Conformal algebra on Fock space and conjugate pairs of operators

Klaus Sibold and Eden Burkhard

Institut für Theoretische Physik, Universität Leipzig, Postfach 100920, D-04009 Leipzig, Germany

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Using the moment construction, we represent the generators of the conformal algebra as bilinear products of creation and annihiliation operators on the Fock space of the massless real scalar field in four dimensions. A complete set of one-particle eigenstates of the dilatation generator is given. Next, a complete set of one-particle eigenstates of the conformal generator is constructed in two distinct ways, once directly and once through an expansion in terms of dilatation eigenstates. The second approach uses an analytic continuation of the dilatation eigenvalue away from the real axis; the validity of the method is illustrated by the consistency with the first approach. Drawing upon this technique, we finally ponder the idea of building conjugates to the four components of the momentum operator by suitably modifying the action of the conformal generators on dilatation eigenstates. The construction of eigenstates of these new operators proceeds as for the conformal generator itself.

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I. MOTIVATION

At extremely short distances, the notion of space and time is supposed to undergo a drastic change as compared to flat Minkowski spacetime. These ideas are very old (see, for instance, Ref. [1,2]) but they came up again and again. Particularly noteworthy is [3] which prompted many quantum field theoretical models where notions of noncommutative geometry [4] have been realized. Among them, the Moyal product of field operators [5] turned out to be a popular instrument. Usually, one introduces without further specification coordinate operators which are Hermitian and whose real eigenvalues label the points of spacetime.

However, even in commutative quantum field theory it is a longstanding puzzle how to construct coordinate operators from the elements of the underlying model. The relation of the Lorentz boost to the coordinates of the center of mass of the system was discussed in [6] in the context of quantum theory generalizing older work on relativity. Let us consider the ratio of M_{0j} over P_0 as a conjugate for P_k. In fact, from the commutation rules of the Poincaré algebra we immediately find

$$[Q_j, P_k] = -i\eta_{jk}, \qquad Q_j = P_0^{-(1/2)} M_{0j} P_0^{-(1/2)}, \quad j, k \in \{1, 2, 3\}. \tag{1}$$

Note that this simple guess for Q_j is similar to some terms of the Newton-Wigner coordinate operator [7] expressed in terms of $M_{\mu\nu}$ and P_μ.

Let us enumerate some features of the "division" prescription (1): First, it clearly fails on states of energy zero, i.e. acting on the vacuum is problematic. Nonetheless, it yields an operator identity on all states with positive energy. Second, the division by P_0 is nonlocal. Third, the special role assigned to P_0 breaks Lorentz invariance and correspondingly it is not clear how to assign a time operator.

Quite generally, the task of finding a conjugate to the Hamiltonian P_0 seems particularly difficult: In quantum mechanics Pauli's theorem [8] asserts that an operator Q_0 conjugate to P_0 (with spectrum bounded below) cannot be self-adjoint if its spectrum is to cover all real numbers. Instead of trying to give a survey of the historical debate, we rather point to some recent publications about ways of side stepping the theorem: One may give up self-adjointness of Q_0 and use only positive spectral measures for it [9]; one may give up a common domain where the commutation relation holds [10]; one may use a nonstandard Fourier transformation relating time and energy [11], and presumably there are many other possibilities.

In [12], it has been proposed to build conjugate operators explicitly as operators on the Fock space of the respective model. This approach is perturbative and based on the hope that eventually deviations of geometry away from the standard flat Minkowski spacetime could also be described perturbatively. The proposal has been further pursued in [13], where all bilinears involving one creation and one annihilation operator and up to two derivatives with respect to (w.r.t.) the momentum were considered. It is well known [14], that the only x moments of the energy-momentum tensor which form conserved currents (at the classical level) are those of the conformal group. Translating into Fock space and searching in the respective class of bilinear products of $a^\dagger(\mathbf{k})a(\mathbf{k})$ with appropriate factors of k, derivatives w.r.t. k and with dimension -1 it was indeed found in [13] that all operators but K_μ are necessarily nonlocal in field space. Second, in the interacting theory the conformal group does become anomalous, but the situation can be controlled by introducing local couplings [15,16].

The present study grew out of an attempt on constructing conjugates to all four components of the momentum P_μ by "dividing" the conformal algebra

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$$[P_\mu, K_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu}), \quad (2)$$

by the dilatation generator D. More precisely, we expand any state in terms of dilatation eigenstates and on each of these we divide out twice the dilatation eigenvalue. The fact that D does not commute over P_μ, K_μ forces us to complexify the dilatation eigenvalue and to introduce a cut into the division prescription. Like in the example (1), we do not achieve

$$[Q_\mu, P_\nu] = -i\eta_{\mu\nu} \quad (3)$$

(here due to the off-diagonal elements $M_{\mu\nu}/D$) but conjugation does hold for each pair $\{Q_\mu, P_\mu\}$ on dilatation eigenstates with real eigenvalue.[1] Lorentz invariance is not broken in this case, and the $\{Q_\mu\}$ commute. An encouraging observation is that the operators posses a complete set of simultaneous eigenstates of exponential type, which might suggest coordinate-like behavior. Some other features closely parallel the first example (1): Dilatation eigenstates with zero eigenvalue are problematic, and the division by D is (mildly) nonlocal.

We remark that in [17], a four-vector position operator was sought amongst the generators of the conformal algebra by an appropriate change of basis. The argument there presented relies on representation theory and remains largely abstract, but for the case of massive spin zero an explicit result is given which contains $K/(2D)$ and some other terms involving $1/P^2$.

Motivated by the above, we present a thorough discussion of the conformal algebra on the Fock space of one real, massless scalar field in four dimensions, of the one-particle eigenstates of the conformal and the dilatation operator including the idea of complexifying the dilatation eigenvalue, and then a section describing our attempt on constructing coordinate operators by the aforementioned division prescription.

In Sec. II we list the complete algebra of conformal transformations, first in field space, thereafter in Fock space. It is based on the improved energy-momentum tensor [14], from which the other currents are derived in terms of x moments. The charges turn out not to contain terms of type $a(\mathbf{p})a(\mathbf{p})$ or the Hermitian conjugate (a being the annihilation operator of the field).[2]

In Sec. III we construct eigenstates $|s\rangle$ of D, the dilatation charge, and show that they are complete by expressing the ordinary one-particle eigenstate $|\mathbf{p}\rangle$ of the energy-momentum operator in terms of $|s\rangle$. It is important to note that the eigenvalue problem does not restrict the eigenvalue s to be real since D is unbounded [18].

In Sec. IV we solve the eigenvalue problems for $\{K_\mu\}$ which in fact reduce to only *one*, since the different components commute with each other. Actually, this goes hand in hand with a restriction of the eigenvalues κ_μ to $\kappa^\mu \kappa_\mu = 0$. As announced in the abstract, we also give a second derivation of the same set of eigenfunctions by expressing the problem in terms of suitably chosen eigenstates of D and allowing for complex eigenvalues. Last, we show that the eigenstates of the conformal operator form a dense set.

In Sec. V we describe how the division idea outlined around Eq. (2) yields a "conjugate" Q_0 of P_0. The definition proceeds via complex eigenvalues of D and it carries over to Q_j. It yields commuting Q's, but still a nondiagonal symplectic structure due the commutators with P_μ for unequal values of the indices. Again, we solve the eigenvalue problem and show the completeness of the eigenstates $|\mathbf{q}\rangle$.

Conclusions and outlook form Sec. VI.

II. SWITCHING TO FOCK SPACE

The dynamics of the massless real scalar field in four dimensions is described by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) \quad (4)$$

leading to the equation of motion

$$\Box \phi = 0. \quad (5)$$

This equation can be solved by the Fourier transform:

$$\phi(x) = \frac{1}{\sqrt{(2\pi)^3}} \int d^3k \left[\frac{e^{ik.x}}{\sqrt{2\omega}} a^\dagger(\mathbf{k}) + \frac{e^{-ik.x}}{\sqrt{2\omega}} a(\mathbf{k}) \right], \quad (6)$$

where $k.x = k^\mu x_\mu$, $\mu \in \{0, 1, 2, 3\}$ and the metric is diagonal with $\eta_{\mu\mu} = (1, -1, -1, -1)$ (no sum). In noncovariant expressions we use Latin indices $l \in 1, 2, 3$ for the spatial directions. Boldface may be used to make explicit the dependence on the three vector k^l only. In the last formula

$$\omega = k_0 = +\sqrt{-k^l k_l}, \qquad d^3k = dk_1 dk_2 dk_3. \quad (7)$$

In the integrand of (6) we may thus identify the first (second) term as relating to the negative (positive) frequency part. We consider Fock space representations in which $a^\dagger(\mathbf{k})$ creates a particle with positive energy from the vacuum $|0\rangle$ while its Hermitian conjugate $a(\mathbf{k})$ is interpreted as an annihilation operator. This is expressed by the statement $a(\mathbf{k})|0\rangle = 0$ and the commutation relation $[a(\mathbf{k}), a^\dagger(\boldsymbol{l})] = \delta^3(\mathbf{k} - \boldsymbol{l})$. The momentum eigenstates

$$|\mathbf{k}_1 \cdots \mathbf{k}_n\rangle = \frac{1}{\sqrt{n!}} a^\dagger(\mathbf{k}_1) \cdots a^\dagger(\mathbf{k}_n)|0\rangle \quad (8)$$

are fundamental in this description. They are complete in the following sense:

$$I_n = \int d^3k_1 \cdots \int d^3k_n |\mathbf{k}_1 \cdots \mathbf{k}_n\rangle\langle \mathbf{k}_1 \cdots \mathbf{k}_n| \quad (9)$$

[1]The violation of the assumptions of Pauli's theorem seems automatic; there is no contradiction to that result.

[2]We thereby correct a calculation mistake in [12].

is a projector onto the n-particle states. The sum

$$I = \sum_0^\infty I_n \tag{10}$$

acts as the identity on any state.

The classical theory is invariant under the infinitesimal transformations of the conformal group:

$$\begin{aligned} \delta^P \phi &= a^\mu \partial_\mu \phi, \\ \delta^M \phi &= \tfrac{1}{2}\omega^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi, \\ \delta^D \phi &= \varepsilon(1 + x^\mu \partial_\mu)\phi, \\ \delta^K \phi &= \alpha_\mu(2x^\mu x^\nu \partial_\nu - x^2 \partial^\mu + 2x^\mu)\phi. \end{aligned} \tag{11}$$

Here a^μ, $\omega^{\mu\nu}$, ε, α^μ are constant infinitesimal parameters. In order to construct the generators of these symmetries, we start from the improved energy-momentum tensor [14–16]

$$\begin{aligned} T_{\mu\nu} &= (\partial_\mu \phi)(\partial_\nu \phi) - \tfrac{1}{2}\eta_{\mu\nu}(\partial^\lambda \phi)(\partial_\lambda \phi) \\ &\quad - \tfrac{1}{6}(\partial_\mu \partial_\nu - \eta_{\mu\nu}\Box)\phi^2, \end{aligned} \tag{12}$$

form all other currents as x moments of it

$$M_{\mu\nu\rho} = x_\mu T_{\nu\rho} - x_\nu T_{\mu\rho} \tag{13}$$

$$\begin{aligned} &= x_\mu\bigg(\partial_\nu \phi \partial_\rho \phi - \frac{1}{2}\eta_{\nu\rho}(\partial\phi)^2 \\ &\quad - \frac{1}{6}(\partial_\nu \partial_\rho - \eta_{\nu\rho}\Box)\phi^2\bigg) - x_\nu\bigg(\partial_\mu \phi \partial_\rho \phi \\ &\quad - \frac{1}{2}\eta_{\mu\rho}(\partial\phi)^2 - \frac{1}{6}(\partial_\mu \partial_\rho - \eta_{\mu\rho}\Box)\phi^2\bigg), \end{aligned} \tag{14}$$

$$D_\nu = x^\mu T_{\mu\nu} \tag{15}$$

$$= x^\mu\bigg(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 - \frac{1}{6}(\partial_\mu \partial_\nu - \eta_{\mu\nu}\Box)\phi^2\bigg), \tag{16}$$

$$K_{\sigma\nu} = (2x_\sigma x^\mu - \eta_\sigma{}^\mu x^2)T_{\mu\nu} \tag{17}$$

$$\begin{aligned} &= (2x_\sigma x^\mu - \eta_\sigma{}^\mu x^2)\bigg(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 \\ &\quad - \frac{1}{6}(\partial_\mu \partial_\nu - \eta_{\mu\nu}\Box)\phi^2\bigg) \end{aligned} \tag{18}$$

integrate their zero components over three-space and obtain thus the charges in terms of the scalar field and its spacetime derivatives:

$$P_0 = \int d^3x\bigg[\frac{1}{2}\partial_0 \phi \partial_0 \phi - \frac{1}{2}\partial_j \phi \partial^j \phi\bigg], \tag{19}$$

$$P_j = \int d^3x \partial_j \phi \partial_0 \phi, \tag{20}$$

$$M_{j0} = \int d^3x\bigg[\frac{1}{2}x_j \partial_0 \phi \partial_0 \phi - \frac{1}{2}x_j \partial_k \phi \partial^k \phi - x_0 \partial_j \phi \partial_0 \phi\bigg], \tag{21}$$

$$M_{jk} = \int d^3x\Big[x_j \partial_k \phi \partial_0 \phi - x_k \partial_j \phi \partial_0 \phi\Big], \tag{22}$$

$$\begin{aligned} D = \int d^3x\bigg[&\frac{1}{2}x_0 \partial_0 \phi \partial_0 \phi - \frac{1}{2}x_0 \partial_k \phi \partial^k \phi + x^j \partial_j \phi \partial_0 \phi \\ &+ \phi \partial_0 \phi\bigg], \end{aligned} \tag{23}$$

$$\begin{aligned} K_0 = 2x_0 D - \int d^3x\bigg[&\frac{1}{2}x^2 \partial_0 \phi \partial_0 \phi - \frac{1}{2}x^2 \partial_j \phi \partial^j \phi \\ &+ \frac{2}{3}x_j \partial^j \phi \phi\bigg], \end{aligned} \tag{24}$$

$$\begin{aligned} K_j = &\int d^3x\bigg[x_j x_0 \partial_0 \phi \partial_0 \phi - x_j x_0 \partial_k \phi \partial^k \phi + \frac{2}{3}x_0 \phi \partial_j \phi\bigg] \\ &+ \int d^3x[2x_j x^l \partial_l \phi \partial_0 \phi + 2x_j \phi \partial_0 \phi - x^2 \partial_j \phi \partial_0 \phi]. \end{aligned} \tag{25}$$

Via equal time commutation relations

$$[\dot\phi(x), \phi(y)] = -i\delta^3(x - y), \tag{26}$$

we may then reproduce the transformation laws. As an example, consider the Lorentz transformations: In order to construct M_{0j}, $j \in \{1, 2, 3\}$, we need

$$\begin{aligned} T_{j0} &= \tfrac{2}{3}\dot\phi \partial_j \phi - \tfrac{1}{3}\phi \partial_j \dot\phi, \\ T_{00} &= \tfrac{1}{2}\dot\phi^2 - \tfrac{1}{6}(\partial^j \phi)(\partial_j \phi) + \tfrac{1}{3}\phi \partial^j \partial_j \phi. \end{aligned} \tag{27}$$

It is easy to verify that

$$\begin{aligned} [T_{j0}(x), \phi(y)] &= -i\delta^3(x - y)\partial_j \phi(y) + \cdots, \\ [T_{00}(x), \phi(y)] &= -i\delta^3(x - y)\dot\phi(y) \end{aligned} \tag{28}$$

(the dots in the first formula denote a total derivative w.r.t. x^j) and thus

$$[M_{\mu\nu}, \phi(y)] = -i(y_0 \partial_j - y_j \partial_0)\phi(y). \tag{29}$$

With the help of (6) we finally convert all charges into functions of a, $a^\dagger$ obtaining

$$P_\mu = \int d^3k k_\mu a^\dagger(\mathbf{k})a(\mathbf{k}), \tag{30}$$

$$M_{0j} = -\frac{i}{2}\int d^3k\omega[(\partial_{k^j} a^\dagger(\mathbf{k}))a(\mathbf{k}) - a^\dagger(\mathbf{k})(\partial_{k^j} a(\mathbf{k}))], \tag{31}$$

$$M_{ij} = -\frac{i}{2}\int d^3k k_i[(\partial_{k^j} a^\dagger(\mathbf{k}))a(\mathbf{k}) - a^\dagger(\mathbf{k})(\partial_{k^j} a(\mathbf{k}))] - (i \leftrightarrow j), \quad (32)$$

$$D = \frac{i}{2}\int d^3k[(k^j\partial_{k^j} a^\dagger(\mathbf{k}))a(\mathbf{k}) - a^\dagger(\mathbf{k})(k^j\partial_{k^j} a(\mathbf{k}))], \quad (33)$$

$$K_0 = \frac{1}{2}\int d^3k\omega[(\partial_{k_j}\partial_{k^j} a^\dagger(\mathbf{k}))a(\mathbf{k}) + a^\dagger(\mathbf{k})(\partial_{k_j}\partial_{k^j} a(\mathbf{k}))] + \frac{1}{4}\int d^3k\frac{1}{\omega}a^\dagger(\mathbf{k})a(\mathbf{k}), \quad (34)$$

$$K_j = \int d^3k[(\partial_{k^j} a^\dagger(\mathbf{k}))(k^l\partial_{k^l} a(\mathbf{k})) + (k^l\partial_{k^l} a^\dagger(\mathbf{k})) \times (\partial_{k^j} a(\mathbf{k}))] - \int d^3k k_j(\partial_{k_l} a^\dagger(\mathbf{k}))(\partial_{k^l} a(\mathbf{k})) - \frac{1}{4}\int d^3k\frac{k_j}{\omega^2}a^\dagger(\mathbf{k})a(\mathbf{k}). \quad (35)$$

Interestingly, without the improvement terms $-(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)\phi^2/6$ in formula (12) for the energy-momentum tensor, the Fock space version for the generators would contain bilinear expressions involving $a(\mathbf{k})a(-\mathbf{k})$ and the Hermitian conjugate terms with two creation operators.

III. EIGENSTATES OF D

Let us construct eigenfunctions of D using its Fock space form (33) and the standard commutator $[a(\mathbf{k}), a^\dagger(\boldsymbol{l})] = \delta^3(\mathbf{k} - \boldsymbol{l})$. With the one-particle ansatz

$$F = \int d^3k f(\mathbf{k})|\mathbf{k}\rangle, \quad (36)$$

one finds the condition

$$DF = -i\int d^3k\left[\left(k^l\partial_{k^l} + \frac{3}{2}\right)f(\mathbf{k})\right]|\mathbf{k}\rangle = sF. \quad (37)$$

The most general eigenfunction is

$$f(\mathbf{k}) = \omega^{is-(3/2)}\tilde{f}(s, \phi, \theta) \quad (38)$$

in spherical coordinates. Here the angular dependence $\tilde{f}$ is completely arbitrary. The corresponding one-particle eigenstate is

$$|s\rangle = \int d^3k\omega^{is-(3/2)}\tilde{f}(s, \phi, \theta)|\mathbf{k}\rangle, \qquad D|s\rangle = s|s\rangle. \quad (39)$$

Likewise, the most general n-particle dilatation eigenstate is

$$|\underline{s}\rangle = |s_1, \ldots, s_n\rangle = \int d^3k_1 \ldots d^3k_n \omega_1^{is_1-(3/2)} \ldots \omega_n^{is_n-(3/2)} * \tilde{f}(s_1, \phi_1, \theta_1; \ldots; s_n, \phi_n, \theta_n)|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle, \quad (40)$$

$$D|\underline{s}\rangle = \sum_{m=1}^{n} s_m|\underline{s}\rangle. \quad (41)$$

Let us return to the discussion of the one-particle case. For the "s-wave" $\tilde{f}(s, \phi, \theta) = 1$, we find

$$\langle t|s\rangle = 8\pi^2\delta(s - t) \quad (42)$$

and similar orthogonality properties hold for the angular part if, for example, the spherical harmonics $Y^{lm}(\phi, \theta)$ are chosen. Functions with more interesting dependence on ω may be constructed as

$$G = \int_{-\infty}^{\infty} dsg(s)|s\rangle, \quad (43)$$

which amounts to an inverse Mellin transform on $g(is)$.

We remark that the dilatation eigenvalue s will usually be real since D is Hermitian. However, the eigenvalue equation clearly does not change if s is complex. The imaginary part of s is independent of its real part; it is in fact fixed by the (momentum space) scaling dimension of any one-particle state that we expand in terms of the dilatation eigenstates by Mellin transform w.r.t. the real part. Complex s does not seem harmful in (39) since the convergence of the integral over $\mathbf{k}$ can only be discussed if $|s\rangle$ is paired with a suitable bra; in any case our eigenfunctions are distributions.

In the following sections we will employ dilatation eigenstates of the special form

$$|s, \mathbf{x}\rangle = \int d^3k\frac{X^{is-1}}{\sqrt{\omega}}|\mathbf{k}\rangle, \qquad X = k^\mu x_\mu, \qquad x_0 = +\sqrt{-x^l x_l}. \quad (44)$$

For the obvious choice of spherical coordinates in $\mathbf{k}$ space, it follows

$$X = x_0\omega(1 + \cos(\theta)) \geq 0. \quad (45)$$

For later use, we define the reference state

$$|\mathbf{x}\rangle = \int d^3k\frac{e^{iX}}{\sqrt{\omega}}|\mathbf{k}\rangle. \quad (46)$$

[This is proportional to the field $\phi(\mathbf{x})$ acting on the vacuum when $\mathbf{x}$ is on the light cone.] By Mellin back transform

$$|\mathbf{x}\rangle = \int_{-\infty}^{\infty} dsg_x(s)|s, \mathbf{x}\rangle, \qquad g_x(s) = \frac{i}{2\pi}e^{\pi s/2}\Gamma(1 - is) \quad (47)$$

as an example of (43). The s integral in the latter formula does not naively converge. Nevertheless, the underlying equation

$$\int_{-\infty}^{\infty} dsg_x(s)X^{is-1} = e^{iX} \quad (48)$$

is certainly an identity for distributions.[3] Here the Mellin back transform involves the argument X, and thus a combination of radial and angular parts. This was our reason to keep s as an argument of $\tilde{f}(s, \phi, \theta)$ in (39).

Finally, we wish to answer the question whether the states $|s, \mathbf{x}\rangle$ defined in (44) are complete. Suppose for the moment that x^μ does not lie on the light cone, but rather on the usual spacelike hypersurface $x_0 = \text{const}$. Otherwise the definition of $|x\rangle$ is as in (46) above. The ansatz ($\leftrightarrow \partial = \rightarrow \partial - \leftarrow \partial$)

$$\tilde{I}_1 = \frac{1}{2(2\pi)^3} \int d^3x |x\rangle i \overleftrightarrow{\partial}_{x_0} \langle x| \tag{49}$$

yields a projector onto one-particle states:

$$\begin{aligned}\tilde{I}_1|\mathbf{p}\rangle &= \frac{i}{2(2\pi)^3} \int \frac{d^3x d^3k_1 d^3k_2}{\sqrt{\omega_{k_1}\omega_{k_2}}} |\mathbf{k}_1\rangle\langle\mathbf{k}_2|\mathbf{p}\rangle \\ &\quad * [e^{ik_1.x}(\partial_{x_0} e^{-ik_2.x}) - (\partial_{x_0} e^{ik_1.x}) e^{-ik_2.x}] \\ &= \frac{1}{2(2\pi)^3} \int d^3k_1 \frac{\omega_p + \omega_{k_1}}{\sqrt{\omega_{k_1}\omega_p}} e^{i(w_{k_1} - \omega_p)x_0} |\mathbf{k}_1\rangle \\ &\quad \times \int d^3x e^{i(k_1 - p)^l x_l} \\ &= |\mathbf{p}\rangle. \end{aligned} \tag{50}$$

However, $\tilde{I}_1$ fails to behave like the identity if $x_0 = \sqrt{-x^l x_l}$ as in the original definition (44): The exponential factor involving x_0 is no longer independent of the x integral, which therefore does not yield a delta-function. Also, we cannot use a time derivative to obtain the extra power of ω needed to compensate the denominator.

The problem is easy to cure, though: Let us rather adopt the definition

$$\check{I}_1 = \frac{1}{(2\pi)^3} \int d^3x P_0^{1/2} e^{-iP_0x_0} |\mathbf{x}\rangle\langle\mathbf{x}| e^{iP_0x_0} P_0^{1/2} \tag{51}$$

with the Hamiltonian

$$P_0 = \int d^3k \omega_k a^\dagger(\mathbf{k}) a(\mathbf{k}), \qquad P_0|\mathbf{p}\rangle = \omega_p|\mathbf{p}\rangle.$$

One quickly checks[4]

$$\check{I}_1|\mathbf{p}\rangle = \frac{1}{(2\pi)^3} \int d^3k_1 \int d^3x e^{i(k_1 - p)^l x_l} |\mathbf{k}_1\rangle = |\mathbf{p}\rangle. \tag{52}$$

Substituting Eq. (47) into (51) demonstrates that the $|s, \mathbf{x}\rangle$ as defined in (44) form a complete set of states.

[3]One may verify this by integrating either side against the test functions Xe^{-uX} over positive X.

[4]This remains valid for the standard case of the spacelike hypersurface $x_0 = \text{const}$, too.

IV. EIGENSTATES OF K

Next [F is as in (36)],

$$K_0F = \int d^3k \bigg[\bigg(\omega \partial_{k_l} \partial_{k^l} - \frac{1}{\omega} k^l \partial_{k^l} - \frac{3}{4\omega} \bigg) f(\mathbf{k}) \bigg] |\mathbf{k}\rangle, \tag{53}$$

$$\begin{aligned} K_jF = \int d^3k \bigg[\bigg(& k_j \partial_{k_l} \partial_{k^l} - 2\partial_{k^j}(k^l \partial_{k^l}) \\ & - \partial_{k^j} - \frac{k_j}{4\omega^2} \bigg) f(\mathbf{k}) \bigg] |\mathbf{k}\rangle. \end{aligned} \tag{54}$$

It is instructive to write the first formula in spherical coordinates:

$$\begin{aligned} K_0F = -\int \omega^2 d\omega d\phi \sin(\theta) d\theta \bigg[\bigg(\mathcal{O}_r + \frac{1}{\omega} \mathcal{O}_a \bigg) \\ \times f(\omega, \phi, \theta) \bigg] a^\dagger(\omega, \phi, \theta)|0\rangle. \end{aligned} \tag{55}$$

Here

$$\begin{aligned} &\mathcal{O}_r = \omega \partial_\omega^2 + 3\partial_\omega + \frac{3}{4\omega}, \qquad [\mathcal{O}_r, \omega] = 2\bigg(\omega\partial_\omega + \frac{3}{2}\bigg), \\ &\mathcal{O}_a = \frac{1}{\sin(\theta)^2} \partial_\phi^2 + \partial_\theta^2 + \cot(\theta)\partial_\theta, \qquad [\mathcal{O}_a, \omega] = 0. \end{aligned} \tag{56}$$

We can immediately check

$$[K_0, P_0]F = -2 \int d^3k \bigg[\bigg(\omega\partial_\omega + \frac{3}{2} \bigg) f(\mathbf{k}) \bigg] |\mathbf{k}\rangle = -2iDF, \tag{57}$$

in particular, on a dilatation eigenstate

$$[K_0, P_0]|s\rangle = -2is|s\rangle. \tag{58}$$

For the most trivial eigenstate with $\tilde{f}(s, \phi, \theta) = 1$, we find

$$P_0|s\rangle = \int d^3k \omega^{is-(1/2)} |\mathbf{k}\rangle = |s - i\rangle, \tag{59}$$

$$K_0|s\rangle = s(s + i)|s + i\rangle, \tag{60}$$

where we have used the possibility to complexify s to write the extra real power of ω resulting from the nonvanishing dimension of P_0, K_0, as a shift in the eigenvalue. The conformal algebra closes on these more general dilatation eigenstates. Let us re-check the commutator (58):

$$\begin{aligned} (K_0P_0 - P_0K_0)|s\rangle &= K_0|s - i\rangle - P_0 s(s + i)|s + i\rangle \\ &= \Big((s - i)s - s(s + i) \Big)|s\rangle = -2is|s\rangle. \end{aligned} \tag{61}$$

Let us now look for eigenfunctions of the operators K_μ. The four components of K commute so that the eigenfunctions will be common to all of them. First, we write the ansatz (c.f. the Appendix)

$$F_K = \int d^3k \frac{f_K(X)}{\sqrt{\omega}} |\mathbf{k}\rangle, \qquad X = \omega\kappa_0 + k^j\kappa_j. \tag{62}$$

The extra root of ω in the denominator reinstates covariance:

$$K_\mu F_K = -2\kappa_\mu \int d^3k \frac{Xf_K''(X) + f_K'(X)}{\sqrt{\omega}} |\mathbf{k}\rangle + \kappa^2 \int d^3k \frac{k_\mu f_K''(X)}{\sqrt{\omega}} |\mathbf{k}\rangle. \tag{63}$$

For $\kappa^2 = 0$,it is now easy to proceed:

$$K_\mu F_K = \kappa_\mu F_K \leftrightarrow Xf_K''(X) + f_K'(X) + \frac{1}{2} f_K(X) = 0. \tag{64}$$

This ordinary differential equation (ODE) on f_K is of Bessel-type and has the two solutions

$$f_{K,1}(X) = J_0(\sqrt{2X}), \qquad f_{K,2}(X) = Y_0(\sqrt{2X}). \tag{65}$$

To sum up, we find the two types of eigenstates

$$|\kappa\rangle_{K,1} = \int d^3k \frac{J_0(\sqrt{2X})}{\sqrt{\omega}} |\mathbf{k}\rangle, \tag{66}$$

$$|\kappa\rangle_{K,2} = \int d^3k \frac{Y_0(\sqrt{2X}}{\sqrt{\omega}} |\mathbf{k}\rangle. \tag{67}$$

Note that both signs of κ_0 are allowed.

Second, let us expand in dilatation eigenstates $|s, \kappa\rangle$ as defined in (44) just with parameter κ instead of $\mathbf{x}$. For $\kappa_0 = +\sqrt{-\kappa^l\kappa_l}$ (so $X \geq 0$) we may equivalently write[5]

$$F_K = \int_{-\infty}^{\infty} ds g_K(s) |s, \kappa\rangle, \tag{68}$$

because the states $|s, \kappa\rangle$ are dense as was shown in Sec. III. By straightforward application of formula (63) with $\kappa^2 = 0$, we find

$$K_\mu F_K = 2\kappa_\mu \int_{-\infty}^{\infty} ds g_K(s)(s + i)^2 |s + i, \kappa\rangle \tag{69}$$

which we want to equate with

$$\kappa_\mu F_K = \kappa_\mu \int_{-\infty}^{\infty} ds g_K(s) |s, \kappa\rangle = \kappa_\mu \int_{-\infty+i}^{\infty+i} ds g_K(s) |s, \kappa\rangle = \kappa_\mu \int_{-\infty}^{\infty} ds g_K(s + i) |s + i, \kappa\rangle, \tag{70}$$

where it is assumed that the contour shift has no effect. The eigenvalue problem is thus solved if $g_K(s)$ obeys the functional equation

$$2(t + i)^2 g_K(t) = g_K(t + i). \tag{71}$$

The shift in the argument of g_K suggests to seek a solution via the Fourier transform

$$g_K(t) = \int_{-\infty}^{\infty} du e^{iut} h_K(u). \tag{72}$$

This translates the functional equation into an ODE on h_K:

$$h_K''(u) + 2h_K'(u) + (1 - \tfrac{1}{2}e^{-u}) h_K(u) = 0. \tag{73}$$

The solutions are

$$h_{K,1}(u) = \frac{1}{2\pi} J_0(\sqrt{2e^{-u}}) e^{-u}, \qquad h_{K,2}(u) = \frac{1}{2\pi} Y_0(\sqrt{2e^{-u}}) e^{-u}, \tag{74}$$

which immediately reproduce the Bessel-type solutions found above, because the integral over s leads to $\delta(u + \log(X))$, so that the u integral then simply replaces $u \to -\log(X)$ (apart from a rescaling by 2π).

On the other hand, if we take the u integral first we find the functions

$$g_{K,1}(s) = \frac{2^{-is}}{\pi} \frac{\Gamma(1 - is)}{\Gamma(is)}, \qquad g_{K,2}(s) = \frac{2^{-is}}{\pi} \cosh(\pi s)\Gamma(1 - is)^2, \tag{75}$$

which both satisfy (71). Both functions have poles only in the lower half plane at $-in$, $n \in \mathbb{N}$. Hence the shift of the integration contour did not cross any singularity.

Next, since

$$|\kappa\rangle_{K,1} = \int d^3k \frac{J_0(\sqrt{2X})}{\sqrt{\omega_k}} |\mathbf{k}\rangle$$

is an eigenstate of K_μ with eigenvalue κ_μ, we also have

$$K_\mu |\lambda^2\kappa\rangle_{K,1} = \lambda^2\kappa_\mu |\lambda^2\kappa\rangle_{K,1}, \qquad \lambda \geq 0. \tag{76}$$

Suppose that this freedom of rescaling the 4-component eigenvalue κ_μ by a scalar factor λ^2 can be used to construct $|\kappa\rangle$ [the reference state (46) with parameter κ instead of x] as a superposition of $|\lambda^2\kappa\rangle_{K,1}$ states: For simplicity, put $Y = \sqrt{2X}$. On the level of the eigenfunctions, we seek $c(\lambda)$ such that

$$e^{(i/2)Y^2} = \int_0^\infty d\lambda c(\lambda) J_0(\lambda Y). \tag{77}$$

The equation may be solved by an order zero Hankel transform:

$$\frac{c(\rho)}{\rho} = \int_0^\infty dY Y e^{(i/2)Y^2} J_0(\rho Y) = i e^{-(i/2)\rho^2}. \tag{78}$$

Upon replacing $\lambda^2 \to \lambda$, we obtain

$$|\kappa\rangle = \frac{i}{2} \int_0^\infty d\lambda e^{-(i/2)\lambda} |\lambda\kappa\rangle_{K,1}, \tag{79}$$

which can in turn be substituted into (51).

This proves that the J_0 class of one-particle eigenstates of K form a complete set. Further, we expect there to be a

[5] If $\kappa_0 < 0$ one should simply replace X by $-X$.

similar expansion of $|\kappa\rangle$ in terms of the second set of eigenstates of K with $Y_0(\sqrt{2X})$ as an eigenfunction. We have not studied the latter problem because the literature does not provide an easy way to invert a convolution onto Bessel Y functions.

V. THE OPERATOR Q

In this section we define a "coordinate operator" Q_0 conjugate to P_0 on dilatation eigenstates:

$$Q_0 = K_0 \frac{1}{2R(s)} |s\rangle \tag{80}$$

with

$$R(s) = s\colon \Im(s) \geq -\tfrac{1}{2}, \qquad R(s) = s + i\colon \Im(s) < -\tfrac{1}{2}. \tag{81}$$

Here the division operation is to be done prior to applying K_0. Inserting this into the little calculation (61), we obtain

$$[Q_0, P_0]|s\rangle = -i|s\rangle \tag{82}$$

if s is real.

In Secs. III and IV we expressed a reference state (47) and eigenstates of the conformal operator (68), respectively, as integrals over dilatation eigenstates like (43). The appropriate basis turned out to be $|s, \mathbf{x}\rangle$ with real s. Similarly, P or K acting on those states could be expanded in generalized eigenstates $|s - i, \mathbf{x}\rangle$ or $s + i, \mathbf{x}\rangle$ with real s, respectively. In such situations, the cut in $R(s)$ in the division operation should be appropriately shifted. We are confident that within each such "slice" the construction here presented goes through accordingly, although it must be emphasized that different slices cannot simultaneously be handled.

Recall the n-particle dilatation eigenstates (40). We introduce the notation $|j-, k+\rangle = |s_1, \ldots, s_j - i, \ldots, s_k + i, \ldots, s_n\rangle$, $j, k \in \{1, \ldots, n\}$, and similar for $R(\underline{s})$. It follows trivially

$$\begin{aligned}[Q_0, P_0]|\underline{s}\rangle = &\sum_{j\neq k}\left(\frac{1}{2R(j-)} - \frac{1}{2R(\underline{s})}\right)(s_k(s_k + i) \\ &+ \mathcal{O}_{a,k})|j-, k+\rangle + \sum_j\left(\frac{s_j(s_j - i) + \mathcal{O}_{a,j}}{2R(j-)}\right. \\ &\left. - \frac{s_j(s_j + i) + \mathcal{O}_{a,j}}{2R(\underline{s})}\right)|\underline{s}\rangle\end{aligned} \tag{83}$$

so that

$$R(j-) = R(\underline{s}) = \sum_k s_k \tag{84}$$

is sufficient to obtain

$$[Q_0, P_0]|\underline{s}\rangle = -i|\underline{s}\rangle \tag{85}$$

for real s_j.

More generally, for real s_j

$$[Q_\mu, P_\nu]|\underline{s}\rangle = -i\eta_{\mu\nu}|\underline{s}\rangle - \frac{i}{R(\underline{s})} M_{\mu\nu}|\underline{s}\rangle \tag{86}$$

because any P_ν on a true dilatation eigenstate yields a sum of generalized eigenstates, all with eigenvalue $(\sum s_j) - i$. A subsequent Q_μ operation on any term thus comes with division by $2R(j-) = 2R(\underline{s})$ by construction.

Two subsequent Q operations on a true dilatation eigenstate $|\underline{s}\rangle$ lead to a division by $2R(\underline{s})$ first and then by $2R(j+) = 2(R(\underline{s}) + i)$ which is again common to all terms. Consequently, for real s_j

$$[Q_\mu, Q_\nu]|\underline{s}\rangle = \frac{1}{4R(\underline{s})(R(\underline{s}) + i)}[K_\mu, K_\nu]|\underline{s}\rangle = 0. \tag{87}$$

The last equation ensures that simultaneous eigenfunctions for the four Q_μ components exist just as they did for the K_μ. In order to obtain Q_μ operators that do not mutually commute one would have to define the various components intrinsically differently. In the current context we have not fully exploited this issue. Suffice it to say that enforcing $[Q_j, P_j] = i$ (no sum) on dilatation eigenstates results into the same division prescription as before, which therefore seems difficult to circumvent.

Last, Hermiticity of the Q operation might be reinstated by the definition

$$\langle \underline{s}^*|Q_\mu = \langle \underline{s}^*| \frac{1}{2R^*(s^*)} K_\mu, \tag{88}$$

where one acts to the left first by the division and then by K.

The construction of eigenfunctions of the K_μ as superpositions of dilatation eigenstates can straightforwardly be adapted to the Q_μ case; the discussion remains limited to eigenvalues on the light cone $q^2 = 0$, of course. The eigenvalue problem for Q_μ is

$$Q_\mu F_Q = 2q_\mu \int_{-\infty}^{\infty} ds g_Q(s)(s + i)^2 \frac{1}{2s}|s + i, \mathbf{q}\rangle = q_\mu F_Q. \tag{89}$$

(In $|s, \mathbf{q}\rangle$ we use the argument $X = k^\mu q_\mu$ etc.) Upon the same change of integration contour on the right-hand side as above, we obtain the condition

$$(t + i)^2 g_Q(t) = t g_Q(t + i). \tag{90}$$

We solve by the Fourier transform as before:

$$g_Q(t) = \int_{-\infty}^{\infty} du e^{iut} h_Q(u) \tag{91}$$

makes the functional equation into

$$h_Q''(u) + (2 + ie^{-u})h_Q'(u) + (1 - ie^{-u})h_Q(u) = 0. \tag{92}$$

Once again, there are two solutions:

$$h_{Q,1} = \frac{1}{2\pi} e^{ie^{-u}-u}(1 + ie^{-u}),$$
$$h_{Q,2} = \frac{1}{2\pi}\left[e^{ie^{-u}-u}(1 + ie^{-u})\Gamma(0, ie^{-u}) - e^{-u}\right] \tag{93}$$

or in the s plane

$$g_{Q,1} = -\frac{1}{2\pi} se\pi^{\pi s/2}\Gamma(1 - is),$$
$$g_{Q,2} = \frac{i}{2}\frac{se^{-(\pi s/2)}\Gamma(1 - is)}{\sinh(\pi s)}. \tag{94}$$

Now, the case $h_{Q,1}$, $g_{Q,1}$ is strictly analogous to the discussion of the eigenfunctions of K_μ itself: There is no obstacle to the shift of the integration contour. The second case deserves a closer look because $g_{Q,2}$ has a pole at $s = i$. Remarkably, both $h_{Q,1}$, $g_{Q,1}$ are square integrable[6] so that we can construct a more rigorous argument. To this end, we start with

$$F_\epsilon = \int_{-\infty-i\epsilon}^{\infty-i\epsilon} ds g_{Q,2}(s)|s, \mathbf{q}\rangle \tag{95}$$

and run through the same steps as above. We must have $0 < \epsilon < \frac{1}{2}$ to avoid the cut in $R(s)$ as well as the singularity at $s = i$ when shifting the contour upwards on the right-hand side (The parameter should be thought of as infinitesimal, though, as we ideally want the s integration to run over the real line.) One may now use the residue theorem to prove that the shifted integral is equal to the original one: The integral over $q_{Q,2}(s)X^{is}$ along the boundary of a rectangle between the points $\{-R - i\epsilon, R - i\epsilon, R + i(1 - \epsilon), -R + i(1 - \epsilon)\}$ must yield zero for some large R because no singularity is encircled. One can easily check numerically that the contributions from the two short sides are indeed exponentially suppressed since $q_{Q,2}$ falls off very quickly. The limit $R \to \infty$ then shows that F_ϵ with $q_{Q,2}$ is an eigenstate of Q_μ. Next,

$$\int_{-\infty-i\epsilon}^{\infty-i\epsilon} ds g(s) X^{is} = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} ds e^{ius} X^{is} e^{u\epsilon} h(u) X^{\epsilon} = 2\pi h(-\log(X)) \tag{96}$$

demonstrating that the infinitesimal shift does not affect the final result.

Upon back substitution, the eigenstates satisfying $Q_\mu F_Q = q_\mu F_Q$, $q^2 = 0$ become

$$|\mathbf{q}\rangle_{Q,1} = \int d^3k \frac{e^{iX}(1 + iX)}{\sqrt{\omega}}|\mathbf{k}\rangle, \tag{97}$$

$$|\mathbf{q}\rangle_{Q,2} = \int d^3k \frac{e^{iX}(1 + iX)\Gamma(0, iX) - 1}{\sqrt{\omega}}|\mathbf{k}\rangle. \tag{98}$$

[6]Despite of this the related eigenstate $|\mathbf{q}\rangle_{Q,2}$ given in (98) is not normalizable.

These enjoy the same completeness properties as the eigenstates of K: The reference state $|\mathbf{q}\rangle$ [c.f. (46)] can be written as a superposition of $|\mathbf{q}\rangle_{Q,1}$

$$|\mathbf{q}\rangle = \int_0^1 d\lambda |\lambda\mathbf{q}\rangle_{Q,1}, \tag{99}$$

and inserting this into (51), we obtain a projector onto one-particle states. The second set of Q eigenfunctions could presumably equivalently be used but as in the case of $|\kappa\rangle_{K,2}$ it is not obvious how to invert the resulting integral transform. In conclusion, w.r.t. its one-particle eigenstates the Q operator is as well-behaved as the conformal generator K.

As already mentioned, by definition [c.f. (46)]

$$\phi(\mathbf{x})|0\rangle = \frac{1}{\sqrt{2(2\pi)^3}}|\mathbf{x}\rangle \tag{100}$$

if $\mathbf{x}$ is lightlike. On the other hand, if the coordinate vector x_μ is timelike and $x_0 > 0$ then the dot product $X = k^\mu x_\mu$ is always greater than zero. In this regime we can take over the Mellin transform (47) to expand the field acting on the vaccum in the usual states $|s, x\rangle$, just that x is not on the future light cone for now. Now,

$$i\int_0^1 d\lambda |s, \lambda x\rangle = i\int_0^1 d\lambda \int d^3k \frac{(\lambda X)^{(is-1)}}{\sqrt{\omega}}|\mathbf{k}\rangle = i\int_0^1 d\lambda \lambda^{(is-1)}|s, x\rangle = \frac{1}{s}|s, x\rangle \tag{101}$$

so that

$$\begin{aligned} Q_\mu \phi(x)|0\rangle &= Q_\mu \frac{1}{\sqrt{2(2\pi)^3}} \int_{-\infty}^{\infty} g_x(s)|s, x\rangle \\ &= K_\mu \frac{1}{\sqrt{2(2\pi)^3}} \int_{-\infty}^{\infty} g_x(s)\frac{1}{2s}|s, x\rangle \\ &= \frac{i}{2}K_\mu \frac{1}{\sqrt{2(2\pi)^3}} \int_{-\infty}^{\infty} g_x(s)\int_0^1 d\lambda |s, \lambda x\rangle \\ &= \frac{i}{2}K_\mu \int_0^1 d\lambda \frac{1}{\sqrt{2(2\pi)^3}} \int_{-\infty}^{\infty} g_x(s)|s, \lambda x\rangle \\ &= \frac{i}{2}K_\mu \int_0^1 d\lambda \phi(\lambda x)|0\rangle. \end{aligned} \tag{102}$$

[The index x on $g_x(s)$ is not a parameter; it is only meant to indicate that the function

$$g_x(s) = \frac{i}{2\pi} e^{\pi s/2}\Gamma(1 - is)$$

occurs in the Mellin transform of $|x\rangle$.] Further, recall that $P_\nu|s, x\rangle$ is a dilatation eigenstate with generalized eigenvalue $s - i$ and we defined $R(s - i) = s$. It then also follows that

$$\begin{aligned} Q_\mu P_\nu \phi(x)|0\rangle &= Q_\mu P_\nu \frac{1}{\sqrt{2(2\pi)^3}} \int_{-\infty}^{\infty} g_x(s)|s, x\rangle \\ &= K_\mu \frac{1}{\sqrt{2(2\pi)^3}} \int_{-\infty}^{\infty} g_x(s) \frac{1}{2R(s-i)} P_\nu |s, x\rangle \\ &= \frac{1}{2} K_\mu P_\nu \frac{1}{\sqrt{2(2\pi)^3}} \int_{-\infty}^{\infty} g_x(s) \frac{1}{s} |s, \lambda x\rangle \\ &= \frac{i}{2} K_\mu P_\nu \int_0^1 d\lambda \phi(\lambda x)|0\rangle. \end{aligned} \tag{103}$$

The nonlocality reproduces the effect of dividing out the dilatation eigenvalue in the Fock space picture. Our cut prescription for $R(s)$ ensures that both equations hold. From the conformal algebra, we obtain

$$\begin{aligned} [Q_\mu, P_\nu]\phi(x)|0\rangle &= (-\eta_{\mu\nu} D - M_{\mu\nu}) \int_0^1 d\lambda \phi(\lambda x)|0\rangle \\ &= -i\eta_{\mu\nu}\phi(x)|0\rangle - M_{\mu\nu} \int_0^1 d\lambda \phi(\lambda x)|0\rangle. \end{aligned} \tag{104}$$

In the last line we have used $[D, \phi(\lambda x)] = i(x.\partial_x + 1)\phi(\lambda x)$ and the integration over λ [which we had already seen in (99)] is actually the inverse of this differential operator:

$$(x.\partial_x + 1) \int_0^1 d\lambda \phi(\lambda x) = \int_0^1 \left(\lambda \frac{d}{d\lambda} + 1\right)\phi(\lambda x) = \phi(x) \tag{105}$$

if $\phi(x)$ is sufficiently regular.

VI. CONCLUSIONS AND OUTLOOK

Following [14], we wrote the operators of the conformal algebra as x moments of the stress energy tensor of the massless real scalar field. Inserting the field's Fourier decomposition we obtained a representation of the generators in terms of $a^\dagger$, a acting directly in Fock space. If the improved stress energy tensor is employed the resulting operators do not contain terms with two creation or two annihilation operators, which reflects the fact that we are dealing with charges.[7]

Next, we derived one-particle eigenstates for D and K and showed that these are (for K more than) complete. In doing so, we showed that it is useful and consistent to pass through an analytic continuation of the dilatation eigenvalues to the complex plane. Naively, one would expect only dilatation eigenstates with real eigenvalue because D is Hermitian. However, since the operator is unbounded this is not necessarily so: Acting on such "true" eigenstates by operators of nonvanishing dilatation weight like P and K produces sums of similar states with complex eigenvalue. More precisely, P lowers the imaginary part of the dilatation eigenvalue by one unit, and K raises it accordingly. The conformal algebra closes on the generalized states.

Interestingly, the various sets of eigenfunctions are all related by one-parameter integral transformations.

The original motivation for this work lay in finding Hermitian operators Q_μ conjugate to the energy-momentum P_μ on dilatation eigenstates:

$$[Q_\mu, P_\mu]|s\rangle = -i\eta_{\mu\mu}|s\rangle, \qquad \text{(no sum)}.$$

Our definition of Q in Sec. V is at most simultaneously consistent with the latter condition on $|s\rangle$ and $|s-i\rangle$ for real s, which means that any Q operation has to be accompanied by P in order to allow for an iteration. Constructions involving

$$f(P^\mu Q^\mu)|s>, \qquad f(Q^\mu P^\mu)|s>, \qquad s \in \mathbb{R} \tag{106}$$

do conserve the conjugacy property. In applications the operators Q should play the role of coordinate operators. In the spirit of [12], the "field"

$$\Phi(Q) = \int d^3k \omega_k^{-(3/2)} [e^{iP^\mu Q_\mu} a^\dagger(\mathbf{k}) + \text{c.c.}] \tag{107}$$

(defined by its matrix elements w.r.t. dilatation eigenstates) could be studied in analogy to the discussion in [3]. A more challenging example (although this requires some extension of our present construction) would be an object like

$$\Phi(Q, \alpha) = \int d^3k e^{i\alpha P^\mu Q_\mu}(a(\mathbf{k}) + a^\dagger(\mathbf{k}))e^{-i\alpha P^\mu Q_\mu}, \tag{108}$$

(compare [13]). Formally, (108) resembles the wedge-local fields of [19] and the warped fields of [20].

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APPENDIX

In this appendix we present a more direct way of solving the eigenvalue problem associated with K_μ in Fock space. In particular, we explain the form of the less general ansatz (62) used in the main text and we prove that the vector of eigenvalues must be lightlike.

We look for a common eigenstate of K_0 and K_j of the form

[7]Charges can be controlled to all orders in perturbation theory by use of the respective Ward identities. This holds even true in the case of conformal charges where anomalies complicate the situation. The additional difficulties become manageable by rendering the coupling local (c.f. [12] for a discussion in this context, and [15,16], for background reading). Therefore, our experiment on coordinate operators in Sec. V may have a generalization to the interacting theory.

$$|\kappa\rangle = \int d^3k f(x, y, \kappa_0, \kappa^j\kappa_j)|\mathbf{k}\rangle, \qquad \text{(A1)}$$

where

$$x = \kappa^j k_j, \qquad y = \kappa_0\omega, \qquad \kappa_0 \neq 0, \qquad \text{(A2)}$$

and κ^μ will play the role of the eigenvalue.

As we have done for D, the action of K_0, K_j on $|\mathbf{k}\rangle$ can be partially integrated onto the eigenfunction f. The resulting operators are given in (53) and (54), in the main text. They contain derivatives with respect to the momentum $\mathbf{k}$, while the eigenvalue κ^μ occurs only as a parameter. We do not wish to construct a function with an open j index. The possible variables are then x and y; if $\kappa_0 \neq 0$ the variable y is as general as ω. Below, we will not explicitly indicate the dependence on the parameters. Further, without loss of generality we may write

$$\kappa^j\kappa_j = -\alpha\kappa_0^2. \qquad \text{(A3)}$$

The eigenvalue problem $K_0|\kappa\rangle = \kappa_0|\kappa\rangle$ yields the equation

$$\mathbf{A} \quad \alpha y^2 f,_{xx} + 2xyf,_{xy} + y^2 f,_{yy} + xf,_x + 3yf,_y + \left(y + \frac{3}{4}\right)f = 0. \qquad \text{(A4)}$$

The K_j eigenequation has two parts with $-k_j/\omega^2$ and $-\kappa_j/x$ carrying the open index, respectively. These are independent vectors in three-space so that we obtain two constraints:

$$\mathbf{B} \quad \alpha y^2 f,_{xx} - y^2 f,_{yy} - yf,_y + \frac{1}{4}f = 0, \qquad \text{(A5)}$$

$$\mathbf{C} \quad 2x^2 f,_{xx} + 2xyf,_{xy} + 3xf,_x + xf = 0. \qquad \text{(A6)}$$

The combination $\mathbf{A}-\mathbf{B}$ has the same homogeneity properties as $\mathbf{C}$:

$$\mathbf{D} \quad 2xyf,_{xy} + 2y^2 f,_{yy} + xf,_x + 4yf,_y + \left(y + \frac{1}{2}\right)f = 0. \qquad \text{(A7)}$$

Next, we put $x = e^X$, $y = e^Y$ to obtain

$$\mathbf{C}' \quad 2f,_{XX} + 2f,_{XY} + f,_X + e^X f = 0, \qquad \text{(A8)}$$

$$\mathbf{D}' \quad 2f,_{XY} + 2f,_{YY} + f,_X + 2f,_Y + \left(e^Y + \frac{1}{2}\right)f = 0. \qquad \text{(A9)}$$

We further substitute

$$a = \frac{X+Y}{2}, \qquad b = \frac{X-Y}{2}. \qquad \text{(A10)}$$

The sum and the difference of the two equations give the conditions

$$\mathbf{P} \quad f,_{aa} + f,_a + (e^a\cosh(b) + \tfrac{1}{4})f = 0, \qquad \text{(A11)}$$

$$\mathbf{M} \quad f,_{ab} - \frac{1}{2}f,_a + \frac{1}{2}f,_b + \left(e^a\sinh(b) - \frac{1}{4}\right)f = 0. \qquad \text{(A12)}$$

The first equation is an ODE in the variable a. Hence we may integrate

$$f = c_1(b)e^{-(a/2)}J_0\left(2\sqrt{e^a\cosh(b)}\right) + c_2(b)e^{-(a/2)}Y_0\left(2\sqrt{e^a\cosh(b)}\right). \qquad \text{(A13)}$$

Upon substituting into equation $\mathbf{M}$,

$$(c_1 - 2c_1')zJ_1(2z) + (c_2 - 2c_2')zY_1(2z) = 0, \quad z = \sqrt{e^a\cosh(b)}. \qquad \text{(A14)}$$

The two Bessel functions are distinct as functions of a so that the two round brackets must separately vanish. It follows:

$$c_{1,2}(b) = e^{b/2}\tilde{c}_{1,2}(\kappa_0, \alpha) \qquad \text{(A15)}$$

Back through the chain of substitutions the two independent eigenfunctions take the form

$$f_1 = \frac{J_0(\sqrt{2(x+y)})}{\sqrt{y}}, \qquad f_2 = \frac{Y_0(\sqrt{2(x+y)})}{\sqrt{y}}. \qquad \text{(A16)}$$

Originally, we started from three equations $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ Putting f_1 into equation $\mathbf{B}$ yields

$$(\alpha - 1)\frac{y^{3/2}J_2(\sqrt{2(x+y)})}{2(x+y)} = 0 \qquad \text{(A17)}$$

and the same with Y_2 for f_2. The functional independence of the two types of Bessel function implies again that these conditions must separately hold. We thus obtain a solution iff $\alpha = 1$. Thus if $\kappa_0 \neq 0$ we must have $\kappa^2 = 0$. The two solutions obtained in the main text are identical up to a rescaling by $\sqrt{\kappa_0}$.

If $\kappa_0 = 0$, we have to distinguish two cases: First, let $\kappa^j \neq 0$. We put $\kappa^j\kappa_j = -\alpha$ and $y = \omega$. The potential term yf in $\mathbf{A}$ drops, but otherwise the equations $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ do not change because all terms involving derivatives w.r.t. y are invariant under rescalings of the variable. The analysis goes through very much as before. However, when verifying the consistency with the third condition we obtain $\alpha = 0$, which implies $\kappa^j = 0$ and thus a contradiction.

It remains to discuss the case $\kappa^\mu = 0$: Here the variable x is absent and $y = \omega$. Thus equation $\mathbf{C}$ disappears, while we have to drop terms with x derivatives in both $\mathbf{A}$, $\mathbf{B}$, and the potential yf in $\mathbf{A}$. The difference of $\mathbf{A}$, $\mathbf{B}$, yields the first order equation $2yf,_y + f = 0$ with the solution $1/\sqrt{\omega}$, which solves also $\mathbf{A}$, $\mathbf{B}$, themselves. This is indeed the limit $\kappa^\mu \to 0$ of the J_0 type solution of the regular scenario. The second solution is singular in this limit. We may conclude that it simply disappears from the solution set; the argument in this paragraph shows how the coupled system reduces to first order.

When considering the complete algebra of the conformal transformations it is also possible to derive constraints on the eigenvalues of K_μ, by using the analogue of the Pauli-Lubanski vector. Furthermore, one finds relations amongst representations as a result of the automorphism $P_\mu \rightarrow K_\mu$, $K_\mu \rightarrow -P_\mu$. We hope to come back to these considerations in a later publication.

[1] H. S. Snyder, Phys. Rev. **71**, 38 (1947).
[2] C. N. Yang, Phys. Rev. **72**, 874 (1947).
[3] S. Doplicher, K. Fredenhagen, and J. E. Roberts, Phys. Lett. B **331**, 39 (1994).
[4] A. Connes, arXiv:0810.2088v1.
[5] V. Gayral, J. M. Gracia-Bondia, B. Iochum, T. Schücker, and J. C. Varilly, Commun. Math. Phys. **246**, 569 (2004).
[6] M. H. L. Pryce, Proc. R. Soc. A **195**, 62 (1948).
[7] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).
[8] W. Pauli, in *Handbuch der Physik*, edited by S. Flügge (Springer, Berlin, 1958), Vol. 5.
[9] R. Brunetti and K. Fredenhagen, Rev. Math. Phys. **14**, 897 (2002).
[10] E. A. Galapon, Proc. R. Soc. London A 458, 451 (2002).
[11] J. M. Isidro, Phys. Lett. A **334**, 370 (2005).
[12] K. Sibold, Nucl. Phys. **B811**, 363 (2009).
[13] K. Sibold and G. Solard, Phys. Rev. D **80**, 124041 (2009).
[14] C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) **59**, 42 (1970).
[15] E. Kraus and K. Sibold, Nucl. Phys. **B372**, 113 (1992).
[16] E. Kraus and K. Sibold, Nucl. Phys. **B398**, 125 (1993).
[17] S. Lagu and H. Laue, Nuovo Cimento Soc. Ital. Fis. A **20**, 217 (1974).
[18] N. Dunford and J. T. Schwartz, *Linear Operators, Part II: Spectral Theory, Self Adjoint Operators in Hilbert Space* (Wiley, New York, 1988).
[19] H. Grosse and G. Lechner, J. High Energy Phys. 11 (2007) 012.
[20] D. Buchholz and S. J. Summers, *Proceedings of the Symposium in Honour of Wolfhart Zimmermann's 80th Birthday*, edited by E. Seiler and K. Sibold (World Scientific, Singapore, 2008).