Fields in nonaffine bundles. IV. Harmonious non-Abelian currents in string defects

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This article continues the study of the category of harmonious field models that was recently introduced as a kinetically nonlinear generalization of the well-known harmonic category of multiscalar fields over a supporting brane world sheet in a target space with a curved Riemannian metric. Like the perfectly harmonious case of which a familiar example is provided by ordinary barotropic perfect fluids, another important subcategory is the simply harmonious case, for which it is shown that as well as "wiggle" modes of the underlying brane world sheet, and sound type longitudinal modes, there will also be transverse shake modes that propagate at the speed of light. Models of this type are shown to arise from a non-Abelian generalization of the Witten mechanism for conducting string formation by ordinary scalar fields with a suitable quartic self-coupling term in the action.

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I. INTRODUCTION

This article continues a series in which a systematic covariant differentiation procedure was developed [1,2] and applied [3] to multiscalar field models for which the relevant target space lacks the usual linear (vectorial or affine) structure, but is curved, as in the prototypical example [4] of a harmonic map, which is governed by dynamical equations involving nonlinear dependence on the fields but only linear dependence on their space-time gradients. As well as the possibility of confinement to string or brane world sheets, and the gauging of internal symmetries of the target space, two different kinds of generalization of the ordinary harmonic category were considered. The first [2,3] was that of forced-harmonic models in which a harmonic type kinetic term is supplemented in the Lagrangian by a self-coupling term $\hat{\mathcal{V}}$ having the form of a predetermined scalar field on the target space. The second kind was that of harmonious models, whose definition will be recapitulated in the next section, and which differ from ordinary harmonic and forced-harmonic models in that their dynamical equations involve nonlinear dependence not just on the fields but also on their gradients.

Nonlinear gradient dependence of the kind in question has long been familiar in the context of irrotational fluid models such as are relevent for superfluidity [5,6] (and perhaps also for cosmology [7,8]). However, the target space in these examples is of the usual flat vectorial kind, as also is that of their cosmic string supported analogues of both the singly [9,10] and multiply [11] conducting kinds that have been studied in recent years.

Interest in such string supported fields began when Witten [12] drew attention to the fact that they would arise naturally by condensation in the cores of string (or other) topological defects in space-time field models of the commonly considered—kinematically linear—kind in which the only nonlinearity is that of a scalar self-coupling term responsible for spontaneous symmetry breaking of the vacuum. It was subsequently recognized that suitable macroscopic models [13,14] for the description of such fields in the thin string limit would need to involve nonlinear field gradient dependence of the type qualifiable as harmonious, though only of the rather trivial kind in which the target space is flat and the relevant internal group Abelian.

The main purpose of the present article is to show that a straightforward—still kinematically linear—extension of the class of models proposed by Witten will give rise to nonlinearly harmonious string models of a more interesting kind, in which the target space is curved and the relevant symmetry group non-Abelian.

Before proceeding it is important to warn that the subject treated under the title "non-Abelian string conductivity" by Kibble and his collaborators [15] needs to be distinguished—e.g. by the insertion of another hyphen, so as to obtain "non-Abelian-string conductivity"—from the subject of the present study, which might appropriately be entitled "non-Abelian string-conductivity." The point of this nuance is that Kibble and his collaborators were concerned with a generalization of Witten's model wherein, instead of being attributable to the spontaneous breakdown of an Abelian U{1} symmetry, the string formation was attributable to the spontaneous breakdown of a non-Abelian SU{2} symmetry, but the currents considered by these authors were nevertheless merely Abelian in the sense of being generated only by a distinct U{1} symmetry subalgebra that had survived the breakdown. In contrast, the present work will be concerned with a different kind of generalization of Witten's model, wherein—although the string itself will just be Abelian, in the sense of being attributable to the spontaneous breakdown only of an Abelian U{1} symmetry subgroup—the currents therein will actually be non-Abelian in the sense of being generated, not just by a U{1} action, but by the action of a surviving non-Abelian SU{2} or higher symmetry algebra.

Although the generalization of the category of underlying kinetically linear space-time field models is straightforward, the Witten mechanism itself cannot be directly employed for the construction of a curved target space, as it depends on a weak cylindrical symmetry ansatz that will not be self-consistently applicable in the non-Abelian case. It will be shown below in Sec. VIII how Witten's weak symmetry ansatz can be replaced for this purpose by a more general local geodicity ansatz that works as a good approximation, and can do the job, so long as the currents involved are not too strong compared with a scale that will be estimated in Sec. IX.

It is to be commented that the evident incompatibility of cylindrical symmetry with a nondegenerate mapping into a curved (spherical or more general) target space means that for a string loop retaining several noncommuting currents it will be impossible (as in the simple Goto-Nambu case) to attain a stationary circular vorton type equilibrium state: such a loop will be condemned to go on oscillating until completion of the dissipation of all currents except those of an Abelian subgroup.

II. HARMONIOUS FIELD MODELS

According to the definition of the preceding article [3], a multiscalar field $\bar{\Phi}$ with local components χ^A in a q-dimensional target space over a brane world sheet $\mathcal{S}$ of dimension d (for $d \leq n$, where n is the dimension of the background space-time) will be of *harmonious* type if it is governed by a scalar Lagrangian $\bar{L}$ that is specified by some equation of state as a diffeomorphically invariant function just of the relevant target-space metric $\hat{g}_{AB}$—which is supposed to have been prescribed in advance—and of the corresponding horizontal projection $\hat{\mathfrak{w}}^{AB}$ of the inverse $\bar{g}^{ij}$ of the underlying space-time metric $\bar{g}_{ij}$ on the world sheet. It is to be understood that the horizontality of the projection is specified with respect to some gauge form $\bar{A}_i$ with vectorial components $\bar{A}_i{}^A$ on the target space. We thereby obtain a prescription of the form

$$\hat{\mathfrak{w}}^{AB} = \bar{g}^{ij}\bar{\Phi}^A{}_{|i}\bar{\Phi}^B{}_{|j}, \tag{1}$$

in which the relevant projector will be specified in terms of the field gradient components $\chi^A_{,i}$ by

$$\bar{\Phi}^A{}_{|i} = \chi^A_{,i} + \bar{A}_i{}^A. \tag{2}$$

In terms of the symmetric target-space tensor κ_{AB} given by the definition

$$\kappa_{AB} = -2\frac{\partial \bar{L}}{\partial \hat{\mathfrak{w}}^{AB}} \tag{3}$$

and therefore such that

$$\kappa_C{}^A\hat{\mathfrak{w}}^{BC} = \kappa_C{}^B\hat{\mathfrak{w}}^{AC} = -2\frac{\partial \bar{L}}{\partial \hat{g}_{AB}}, \tag{4}$$

it was shown in the preceding article [3] that, for a system of this harmonious type, the intrinsic dynamical equations will be expressible in terms of a set of surface currents with components given by

$$\bar{J}_A{}^i = \bar{g}^{ij}\kappa_{AB}\bar{\Phi}^B{}_{|j}. \tag{5}$$

as the pseudoconservation laws

$$\bar{D}_i\bar{J}_A{}^i = 0, \tag{6}$$

in which $\bar{D}_i$ is a bitensorially covariant differentiation operator of the not so simple kind introduced in earlier work [1,2], and which will give rise to genuine current conservation laws [3] only when suitable internal symmetry conditions are satisfied.

One of the questions that arises in the study of any system of differential equations of motion is the orientation of the characteristic surfaces, with normal direction λ_i say, along which infinitesimal discontinuities can be propagated. For the transverse "wiggle" perturbation modes of the extrinsic evolution of the supporting world sheet, it is easily shown [10] that—regardless of the internal dynamics—the relevant characteristic equation will always have the simple form

$$\bar{T}^{ij}\lambda_i\lambda_j = 0, \tag{7}$$

where $\bar{T}^{ij}$ are the components of the surface stress-energy tensor, which for a harmonious system of the kind under consideration here will be given [3] by

$$\bar{T}^{ij} = \kappa_{AB}\bar{\Phi}^{A|i}\bar{\Phi}^{B|j} + \bar{L}\bar{g}^{ij}. \tag{8}$$

However, for the internal perturbation modes, within the world sheet, the form of the characteristic equation will depend on the specific details of the system. In particular, for the acoustic type modes of the harmonious system (6) the characteristic equation will in general be rather complicated, reducing to a simple quadratic form like that of (7) only under special conditions, such as those of the simply harmonious and perfectly harmonious categories that will be presented in the following sections.

III. HARMONIOUSLY ELASTIC MODELS

The category of harmonious models includes models of the perfect solid type [16] which belong to the extensive *harmoniously elastic* subcategory that is characterized by the condition that the component matrix $\hat{\mathfrak{w}}^{AB}$ should have a well-defined inverse $\hat{\gamma}_{AB} = \hat{\mathfrak{w}}^{-1}_{AB}$, which, if it exists, will be interpretable as the tensorially well-behaved metric that is locally induced on the target space by the section Φ according to the specification

$$\hat{\gamma}_{AC}\hat{\mathfrak{w}}^{CB} = \delta^B_A, \tag{9}$$

This is something that will be possible only if the target-space dimension q does not exceed the dimension $d = p + 1$ of the supporting base space, which if it is an embedded p-brane world sheet can itself not exceed the dimension n of the background space-time: $q \leq p + 1 \leq n$. The dimensionally maximal case $q = p + 1$ includes various models of the recently investigated kind [17] referred to a hyperelastic, while perfect solids [16] and so, in particular, ordinary fluid models of the barotropic type are included in the case $q = p$.

Harmoniously elastic models are perfectly elastic in the usual sense, meaning [18,19] that they are governed by a Lagrangian that is determined by some prescribed intrinsic structure as a function just of the induced metric $\hat{\gamma}_{AB}$ on the relevant target space, but they are not elastic models of the most general kind: for nonharmoniously elastic models the prescribed intrinsic structure can include various other vectorial or tensorial fields (for example, to allow for the anisotropic grain in wood) as well as the prescribed metric $\hat{g}_{AB}$ which is all that is allowed in the harmonious case. In the usual approach [16,18–20] to the treatment of elastic solid models, the locally induced metric $\hat{\gamma}_{AB}$ is what is used for lowering and raising of target-space indices. It is therefore important to remember that in the dimensionally unrestricted approach [1–3] followed here it is instead the globally prescribed target-space metric $\hat{g}_{AB}$ that it is used for this purpose.

It will be convenient for the following discussion to introduce a new kind of elasticity tensor that is defined, for any harmonious model, by

$$\mathfrak{E}_{ABCD} = \mathfrak{E}_{(CD)(AB)} = 2\frac{\partial \kappa_{AB}}{\partial \hat{\mathfrak{w}}^{CD}}. \tag{10}$$

In terms of this quantity the ordinary elasticity tensor of the usual treatment [16,18–20] will be given by the expression

$$E_{ABCD} = \mathfrak{E}_{ABCD} + 2\kappa_{A(C}\gamma_{D)B} - \kappa_{AB}\gamma_{CD} + 2\gamma_{A(C}P_{D)B} - \gamma_{AB}P_{CD}, \tag{11}$$

in which the pressure tensor is given in terms of the energy density $\rho = -L$ by

$$P_{AB} = \kappa_{AB} - \rho\gamma_{AB}. \tag{12}$$

A simple example [3] is that of the baby Skyrme model [21] for which the target space is a 2-sphere on which $\kappa_{AB} = \kappa_\star g_{AB} + \alpha_\star(\hat{\mathfrak{w}} g_{AB} - \hat{\mathfrak{w}}_{AB})$, where $\kappa_\star$ and $\alpha_\star$ are constants, so that one obtains $\mathfrak{E}_{ABCD} = 2\alpha_\star(g_{AB}g_{CD} - g_{A(C}g_{D)B})$.

IV. THE SOUND CONE IN SIMPLY HARMONIOUS MODELS

Instead of working through the details of a complete perturbation analysis, the characteristic equation governing the propagation of infinitesimal discontinuities is obtainable efficiently by adaptation, from the rather similar case of an elastic solid [20,22], of a method due originally to Hadamard, of which the simplest illustration is provided by the Dalembertian wave equation for a scalar φ say, namely, $\bar{\nabla}_j\varphi^j = 0$ where $\varphi^j = \bar{g}^{ji}\bar{\nabla}_i\varphi$. The idea of the Hadamard method is to use the fact that the discontinuity of the gradient of a continuous quantity will be aligned with the normal covector λ_i of the discontinuity surface. Applying this to the components φ^j, one sees that the discontinuity of their gradients will be given in terms of corresponding discontinuity amplitude components $\tilde{\varphi}^j$ by an expression of the form $[\nabla_i\varphi^j] = \lambda_i\tilde{\varphi}^j$. Moreover, taking the discontinuity of the integrability condition $\bar{\nabla}_{[i}\varphi_{j]} = 0$, one sees that the discontinuity amplitude will have to satisfy $\lambda_{[i}\tilde{\varphi}_{j]} = 0$, and hence that it will be given in terms of some scalar amplitude $\tilde{\varphi}$ by $\tilde{\varphi}_i = \tilde{\varphi}\lambda_i$. Thus finally, taking the discontinuity of the Dalembert equation itself, one obtains the well-known light-cone tangency condition

$$\bar{g}^{ij}\lambda_i\lambda_j = 0. \tag{13}$$

Applying the same line of reasoning to the multiscalar field $\bar{\Phi}$, one sees that the discontinuity of the gradient of its covariant derivative (2) will be given in terms of some set of amplitude components $\tilde{\chi}^A$ by

$$[\bar{\nabla}_i\bar{\Phi}^A{}_{|j}] = \lambda_i\lambda_j\tilde{\chi}^A. \tag{14}$$

It therefore follows from the definition (1) of the horizontally induced metric $\hat{\mathfrak{w}}^{AB}$ that discontinuity of its gradient will be given by

$$[\bar{\nabla}_i\hat{\mathfrak{w}}^{AB}] = 2\lambda_i\lambda_j\tilde{\chi}^{(A}\bar{\Phi}^{B)|j}. \tag{15}$$

These quantities are needed for the evaluation of the discontinuity of the gradient of the current (5), which will be given by the formula

$$[\bar{D}_i\bar{J}_A{}^j] = \kappa_{AB}\bar{g}^{jk}[\bar{\nabla}_i\bar{\Phi}^B{}_{|k}] + \frac{\partial\kappa_{AB}}{\partial\hat{\mathfrak{w}}^{CD}}\bar{\Phi}^{B|j}[\bar{\nabla}_i\hat{\mathfrak{w}}^{CD}], \tag{16}$$

in which the distinction between $\bar{D}_i$ and $\bar{\nabla}_i$ disappears, as the relevant [3] affine and gauge connection terms (but not their derivatives) will be continuous. The discontinuity of the set of pseudoconservation equations (6) thereby provides the required characteristic equation in the form

$$\lambda_i\lambda_j\mathcal{Q}^{ij}{}_{AB}\tilde{\chi}^B = 0, \tag{17}$$

in which, using the notation (10), we shall have

$$\mathcal{Q}^{ij}{}_{AB} = \bar{g}^{ij}\kappa_{AB} + \mathfrak{E}_{ADBC}\bar{\Phi}^{C|i}\bar{\Phi}^{D|j}. \tag{18}$$

It is to be remarked that this formula is simpler than the corresponding expression using the usual elasticity tensor (11) of the traditional approach [16,18–20].

The eigenvalue equation ensuing from (17) may take a rather complicated quartic or higher polynomial form when the target-space dimension is two or more, with a generic equation of state involving dependence not just on the trace

$$\hat{\mathfrak{w}} = \hat{\mathfrak{w}}_A{}^A = \hat{g}_{AB}\hat{\mathfrak{w}}^{AB}, \tag{19}$$

but also on higher order invariants starting with $\hat{\mathfrak{w}}_A{}^B\hat{\mathfrak{w}}_B{}^A$. However, it will conveniently separate into merely quadratic subsystems in what will be referred to as the *simply harmonious* case, namely, that for which the Lagrangian depends *only* on the trace invariant $\hat{\mathfrak{w}}$, so that one obtains

$$\kappa_{AB} = \kappa \hat{g}_{AB}, \tag{20}$$

with

$$\kappa = -2\frac{d\bar{L}}{d\hat{\mathfrak{w}}}. \tag{21}$$

In this simply harmonious case one obtains

$$\mathfrak{E}_{ABCD} = 2\frac{d\kappa}{d\hat{\mathfrak{w}}}\hat{g}_{AB}\hat{g}_{CD}, \tag{22}$$

which gives a characteristic equation of the form

$$\lambda_i\lambda^i\tilde{\chi}^A + \frac{2}{\kappa}\frac{d\kappa}{d\hat{\mathfrak{w}}}\lambda^i\bar{\Phi}^A{}_{|i}\lambda^j\bar{\Phi}^B{}_{|j}\tilde{\chi}_B = 0. \tag{23}$$

It is evident that this will be trivially satisfied by a set of shake modes, propagating at the speed of light, with polarization $\tilde{\chi}^A$ that is transverse to the current across the discontinuity, in the sense that

$$\tilde{\chi}^A\bar{J}_A{}^i\lambda_i = 0, \tag{24}$$

since for such a mode—regardless of the particular linear or nonlinear functional form of the equation of state—the characteristic equation will evidently reduce just to the same nullity condition as in the ordinary Dalembertian case (13), namely,

$$\lambda_i\lambda^i = 0. \tag{25}$$

There will also be a less trivial set of sound type modes with polarization that is longitudinal in the sense of being aligned with the current across the discontinuity, so that for such a mode the discontinuity amplitude vector $\tilde{\chi}^A$ will be given (modulo a multiplicative factor that can be absorbed into the normalization of the characteristic covector λ_i) by the prescription

$$\tilde{\chi}^A = \bar{\Phi}^A{}_{|i}\lambda^i. \tag{26}$$

This reduces the characteristic equation to a simple quadratic form—specifying what is describable as a sound cone—that will be given by

$$\left(\bar{g}_{ij} + \frac{2}{\kappa}\frac{d\kappa}{d\hat{\mathfrak{w}}}\bar{\mathfrak{w}}_{ij}\right)\lambda^i\lambda^j = 0 \tag{27}$$

using the notation

$$\bar{\mathfrak{w}}_{ij} = \hat{g}_{AB}\bar{\Phi}^A{}_{|i}\bar{\Phi}^B{}_{|j} \tag{28}$$

for the gauge covariant pullback of the target-space metric.

It will be shown below how nontrivially nonlinear models of this simply harmonious type arise naturally in the treatment of string defects of multiscalar field theories of the common kinetically linear kind. However, before doing that it is instructive, for the sake of comparison, to describe another noteworthy subcategory, namely, that of *perfectly harmonious* models for which the characteristic equation will be similarly simplifiable.

V. THE SOUND CONE IN PERFECTLY HARMONIOUS MODELS

The subcategory of what are describable as *perfectly harmonious* models is physically important because it includes the generic (not necessarily irrotational) case of an ordinary barotropic perfect fluid. The perfectly harmonious subcategory is defined by the requirement that the dependence of the Lagrangian on the target-space tensor $\hat{\mathfrak{w}}^{AB}$ should again involve only a single scalar invariant, but with the latter now chosen to be the determinant $\det\{\hat{\mathfrak{w}}\}$ of the matrix with components

$$\hat{\mathfrak{w}}_A{}^B = \hat{g}_{AC}\hat{\mathfrak{w}}^{CB}, \tag{29}$$

which will be admissible so long as the target-space dimension q does not exceed the base-space–time dimension $d = p + 1$ (whereas for $q > p + 1$ this determinant would vanish identically). Ordinary perfect fluids are of the particular kind for which the target-space dimension is the same, $q = p$, as the space (as distinct from space-time) dimension of the supporting base, which is three in the usual terrestrial and astrophysical applications, but might be higher for exotic cosmological theories in which the space-time dimension is not four but five or more.

In terms of the tensorial inverse matrix $\hat{\gamma}_{AB} = \hat{\mathfrak{w}}^{-1}{}_{AB}$ of $\hat{\mathfrak{w}}^{AB}$ (which is interpretable as the metric locally induced on the target space by the section Φ) as defined by (9), the stipulation that L should depend only on $\det\{\hat{\mathfrak{w}}\}$ leads to the expression

$$\kappa_{AB} = h\hat{\gamma}_{AB} \tag{30}$$

with

$$h = -2\frac{\det\{\hat{\mathfrak{w}}\}dL}{d(\det\{\hat{\mathfrak{w}}\})}. \tag{31}$$

This quantity h will be interpretable simply as the enthalpy density in the case of an ordinary perfect fluid, for which the pressure tensor (12) takes the form $P_{AB} = P\hat{\gamma}_{AB}$, in which the pressure is given in terms of the energy density $\rho = -L$, and the enthalpy density h by the well-known formula $P = h - \rho$.

The ensuing formula

$$\frac{\partial\kappa_{AB}}{\partial\hat{\mathfrak{w}}^{CD}} = \frac{\det\{\hat{\mathfrak{w}}\}dh}{d(\det\{\hat{\mathfrak{w}}\})}\hat{\gamma}_{AB}\hat{\gamma}_{CD} - h\hat{\gamma}_{A(C}\hat{\gamma}_{D)B}, \tag{32}$$

can be used to reduce the characteristic matrix (18) to the form

$$\mathcal{Q}^{ij}{}_{AB} = \hat{\gamma}_{AB}(h\bar{g}^{ij} - \kappa_{CD}\bar{\Phi}^{C|i}\bar{\Phi}^{D|j}) + \left(2\frac{\det\{\hat{\mathfrak{w}}\}dh}{d(\det\{\hat{\mathfrak{w}}\})} - h\right)\hat{\gamma}_{AC}\bar{\Phi}^{C|i}\hat{\gamma}_{BD}\bar{\Phi}^{D|j}. \quad (33)$$

As before, this will be satisfied by trivial shake modes, with polarization $\tilde{\chi}^A$ that is transverse to the current across the discontinuity in the sense specified by (24), since for such a mode—regardless of the particular linear or nonlinear functional form of the equation of state—the characteristic equation will reduce to the quadratic form

$$((h + \bar{L})\bar{g}^{ij} - \bar{T}^{ij})\lambda_i\lambda_j = 0, \quad (34)$$

with $\bar{T}^{ij}$ as given by (8).

In the ordinary perfect fluid case this will take the degenerate form $(\bar{u}^i\lambda_i)^2 = 0$, where $\bar{u}^i$ is the timelike (and physically well-defined) unit fluid flow tangent vector that is characterized by the condition $\bar{J}^A{}_i\bar{u}^i = 0$, meaning orthogonality to all the (separately unphysical, since target coordinate dependent) currents, and in terms of which the stress-energy tensor will take the familiar form $\bar{T}^{ij} = h\bar{u}^i\bar{u}^j + P\bar{g}^{ij}$.

As before, there will also be a set of nontrivial sound type modes with polarization that is longitudinal in the sense specified by (26), for which the characteristic equation will reduce to the quadratic form

$$\left(h\bar{g}^{ij} + 2\left(\frac{\det\{\hat{\mathfrak{w}}\}dh}{hd(\det\{\hat{\mathfrak{w}}\})} - 1\right)(\bar{T}^{ij} - L\bar{g}^{ij})\right)\lambda_i\lambda_j = 0, \quad (35)$$

which is what characterizes the ordinary sound cone in the familiar perfect fluid case.

VI. EXTENDED WITTEN MODELS

The physical relevance of the perfectly harmonious category presented in the immediately preceding section is obvious, at least in the case of the ordinary elastic solid and fluid applications for which the target-space dimension is $q = 3$. However, for the study of the simply harmonious category, as presented in the section before that, some physical motivation needs to be provided. In the case for which the target space is one dimensional (and for which simply harmonious means the same thing as perfectly harmonious) such a justification was provided by the demonstration [13,14] that such models are what is appropriate for the macroscopic description of string defects in simple kinetically linear field models of a subcategory proposed by Witten [12]. The purpose of the present section is to present a straightforward extension of Witten's subcategory that can form string defects which will be shown in the following section to be macroscopically describable by simply harmonious models of a less trivial kind, with two—or higher—dimensional target spaces that are curved.

Within the category characterized by a Lagrangian of the forced-harmonic type [3], the original Witten subcategory and the extensions considered here are characterized by two essential properties of which the first is that of having a $(3 + q)$-dimensional target space $\mathcal{X}$ of the ordinary flat kind, so that the symmetry group of the kinetic part $L_{\mathbf{kin}}$ of the Lagrangian is $0\{3 + q\}$. The second property is that the target space is, however, endowed with a potential function $\hat{\mathcal{V}}$ depending on just two scalar combinations, namely, a squared "Higgs amplitude" Ψ^2 and a squared "carrier amplitude" X^2. These are obtained by decomposing the target space as a direct product of a two-dimensional "Higgs field" space and a $(q + 1)$-dimensional "carrier field" space, so that the symmetry group of the whole Lagrangian,

$$L = L_{\mathbf{kin}} - \hat{\mathcal{V}}, \quad (36)$$

will be generically broken down to the direct product, $0\{2\} \times 0\{q + 1\}$, of a Higgs field symmetry group having the form $0\{2\}$ with a "carrier" symmetry group having the form $0\{q + 1\}$. The idea is that the potential should be such that the $0\{2\}$ symmetry of the Higgs part is spontaneously broken, so that the vacuum will admit the occurrence of string type topological defects (which will be "local" if the symmetry algebra of the Higgs field part is "gauged") containing fields whose internal symmetries are just those of the carrier group $0\{q + 1\}$.

The Higgs field space can be taken to have flat coordinates Ψ^1 and Ψ^2, say, which can be conveniently thought of as the real and imaginary parts of a complex field

$$\Psi^1 + i\Psi^2 = \Psi e^{i\psi}, \quad (37)$$

in terms of which the two-dimensional Higgs field part of the target-space metric will be

$$d\hat{s}^2_{\mathbf{hig}} = d\Psi^{1^2} + d\Psi^{2^2} = d\Psi^2 + \Psi^2 d\psi^2, \quad (38)$$

and corresponding Higgs field amplitude will be given by

$$\Psi^2 = \Psi^{1^2} + \Psi^{2^2}. \quad (39)$$

Similarly the carrier field space can be taken to have flat target-space coordinates, X^a say, $a = 1, \ldots, q + 1$ in terms of which the carrier metric contribution will be

$$d\hat{s}^2_{\mathbf{car}} = \delta_{ab}dX^a dX^b, \quad (40)$$

where δ_{ab} is the unit matrix, and the carrier amplitude itself will be given by

$$X^2 = \delta_{ab}X^a X^b. \quad (41)$$

These contributions combine to give the complete metric on the flat target space $\mathcal{X}$ as $d\hat{s}^2_{\mathbf{hig}} + d\hat{s}^2_{\mathbf{car}}$, which means that the kinetic part of the Lagrangian will take the form

$$L_{\mathbf{kin}} = L_{\mathbf{hig}} + L_{\mathbf{car}}, \quad (42)$$

with

$$L_{\mathbf{hig}} = -\frac{1}{2}((D_\mu \Psi^1) D^\mu \Psi^1 + (D_\mu \Psi^2) D^\mu \Psi^2) - \frac{\mathfrak{F}_{\mu\nu}\mathfrak{F}^{\mu\nu}}{16\pi \mathfrak{e}^2}, \qquad (43)$$

and

$$L_{\mathbf{car}} = -\tfrac{1}{2}\delta_{ab}(\nabla_\mu X^a)\nabla^\mu X^b, \qquad (44)$$

where the internal gauge coupling of the Higgs field part has been incorporated by the use of the covariant differentiation operation that is given by

$$D^\mu \Psi^1 = \nabla_\mu \Psi^1 - \mathfrak{A}_\mu \Psi^2,$$
$$D^\mu \Psi^2 = \nabla_\mu \Psi^2 + \mathfrak{A}_\mu \Psi^1,$$

where $\mathfrak{A}_\mu$ is a U{1} gauge form with curvature $\mathfrak{F}_{\mu\nu} = 2\nabla_{[\mu}\mathfrak{A}_{\nu]}$, for which Gothic letters have been used to indicate that, although mathematically analogous, this internal gauge field is not meant to be physically interpretable as the ordinary electromagnetic field. For a small but nonzero value of the coupling constant $\mathfrak{e}$ this gauge field enables the vortex defects of the model to be locally confined—without the logarithmic energy divergence for which a long range ''infrared'' cutoff would otherwise be needed.

In his original formulation [12] Witten made the further postulate that, as well as this internal gauge coupling of the Higgs field, there would also be an external gauge coupling of the carrier field part to an analogous U{1} gauge form A_ν, with curvature $F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]}$, that was meant to be interpreted as that of an ordinary electromagnetic field, with its own extra Lagrangian contribution $F_{\mu\nu}F^{\nu\mu}/16\pi e^2$. However—unless the corresponding coupling constant, e say, is set to zero—such a coupling engenders technical trouble by reintroducing the logarithmic infrared divergence that had been removed by the other gauge coupling.

The present treatment will be based on the supposition that gauge self-coupling of the carrier field part is weak enough to be neglected, so that the divergence problem is avoided, but this does not exclude allowance for passive coupling to an external background of electromagnetic or other conceivable radiation. It will nevertheless be supposed that such radiation is sufficiently weak to allow the gauge to be chosen so that the corresponding gauge form (namely, A_ν in the electromagnetic case) to be taken to be zero in a neighborhood that is large compared with the internal dimensions of the defect, so that within this neighborhood there will be no further loss of generality in taking the kinetic part of the Lagrangian to have the simple form (43).

The original Witten model was characterized by $q = 1$, so that the carrier target-space coordinates could be considered as components of a complex field $X^1 + iX^2 = X\mathrm{e}^{i\chi}$. What I refer to as the minimally extended Witten model is characterized by $q = 2$, so that the carrier symmetry group will have the non-Abelian form 0{3}, instead of the Abelian form 0{2} that it had in the original Witten model.

In a more elaborate—nonminimal—extension proposed for consideration by Lilley *et al.* [23] the carrier space dimension is taken to be given by $q + 1 = 4$. Instead of retaining its full symmetry group 0{4}, these authors took the carrier space to be endowed with a complex structure by grouping its coordinates into a pair of complex fields $X^1 + iX^2$ and $X^3 + iX^4$, so that the carrier symmetry group is reduced to the form U{2}. The latter has the structure of a direct product of an SU{2} group (which Lilley *et al.* took to be gauged) with a U{1} group (which they left as merely ''global,'' but which could just as well be taken to be coupled to ordinary electromagnetism).

In all these cases the flat metric (40) of the $(q + 1)$-dimensional carrier part can be rewritten as

$$d\hat{s}^2_{\mathbf{car}} = dX^2 + X^2 d\hat{\Omega}^2, \qquad (45)$$

where $d\hat{\Omega}^2$ is the metric on the relevant symmetry-orbit space $\bar{\mathcal{X}}$, which will be the unit q-sphere as given in terms of some system of coordinates χ^A by an expression of the form

$$d\hat{\Omega}^2 = \hat{g}_{AB} d\chi^A d\chi^B. \qquad (46)$$

More particularly, for the minimally extended model characterized by $q = 2$, the standard choice $\chi^1 = \hat{\theta}$ and $\chi^2 = \hat{\varphi}$ will be obtained by setting $X^1 = X\sin\hat{\theta}\cos\hat{\varphi}$, $X^2 = X\sin\hat{\theta}\sin\hat{\varphi}$, and $X^3 = X\cos\hat{\theta}$. In terms of these, the spherical metric components will be given by a prescription of the familiar form $\hat{g}_{11} = 1$, $\hat{g}_{12} = 0$, $\hat{g}_{22} = \sin^2\hat{\theta}$.

The basic idea behind cosmic string theory, as developed at first most notably by Kibble [15,24], was that short, effectively straight string segments in a locally uniform background neighborhood could be approximated by Nielsen-Olesen type vortex solutions of the underlying field model. The presence of longitudinal currents in the vortex was excluded by the rather strong kind of cylindrical symmetry postulated by an ansatz of the Nielsen-Olesen type, but was admitted by a weaker kind of cylindrical symmetry ansatz that was subsequently introduced by Witten [12]. As shown by the recent work of Lilley *et al.* [23] even the weaker kind of cylindrical symmetry ansatz proposed by Witten is incompatible with the simultaneous presence of several noncommuting longitudinal currents, whose treatment will therefore require an ansatz of an even weaker kind that will be introduced in the next section, whereby it is required that the cylindrical symmetry should hold only approximately in the relevant locally uniform background neighborhood.

Provided the external gauge coupling is sufficiently weak for its self-coupling to be neglected, it will be possible to choose the gauge so that the corresponding (electromagnetic) gauge form A_ν vanishes in the neighborhood under consideration. The field equations for the remaining internal gauge form A_ν and for the flat target-space components will then be expressible in kinetically decoupled form as a first subsystem consisting of

$$\nabla_\nu \mathfrak{F}^{\mu\nu} = 4\pi e^2 (\Psi^2 D^\mu \Psi^1 - \Psi^1 D^\mu \Psi^2), \tag{47}$$

and

$$D_\mu D^\mu \Psi^1 = 2\frac{\partial \hat{\mathcal{V}}}{\partial(\Psi^2)}\Psi^1, \qquad D_\mu D^\mu \Psi^2 = 2\frac{\partial \hat{\mathcal{V}}}{\partial(\Psi^2)}\Psi^2, \tag{48}$$

for the gauge form and the Higgs field, and a second subsystem given simply by

$$\nabla_\mu \nabla^\mu X^a = 2\frac{\partial \hat{\mathcal{V}}}{\partial(X^2)} X^a \tag{49}$$

(in which $a = 1, \ldots, q+1$) for the carrier field.

VII. WITTEN'S WEAK SYMMETRY ANSATZ

Following Witten [12], attention will now be restricted to configurations in which the Higgs subsystem is subject to the same symmetry conditions as in a simple Nielsen-Olesen type vortex (for which the carrier subsystem is absent). This means that in a flat space-time, with respect to cylindrical coordinates for which the metric is

$$ds^2 = d\varrho^2 + \varrho^2 d\phi^2 + dz^2 - dt^2, \tag{50}$$

the Higgs field and its gauge form are postulated to be longitudinally symmetric in the strong sense, to the effect that Ψ^1 and Ψ^1 are independent of z and t, but they are required to be axially symmetric only in the weak (albeit strict [3]) sense, meaning modulo an action of the primary symmetry group O{2}, so that the phase ψ in (37) is allowed to have an angle dependence of the form

$$\psi = n\phi, \tag{51}$$

where n is a fixed integer winding number, while the amplitude Ψ can depend only on ϱ. The corresponding ansatz for the internal gauge field is that it should have the form

$$\mathfrak{A}_\mu dx^\mu = \mathfrak{A} d\phi, \tag{52}$$

in which the quantity $\mathfrak{A}$ is also a function only of ϱ.

Still following Witten [12], it will be postulated that the carrier subsystem is axisymmetric in the strong sense—meaning that the field components X^a are all independent of ϕ—so that using a prime for differentiation with respect to z and a dot for differentiation with respect to t, their dynamical equations (49) will take the form

$$\frac{1}{\varrho}\frac{d}{d\varrho}\left(\varrho \frac{dX^a}{d\varrho}\right) = \ddot{X}^a - X''^a + 2\frac{\partial \hat{\mathcal{V}}}{\partial(X^2)} X^a. \tag{53}$$

An ansatz of the restrictive Nielsen-Olesen type postulated by Kibble [24] would also require staticity and cylindrical symmetry in the strong sense, meaning $\dot{X}^a = 0$ and $X'^a = 0$, so that the components X^a should depend only on ϱ. The system of field equations would thereby be reduced from four to two dimensions, namely, those of a flat cross section with fixed longitudinal coordinate values that can, without loss of generality, be taken to be $t = 0$ and $z = 0$. Macroscopic quantities such as the string energy per unit length will then be obtainable by integration over the cross section.

Witten's innovation [12] was to recognize that such a reduction to a two-dimensional system on a flat cross section will still be obtainable from a less restrictive ansatz whereby the longitudinal symmetry of the carrier field is required to be only of the weak type. The Witten ansatz can be decomposed into two successive conditions, of which the first is that the amplitude X can depend only on ϱ, so that

$$\dot{X} = X' = 0 \tag{54}$$

but that subject, of course, to the ensuing restaints, namely,

$$\delta_{ab} X^a \dot{X}^b = 0, \qquad \delta_{ab} X^a X'^b = 0, \tag{55}$$

the longitudinal gradients $\dot{X}^b$ and X'^b are allowed to have nonvanishing values. The second condition of the Witten ansatz is that these gradient fields themselves should be longitudinally symmetric in the strong sense. It is this second condition that will have to be relaxed in the work that follows.

There will be no obstacle to the implementation of such a Witten type symmetry ansatz provided the relevant symmetry-orbit space $\bar{\mathcal{X}}$, say (meaning the quotient of the carrier field space by the action of its symmetry group), happens to be *flat*—as in the single carrier component case originally considered by Witten, as well as in multicomponent Abelian cases that have been considered more recently [11]. In such cases a linear symmetry-preserving map from the z, t plane to $\bar{\mathcal{X}}$ will be available as a framework for parallel propagation of the field on the sample cross section at $t = 0$, $z = 0$, so as to construct a solution that remains valid for all values of t and z. Subject to the choice of a system of *flat* coordinates χ^A on the symmetry-orbit space $\bar{\mathcal{X}}$, the Witten ansatz simply amounts to taking uniformly constant values for χ'^A and $\dot{\chi}^A$. The supplementary requirement of invariance under the discrete symmetries of time and parity reversal imply the further restriction that the initial value of χ^A at $t = 0$, $z = 0$ be uniform over the cross section, meaning independent not just of ϕ but also of ϱ.

Such an ideal procedure will unfortunately be available only for a restricted choice [23] of the values of χ'^A and $\dot{\chi}^A$ if—as in the cases we are concerned with here—the relevant q-dimensional symmetry-orbit space $\bar{\mathcal{X}}$ is *curved* so that its symmetry algebra is non-Abelian, with the implication that Lie transport operations with respect to different generators will fail to be mutually consistent. In order to obtain an effectively two-dimensional description on a sample cross section at $t = 0$, $z = 0$, in cases involving currents aligned with symmetry generators that do not commute, the Witten type weak symmetry ansatz will need to be replaced by something less restrictive.

VIII. THE LOCAL GEODICITY ANSATZ

For the treatment of a generic current configuration, what I propose is something describable as the *local geodicity* ansatz, whereby one is enabled to start from *arbitrarily* chosen uniform values of χ^A and its derivatives χ'^A and $\dot{\chi}^A$ on an initial cross section at $t = 0$, $z = 0$. This ansatz retains the first condition of the Witten ansatz, as embodied in (54) and (55), but instead of a further symmetry requirement the second condition of the local geodicity ansatz is that the value of χ^A throughout a finite space-time neighborhood of the cross section should be obtained by the standard process of geodesic extrapolation with respect to the metric $\hat{g}_{AB}$ on $\bar{\mathcal{X}}$. According to this prescription, a coordinate pair $\{z, t\}$ maps to a position specified by unit parameter value $\tau = 1$ on the affinely parametrized geodesic $\chi^A\{\tau\}$ specified at $\tau = 0$ by the tangent

$$\frac{d\chi^A}{d\tau} = t\dot{\chi}^A + z\chi'^A. \tag{56}$$

So long as one is concerned with derivatives of at most second order, which is all that is needed for the field equations in question, the application of this ansatz is very easily achievable by choosing to work with local coordinates such that the relevant connection components vanish. In such a system the local geodicity ansatz simply means that all the second derivatives will also vanish:

$$\hat{\Gamma}_A{}^B{}_C = 0 \Rightarrow \ddot{\chi}^A = \dot{\chi}'^A = \chi''^A = 0. \tag{57}$$

When the symmetry-orbit space $\bar{\mathcal{X}}$ is flat, this ansatz is evidently equivalent to a weak symmetry ansatz of the kind introduced by Witten. The advantage of the prescription (57) is that it is applicable even when $\bar{\mathcal{X}}$ is curved, and its disadvantage in that case is that it is not exactly applicable everywhere simultaneously, but only on the chosen cross section at $t = 0$, $z = 0$. It can, however, be adopted as a very good approximation so long as χ^A is restricted to a range that is small compared with the curvature scale of $\bar{\mathcal{X}}$.

The concrete implementation of such an approximation procedure is conveniently achievable, for the extended Witten models introduced above, by taking the local coordinates on the symmetry-orbit space $\bar{\mathcal{X}}$ to be specified by simply setting

$$\chi^A = \frac{X^a}{X}, \qquad A = a - 1, \qquad a = 2, \ldots, q + 1. \tag{58}$$

By substituting this in (40), one obtains the q-spherical metric (40) in the explicit form given by

$$d\hat{\Omega}^2 = \delta_{AB} d\chi^A d\chi^B + \frac{(\delta_{AB}\chi^A d\chi^B)^2}{1 - \chi^2}, \tag{59}$$

in which the deviation from flatness is attributable just to the last term, which will be negligible so long as the dimensionless quantity

$$\chi^2 = \delta_{AB}\chi^A\chi^B \tag{60}$$

is very small compared with unity.

In the minimally extended case, for which $q = 2$, this metric will be that of an ordinary 2-sphere, with coordinates expressible as

$$\chi^1 = \sin\hat{\theta}\sin\hat{\varphi}, \qquad \chi^2 = \cos\hat{\theta}, \tag{61}$$

in terms of spherical coordinates of the usual kind, for which

$$d\hat{\Omega}^2 = d\hat{\theta}^2 + \sin^2\hat{\theta}d\hat{\varphi}^2. \tag{62}$$

Adoption of the convention that $\chi^A = 0$, on the chosen initial cross section, as specified by $t = 0$, $z = 0$, is equivalent to requiring there that

$$X^1 = X, \qquad X^a = 0 \quad \forall\ a \neq 1, \tag{63}$$

which implies by (54) that the first derivatives of X^1 will vanish there,

$$\dot{X}^1 = 0, \qquad X'^1 = 0, \tag{64}$$

and by (55) that its second derivatives will be given there by

$$\begin{aligned} \frac{\ddot{X}^1}{X} &= -\delta_{AB}\dot{\chi}^A\dot{\chi}^B, \\ \frac{\dot{X}'^1}{X} &= -\delta_{AB}\dot{\chi}^A\chi'^B, \\ \frac{X''^1}{X} &= -\delta_{AB}\chi'^A\chi'^B. \end{aligned} \tag{65}$$

Since the metric (59) has the form required for the local geodicity ansatz to take the form (57), whereby the second derivatives of χ^A all vanish where $\chi^A = 0$, it follows that the second derivatives of the other components of X^a will also vanish there,

$$\ddot{X}^a = \dot{X}'^a = X''^a = 0 \quad \forall\ a \neq 1. \tag{66}$$

We thereby obtain

$$\ddot{X}^1 - X''^1 = \hat{\mathfrak{w}}X, \qquad \ddot{X}^a - X''^a = 0 \quad \forall\ a \neq 1, \tag{67}$$

where $\hat{\mathfrak{w}}$ is defined according to (19) as the trace of the target-space tensor

$$\hat{\mathfrak{w}}^{AB} = \chi'^A \chi'^B - \dot{\chi}^A \dot{\chi}^B,$$

so that, where $\chi^A = 0$, it will be given, independently of ϱ, by

$$\hat{\mathfrak{w}} = \delta_{AB}(\chi'^A \chi'^B - \dot{\chi}^A \dot{\chi}^B) = \frac{1}{X^2}\delta_{ab}(X'^a X'^b - \dot{X}^a \dot{X}^b). \tag{68}$$

Under these circumstances, the dynamical equations of the carrier subsystem (53) will be satisfied automatically for $a \neq 1$, so this subsystem reduces to just a single nontrivial equation, which will take the form

$$\frac{1}{\varrho}\frac{d}{d\varrho}\left(\varrho\frac{dX}{d\varrho}\right) = \left(\hat{\mathfrak{w}} + 2\frac{\partial \hat{\mathcal{V}}}{\partial(X^2)}\right)X. \tag{69}$$

This can be seen to be independent of the internal dimension q, and thus exactly the same as that of its analogue for the original unextended Witten model. It is evident that the carrier contribution to the kinetic part (44) of the Lagrangian density (44) will also reduce to the same form as for the original Witten model, namely,

$$L_{\mathbf{car}} = -\tfrac{1}{2}\hat{\mathfrak{w}}X^2. \tag{70}$$

Within the framework of the local geodicity ansatz, the preceding work establishes that in terms of the constant parameter $\hat{\mathfrak{w}}$ and the radially dependent field variable X^2 the carrier contribution in the extended model is indistinguishable from the carrier contribution in the original model. It follows that, in terms of the scalar trace $\hat{\mathfrak{w}}$, the extended model will share, with the original Witten model, an equation of state of exactly the same simply harmonious form for the ensuing string model Lagrangian,

$$\bar{L} = \int 2\pi\varrho L d\varrho, \tag{71}$$

as obtained [13,14] by integration over the cross section using field values satisfying the dynamical equations consisting of (69) for the carrier amplitude in conjunction with the Higgs subsystem consisting of (47) and (48).

It is to be remarked that instead of interpreting the independent variable $\hat{\mathfrak{w}}$ in the simply harmonic equation of state as the trace of the target-space tensor $\hat{\mathfrak{w}}^{AB}$, it can equivalently be characterized as the trace

$$\bar{\mathfrak{w}} = \bar{\mathfrak{w}}_i{}^i, \tag{72}$$

of the world sheet tensor defined by (28), which in the local neighborhood where $\bar{A}_i{}^A$ vanishes will take the simple form $\bar{\mathfrak{w}}_{ij} = \hat{g}_{AB}\chi^A_{,i}\chi^B_{,j}$. As well as having the same trace,

$$\bar{\mathfrak{w}} = \hat{\mathfrak{w}}, \tag{73}$$

the target-space tensor $\hat{\mathfrak{w}}^{AB}$ and base-space tensor $\bar{\mathfrak{w}}_{ij}$ share their quadratic invariant

$$\bar{\mathfrak{w}}_i{}^j\bar{\mathfrak{w}}_j{}^i = \hat{\mathfrak{w}}_A{}^B\hat{\mathfrak{w}}_B{}^A, \tag{74}$$

which can thereby be seen to be positive definite, due to the Euclidean signature of $\hat{g}_{AB}$ (whereas the signature of the two-dimensional world sheet metric $\bar{g}_{ij}$ is Lorentzian).

IX. CONCLUSIONS

The upshot of the foregoing work is that the extension of Witten's field model will have vortices macroscopically describable by a conducting string model of simply harmonic type, in the sense of being governed by a Lagrangian depending just on the trace $\bar{\mathfrak{w}}$ of the base-space tensor $\bar{\mathfrak{w}}_{ij}$ defined by (28) as the pullback of the prescribed target-space metric $\hat{g}_{AB}$. It is instructive to present the properties of such models using a preferred world sheet reference frame of orthonormal type, meaning $\bar{g}_{00} = -1$, $\bar{g}_{11} = 1$, $\bar{g}_{01} = 0$, that is chosen so as to diagonalize the pullback $\bar{\mathfrak{w}}_{ij}$, of which the nonvanishing components will be the eigenvalues that are given in terms of the invariants $\bar{\mathfrak{w}}$ and $\bar{\mathfrak{w}}_i{}^j\bar{\mathfrak{w}}_j{}^i$ by the expressions

$$\bar{\mathfrak{w}}_{00} = \sqrt{\frac{\bar{\mathfrak{w}}^2}{4} - \det\{\bar{\mathfrak{w}}\}} + \frac{\bar{\mathfrak{w}}}{2}, \qquad \bar{\mathfrak{w}}_{11} = \sqrt{\frac{\bar{\mathfrak{w}}^2}{4} - \det\{\bar{\mathfrak{w}}\}} - \frac{\bar{\mathfrak{w}}}{2}, \tag{75}$$

in which the relevant invariant determinant, namely, that of $\bar{\mathfrak{w}}_i{}^j$, is defined as

$$\det\{\bar{\mathfrak{w}}\} = \bar{\mathfrak{w}}_0{}^0\bar{\mathfrak{w}}_1{}^1 - \bar{\mathfrak{w}}_0{}^1\bar{\mathfrak{w}}_1{}^0 = \tfrac{1}{2}(\bar{\mathfrak{w}}^2 - \bar{\mathfrak{w}}_i{}^j\bar{\mathfrak{w}}_j{}^i). \tag{76}$$

It is to be noticed that positive definite character of the target metric $\hat{g}_{AB}$ entails the non-negativity of the eigenvalues,

$$\bar{\mathfrak{w}}_{00} \geq 0, \qquad \bar{\mathfrak{w}}_{11} \geq 0, \tag{77}$$

and hence the nonpositivity of the determinant:

$$\hat{\mathfrak{w}}_A{}^B\hat{\mathfrak{w}}_B{}^A - \hat{\mathfrak{w}}^2 = -2\det\{\bar{\mathfrak{w}}\} \geq 0. \tag{78}$$

It is also to be observed that the determinant will necessarily vanish—meaning that the equality $\hat{\mathfrak{w}}_A{}^B\hat{\mathfrak{w}}_B{}^A = \hat{\mathfrak{w}}^2$ will automatically be satisfied so that either $\bar{\mathfrak{w}}_{00}$ or $\bar{\mathfrak{w}}_{11}$ will be zero—in the special case [23] for which the currents happen to be all aligned with the *same* one-parameter subgroup generator in $\bar{\mathcal{X}}$, which is the only kind of configuration for which a stationary circular vorton type equilibrium state will be possible.

In terms of the eigenvalues $\bar{\mathfrak{w}}_{00}$ and $\bar{\mathfrak{w}}_{11}$, the corresponding nonvanishing components of the surface stress-energy tensor (8) will be given by the formulas

$$\bar{T}_{00} = \kappa\bar{\mathfrak{w}}_{00} - \bar{L}, \qquad \bar{T}_{11} = \kappa\bar{\mathfrak{w}}_{11} + \bar{L}, \tag{79}$$

in which $\bar{L}$ and κ depend only on the trace, $\bar{\mathfrak{w}} = \bar{\mathfrak{w}}_{11} - \bar{\mathfrak{w}}_{00}$.

The original Witten field model gave rise to an ensuing conducting string model that was of ordinarily elastic type in the sense of having a target-space dimension q that was the same as the space dimension p (not the space-time dimension $p+1$) of the supporting world sheet, which in the string case is simply $p=1$. It was found [13] that a good approximation for the equation of state of the ensuing string model could be provided in terms of a microscopic length scale $\delta_\star$ and a couple of mass scales m and $m_\star$—depending [14] on the particular functional form of the interaction potential $\hat{\mathcal{V}}$—by the formula

$$\bar{L} = -m^2 - \tfrac{1}{2}m_\star^2 \ln\{1 + \delta_\star^2 \bar{\mathfrak{w}}\}, \tag{80}$$

which gives

$$\kappa = \frac{m_\star^2 \delta_\star^2}{1 + \delta_\star^2 \bar{\mathfrak{w}}}, \qquad \frac{1}{\kappa}\frac{d\kappa}{d\bar{\mathfrak{w}}} = \frac{-\delta_\star^2}{1 + \delta_\star^2 \bar{\mathfrak{w}}}. \tag{81}$$

The work in the preceding sections implies that the minimally extended Witten model with the same interaction potential $\hat{\mathcal{V}}$ will give rise to an ensuing conducting string model that will be governed by the *same* equation of state, even though it will not be of the ordinary elastic type with $q=p$, but of the hyperelastic type [17] with $q=p+1$, that is to say with a target space having a dimension that is equal to the space-time dimension of the supporting world sheet, namely, $p+1=2$ in the string case under consideration. It can be seen from the characteristic Eq. (27) that, according to (81), the speed $c_{\rm L}$ of longitudinal sound type perturbations relative to the preferred reference system will be given in this hyperelastic case by

$$c_{\rm L}^2 = \frac{1 - \delta_\star^2(\bar{\mathfrak{w}}_{00} + \bar{\mathfrak{w}}_{11})}{1 + \delta_\star^2(\bar{\mathfrak{w}}_{00} + \bar{\mathfrak{w}}_{11})}, \tag{82}$$

where $\bar{\mathfrak{w}}_{00}$ and $\bar{\mathfrak{w}}_{11}$ are the—necessarily non-negative—eigenvalues given by (75), of which the sum in (82) will be expressible as $\bar{\mathfrak{w}}_{00} + \bar{\mathfrak{w}}_{11} = \sqrt{2\hat{\mathfrak{w}}_A{}^B\hat{\mathfrak{w}}_B{}^A - \hat{\mathfrak{w}}^2}$.

It is to be remarked that the dually related [10,25] "electric" and "magnetic" varieties of the ordinarily elastic case obtained from the original unextended Witten model can be considered as degenerate limits of this hyperelastic case: the electric variety is characterized by $\bar{\mathfrak{w}} = -\bar{\mathfrak{w}}_{00}$ with $\mathfrak{w}_{11}=0$, and the magnetic variety is characterized by $\bar{\mathfrak{w}} = \bar{\mathfrak{w}}_{11}$ with $\bar{\mathfrak{w}}_{00} = 0$.

As well as the longitudinal modes characterized by (81), there will, of course, be extrinsic wiggle perturbation of the world sheet, with propagation speed $c_{\rm E}$, which according to (7) will be given in terms of the energy density $\mathcal{U} = T_{00}$ and the string tension $\mathcal{T} = -T_{11}$ by the universally [10,26] valid formula

$$c_{\rm E}^2 = \frac{\mathcal{T}}{\mathcal{U}}. \tag{83}$$

The novelty in the minimally extended Witten model, as contrasted with the original Witten model, is that the ensuing string model has a third kind of perturbation mode. As well as the longitudinally polarized sound type modes governed by (82) and the extrinsic wiggle modes governed by (83), there will also be intrinsic "shake" modes characterized (with respect to the curved target space $\bar{\mathcal{X}}$) by the transverse polarization condition (24) with propagation speed $c_{\rm T}$ given according to (25) by

$$c_{\rm T}^2 = 1, \tag{84}$$

which simply means that (like the perturbation modes of the underlying field model with the flat target space $\mathcal{X}$) these shear type modes will travel at the speed of light.

Having completed the presentation of this simply harmonious hyperelastic string model, we still need to consider the extent to which its use is justifiable as a macroscopic string description of vortex defects in the minimal extension of the Witten field model. The foregoing derivation relied on a local geodicity ansatz whose applicability depended on a local flatness approximation whose range of validity is limited by the requirement that the dimensionless magnitude χ^2 defined by (60) should remain small.

Subject to the conditions of (57), the local geodicity ansatz means that for small but nonzero values of t and z the value of χ^A will be given approximately by $\chi^A = \dot{\chi}^A t + \chi'^A z$, or more concisely, in terms of the world sheet coordinates $\sigma^0 = t$ and $\sigma^1 = z$, by $\chi^A = \chi^A_{,i}\sigma^i$. It follows that the required magnitude χ^2 will be given in this approximation by the formula $\chi^2 = \bar{\mathfrak{w}}_{ij}\sigma^i\sigma^j$. Thus more particularly, with respect to the preferred longitudinal coordinate system characterized by (75), it will be given by

$$\chi^2 = \bar{\mathfrak{w}}_{00}t^2 + \bar{\mathfrak{w}}_{11}z^2. \tag{85}$$

For the approximate flatness approximation to be utilizable, the necessary condition,

$$\chi^2 \ll 1, \tag{86}$$

must be satisfied in a range of t and z that is large compared with the thickness δ of the vortex defect, which entails the requirements $\bar{\mathfrak{w}}_{00}\delta^2 \ll 1$ and $\bar{\mathfrak{w}}_{11}\delta^2 \ll 1$.

This leads us to the final conclusion, which is that the condition for negligibility of the curvature term in (59)—and thus for the validity of the simply harmonious conducting string model—will be expressible simply as an upper bound on the quadratic field gradient invariant $\hat{\mathfrak{w}}_A{}^B\hat{\mathfrak{w}}_B{}^A$ [and hence by (78) also on $\hat{\mathfrak{w}}^2$] in the form

$$\hat{\mathfrak{w}}_A{}^B\hat{\mathfrak{w}}_B{}^A\delta^4 \ll 1. \tag{87}$$

So long as the string thickness δ is small enough, this condition will hold automatically by (78) as a consequence of the requirement for stability against longitudinal perturbations, namely, the condition $c_{\rm L}^2 > 0$, which according to (82) will take the form

$$(2\hat{\mathfrak{w}}_A{}^B\hat{\mathfrak{w}}_B{}^A - \hat{\mathfrak{w}}^2)\delta_\star^4 < 1. \tag{88}$$

Within this allowed range there will clearly be no danger of violation of (87) provided that (as is usually supposed in the cosmological context envisaged by Witten) the length scales associated with the carrier field are relatively large, and the mass scales correspondingly small, compared with those associated with the Higgs field:

$$\delta_\star^2 \gg \delta^2, \qquad m_\star^2 \ll m^2. \tag{89}$$

There is never any possibility of instability of the shake modes characterized by (84), and the usual extra requirement $c_{\rm E}^2 > 0$, for stability against extrinsic wiggle perturbations—which is equivalent to the condition that the tension $\mathcal{T}$ given by (79) should be positive—makes little difference in the circumstances of (89), as it can be seen that it will hold automatically if (88) is replaced by the only marginally stronger condition $(2\hat{\mathfrak{w}}_A{}^B\hat{\mathfrak{w}}_B{}^A - \hat{\mathfrak{w}}^2)\delta_\star^4 < 1 - \exp\{-2m^2/m_\star^2\}$.

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