, C, \ldots$ in Fig. 1). Although these pinched black holes have the same isometries as the original MP black hole, their spherical-topology horizons are distorted by ripples along the polar direction. When continued in the full nonlinear regime, these new branches of solutions are conjectured to connect to the black ring and black Saturn families [40]. Although we have identified the critical values of the rotation where these new branches of pinched black holes appear, unfortunately with our methods we cannot determine if they will have larger or lower entropy in the phase diagram S vs J at fixed total mass. This would require going beyond linear order in perturbation theory.

We have limited our study to *stationary* perturbations, and therefore we cannot claim to have found an unstable mode, i.e., a linear perturbation (satisfying our boundary conditions) that grows exponentially with time; we have found only the stationary zero-modes that signal the onset of the instability. Including time dependence is not conceptually difficult; it is only technically harder. As pointed out in footnote 9, consistency of (2.20) requires turning on extra components of the perturbed metric. This problem is however of fundamental interest since its solution would provide a definite proof of the ultraspinning instability together with information on its time scale. The analogous stability problem, including time dependence, was studied in [24] in the context of odd-dimensional MP black holes with equal angular momenta in all rotation planes. There, the analogue of the $\ell = 1$ zero-mode is also present, and the time-dependent analysis confirmed the absence of a black hole instability in this sector. We take this result as good evidence in support of a similar interpretation in the singly spinning case.

The $d = 5$ case is special. As [45] already pointed out, the $d = 5$ singly spinning MP black hole does not have an ultraspinning regime (in the sense of Sec. II B) and therefore *a priori* there is no argument that suggests the existence of an instability within the class of perturbations that preserve the isometries of the background. In this paper we have confirmed this picture since in $d = 5$ we found only one negative mode (the $\ell = 0$ branch) which has $k_c \neq 0$ for all values of a such that $a < r_m$. The $d = 4$ case has already been discussed in [22].

An interesting open question to be addressed is whether the ultraspinning instability is also present in MP black holes in AdS. This background might introduce new features and physics that deserve a study. In particular, AdS might exhibit new black hole phases with an interesting interpretation in the holographic dual field theory. In fact, pinched plasma balls [58], new kinds of deformed plasma tubes [59] and rotating plasma-ball instabilities [60] have been found in the context of Scherk-Schwarz-AdS. The spectral methods that we have employed can also be applied in AdS spacetimes (see [22] for an application in $d = 4$).

The existence of an ultraspinning regime is not unique to MP black holes. For instance, the five-dimensional black

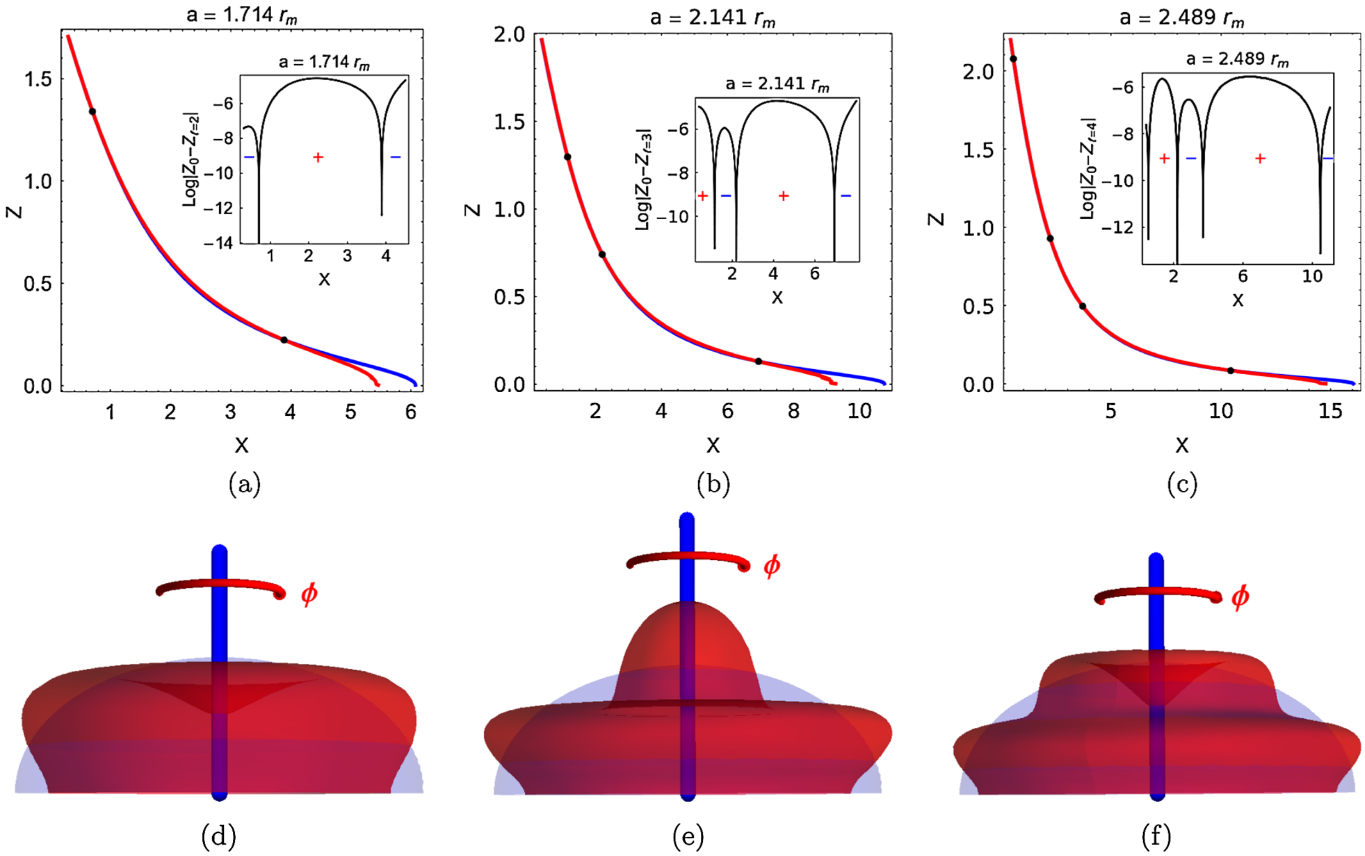

FIG. 6 (color online). Embedding diagrams [Figs. 6(a)–6(c)] at $(a/r_m)_{\rm crit}$ of the $d = 7$ black hole horizon, unperturbed (light blue), and perturbed (dark red) with the $\ell = 2, 3, 4$ harmonics (in this order). The embedding Cartesian coordinates Z and X lie along the rotation axis $\theta = 0$ and the rotation plane $\theta = \pi/2$, respectively. We also show the logarithmic difference between the embeddings of the perturbed ($Z_{\ell=2,3,4}$) and unperturbed (Z_0) horizons. The number of spikes corresponds to the number of crossings between the two embeddings. Each + sign indicates that the perturbation bulges out relative to the background, and the − signs indicate the opposite situation. Figures 6(d)–6(f) provide a three-dimensional illustration of the spatial sections of the horizon, with the $(d - 4)$-sphere suppressed at every point, with the same color code.

rings also admit a certain ultraspinning regime. Namely, in the limit of arbitrarily large angular momentum, the black ring resembles a boosted black string [61], which is known to suffer from GL instabilities [62]. Moreover, the arguments in [28,62,63] suggest that black rings should suffer from this GL instability for relatively low values of the angular momentum. Although these GL modes would break the symmetries of the background, it is interesting to ask whether there are zero-modes which can be captured by our methods. In fact, black rings are expected to suffer from other types of instabilities. For instance, fat rings are expected to be unstable under changes of their radius [63], and doubly spinning black rings are conjectured to suffer from similar instabilities as well as from superradiant instabilities associated with the rotation along S^2 [64].

Rotating asymptotically flat black holes can also suffer from instabilities which break the axisymmetry. Recently, the authors of [19,20] performed a remarkable fully nonlinear numerical evolution of a perturbed singly spinning MP solution in $d \geq 5$ and found that these black holes are dynamically unstable against *nonaxisymmetric* bar-mode perturbations. The endpoint of this instability (at least in the regime of rotations explored in [20]) seems to be another MP black hole with a smaller angular momentum. Notice however that, because the axisymmetry is broken, the threshold of the instability is not expected to be stationary, and thus there will be no bifurcation to a new stationary family of black holes. The existence of such an instability had also been conjectured by Emparan and Myers in [45] using a thermodynamic argument of a different type to the one explored in the current paper.[15] This bar-mode instability is certainly different in nature from the axisymmetric instability, which is not present in the $d = 5$ case. Moreover, the bar-mode instability becomes active at slightly lower rotations: compare the critical

[15]The original argument of [45] for the bar-mode instability is the following: The entropy of a highly rotating black hole can be smaller than that of two boosted Schwarzchild-Tangherlini black holes in orbital motion. Therefore it should be entropically favored to the system to develop a nonaxisymmetric configuration.

values of a/r_m for the onset of the bar instability in Table 1 of [20] with the critical values of a/r_m for the onset of the ultraspinning instability in our Table I. It would certainly also be interesting to study the nonlinear time evolution of the axisymmetric ultraspinning instability.

In addition, in the presence of certain matter or a cosmological constant, rotating or charged black holes (including the Kerr solution) can be unstable due to the superradiant phenomena [47,65–67]. Consider a black hole with angular velocity Ω_H (or with chemical potential μ_H). If the frequency ω and the angular momentum m (or charge e) of a scattering wave satisfy the relation $\omega < m\Omega_H$ (or $\omega < e\mu_H$), the scattering is superradiant: the wave extracts rotational energy (or electromagnetic energy) from the background and gets amplified. The presence of an effective reflective potential barrier, which may be due to the mass in a massive scalar field case or due to the AdS gravitational potential, combined with superradiance can then lead to multiple wave amplification and reflection that render the system unstable. This can occur for scalar, electromagnetic or gravitational (but not fermionic) waves. Such an unstable system is often called a "black hole bomb" [65]. It would be interesting to determine the endpoint of this superradiant instability [47,66], and there is active ongoing research in this direction [68]. The spectral methods might also be useful to tackle this problem.

Returning to Smarr's expectation of an instability in rotating black holes based on an analogy between rotating black holes and fluid droplets [43], it is interesting that the expectation is realized in higher dimensions with the ultraspinning instability. That this happens is no longer seen as a mere analogy but a consequence of the gauge/gravity correspondence. Black holes are thermodynamic objects, and as such one should expect that, in a long wavelength regime (compared to the energy scale set by the temperature), they should have an effective hydrodynamic description. The first attempt to materialize this idea was the membrane paradigm [69]. More recently, the AdS/CFT duality motivated further research in this direction which culminated with the formulation of the fluid/gravity correspondence [12], which provides a precise hydrodynamic description of asymptotically large AdS black holes. In addition, we should mention the blackfold approach of [17,18] which also provides a hydrodynamic description of *asymptotically flat* black holes in the regime where this approximation applies. In both latter cases, the formal connection between gravity and fluid dynamics is established through a derivative expansion of the Einstein equations.

In this context, it is therefore no surprise that many of the aforementioned black hole instabilities have a hydrodynamic description. The Gregory-Laflamme instability [48] of black strings is in correspondence with the Rayleigh-Plateau instability in fluid tubes [59] and with damped unstable sound wave oscillations [70]. The subject of our study, the ultraspinning instability and the associated new phases of pinched stationary black holes, also possesses a fluid description as ultraspinning pinched plasma balls [58]. Finally, the bar-mode nonaxisymmetric instability [19] was also conjectured to exist in black holes due to its presence in rotating fluids [60].

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the stimulating discussions with our collaborators in the ultraspinning project, Roberto Emparan and Harvey Reall. Ó. J. C. D. acknowledges financial support provided by the European Community through the Intra-European Marie Curie Contract No. PIEF-GA-2008-220197. P. F. is supported by an STFC rolling grant. R. M. and J. E. S. acknowledge support from the Fundação para a Ciência e Tecnologia (FCT-Portugal) through Grants No. SFRH/BD/22211/2005 (R. M.) and No. SFRH/BD/22058/2005 (J. E. S.). This work was partially funded by FCT-Portugal through Projects No. PTDC/FIS/099293/2008, No. CERN/FP/83508/2008, and No. CERN/FP/109306/2009.

APPENDIX: EMBEDDING DIAGRAMS

In this Appendix, we summarize the main ingredients of the embeddings presented in Fig. 6. Since the kind of perturbations we are considering preserve the transverse $(d-4)$-sphere, we will henceforth suppress it and concentrate on the nontrivial four-dimensional part of the metric. In order to be able to visualize the geometry of the horizon for *all* values of the rotation parameter, we will adopt the embedding proposed in [71] in the context of the Kerr-Newman black hole.

Recall that for both the perturbed and the unperturbed geometries, the spatial cross sections of the future event horizon $\mathcal{H}^+$ are hypersurfaces of constant t and $r = r_+$. Here r_+ denotes the location of the event horizon of the background MP black hole. The induced metric on this hypersurface [suppressing the transverse $(d-4)$-sphere] can be written as

$$ds^2_{2d} = V(\sigma)d\sigma^2 + \sigma^2 d\phi^2, \quad \text{with} \quad \sigma = \sqrt{g_{\phi\phi}}|_{r_+}, \quad \text{and} \quad V(\sigma) = g_{xx}|_{r_+}\left(\frac{dx}{d\sigma}\right)^2, \tag{A1}$$

where $x = \cos\theta$. This two-dimensional surface of revolution can be embedded into Euclidean four-dimensional space $\mathbb{E}^4$ via the map $(\sigma, \phi) \mapsto (X, Y, Z, R)$ defined by

$$(X, Y, Z) = \frac{\sigma}{\Phi_0}(F(\phi), G(\phi), H(\phi)), \quad R = R(\sigma), \quad \text{and} \quad F(\phi)^2 + G(\phi)^2 + H(\phi)^2 = 1, \tag{A2}$$

and the induced metric for the associated 2-surface is

$$ds^2_{\mathbb{E}^4} = dX^2 + dY^2 + dZ^2 + dR^2 = \left[\Phi_0^{-2} + \left(\frac{dR}{d\sigma}\right)^2\right]d\sigma^2 + \sigma^2 d\phi^2. \tag{A3}$$

We ask the reader to see [71] for a detailed understanding of the second equality.

Introducing the scale parameter η and the distortion parameter β,

$$\eta = (r_+^2 + a^2)^{1/2}, \qquad \beta = a(r_+^2 + a^2)^{-1/2}, \tag{A4}$$

it follows that

$$g_{xx}|_{r_+} = \eta^2 f(x),$$
$$g_{\phi\phi}|_{r_+} = \eta^2 f(x)^{-1}, \quad \text{with} \quad f(x) = \frac{1 - x^2}{1 - \beta^2(1 - x^2)}. \tag{A5}$$

Then, from $ds^2_{\mathbb{E}^4} = ds^2_{2d}$, we get

$$R'(x) = \eta f(x)^{-1/2}\sqrt{1 - \frac{f'(x)^2}{4\Phi_0}}. \tag{A6}$$

Reality of this expression requires that $\Phi_0 \geq |f'(x)|/2$ for $0 \leq x \leq 1$ which is satisfied if

$$\Phi_0 \geq \frac{1}{2}\max_{x\in[0,1]} |f'(x)|. \tag{A7}$$

In the embedding diagrams shown in Fig. 6, the Cartesian coordinate Z was chosen to be given by the $R(x)$ that solves (A6). Similarly, X was chosen to be given by the function $f(x)^{1/2}$ defined in (A5).

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