

**Slowly rotating black hole solutions to Hořava-Lifshitz gravity**

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We present a new stationary solution to the field equations of Hořava-Lifshitz gravity with the detailed balance condition and for any value of the coupling constant  $\lambda > 1/3$ . This is the generalization of the corresponding spherically symmetric solution earlier found by Lü, Mei, and Pope to include a small amount of angular momentum. For the relativistic value  $\lambda = 1$ , the solution describes slowly rotating AdS type black holes. With a soft violation of the detailed balance condition and for  $\lambda = 1$ , we also find such a generalization for the Schwarzschild type black hole solution of the theory. Finally, using the canonical Hamiltonian approach, we calculate the mass and the angular momentum of these solutions.

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**I. INTRODUCTION**

General relativity being the best classical theory of gravitation fails to describe gravity at high enough energy scales, where gravitational dynamics becomes essentially quantum in nature. This fact has led to the idea of modification of general relativity which, in physical terms, originates in the first studies of perturbative quantum gravity (see a review paper [1]). In the quantum context, general relativity results in austere ultraviolet (UV) divergences in perturbation theory, defying the renormalizability. Curing the problem of UV divergences can be achieved by adding to the Einstein-Hilbert action higher-derivative correction terms [2]. Unfortunately, this procedure generically entails violation of unitarity and disastrous instabilities appear in the theory [3].

Recently, Hořava proposed a new intriguing way of curing these pathologies by endowing the theory with scaling properties which at short distances lead to an anisotropy between space and time [4] (see also [5]). This type of anisotropic scaling is well known in condensed matter systems and historically traces back to the classic work of Lifshitz [6]. The degree of the anisotropy between space and time is characterized by a number  $z$ , called *dynamical critical exponent*, the value of which is of crucial importance for the well-defined UV behavior of the theory. Hořava [4] developed a gravity model which in the UV limit exhibits an anisotropic scaling with  $z = 3$  Lifshitz fixed point and leads to a power-counting renormalizable theory of interacting nonrelativistic gravitons in  $3 + 1$  dimensions. As for the relativistic invariance, it appears as an emergent invariance in the infrared (IR) limit through the relevant deformations of the theory which realize the scaling with  $z = 1$ . This theory is now known as Hořava-Lifshitz (HL) gravity and is supposed to be a promising candidate for a UV completion of general relativity or its IR modification.

Shortly after its advent, HL gravity has been an active arena for many investigations. Some fundamental issues related to the internal structure of the theory as well as its

physically interesting extensions were considered in [7–11]. Cosmological implications of HL gravity have been studied in a number of papers (see for instance, [12–16]). In particular, it was shown that the higher-order spatial derivatives, which are inherent in the theory, have a drastic effect on the early history of the Universe, resulting in regular cyclic and bouncing solutions [12–15]. It is also interesting that HL gravity provides a new mechanism for scale-invariant cosmological perturbations without invoking the idea of inflation [12,16] (see also a recent review [17] and references therein). However, it should be emphasized that HL gravity suffers from a number of drawbacks as well. Without the projectability condition, but with the detailed balance condition, a scalar field becomes unstable in the UV limit [13], the theory exhibits scalar instabilities in the IR limit [18] and also suffers from the strong coupling problem [7,9,19], resulting in serious inconsistencies [20]. Furthermore, the theory does not give rise to scale-invariant perturbations [21] until the detailed balance condition is broken in the UV limit [22].

Another issue of fundamental significance is black hole solutions in HL gravity. The authors of [23] found a class of static and spherically symmetric solutions with a cosmological constant. Among these solutions, the AdS type black hole solution exhibits an asymptotic behavior, which is essentially different from the Schwarzschild-AdS black hole in general relativity. This means that general relativity is not always recovered in the IR limit. With a relevant deformation of the action, HL gravity also admits an asymptotically flat static black hole solution [24], which is a counterpart of the usual Schwarzschild black hole. Various extensions and physical properties of these solutions were studied [25–32]. Meanwhile, possible observational signatures of the asymptotically flat black holes in HL gravity were examined in [33–35] through the classical tests of gravitational effects in both the Solar System and the strong gravity regime near the black holes.

In light of these developments, it makes sense to ask the following question: *What is the rotating counterparts of the static black hole solutions, with or without cosmological*

*constant, in HL gravity?* To the best of our knowledge, today this question remains largely open. There is the only case of [36], where the authors managed to discuss near horizon geometries of putative extremal Kerr-AdS type black holes, for special range of parameters of the HL theory and for special values of the parameters of these black holes. On the other hand, it is clear that in strive to answer the above question, it would be of great importance to find genuinely HL analogs of the familiar Kerr and Kerr-AdS solutions of general relativity. In this paper, we consider a rotating solution to HL gravity (with the detailed balance condition) in the limit of slow rotation, but for any value of the dynamical coupling constant  $\lambda > 1/3$ . For the relativistic value  $\lambda = 1$ , this solution describes slowly rotating AdS type black holes. With a soft violation of the detailed balance condition and  $\lambda = 1$ , we also consider slowly rotating asymptotically flat black holes. In Sec. II we briefly recall the basic ingredients of HL gravity, giving the full set of the field equations, which involves an arbitrary parameter of the soft violation. Here we also discuss static and spherically symmetric solutions to these equations. In Sec. III we solve the field equations by linearizing them with respect to a small rotation parameter. For this purpose, we use the metric ansatz, involving the only off-diagonal component governed by a small rotation parameter. In Sec. IV using the canonical Hamiltonian approach, we calculate the mass and the angular momentum for the slowly rotating solutions.

## II. HORÁVA-LIFSHITZ GRAVITY AND STATIC BLACK HOLES

We begin with a brief review of the basic ingredients and the field equations of HL gravity. As the basic idea behind Hořava's proposal is to choose different scalings of space and time in the UV limit, it is fitting to formulate the theory in terms of ADM type coordinates. As is known [37], the use of such coordinates implies a (3 + 1) decomposition of the spacetime metric in the form

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (1)$$

where  $N$  is the lapse function,  $N^i$  is the shift vector, and  $g_{ij}$  is the three-dimensional spatial metric. In HL gravity [4] two conditions were invoked: (i) the ‘‘detailed balance’’ condition, which implies that the potential term in the action originates from Euclidean three-dimensional topological massive gravity, and (ii) the ‘‘projectability’’ condition, requiring that the lapse function does not depend on spatial coordinates. In our description below, as in a number of cases before (see for instance, [23,24]) we will relax both of these conditions.

With the metric decomposition given in Eq. (1), the Einstein-Hilbert action is written in the form

$$I_{\text{EH}} = \frac{1}{16\pi G} \int dt d^3x \sqrt{g} N (K_{ij} K^{ij} - K^2 + R - 2\Lambda), \quad (2)$$

where  $G$  is the Newtonian gravitational constant,  $R = g^{ij} R_{ij}$  is the three-dimensional Ricci scalar,  $\Lambda$  is the cosmological constant. The extrinsic curvature of the ADM decomposition is given by

$$K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - D_i N_j - D_j N_i), \quad K = g^{ij} K_{ij}, \quad (3)$$

where the dot stands for the derivative with respect to time and the covariant derivative operator  $D$  is defined with respect to the spatial metric  $g_{ij}$ .

The total action of HL gravity is given by (see [4,24])

$$I = \int dt d^3x \sqrt{g} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda_W + \nu)}{8(1-3\lambda)} \right. \\ \times \left( R - \frac{3\Lambda_W^2}{\Lambda_W + \nu} \right) + \frac{\kappa^2 \mu^2 (1-4\lambda)}{32(1-3\lambda)} R^2 \\ \left. - \frac{\kappa^2}{2\omega^4} \left( C_{ij} - \frac{\mu \omega^2}{2} R_{ij} \right) \left( C^{ij} - \frac{\mu \omega^2}{2} R^{ij} \right) \right\}, \quad (4)$$

where  $\kappa$ ,  $\lambda$ ,  $\mu$ ,  $\omega$  are coupling constants and  $\Lambda_W$  is a three-dimensional cosmological constant and  $C^{ij}$  is the symmetric, traceless, and covariantly conserved Cotton tensor defined as

$$C^{ij} = \frac{\epsilon^{ikl}}{\sqrt{g}} D_k \left( R^j_l - \frac{1}{4} \delta^j_l R \right). \quad (5)$$

Here we have also introduced the additional parameter

$$\nu = \frac{8\mu^2(1-3\lambda)}{\kappa^2}, \quad (6)$$

characterizing a soft violation of the detailed balance condition [24]. This leads to the modification of the IR behavior of the theory to allow a Minkowski vacuum solution for  $\Lambda_W \rightarrow 0$ . Next, rescaling the time coordinate as  $t \rightarrow ct$  and comparing the IR limit of action (4) with that given in Eq. (2), where  $\lambda = 1$ , we deduce that the speed of light, the Newtonian constant, and the cosmological constant appear as emergent quantities which are given by the relations

$$c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda_W + \nu}{1-3\lambda}}, \quad G = \frac{\kappa^2 c^2}{32\pi}, \quad \Lambda = \frac{3}{2} \frac{\Lambda_W^2}{\Lambda_W + \nu}. \quad (7)$$

We recall that  $\lambda$  is the dynamical coupling constant of HL gravity which governs the contribution to the theory coming from the trace of the extrinsic curvature. From Eq. (7) it follows that the sign of the cosmological constant is sensitive to the value of  $\lambda$ . We see that for  $\lambda > 1/3$  and  $\nu = 0$  it must be negative. However, for the same  $\lambda$ , as noted in [23], one can also consider the positive cosmological constant by making an analytical continuation with  $\mu \rightarrow i\mu$ ,  $\omega^2 \rightarrow -i\omega^2$ . In what follows, we shall consider the case of the negative cosmological constant.

The field equations of HL gravity are obtained by varying action (4) with respect to variables  $N$ ,  $N^i$ , and  $g^{ij}$ . The result consists of the Hamiltonian constraint

$$-\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda_W + \nu)}{8(1-3\lambda)} \left( R - \frac{3\Lambda_W^2}{\Lambda_W + \nu} \right) + \frac{\kappa^2 \mu^2 (1-4\lambda)}{32(1-3\lambda)} R^2 - \frac{\kappa^2}{2\omega^4} Z_{ij} Z^{ij} = 0, \quad (8)$$

and the momentum constraint

$$\frac{4}{\kappa^2} D_j (K^{ij} - \lambda g^{ij} K) = 0, \quad (9)$$

due to the variations  $\delta N$  and  $\delta N^i$ , respectively, and the dynamical equation, due to the variation  $\delta g^{ij}$ , which is given by

$$E_{ij} = 0, \quad (10)$$

where

$$E_{ij} = \frac{2}{\kappa^2} E_{ij}^{(1)} - \frac{2\lambda}{\kappa^2} E_{ij}^{(2)} + \frac{\kappa^2 \mu^2 (\Lambda_W + \nu)}{8(1-3\lambda)} E_{ij}^{(3)} + \frac{\kappa^2 \mu^2 (1-4\lambda)}{32(1-3\lambda)} E_{ij}^{(4)} + \frac{\kappa^2 \mu}{4\omega^2} E_{ij}^{(5)} - \frac{\kappa^2}{2\omega^4} E_{ij}^{(6)} \quad (11)$$

with

$$\begin{aligned} E_{ij}^{(1)} &= 2N_{(i} D_{|k|} K^k_{j)} - 2K^k_{(i} D_{j)} N_k - N^k D_k K_{ij} - 2NK_{ik} K^k_j - \frac{1}{2} g_{ij} N K_{kl} K^{kl} + N K K_{ij} + \dot{K}_{ij}, \\ E_{ij}^{(2)} &= \left( \frac{1}{2} N K^2 - N^k \partial_k K + \dot{K} \right) g_{ij} + 2N_{(i} \partial_{j)} K, \\ E_{ij}^{(3)} &= \left[ R_{ij} - \frac{1}{2} g_{ij} \left( R - \frac{3\Lambda_W^2}{\Lambda_W + \nu} \right) - D_i D_j + g_{ij} D^2 \right] N, \\ E_{ij}^{(4)} &= 2 \left( R_{ij} - \frac{1}{4} g_{ij} R - D_i D_j + g_{ij} D^2 \right) N R, \\ E_{ij}^{(5)} &= -2D_k D_{(i} [Z_{j)}^k N] + D^2 (N Z_{ij}) + g_{ij} D_k D_l (N Z^{kl}), \\ E_{ij}^{(6)} &= \left( -\frac{1}{2} g_{ij} Z_{kl} Z^{kl} + 2Z_{ik} Z^k_j - 2Z_{k(i} C_{j)}^k + g_{ij} Z_{kl} C^{kl} \right) N - D_k [T^{kl}_{(i} R_{j)l}] + R^n_i D_n [T^{kl}_{(i} g_{j)k}] - D^n [T^{kl}_n g_{k(i} R_{j)l}] \\ &\quad - D^2 D_k [T^{kl}_{(i} g_{j)l}] + D^n [g_{l(i} D_{j)}] D_k T^{kl}_n + D_l D_{(i} D_{|k|} T^{kl}_{j)} + g_{ij} D^n D_k D_l T^{kl}_n. \end{aligned} \quad (12)$$

Here we have used the notations  $D^2 = D_i D^i$ ,

$$Z_{ij} = C_{ij} - \frac{\mu \omega^2}{2} R_{ij}, \quad T^{ij}_k = \frac{N}{\sqrt{g}} \epsilon^{ijl} Z_{lk} \quad (13)$$

and a symmetrization procedure over the indices enclosed in round parentheses is implied. These equations are slightly different from those obtained in [12,23], but precisely recover them for the vanishing parameter ( $\nu \rightarrow 0$ ) of the soft violation of the detailed balance condition.

Exact solutions to these equations which describe static black holes were obtained in [23,24], using the standard spherically symmetric ansatz for the spacetime metric

$$ds^2 = -\tilde{N}^2(r) f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (14)$$

for which the Cotton tensor  $C_{ij}$  and the extrinsic curvature  $K_{ij}$  vanish identically. In [23], it was shown that with this metric ansatz the field equations given in (8)–(10) admit three different solutions. For the vanishing soft violation parameter,  $\nu \rightarrow 0$ , but with arbitrary dynamical coupling  $\lambda$  one has the solution

$$f = 1 - \Lambda_W r^2, \quad (15)$$

which leaves the function  $\tilde{N}$  unconstrained. The other two solutions are given by

$$f = 1 - \Lambda_W r^2 - \alpha r^{p_{\pm}}, \quad \tilde{N} = \beta r^{1-2p_{\pm}}, \quad (16)$$

where  $\alpha$  and  $\beta$  are integration constants and

$$p_{\pm} = \frac{2\lambda \pm \sqrt{6\lambda - 2}}{\lambda - 1}. \quad (17)$$

We shall further focus on the solution, which exhibits a well-defined asymptotic behavior in the  $\lambda = 1$  limit. This corresponds to the case with  $p = p_-$ . Furthermore, for the solution to be real, it is required that  $\lambda > 1/3$ . Since for  $\lambda \in (1/3, \infty)$  we have  $p \in (-1, 2)$ , the  $r^2$  term in the metric function  $f$  given in (16) becomes dominant at large distances. In the limit  $\lambda = 1$  ( $p = 1/2$ ), we arrive at the solution with

$$f = 1 - \Lambda_W r^2 - \alpha \sqrt{r}, \quad \tilde{N} = \beta, \quad (18)$$

where the constant  $\alpha$  can be related to the ADM mass of the spacetime [26] and the constant  $\beta$  can be set equal to unity by a redefinition of the time coordinate. This gives us the static AdS type black hole solution to HL gravity [23]. We recall that this solution was obtained within the detailed balance condition ( $\nu = 0$ ). This explains the reason

why the solution is significantly different from the usual AdS-Schwarzschild black hole solution. The authors of [23] argued that a deviation slightly from the detailed balance condition provides us with the AdS-Schwarzschild black hole of HL gravity.

On the other hand, for an arbitrary soft violation parameter,  $\nu \neq 0$ , there exists an asymptotically flat non-rotating black hole solution [24] for  $\lambda = 1$  and  $\Lambda_W = 0$ . In this case, we have

$$f = 1 - \nu r^2 - \sqrt{\nu^2 r^4 + \tilde{\alpha} r}, \quad \tilde{N} = \tilde{\beta}. \quad (19)$$

With this function, choosing  $\tilde{\alpha} = -4\nu GM$  and  $\tilde{\beta} = 1$ , it is easy to see that the metric in (14) becomes the standard Schwarzschild one in the limit  $4GM/\nu r^3 \ll 1$ . We see that there also exist two event horizons located at

$$r_{\pm} = GM \left[ 1 \pm \sqrt{1 + \frac{1}{2\nu(GM)^2}} \right]. \quad (20)$$

With Eq. (6) in mind, it follows that the requirement of cosmic censorship results in the bound  $\nu(GM)^2 \leq 1/2$ . In the regime of general relativity for which  $\nu(GM)^2 \ll 1$ , the outer horizon approaches the usual Schwarzschild radius,  $r_+ \rightarrow 2GM$ , whereas, the inner horizon shrinks to the singularity at  $r = 0$ .

### III. SLOWLY ROTATING SOLUTIONS

It is clear that the static and spherically symmetric solutions, which have been discussed above, are only

special cases of more general solutions with rotational dynamics. However, such exact solutions to HL gravity have not been found yet. On this route, it would be certainly of great interest to find the HL counterparts of the familiar Kerr and Kerr-AdS black holes. An attempt undertaken in [36] succeeded to present only near horizon limits of extremal Kerr-AdS type black holes for a special range of parameters of both HL gravity and the black holes. Here we wish to consider the rotating solutions by imposing the restriction only on their angular momentum. Namely, we restrict ourselves to slow rotation, considering only linear order in rotation parameter perturbations around the known spherically symmetric metrics [23,24]. In doing this, we assume that the spectrum of small gravitational perturbations of a spherically symmetric solution in HL gravity, as that of in the case of ordinary general relativity [38], contains the only zero mode which descends from the presence of slow rotation. This allows us to employ the following ansatz for the stationary metric:

$$ds^2 = -\tilde{N}^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) + 2ag(r)\sin^2\theta dt d\phi, \quad (21)$$

which involves the only off-diagonal component with a small rotation parameter  $a$ .

Next, for this metric ansatz we calculate the nonvanishing components of the spatial tensor in (11). Discarding all terms involving quadratic and higher powers in  $a$ , we obtain that

$$\begin{aligned} E_{rr} &= \frac{\kappa^2 \mu^2 \sqrt{f} \tilde{N}}{8(3\lambda - 1)r^2} \left\{ (\ln \tilde{N})' \left[ (\lambda - 1)f' - 2\lambda \frac{f-1}{r} - 2r(\Lambda_W + \nu) \right] + (\lambda - 1) \left( f'' - 2\frac{f-1}{r^2} \right) + \frac{1}{2f} \left[ (2\lambda - 1) \frac{(f-1)^2}{r^2} \right. \right. \\ &\quad \left. \left. - 2\lambda \frac{f-1}{r} f' + \frac{\lambda-1}{2} f'^2 + 2(\Lambda_W + \nu)(1-f-rf') - 3r^2 \Lambda_W^2 \right] \right\}, \\ E_{\theta\theta} &= \frac{\kappa^2 \mu^2 r f^{3/2} \tilde{N}}{16(3\lambda - 1)} \left\{ \frac{\tilde{N}''}{\tilde{N}} \left[ (\lambda - 1)f' - 2\lambda \frac{f-1}{r} - 2r(\Lambda_W + \nu) \right] - \frac{(\ln \tilde{N})'}{2r^2 f} [3rf'[-2\lambda + 2r^2(\Lambda_W + \nu) + r(1-\lambda)f']] \right. \\ &\quad \left. + 2f[2(1-f) + 5\lambda rf' + 2r^2(\Lambda_W + \nu) + 2r^2(1-\lambda)f''] - \frac{1}{r^3 f} [-1 + 2\lambda + 3r^4 \Lambda_W^2 + (3-2\lambda)f^2 + \lambda r^2(f'^2 - f'')] \right. \\ &\quad \left. + r^4(\Lambda_W + \nu)f'' + 2rf'[1 + r^2(\Lambda_W + \nu) + r^2(1-\lambda)f''] + f[-2 + 2r(\lambda - 2)f' + \lambda r^2 f'' + r^3(1-\lambda)f'''] \right\}, \\ E_{\phi\phi} &= \sin^2\theta E_{\theta\theta}. \end{aligned} \quad (22)$$

Here and in the following the prime denotes differentiation with respect to the radial coordinate  $r$ . We note that to the first order in rotation parameter  $a$ , the above expressions do not depend on  $a$  at all. It is also not difficult to check that with the metric ansatz (21), the Hamiltonian constraint (8) does not contain any term involving the rotation parameter  $a$  as well. As a consequence, we have

$$\begin{aligned} &\frac{\kappa^2 \mu^2}{8(1-3\lambda)r^2} \left\{ (2\lambda - 1) \frac{(f-1)^2}{r^2} - 2\lambda \frac{f-1}{r} f' \right. \\ &\quad \left. + \frac{\lambda-1}{2} f'^2 + 2(\Lambda_W + \nu)(1-f-rf') - 3r^2 \Lambda_W^2 \right\} = 0. \end{aligned} \quad (23)$$

In obtaining the above expressions we have used the fact that in the linear approximation in  $a$ , the extrinsic curvature has the only nonvanishing component  $K_{r\phi} = \mathcal{O}(a)$ . Therefore, its trace vanishes,  $K = 0$ . Furthermore, in this approximation the Cotton tensor  $C_{ij}$  vanishes as well. With all this in mind, it is straightforward to check that the metric functions given in (16) for  $\lambda > 1/3$  and  $\nu = 0$  as well as those given in (19) for  $\nu \neq 0$ ,  $\lambda = 1$ , and  $\Lambda_W = 0$ , satisfy the dynamical equations

$$E_{rr} = 0, \quad E_{\theta\theta} = 0, \quad E_{\phi\phi} = 0, \quad (24)$$

along with the constraint Eq. (23). Similarly, substituting the metric form (21) in the momentum constraint (9) and keeping only linear in  $a$  terms, we arrive at the equation

$$-\frac{2a\sqrt{f}}{\kappa^2 r^4} \left[ r^4 \left( \frac{g}{r^2} \right)' \right]' = 0. \quad (25)$$

We note that this equation is valid for any value of  $\lambda$ , as there is no contribution from the trace of the extrinsic curvature ( $K = 0$ ). Integrating (25), using the explicit form of  $\tilde{N}(r)$  given in (16), we find that

$$g = \sigma r^2 + \frac{\gamma}{r^{2p}}, \quad (26)$$

where  $\sigma$  and  $\gamma$  are constants of integration. In the limiting case  $\lambda \rightarrow 1$ , we have

$$g = \sigma r^2 + \frac{\gamma}{r}. \quad (27)$$

In the asymptotically flat case, when  $\nu \neq 0$ ,  $\lambda = 1$ , and  $\Lambda_W = 0$ , one can use  $\tilde{N} = \text{const}$  to integrate Eq. (25). As a consequence, we again arrive at Eq. (27).

In summary, we conclude that the metric

$$ds^2 = -\tilde{N}(r)^2 f(r) dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) + 2N_\phi(r) dt d\phi \quad (28)$$

is the solution to the field equations of HL gravity linearized in rotation parameter  $a$ , for any  $\lambda > 1/3$  and with the detailed balance condition,  $\nu = 0$ . The metric functions are given by

$$\begin{aligned} f &= 1 - \Lambda_W r^2 - \alpha r^p, \\ \tilde{N} &= \tilde{N}(\infty) r^{1-2p}, \\ N_\phi &= g a \sin^2\theta, \end{aligned} \quad (29)$$

and we recall that

$$p = \frac{2\lambda - \sqrt{6\lambda - 2}}{\lambda - 1}. \quad (30)$$

This metric generalizes the static and spherically symmetric solution, earlier found in [23] to include a small amount of angular momentum. For the relativistic value of the coupling constant,  $\lambda = 1$  ( $p = 1/2$ ), this metric describes a slowly rotating AdS type black hole in HL gravity. Meanwhile, with a soft violation of the detailed balance

condition,  $\nu \neq 0$ , the metric in (28) generalizes asymptotically flat nonrotating black hole solution of [24] with

$$\begin{aligned} f &= 1 - \nu r^2 - \sqrt{\nu^2 r^4 + \tilde{\alpha} r}, \\ \tilde{N} &= \tilde{N}(\infty), \\ N_\phi &= g a \sin^2\theta, \end{aligned} \quad (31)$$

and for  $\lambda = 1$  and  $\Lambda_W = 0$ . Clearly, the function  $g$  can alternatively be written as

$$g = \frac{N^\phi(\infty)}{a} r^2 + \frac{\gamma}{r^{2p}}. \quad (32)$$

In the above expressions, for a further convenience, we have renamed the constants of integration as asymptotic values  $\tilde{N}(\infty)$  and  $N^\phi(\infty)$ , keeping in mind  $\lambda = 1$ , for which  $\tilde{N} = \tilde{N}(\infty)$ . In this case, without loss of generality, we can set  $\tilde{N}(\infty) = 1$  and  $N^\phi(\infty) = 0$ . However, in the following we will keep them general to calculate the physical parameters of the solutions in the framework of the Hamiltonian approach.

#### IV. MASS AND ANGULAR MOMENTUM

In this section we calculate the mass and the angular momentum of the solution in (28) using the canonical Hamiltonian approach [39]. This approach was earlier employed in [25,26] to obtain the mass and entropy of the static AdS type black holes in HL gravity. In the Hamiltonian approach, the action of HL gravity (4) takes the form

$$I = \int dt d^3x (\pi^{ij} \dot{g}_{ij} - N\mathcal{H} - N^i \mathcal{H}_i) + B, \quad (33)$$

where

$$\pi^{ij} = \frac{2}{\kappa^2} \sqrt{g} (K^{ij} - \lambda K g^{ij}), \quad (34)$$

$$\begin{aligned} \mathcal{H} &= \sqrt{g} \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) - \frac{\kappa^2 \mu^2 (\Lambda_W + \nu)}{8(1-3\lambda)} \right. \\ &\quad \left. \times \left( R - \frac{3\Lambda_W^2}{\Lambda_W + \nu} \right) - \frac{\kappa^2 \mu^2 (1-4\lambda)}{32(1-3\lambda)} R^2 + \frac{\kappa^2}{2\omega^4} Z_{ij} Z^{ij} \right\}, \end{aligned} \quad (35)$$

$$\mathcal{H}_i = -2D_j \pi_i^j, \quad (36)$$

and  $B$  is a surface term, which is needed for a well-defined variational principle. We recall that in our case  $\dot{g}_{ij} = 0$  and the lapse function  $N = \tilde{N} \sqrt{f}$ . We shall focus on solution (28) with the metric functions given in (29) and (32). With this in mind, it is straightforward to show that the action in (33) can be written in the form

$$I = -2\pi(t_2 - t_1) \int dr d\theta (\tilde{N} \sqrt{f} \mathcal{H} + N^\phi \mathcal{H}_\phi) + B \quad (37)$$

with

$$\mathcal{H} = \frac{\kappa^2 \mu^2}{8(3\lambda - 1)} \frac{\sin\theta}{\sqrt{f}} \left[ \frac{\lambda - 1}{2} b'^2 - \frac{2\lambda}{r} b b' + \frac{2\lambda - 1}{r^2} b^2 \right], \quad (38)$$

$$\mathcal{H}_\phi = \frac{2a \sin^3\theta}{\kappa^2 \tilde{N}^2} [\tilde{N}'(r^2 g'' - 2g) - \tilde{N}(r^2 g' - 2rg)], \quad (39)$$

where we have introduced the new function

$$b = 1 - \Lambda_W r^2 - f. \quad (40)$$

Next, taking into account that  $N^\phi = ag/r^2$  we perform in (37) the integration over  $\theta$ . Varying the resulting expression with respect to the functions  $b$  and  $g$ , we find that

$$\begin{aligned} \delta I = -(t_2 - t_1) & \left\{ \frac{\kappa^2 \mu^2 \pi}{2(3\lambda - 1)} \left[ (\lambda - 1) \tilde{N} b' \delta b - \frac{2\lambda}{r} \tilde{N} b \delta b \right] \right. \\ & \left. + \frac{16\pi}{3\kappa^2} \left[ \frac{g}{\tilde{N}} \delta g' - \frac{g'}{\tilde{N}} \delta g - \frac{g}{\tilde{N}^2} \left( g' - \frac{2}{r} g \right) \delta \tilde{N} \right] a^2 \right\} + \delta B, \end{aligned} \quad (41)$$

where we have omitted the terms vanishing when the equations of motion hold. Furthermore, it is clear that the contribution from the boundary term must cancel the first two terms in this expression. In the approach under consideration, the mass and angular momentum are defined as the conjugates to the asymptotic displacements  $\tilde{N}(\infty)$  and  $N^\phi(\infty)$ . Therefore, after some manipulations, we find that at  $r \rightarrow \infty$

$$\begin{aligned} B = (t_2 - t_1) & \left\{ -\tilde{N}(\infty) \left( \frac{\pi \kappa^2 \mu^2}{2\sqrt{6\lambda - 2}} \alpha^2 \right) \right. \\ & \left. + N^\phi(\infty) \left( -\frac{32\pi(p+1)}{3\kappa^2 \tilde{N}(\infty)} \gamma a \right) \right\} + B_0, \end{aligned} \quad (42)$$

where  $B_0$  is an arbitrary constant of integration that can be set equal to zero. In obtaining this expression we have used the formulas

$$[(\lambda - 1) \tilde{N} b' \delta b]_\infty = \frac{\tilde{N}(\infty)}{2} (2\lambda - \sqrt{6\lambda - 2}) \delta \alpha^2, \quad (43)$$

$$\left[ \frac{2\lambda}{r} \tilde{N} b \delta b \right]_\infty = \lambda \tilde{N}(\infty) \delta \alpha^2, \quad (44)$$

$$\left[ \frac{g}{\tilde{N}} \delta g' \right]_\infty = -2p \frac{N^\phi(\infty)}{\tilde{N}(\infty)a} \delta \gamma, \quad (45)$$

$$\left[ \frac{g'}{\tilde{N}} \delta g \right]_\infty = 2 \frac{N^\phi(\infty)}{\tilde{N}(\infty)a} \delta \gamma, \quad (46)$$

$$\left[ \frac{g}{\tilde{N}^2} \left( g' - \frac{2}{r} g \right) \delta \tilde{N} \right]_\infty = 0, \quad (47)$$

which are obtained by means of Eqs. (29) and (32). It is now easy to identify the mass and the angular momentum from (42). We have

$$M = \frac{\pi \kappa^2 \mu^2}{2\sqrt{6\lambda - 2}} \alpha^2, \quad (48)$$

$$J = -\frac{32\pi(p+1)}{3\kappa^2 \tilde{N}(\infty)} \gamma a, \quad (49)$$

where we can now set  $\tilde{N}(\infty) = 1$  for  $\lambda = 1$ . It also follows that one can set  $N^\phi(\infty) = 0$ , as it does not contribute to the physical parameters of the solution. For  $a = 0$ , this result agrees with that found in [25,26].

Performing similar calculations for a slowly rotating asymptotically flat black hole which is described by metric (28) with the functions given in (31), we find that

$$M = \frac{\pi \kappa^2 \mu^2}{4} \tilde{\alpha}, \quad J = -\frac{16\pi}{\kappa^2} \gamma a. \quad (50)$$

We note that these expressions are in agreement with those given in (48) and (49) for  $\lambda = 1$  ( $p = 1/2$ ) and  $\alpha^2 \rightarrow \tilde{\alpha}$ . Taking into account the emergent quantities in (7) along with Eq. (6) for  $\lambda = 1$ , we deduce that

$$\tilde{\alpha} = -4\nu GM, \quad \gamma a = -2GJ \quad (51)$$

in the regime of general relativity. We note that the value of  $\tilde{\alpha}$  in (51) agrees with that given in [24]. Next, choosing in the second equation of (50) the integration constant as

$$\gamma = -\frac{\kappa^4 \mu^2}{64} \tilde{\alpha} = -\frac{\kappa^2}{16\pi} M, \quad (52)$$

it is easy to see that

$$J = Ma, \quad (53)$$

just as for a slowly rotating Kerr black hole in general relativity.

## V. CONCLUSION

Hořava-Lifshitz gravity is a new power-counting renormalizable theory of interacting nonrelativistic gravitons in 3 + 1 dimensions. Many attractive features as well as many intriguing physical and cosmological implications of this theory make sense to consider it as a possible candidate for a UV completion of general relativity. Among other implications, the black hole solutions to the theory are certainly of particular interest. In the existing literature, there are a number of static and spherically symmetric black hole solutions to HL gravity. However, rotating counterparts of these solutions, including both exact Kerr and Kerr-AdS type solutions have not been found yet. In this paper, as a first step towards this goal, we

have found a new stationary solution to HL gravity with the detailed balance condition and for any value of the coupling constant  $\lambda > 1/3$ . This solution generalizes the corresponding static solution of [23] to include a small amount of angular momentum. For the relativistic value of the coupling constant,  $\lambda = 1$ , the solution corresponds to a slowly rotating AdS type black hole of HL gravity. Abandoning the detailed balance condition and going over into the value of  $\lambda = 1$ , we have also found a slowly rotating and asymptotically flat black hole solution, thereby generalizing the Schwarzschild type solution of [24]. Using the canonical Hamiltonian approach, we have

calculated the mass and the angular momentum of these solutions.

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*Note added.*—While completing this paper, the work of [40] appeared, where for the case of  $\lambda = 1$  a slowly rotating counterpart of the static and asymptotically flat black hole solution of [24] is also discussed.

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