How to adapt broad-band gravitational-wave searches for *r*-modes

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Up to now there has been no search for gravitational waves from the *r*-modes of neutron stars in spite of the theoretical interest in the subject. Several oddities of *r*-modes must be addressed to obtain an observational result: The gravitational radiation field is dominated by the mass current (gravitomagnetic) quadrupole rather than the usual mass quadrupole, and the consequent difference in polarization affects detection statistics and parameter estimation. To astrophysically interpret a detection or upper limit it is necessary to convert the gravitational-wave amplitude to an *r*-mode amplitude. Also, it is helpful to know indirect limits on gravitational-wave emission to gauge the interest of various searches. Here I address these issues, thereby providing the ingredients to adapt broad-band searches for continuous gravitational waves to obtain *r*-mode results. I also show that searches of existing data can already have interesting sensitivities to *r*-modes.

I. INTRODUCTION

In recent years the LIGO Scientific Collaboration and Virgo Collaboration have performed numerous searches for continuous gravitational-wave (GW) emission from rapidly rotating neutron stars. These include narrow-band searches for known pulsars [1–5] (i.e. on or near a timing solution), broad-band searches for the (nonpulsing) accreting neutron star in the low-mass x-ray binary Sco X-1 [6,7], and all-sky broad-band surveys for previously unknown neutron stars [6,8–12]. Also, a broad-band search for the nonpulsing central compact object in supernova remnant Cas A is underway [13]. (After this paper was submitted, the Cas A search was presented in Ref. [14].)

Cas A is the youngest object targeted by a continuous GW search so far, and at 300 years old it may still be subject to the *r*-mode GW emission mechanism. The *r*-modes [15] are rotation-dominated oscillations which are driven unstable by GW emission [16,17] (the "Chandrasekhar-Friedman-Schutz" version of the two-stream instability [18,19]) even in the presence of nuclear-matter viscosity [20–22]. This makes *r*-modes observationally a very interesting possibility for GW emission in newborn neutron stars [23] and accreting neutron stars in low-mass x-ray binaries [24,25]. Nonlinear hydrodynamic saturation of the *r*-mode instability is now thought (under most conditions) to occur at amplitudes several orders of magnitude below those originally guessed [26], making unlikely the most optimistic scenario of extragalactically detectable highly chirping signals from young neutron stars [23]. However a low saturation amplitude would also keep *r*-modes active in neutron stars up to thousands of years old under the right conditions [26]. Therefore, it is interesting to search for continuous nearly periodic *r*-mode GW emission from Cas A and other very young pulsing or nonpulsing neutron stars.

However up to now the continuous GW observational literature has assumed that GW emission is from "ellipticity," i.e. an $\ell = m = 2$ mass quadrupole which rotates with the star. By contrast *r*-modes are dominated by $\ell = m = 2$ current quadrupole emission [20]. The multipole structure of the radiation field affects the wave polarization and detector response to a signal. These in turn affect detection statistics and parameter estimation, most notably the procedure used to convert the detector response $h(t)$ into an intrinsic GW strain amplitude h_0 (or an observational upper limit on it). Also for an *r*-mode the astrophysical interpretation of a measured h_0 or upper limit must lead to a mode amplitude rather than an ellipticity. (If the mode amplitude evolves in a complicated way, as predicted under some conditions [27], an observational upper limit would apply to a time-rms amplitude.) Finally, indirect limits on GW emission used as milestones for search sensitivities and to evaluate potential searches are different for *r*-modes than for ellipticity.

In this paper I address these amplitude- and polarization-related issues for *r*-mode GW emission. This is sufficient for broad-band searches, i.e., those not targeting a known pulsar. I do not address two important issues related to frequency and phase: The *r*-mode frequency as a function of neutron-star spin frequency depends on the equation of state and relativistic effects (e.g. [28]). Frequency estimates are needed for GW searches targeted at known pulsars and will be addressed elsewhere [29]. Also, in some scenarios [27] the *r*-modes may not maintain phase coherence over likely observing times. This also will be addressed elsewhere [30].

The rest of this paper goes as follows: In Sec. II, I briefly review the basics of *r*-modes and the multipole formalism of GW generation. In Sec. III, I show how these issues relate to GW observations, translating the theory-oriented quantities in Sec. II to those used in the observational

literature. I present indirect limits on r-mode GW emission in Sec. IV. I summarize in Sec. V. Unless otherwise noted I use geometrized units, where Newton's gravitational constant and the speed of light are unity, and $1 M_\odot = 4.92 \times 10^{-6}$ s. I also use "neutron star" in a broad sense which includes the possibility of quark matter, meson condensates, and other exotica.

II. MULTIPOLE STRUCTURE

Here I describe ellipticity and r-mode GW emission in the context of the multipole formalism for GW generation, particularly the canonical review by Thorne [31].

The LIGO and Virgo continuous-wave observational literature characterizes the mass quadrupole moment of a neutron star in terms of an equatorial ellipticity

$$\epsilon = (I_{xx} - I_{yy})/I_{zz}, \qquad (1)$$

where the z axis is the rotation axis of the star and I_{ab} is the moment of inertia tensor. The latter is identical to the mass quadrupole tensor up to a trace, which is not important here. The x and y axes are chosen to corotate with the star so that ϵ does not oscillate. It is convenient to relate ϵ to the scalar mass multipoles $I^{\ell,m}$, which are given in a nonrotating frame and the Newtonian limit by [31]

$$I^{\ell,m} = \frac{16\pi}{(2\ell+1)!!}\sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}} \int d^3 r \rho r^\ell Y^{\ell,m*}, \qquad (2)$$

with ρ representing the mass density, r the radial coordinate, and $Y^{\ell,m}$ the standard spherical harmonic. For a real-valued density perturbation fixed with respect to a star rotating with angular velocity Ω, the time dependence of these is $I^{\ell,m} = I^{\ell,-m*} \propto e^{-im\Omega t}$. From, e.g., Eq. (4.7a) of Ref. [31] we obtain the relation

$$|I^{2,2}| = \sqrt{\frac{8\pi}{5}} I_{zz}\epsilon. \qquad (3)$$

This can be checked by examining the GW luminosity: Inserting Eq. (3) into Eq. (4.16) of Ref. [31] obtains, e.g., Eq. (20) of Ref. [6]:

$$\frac{dE}{dt} = \frac{\omega^6}{16\pi}|I^{2,2}|^2 = \frac{\omega^6}{10}I_{zz}^2\epsilon^2, \qquad (4)$$

where ω is the angular frequency of the wave, equal to $m\Omega$ for a perturbation rotating with the star. (The above expressions assume a negligible contribution from all mass multipoles except $I^{2,2} = I^{2,-2*}$, but there are arguments that $I^{2,1}$ also can be significant in some stars—see Jones [32] for a new one and a summary of older ones.)

In Newtonian gravity and the slow-rotation approximation, an r-mode is an Eulerian velocity perturbation [33]

$$\delta v_j = \alpha \Omega R (r/R)^\ell Y_j^{B,\ell,\ell} e^{i\omega t}, \qquad (5)$$

where α is a dimensionless amplitude, R is the stellar radius, and the magnetic-parity vector spherical harmonic is defined in terms of scalar spherical harmonics $Y^{\ell,m}$ as [31]

$$Y_j^{B,\ell,m} = [\ell(\ell+1)]^{-1/2}\epsilon_{jkp}rN^k\nabla^p Y^{\ell,m}, \qquad (6)$$

where N^k is the unit vector pointing from the center of the star. The GW frequency $f = \omega/(2\pi)$ is identical to the mode frequency, which is related to the spin frequency by [15]

$$\omega = -\frac{(\ell+2)(\ell-1)}{\ell+1}\Omega \qquad (7)$$

(using $\ell = m$ for the proper r-modes).

The GW strain tensor can be expanded as [31]

$$h_{jk} = \frac{1}{r}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{+\ell}\left(\frac{d}{dt}\right)^\ell (I^{\ell,m}T_{jk}^{E2,\ell,m} + S^{\ell,m}T_{jk}^{B2,\ell,m}), \qquad (8)$$

where the spin-2 tensor spherical harmonics of electric (E) and magnetic (B) parity are defined [31]

$$T_{jk}^{E2,\ell,m} = \sqrt{\frac{2(\ell-2)!}{(\ell+2)!}}r^2\nabla_j\nabla_k Y^{\ell,m} - \text{trace}, \qquad (9)$$

$$T_{jk}^{B2,\ell,m} = N_p T_{q(j}^{E2,\ell,m}\epsilon_{k)pq}. \qquad (10)$$

As well as the mass multipoles $I^{\ell,m}$ we encounter the gravitomagnetic or current multipoles

$$S^{\ell,m} = \frac{-32\pi}{(2\ell+1)!!}\sqrt{\frac{\ell+2}{2(\ell-1)}}\int d^3 r\rho r^\ell v_j Y_j^{B,\ell,m*}. \qquad (11)$$

The time derivatives in Eq. (8) suppress GW emission from higher-ℓ multipoles by powers of a characteristic velocity, and thus the $\ell = 2$ multipoles tend to be dominant. For r-modes, the $\ell = 2$ mode is also the most unstable and the least damped by viscosity [20].

It is also useful to write the full GW luminosity including all multipoles [31]

$$\frac{dE}{dt} = \frac{1}{32\pi}$$
$$\times\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{+\ell}\left\langle\left|\left(\frac{d}{dt}\right)^{\ell+1}I^{\ell,m}\right|^2 + \left|\left(\frac{d}{dt}\right)^{\ell+1}S^{\ell,m}\right|^2\right\rangle, \qquad (12)$$

where the angle brackets denote a time average.

By using Eq. (11) and orthonormality relations from Thorne [31], we see that an $\ell = 2$ r-mode produces a current quadrupole [23]

$$S^{2,2} = -\frac{32\sqrt{2}\pi}{15}\alpha M\Omega R^3 \tilde{J} e^{i\omega t}, \qquad (13)$$

where $\tilde{J}$ is a dimensionless functional of the neutron-star equation of state and M and R are the mass and radius of

the (unperturbed) star. Neglecting the r-mode density perturbation, all multipoles other than $S^{2,2}$ vanish—including $I^{2,2}$ and $I^{2,-2}$.

The r-mode amplitude α which I use here is related to others as follows: The results of Arras $et\ al.$ [26] are expressed in terms of $A_1 = \alpha\sqrt{\tilde{J}/2}$, sometimes called c_α in other papers, which is about 0.1α for fiducial neutron-star parameters (see below). Bondarescu $et\ al.$ [27] express most of their results in terms of $\bar{C}_\alpha$, which removes the adiabatic change of amplitude as the star spins down. For the example mode triplet they use, the zero-viscosity parametric instability threshold is normalized to $|C_\alpha|_0 = 0.011\ \mathrm{s}^{-1/2}$. This makes their $\bar{C}_\alpha$ amplitudes about $1.1 \times 10^3 \alpha\sqrt{\Omega/8.4 \times 10^3\ \mathrm{s}^{-1}}$ or about 200α at a spin frequency of 100 Hz.

In Eqs. (5) and (7) and the others derived from them, I have neglected corrections of relative order $\Omega^2/(\pi\bar{\rho})$ where $\bar{\rho}$ is the mean density of the nonrotating star. These include the density perturbation associated with an r-mode, which contributes to the GW tensor and mass quadrupole also at relative order $\Omega^2/(\pi\bar{\rho})$. These corrections rise to the 10% level only for stars rotating faster than the fastest known pulsars [22]. In view of the uncertain factors of 2 or more in quantities such as the moment of inertia (see below), these and the somewhat larger corrections due to relativistic gravity (e.g., [34]) are not important. The exception is the correction to Eq. (7), which affects narrow-band searches for known pulsars and will be addressed elsewhere [29].

III. DIRECT OBSERVATIONS

Here I discuss the effects of the r-mode multipole structure on aspects of direct GW observations, including polarizations, detection statistics, waveform parameter estimation (especially amplitude), and upper-limit procedures.

Equation (8) implies that GWs from r-modes will have different polarizations than GWs from ellipticity. To see this compare two GWs with equal luminosity (12), one dominated by a mass multipole $I^{\ell,m}$ and one dominated by the corresponding current multipole $S^{\ell,m}$. The linear $+$ and $\times$ polarizations are projected out of the GW strain tensor as

$$h_+ = h_{jk}(p^j p^k - q^j q^k)/2 = h_{jk}e_+^{jk},$$
$$h_\times = h_{jk}(p^j q^k + q^j p^k)/2 = h_{jk}e_\times^{jk}, \qquad (14)$$

where the unit vectors p^j and q^j are orthogonal to each other and to N^j, and the basis tensors e^{jk} are not important here. Using the identity $q_j = \epsilon_{jab}N_a p_b$ with Eq. (10), we find that

$$T_{jk}^{B2,\ell,m}e_+^{jk} = -T_{jk}^{E2,\ell,m}e_\times^{jk},$$
$$T_{jk}^{B2,\ell,m}e_\times^{jk} = T_{jk}^{E2,\ell,m}e_+^{jk}. \qquad (15)$$

For a fixed luminosity, the GW tensor h_{jk} of the current-dominated GW is obtained from that of the mass-dominated signal by taking $T_{jk}^{E2,\ell,m} \to T_{jk}^{B2,\ell,m}$. Substituting Eq. (8) into Eq. (14) and combining with Eq. (15), we see that this takes

$$(h_+, h_\times) \to (-h_\times, h_+). \qquad (16)$$

Since the $+$ and $\times$ polarizations are positive and negative parity eigenstates, and the difference between mass and current multipoles is parity, this was to be expected. Note that the GW power radiated at a given angle from the star's rotation axis is proportional to $h_+^2 + h_\times^2$—e.g., Eqs. (4.12) and (4.13) of Thorne [31]—and thus is not affected.

The strain response of a detector is the contraction of the GW tensor h_{jk} with a tensor describing the detector's beam pattern. It can be written by combining Eqs. (10)–(11) and (20)–(22) of Ref. [35] as (making the time dependence explicit)

$$h(t) = [a(t)\cos2\psi + b(t)\sin2\psi]h_+(t)$$
$$+ [b(t)\cos2\psi - a(t)\sin2\psi]h_\times(t), \qquad (17)$$

where ψ is the polarization angle, explained simply in footnote 4 of Ref. [1], whose definition is equivalent to fixing p^j and q^j in Eq. (14). (The precise forms of the modulation functions a and b arising from the rotation of the Earth-based detector are not needed here.) From this we see that Eq. (16) is equivalent to taking

$$\psi \to \psi + \pi/4. \qquad (18)$$

The transformation (18) lets us quickly examine the suitability for r-modes of data analysis methods developed for ellipticity.

Many GW search methods assume a uniform Bayesian prior on ψ and thus are not affected by the transformation (18). The $\mathcal{F}$-statistic of Jaranowski $et\ al.$ [35] is one, as can be seen by the lack of ψ in their Eq. (55) and its derivation. The $\mathcal{F}$-statistic was derived as a frequentist maximization over ψ (and other angles); but when deriving a Bayesian alternative $\mathcal{B}$-statistic, Prix and Krishnan [36] explicitly showed that $\mathcal{F}$ has an implicit uniform prior on ψ. The $\mathcal{B}$-statistic itself has an explicit uniform prior on ψ. The multi-interferometer $\mathcal{F}$-statistic [37] performs a weighted sum of $\mathcal{F}$ over detectors and thus also has an implicit uniform prior in ψ. The heterodyning method used in most known-pulsar GW searches is Bayesian and usually uses an explicit uniform prior in ψ, as in Eq. (15) of Dupuis and Woan [38]. Semicoherent searches for unknown pulsars based on combining the raw power [8,9,11] or $\mathcal{F}$-statistics [10,12] of short stretches of data also effectively use implicit uniform priors on ψ (see Abbott $et\ al.$ [9], especially the Appendices, for details of the former). The radiometer search for Sco X-1 [7] was adapted from a stochastic background search and explicitly assumes no preferred polarization angle.

Some recent known-pulsar searches [4,5] use values of ψ from measurements of pulsar wind nebulae. The measurement and error estimates can be folded into a Bayesian method [38] as nonuniform priors on ψ, or the best ψ value can be inserted into the $\mathcal{F}$-statistic to obtain the $\mathcal{G}$-statistic [39]. In order to cover the possibility of r-modes, such known-pulsar searches will need to target not only different GW frequencies from the ellipticity case but also $\psi \rightarrow \psi + \pi/4$.

Parameter estimation of candidate signals is also affected by Eq. (18). The observational literature characterizes the continuous GW amplitude with an intrinsic strain amplitude defined in terms of ellipticity as

$$h_0 = r^{-1}\omega^2 I_{zz}\epsilon = r^{-1}\omega^2\sqrt{\frac{5}{8\pi}}|I^{2,2}|, \qquad (19)$$

where the second equality uses Eq. (3). This h_0 is the amplitude of the response of a hypothetical detector at either of the Earth's poles to a signal originating from a star over either pole whose rotation axis is parallel to that of the Earth. It is also simply related to the GW luminosity. The full parameter estimation problem is lengthy and I do not address it here, but I note the following simple and useful approximation. A detected signal will be integrated for much more than one day, since even semi–coherent-search candidates will be followed up coherently. In this limit Jaranowski and Krolak [40] performed detailed simulations confirming that the beam-pattern modulation averages out and the signal is effectively replaced by an unmodulated sine wave with amplitude

$$h_{\text{eff}} = h_0\sqrt{A + B\cos(4\psi)}, \qquad (20)$$

where the lengthy expressions for the functions A and B are given in their Eqs. (36) *et seq.* Thus, $\psi \rightarrow \psi + \pi/4$ simply flips the sign of B, which for most source parameters is a few percent correction to h_{eff}. (It may seem strange that h_0 is affected; this is because we are no longer transforming ψ at fixed luminosity but rather for fixed detector response.)

An accurate measurement of ψ could yield information on whether a signal is from ellipticity or an r-mode: For a known pulsar the ratio of GW frequency to spin frequency already distinguishes between mechanisms and thus ψ is redundant. But if the GW signal comes from a pulsar wind nebula without a timed pulsar, and the GW-measured ψ (assuming ellipticity) is inconsistent with the orientation of a jet or torus, this would indicate r-mode emission. Also, a GW signal coming from Sco X-1 or another non-pulsing accreting neutron star could be compared with possible orientations from radio jets [41] or x-ray reprocessing [42] (since the difference between these estimates is less than $\pi/4$).

The practicality of a GW-only measurement of ψ can be roughly estimated as follows: A GW detection (or candidate) with a semi-coherent search (which is needed for the

objects described above) will be followed up with a coherent search, producing a result of comparable sensitivity to a known-pulsar search. From Fig. 2 of the latest known-pulsar search [5], we see that ψ is measured for the loudest hardware-injected signals to an accuracy of a few times 10^{-3} radians. Comparing Fig. 2 and Table 1 of Ref. [5], we see that these loud injections had $h_0 \sim 10^{-23}$, a few hundred times the upper limits on h_0 (a few times 10^{-26}) for real pulsars at comparable frequencies. Therefore, since measurement uncertainties scale roughly as $1/h_0$, the uncertainty in ψ would be of order 1 rad near the sensitivity threshold of the coherent follow-up. Since a coherent search is typically a few times more sensitive than a semi-coherent one, a signal found by a coherent follow-up is likely to be a few times stronger than the coherent-search threshold. Thus, the uncertainty in ψ should be somewhat less than a radian and the $\pi/4$ difference in ψ between ellipticity and r-modes should be borderline detectable.

Upper limits on h_0 from the uniform-ψ searches are obtained in terms of populations of software-injected signals uniformly distributed in ψ, and hence the upper-limit procedures for ellipticity remain valid for r-modes. Upper limits on h_0 from searches with a given ψ determined, e.g., by a pulsar wind nebula will need to use populations of injections taking Eq. (18) into account to produce separate limits for ellipticity and r-modes.

To convert an estimated or upper-limit h_0 to an r-mode amplitude, it is convenient to write the GW luminosity as

$$\frac{dE}{dt} = \frac{1}{10}\omega^2 r^2 h_0^2, \qquad (21)$$

and compare it to Eq. (12),

$$\frac{dE}{dt} = \frac{\omega^6}{32\pi}|S^{2,2}|^2 \qquad (22)$$

if $S^{2,2}$ is the only nonvanishing multipole. [Note that the numerical coefficient is $1/2$ that of Eq. (4) because the r-mode is traditionally defined as a complex perturbation while the ellipticity is defined as real, and $e^{i\omega t}$ contains twice as much power as $\cos(\omega t)$ which lacks the imaginary part.] Equating (21) and (22) obtains

$$h_0 = \sqrt{\frac{8\pi}{5}}r^{-1}\omega^3\alpha M R^3\tilde{J}. \qquad (23)$$

This equation allows us to convert h_0 to α for a fiducial value of $MR^3\tilde{J}$, in the same way that h_0 is converted to ϵ for a fiducial value of $I_{zz} = 10^{45}$ g cm^2. (In both cases what is really measured is a quadrupole—and, if the star's rotation axis is known, the parity which determines if it is a mass or current quadrupole.) Much of the theoretical neutron-star literature uses a polytropic equation of state with polytropic index $n = 1$, which (for Newtonian gravity) fixes $\tilde{J} \approx 0.0164$ [23]. Combined with the above choice of I_{zz} and the usual choice of $M = 1.4M_\odot$, this fixes $R \approx 11.7$ km rather than the common choices of 10 km

and 12.53 km. Inverting Eq. (23) and substituting the fiducial neutron-star values above, we obtain

$$\alpha = 0.028 \left(\frac{h_0}{10^{-24}}\right) \left(\frac{r}{1 \text{ kpc}}\right) \left(\frac{100 \text{ Hz}}{f}\right)^3. \quad (24)$$

Note that modern investigations of mode saturation [27] indicate that α may vary substantially over a typical observing time. Since detection statistics respond to integrated signal power, the α inferred from them is really a time-rms average.

Here and below, when I give numerical values I assume the fiducial neutron-star parameters above and I do not write scalings with M, R, I_{zz}, and $\tilde{J}$. That is because, even if the equation of state (EOS) is not specified, as long as it is barotropic only two of the quantities are independent. By noting that I_{zz} can be written as MR^2 times a relatively EOS-independent function (e.g. Lattimer and Prakash [43]) except for very low-mass stars, and that $\tilde{J}$ is less EOS-dependent than MR^3, I estimate that the EOS-related uncertainty in these quantities is dominated by uncertainties in M and R and is therefore usually a factor 2–3.

IV. INDIRECT LIMITS

Indirect limits on GW emission inferred from electromagnetic astronomical observables are useful to gauge the astrophysical interest of existing and planned GW searches. As in Ref. [44], where I summarize this and related issues in more detail, I divide the sources into the four categories of accreting neutron stars, known pulsars and nonpulsing stars without accretion, and the large population of unseen neutron stars sought by all-sky surveys.

For known pulsars the primary indirect limit is the "spin-down limit (sd)," obtained by assuming that all of the observed change in spin frequency is due to GW emission. Thus Eq. (21) is equated to the kinetic energy loss

$$\frac{dE}{dt} = I_{zz}\Omega\dot{\Omega} \approx (9/16)I_{zz}\omega\dot{\omega} \quad (25)$$

to obtain

$$h_0^{\text{sd}} = \frac{1}{r}\sqrt{\frac{45 I_{zz}\dot{P}}{8P}}, \quad (26)$$

where $P = 2\pi/\Omega$ is the observed pulse period. For r-modes this is $3/2$ times the value for ellipticity-dominated emission, given for instance by Eq. (2) of Wette *et al.* [13]. Substituting Eq. (23) obtains

$$\alpha_{\text{sd}} = \frac{405}{4096\pi^{7/2}}\sqrt{I_{zz}\dot{P}P^5}M^{-1}R^{-3}\tilde{J}^{-1}$$

$$= 1.4\left(\frac{P}{10 \text{ ms}}\right)^{5/2}\left(\frac{\dot{P}}{10^{-10}}\right)^{1/2}$$

$$= 0.033\left(\frac{100 \text{ Hz}}{f}\right)^{7/2}\left(\frac{|\dot{f}|}{10^{-10} \text{ Hz s}^{-1}}\right)^{1/2}, \quad (27)$$

where the last equality uses $f = 4/3/P$ from Eq. (7) with $\ell = 2$. Again, these are time-rms α limits if α fluctuates over the observation time as possible in many young neutron-star scenarios [27]. Based on numbers from the online version 1.40 of the Australian Telescope National Facility catalog [45], the pulsars with dipole spin-down ages $P/(2\dot{P})$ less than 10^4 yr generally have α_{sd} of order unity, except for J0537–6910 which has about 0.1. Reisenegger and Bonacic [46] also pointed out that, under the right conditions of cooling and viscosity, r-modes could remain active in millisecond pulsars for a long time (comparable to the spin-down age) after the star stops accreting as a low-mass x-ray binary. The millisecond pulsars typically have α_{sd} of order 10^{-6}–10^{-4}, with a few as high as order 10^{-2}.

A numerically stricter but more model-dependent indirect limit was obtained by Palomba [47] for a few known pulsars by incorporating additional information on the age and braking index

$$n = \Omega\ddot{\Omega}/\dot{\Omega}^2. \quad (28)$$

If $\dot{\Omega}$ is proportional to a power of Ω, the braking index picks out that power. For radiative braking from a static (electromagnetic or GW) multipole ℓ, the power is $n = 2\ell + 1$, while for particle winds it can be lower. The highest value usually considered is $n = 7$, which corresponds to constant α. As first argued from adiabatic invariance [48] and later seen explicitly in nonlinear hydrodynamic saturation calculations [27], the long-term average of α might scale as $\Omega^{-1/2}$, resulting in $n = 6$. Palomba [47] took $\dot{\Omega}$ to be a sum of two powers of Ω, one power 5 (GWs from ellipticity) and one a free parameter; and performed numerical spin-down evolutions of the pulsars over a wide parameter space constrained to be consistent with their known ages and present values of Ω, $\dot{\Omega}$, and n. Because all observed braking indices are less than 3, the Ω^5 GW component of $\dot{\Omega}$ is constrained to be less than the spin-down limit by some factor—in the case of the Crab pulsar, this limit on h_0 is 2.5 times stricter than the spin-down limit. Without performing such detailed simulations, it is clear that since r-modes have $\dot{\Omega}$ proportional to higher powers of Ω, this type of limit on the Crab would be stricter than the spin-down limit by more than the factor 2.5 for ellipticity.

Compare α_{sd} to the theory of r-mode nonlinear hydrodynamical saturation: The lowest zero-viscosity parametric instability threshold ($\bar{C}_\alpha = 1$ in the notation of Bondarescu et al. [27]) corresponds to α a few times 10^{-3} for the frequencies of interest. The real threshold, which tends to serve as an attractor for r-mode evolutions, depends on temperature as well as frequency and can be an order of magnitude higher (e.g. their Fig. 6). But it is still below the values of α_{sd} for young pulsars, which are high enough to lie in the "run-away" regime of nonlinear hydrodynamics which requires further study [27]. For

milliseconds pulsars, the appropriate comparison is to lower values (see the discussion of accreting stars below).

For nonaccreting neutron stars without observed spins, such as the central compact object in Cas A, age-based indirect limits (age) are obtained by substituting the age of the object into the spin-down limits under the assumption that the object has spun down significantly and predominantly by GW emission [13]. In this case we have the relation

$$P/\dot{P} = (n-1)a, \tag{29}$$

where a is the age of the neutron star and the braking index n is assumed to be constant or appropriately averaged. In parallel to the derivation of the ellipticity version of this limit [13], we substitute Eq. (29) into Eq. (26) to obtain $\sqrt{3/2}$ the value for ellipticity, or

$$h_0^{\text{age}} = 1.5 \times 10^{-24} \left(\frac{300 \text{ yr}}{a}\right)^{1/2} \left(\frac{3.4 \text{ kpc}}{r}\right) \left(\frac{6}{n-1}\right)^{1/2}. \tag{30}$$

(The fiducial values are for Cas A and constant-α evolution.) Similarly for the age-based indirect limit on r-mode amplitude we obtain

$$\alpha_{\text{age}} = \frac{15}{8\sqrt{\pi}} \tilde{J}^{-1} I^{1/2} M^{-1} R^{-3} \omega^{-3} a^{-1/2} (n-1)^{-1/2}$$
$$= 0.14 \left(\frac{300 \text{ yr}}{a}\right)^{1/2} \left(\frac{100 \text{ Hz}}{f}\right)^3 \left(\frac{6}{n-1}\right)^{1/2}, \tag{31}$$

where again the limit is a time-rms value if α fluctuates quickly compared to the spin-down timescale. For Cas A, the youngest known neutron star, the values of α_{age} are 0.14–0.005 over the 100–300 Hz band considered by Wette et al. [13] and searched by LIGO [14]. Several similar objects, e.g., the nonpulsing ones listed by Halpern and Gotthelf [49], have indirect limits of the same order of magnitude. At a few hundred Hz these values are comparable to the parametric instability thresholds in several scenarios analyzed by Bondarescu et al. [27], with zero-viscosity α thresholds a few times 10^{-3} and true thresholds several times higher.

For accreting neutron stars the standard indirect limit is derived from the argument that accretion torque (acc) and GW torque are in equilibrium, originally made by Papaloizou and Pringle [50], put on firm astrophysical footing by Wagoner [51], and later tied to observations of spins of accreting neutron stars [24,25]. From $dJ/dt = (2/\omega)dE/dt$—as in Eq. (4.23) of Thorne [31]—we have

$$h_0^2 = 5\omega^{-1}r^{-2}dJ/dt, \tag{32}$$

equivalent to Eq. (4) of Watts et al. [52]. The indirect limit on h_0 is obtained by assuming accretion of Keplerian angular momentum at the stellar surface and 100% conversion of gravitational potential energy to x-ray flux F_x measured at Earth to obtain

$$h_0^{\text{acc}} = \sqrt{20\pi}\omega^{-1/2}F_x^{1/2}M^{-1/4}R^{3/4}$$
$$= 3.3 \times 10^{-26} \left(\frac{F_x}{3.9 \times 10^{-7} \text{ erg cm}^{-2}}\right)^{1/2}$$
$$\times \left(\frac{800 \text{ Hz}}{f}\right)^{1/2}. \tag{33}$$

Here the numerical value is scaled to the average bolometric flux of Sco X-1 (which does not pulse) and the highest spin rate observed in an accreting millisecond pulsar, both from Watts et al. [52]. For a given GW frequency h_0^{acc} is the same as the indirect limit for ellipticity GW emission, but it is different for a fixed spin period. Combining with Eq. (23) we obtain

$$\alpha_{\text{acc}} = \frac{135\sqrt{3}}{2048\pi^{7/2}} F_x^{1/2} P^{7/2} r M^{-5/4} R^{-9/4} \tilde{J}^{-1}$$
$$= 5.1 \times 10^{-6} \left(\frac{F_x}{3.9 \times 10^{-7} \text{ erg cm}^{-2}}\right)^{1/2}$$
$$\times \left(\frac{r}{2.8 \text{ kpc}}\right) \left(\frac{800 \text{ Hz}}{f}\right)^{7/2}. \tag{34}$$

For comparison, Bondarescu et al. [53] find that r-mode evolutions of accreting neutron stars have $\alpha \approx 10^{-4}$ if there is no runaway (here the viscosities and thus the parametric instability thresholds differ from those of young neutron stars). That nominally corresponds to $P = 4$ ms or $f \approx 300$ Hz for Sco X-1 at the indirect limit. Since the r-mode amplitudes in these saturation studies are uncertain by at least a factor of a few due to uncertainties in stellar structure and damping rates, this is not a precise prediction of the spin period but rather a statement that Sco X-1 is consistent with having a short spin period regulated by r-modes at or near equilibrium. (This was independently found in a more detailed examination of possible r-mode evolutions [54].) However nonlinear mode evolutions may avoid accretion torque equilibrium altogether, may spin up as well as down, and may go through intermittent episodes of r-mode activity; and realistic accretion may be more complicated than what is usually assumed. Therefore, indirect limits from accretion equilibrium are much softer than the spin-down and age-based limits, which are derived from energy conservation. (This is a good thing, since the former are much more pessimistic than the latter [44].)

All-sky surveys for continuous GWs are subject to statistically estimated indirect limits derived from assumptions about the galactic supernova rate and distribution of neutron-star parameters at birth. A simple version due originally to Blandford (unpublished) is described in Abbott et al. [6] and is not changed much by r-modes since it essentially relies on h_0^{age} and a planar galactic geometry to yield a population estimate (pop) of the brightest signal h_0^{pop} a few times 10^{-24} independent of frequency and the poorly known ellipticity. Simulations by Knispel and Allen [55] using realistic spatial distributions of neutron stars

produce h_0^{pop} values which do depend on ellipticity and frequency. Investigating the effect of r-mode emission on this type of indirect limit estimate is a substantial undertaking which I do not attempt here. However r-mode emission has a steeper frequency dependence than ellipticity emission, resulting in r-modes dominating dE/dt for

$$\alpha > \frac{3\sqrt{5}}{16\pi} \frac{I_{zz}}{M\Omega R^3 \tilde{J}} \epsilon = 87\epsilon \left(\frac{P}{10 \text{ ms}}\right) \qquad (35)$$

(obtained by equating $I^{2,2}$ to $S^{2,2}$). This condition is satisfied even by the lowest parametric instability threshold for the frequencies and ellipticities contributing to the main result (Fig. 5) of Knispel and Allen [55]. Thus, the presence in a similar simulation of a significant population with active r-modes should spin down stars more quickly into the best frequencies for GW detection and produce a higher h_0^{pop}, even assuming the scenarios of [27] and not invoking a low-viscosity runaway.

The last form of the indirect limit (27) for known pulsars can also be used to interpret direct upper limits from all-sky surveys by overlaying the contours of $\alpha_{\text{sd}}(f, \dot{f})$ on a plot similar to Fig. 41 of Abbott et al. [9]. Following the discussion around that figure we can say: The range of the multi-interferometer Hough search from Abbott et al. [9] was about 1 kpc at $f = 100$ Hz and $|\dot{f}| = 10^{-8}$ Hz/s, if $\dot{f}$ for a star is GW dominated—and Eq. (27) tells us that would require $\alpha \approx 0.3$. A star at $f = 1$ kHz with the same $\dot{f}$ would have been detectable up to 100 pc (comparable to the closest known neutron stars) if GW dominated, and this would require $\alpha \approx 1 \times 10^{-4}$, which is well within the range of possibilities for nonlinear evolutions.

V. DISCUSSION

I have shown that current data analysis methods can detect or set valid upper limits on continuous GW emission from r-modes, in some cases with small modifications. I have derived relations needed to interpret GW observational results in terms of r-mode emission. This allows broad-band searches for continuous GWs to infer r-mode amplitudes or upper limits. (After this paper was submitted, the LIGO Cas A paper [14] showed the first such limits.)

Searches for r-modes in known pulsars will also require information on the ratio of GW frequency to spin frequency [29], and will be more sensitive to the issue of r-mode phase coherence time due to their long integration times [30].

I have also rederived some commonly used indirect limits on GW emission for the case of r-modes. Several young pulsars and nonpulsing neutron stars are interesting targets for searches for r-mode GW emission, and all-sky surveys can have interesting ranges, even with presently available LIGO and Virgo data. More young objects further away will be interesting with advanced LIGO and Virgo.

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[1] B. Abbott et al., Phys. Rev. D **69**, 082004 (2004).
[2] B. Abbott et al., Phys. Rev. Lett. **94**, 181103 (2005).
[3] B. Abbott et al., Phys. Rev. D **76**, 042001 (2007).
[4] B. Abbott et al., Astrophys. J. **683**, L45 (2008).
[5] B. P. Abbott et al., Astrophys. J. **713**, 671 (2010).
[6] B. Abbott et al., Phys. Rev. D **76**, 082001 (2007).
[7] B. Abbott et al., Phys. Rev. D **76**, 082003 (2007).
[8] B. Abbott et al., Phys. Rev. D **72**, 102004 (2005).
[9] B. Abbott et al., Phys. Rev. D **77**, 022001 (2008).
[10] B. Abbott et al., Phys. Rev. D **79**, 022001 (2009).
[11] B. P. Abbott et al., Phys. Rev. Lett. **102**, 111102 (2009).
[12] B. P. Abbott et al., Phys. Rev. D **80**, 042003 (2009).
[13] K. Wette et al., Classical Quantum Gravity **25**, 235011 (2008).
[14] J. Abadie et al., Astrophys. J. **722**, 1504 (2010).
[15] J. Papaloizou and J. E. Pringle, Mon. Not. R. Astron. Soc. **182**, 423 (1978).
[16] N. Andersson, Astrophys. J. **502**, 708 (1998).
[17] J. L. Friedman and S. M. Morsink, Astrophys. J. **502**, 714 (1998).
[18] S. Chandrasekhar, Phys. Rev. Lett. **24**, 611 (1970).
[19] J. L. Friedman and B. F. Schutz, Astrophys. J. **222**, 281 (1978).
[20] L. Lindblom, B. J. Owen, and S. M. Morsink, Phys. Rev. Lett. **80**, 4843 (1998).
[21] N. Andersson, K. D. Kokkotas, and B. F. Schutz, Astrophys. J. **510**, 846 (1999).
[22] L. Lindblom, G. Mendell, and B. J. Owen, Phys. Rev. D **60**, 064006 (1999).
[23] B. J. Owen et al., Phys. Rev. D **58**, 084020 (1998).
[24] L. Bildsten, Astrophys. J. **501**, L89 (1998).
[25] N. Andersson, K. D. Kokkotas, and N. Stergioulas, Astrophys. J. **516**, 307 (1999).
[26] P. Arras et al., Astrophys. J. **591**, 1129 (2003).

[27] R. Bondarescu, S. A. Teukolsky, and I. Wasserman, Phys. Rev. D **79**, 104003 (2009).

[28] S. Yoshida, S. Yoshida, and Y. Eriguchi, Mon. Not. R. Astron. Soc. **356**, 217 (2005).

[29] D. I. Jones and B. J. Owen (unpublished).

[30] I. Wasserman (unpublished).

[31] K. S. Thorne, Rev. Mod. Phys. **52**, 299 (1980).

[32] D. I. Jones, Mon. Not. R. Astron. Soc. **402**, 2503 (2010).

[33] J. Provost, G. Berthomieu, and A. Rocca, Astron. Astrophys. **94**, 126 (1981).

[34] K. H. Lockitch, J. L. Friedman, and N. Andersson, Phys. Rev. D **68**, 124010 (2003).

[35] P. Jaranowski, A. Krolak, and B. F. Schutz, Phys. Rev. D **58**, 063001 (1998).

[36] R. Prix and B. Krishnan, Classical Quantum Gravity **26**, 204013 (2009).

[37] C. Cutler and B. F. Schutz, Phys. Rev. D **72**, 063006 (2005).

[38] R. J. Dupuis and G. Woan, Phys. Rev. D **72**, 102002 (2005).

[39] P. Jaranowski and A. Krolak, Classical Quantum Gravity **27**, 194015 (2010).

[40] P. Jaranowski and A. Krolak, Phys. Rev. D **59**, 063003 (1999).

[41] E. B. Fomalont, B. J. Geldzahler, and C. F. Bradshaw, Astrophys. J. **553**, L27 (2001).

[42] T. Muñoz-Darias, I. G. Martínez-Pais, J. Casares, V. S. Dhillon, T. R. Marsh, R. Cornelisse, D. Steeghs, and P. A. Charles, Mon. Not. R. Astron. Soc. **379**, 1637 (2007).

[43] J. M. Lattimer and M. Prakash, Astrophys. J. **550**, 426 (2001).

[44] B. J. Owen, Classical Quantum Gravity **26**, 204014 (2009).

[45] R. N. Manchester, G. B. Hobbs, A. Teoh, and M. Hobbs, Astron. J. **129**, 1993 (2005), see http://www.atnf.csiro.au/research/pulsar/psrcat/.

[46] A. Reisenegger and A. A. Bonacic, Phys. Rev. Lett. **91**, 201103 (2003).

[47] C. Palomba, Astron. Astrophys. **354**, 163 (2000).

[48] W. C. G. Ho and D. Lai, Astrophys. J. **543**, 386 (2000).

[49] J. P. Halpern and E. V. Gotthelf, Astrophys. J. **709**, 436 (2010).

[50] J. Papaloizou and J. E. Pringle, Mon. Not. R. Astron. Soc. **184**, 501 (1978).

[51] R. V. Wagoner, Astrophys. J. **278**, 345 (1984).

[52] A. Watts, B. Krishnan, L. Bildsten, and B. F. Schutz, Mon. Not. R. Astron. Soc. **389**, 839 (2008).

[53] R. Bondarescu, S. A. Teukolsky, and I. Wasserman, Phys. Rev. D **76**, 064019 (2007).

[54] R. Bondarescu and I. Wasserman (unpublished).

[55] B. Knispel and B. Allen, Phys. Rev. D **78**, 044031 (2008).