Analytic formulae of the CMB bispectra generated from non-Gaussianity in the tensor and vector perturbations

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We present a complete set of formulae for calculating the bispectra of CMB temperature and polarization anisotropies generated from non-Gaussianity in the vector and tensor mode perturbations. In the all-sky analysis, it is found that the bispectrum formulae for the tensor and vector-mode non-Gaussianity formally take complicated forms compared to the scalar mode one because the photon transfer functions in the tensor and vector modes depend on the azimuthal angle between the direction of the wave number vector of the photon's perturbation and that of the line of sight. We demonstrate that flat-sky approximations remove this difficulty because this kind of azimuthal angle dependence apparently vanishes in the flat-sky limit. Through the flat-sky analysis, we also find that the vector or tensor bispectrum of B-mode polarization vanishes in the squeezed limit, unless the cosmological parity is violated at the nonlinear level.

I. INTRODUCTION

Recently, the primordial non-Gaussianity of curvature perturbations has been a focus of constant attention all over the world. One of the main reasons for attracting so much attention is that meaningful measurement of this quantity will become observationally available in the near future, which brings us valuable information about the dynamics of inflaton. Bispectrum (three point correlation functions) of the CMB temperature anisotropies has been most commonly used to investigate primordial non-Gaussianity [1,2].

As is well known, if the primordial curvature perturbation deviates from the pure Gaussian statistics, then it produces the nonzero bispectrum of the CMB temperature anisotropies. However, there are a lot of sources of the bispectrum that not only include the primordial non-Gaussianity of the curvature perturbations but also the nonlinearities of the Sachs-Wolfe effect [3–6] and the radiative transfer [7–11], cosmological recombination [12–15], the nonlinear gravitational clustering of dark matter [16], the cosmic strings [17,18], the magnetic fields [19–21], and so on. Hence, in order to evaluate the magnitude of the primordial non-Gaussianity of the curvature perturbations precisely, it is very important to identify these nonlinear effects.

In fact, these effects will also induce the tensor and vector-mode perturbations, and the modes may generate more characteristic features in the CMB angular spectra than in the scalar one. Some models have been proposed in which vector modes are produced at the inflationary phase as well as the scalar and tensor modes, by breaking the conformal invariance at that phase [22]. In such cases, the non-Gaussian vector mode with an interesting amplitude

could also be generated as in the scalar case. For example, if primordial magnetic fields are considered, the magnetic stresses depend quadratically on the primordial Gaussian magnetic field (PMF); hence, their vector or tensor components of their bispectra also have finite values. As the vector mode of the CMB transfer function sourced from magnetic fields is dominant at small scale, one can expect that the vector-mode bispectrum dominates there in the same manner as the power spectra discussed in Refs. [23–25]. Therefore, if one adds the effects of the vector mode in constraining the amplitude of PMF by using the CMB bispectra, one will obtain a tighter bound than the current one as $\mathcal{O}(10)$ nG [20,21]. Furthermore, the cosmic strings or the magnetic fields give more characteristic effects also in the polarization spectra than in the temperature one [26–29]. Hence, for the identification of the sources of the bispectrum, information of temperature and polarization fluctuation generated from tensor and vector-mode ones should be used, not only from the scalar mode perturbations. However, there are not enough studies about their effects yet.

In this paper, we newly present the bispectrum formulae of the CMB temperature and polarization anisotropies sourced from non-Gaussianity in the tensor and vector-mode perturbations. First, we formulate all-sky bispectra generated from scalar, vector, and tensor modes and find that the bispectrum formulae for vector and tensor modes in all-sky analysis formally take complicated forms compared to the scalar mode case due to the dependence of the photon transfer functions on the azimuthal angle between the wave vector of photon perturbation **k** to the unit vector specifying the line of sight direction $\hat{n}$. Next, by using the flat-sky approximation, we simplify the equations of bispectra of the CMB anisotropies to solve the above difficulty because no azimuthal dependence arises in this limit. In addition, in our flat-sky formulae, we find that if the

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bispectra of B-mode polarization is a nonzero value, it infers the parity violation in the nonlinear sector.

This paper is organized as follows: In Sec. II, we define the primordial non-Gaussianity from tensor and vector perturbations. In Sec III, we discuss the formulae of the CMB bispectrum generated from tensor and vector perturbations in the flat-sky analysis. In Sec IV, we explain the formulae of their bispectra in the flat-sky approximation. Finally, in Sec V, we provide the summary of this paper. In the Appendices, we derive the formulae of the CMB 1-point function used in the discussion in Secs. III and IV.

Throughout this paper, we assume that the Universe is spatially flat and use the definition of Fourier transformation:

$$f(\mathbf{x}) \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{1}$$

$$f(\mathbf{\Theta}) \equiv \int \frac{d^2\ell}{(2\pi)^2} \tilde{f}(\ell) e^{i\ell\cdot\mathbf{\Theta}}, \tag{2}$$

where $\mathbf{\Theta}$ and $\mathbf{x}$ are, respectively, 2D and 3D vectors in the configuration space and ℓ and $\mathbf{k}$ are, respectively, their Fourier conjugate variables.

II. PRIMORDIAL POWER SPECTRA AND BISPECTRA OF THE SCALAR, TENSOR, AND VECTOR PERTURBATIONS

In this section, we parametrize the primordial non-Gaussianity in the tensor and vector perturbations. As mentioned in the introduction, in order to discuss the primordial non-Gaussianity, the bispectrum of the fluctuations is commonly used. In this paper, we consider a general expression of the bispectrum of the tensor/vector perturbations which is given by

$$\langle \xi^{s_1}(\mathbf{k_1})\xi^{s_2}(\mathbf{k_2})\xi^{s_3}(\mathbf{k_3})\rangle$$
$$= (2\pi)^3 F^{s_1 s_2 s_3}(k_1, k_2, k_3)\delta^{(3)}(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3}), \tag{3}$$

where s_i expresses two helicity states: ± 1 for a vector mode, ± 2 for a tensor mode. Here, for simplifying numerical calculation, we neglect the angular dependence of three wave number vectors in the bispectrum in the bispectrum F. This expression includes the so-called "squeezed" or "equilateral" type of the non-Gaussianity [30,31].

For example, the squeezed-type of the bispectrum is given as follows: As in Refs. [1,2,32,33], the primordial power spectrum and bispectrum of the scalar curvature perturbations are introduced as

$$\langle \Phi_L(\mathbf{k_1})\Phi_L(\mathbf{k_2})\rangle = (2\pi)^3 P_\Phi(k_1)\delta^{(3)}(\mathbf{k_1} + \mathbf{k_2}), \tag{4}$$

$$\langle \Phi_L(\mathbf{k_1})\Phi_L(\mathbf{k_2})\Phi_{NL}(\mathbf{k_3})\rangle$$
$$= (2\pi)^3 P_\Phi(k_1)P_\Phi(k_2)2f_{NL}\delta^{(3)}(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3}), \tag{5}$$

where f_{NL} is the nonlinear parameter of the scalar perturbation and $\Phi(\mathbf{k})$ denotes the Fourier component of the

primordial curvature perturbation which is decomposed into Gaussian and non-Gaussian part as

$$\Phi(\mathbf{x}) \equiv \Phi_L(\mathbf{x}) + \Phi_{NL}(\mathbf{x}), \tag{6}$$

$$\Phi_{NL}(\mathbf{x}) \equiv f_{NL}[\Phi_L(\mathbf{x})^2 - \langle \Phi_L(\mathbf{x})^2\rangle] \tag{7}$$

in real space.

This parametrization can be readily extended to the tensor and vector cases. In contrast to the scalar perturbation, Fourier modes of tensor and vector perturbations have two independent polarizations. For the convenience of calculating the CMB power spectrum as discussed in Refs. [34–36], we use two helicity states (± 1 for a vector mode, ± 2 for a tensor mode) to decompose the initial stochastic fields, ξ^s, where s represents a helicity state. We apply this description to the definition of the initial non-Gaussianity of the tensor and vector perturbations as[1]

$$\xi^s(\mathbf{x}) \equiv \xi_L^s(\mathbf{x}) + \xi_{NL}^s(\mathbf{x}), \tag{8}$$

$$\xi_{NL}^{s_k}(\mathbf{x}) \equiv \tfrac{1}{2} f_{Z,s_i s_j}^{s_k}[\xi_L^{s_i}(\mathbf{x})\xi_L^{s_j}(\mathbf{x}) - \langle \xi_L^{s_i}(\mathbf{x})\xi_L^{s_j}(\mathbf{x})\rangle], \tag{9}$$

where the index $Z = T$ is used for the tensor mode (s_i, s_j, $s_k = \pm 2$) and $= V$ for the vector mode (s_i, s_j, $s_k = \pm 1$). Here, we have introduced new nonlinear parameters for the tensor and vector perturbations denoted by $f_{Z,s_i s_j}^{s_k}$. These three indices, s_i, s_j, s_k, allow the correlation between each field of the different helicity states in the nonlinear level. Because of the symmetry, we have $f_{Z,s_i s_j}^{s_k} = f_{Z,s_j s_i}^{s_k}$. By using these expressions, the primordial power spectra and bispectra of the tensor and vector perturbations are expressed as[2]

$$\langle \xi_L^{s_1}(\mathbf{k_1})\xi_L^{s_2}(\mathbf{k_2})\rangle = (2\pi)^3 \frac{P_Z(k_1)}{2}\delta_{s_1 s_2}\delta^{(3)}(\mathbf{k_1} + \mathbf{k_2}), \tag{10}$$

$$\langle \xi_L^{s_1}(\mathbf{k_1})\xi_L^{s_2}(\mathbf{k_2})\xi_{NL}^{s_3}(\mathbf{k_3})\rangle$$
$$= (2\pi)^3 \frac{P_Z(k_1)}{2}\frac{P_Z(k_2)}{2}f_{Z,s_1 s_2}^{s_3}\delta^{(3)}(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3}). \tag{11}$$

Then, the squeezed type of the non-Gaussianity can be expressed as

$$F^{s_1 s_2 s_3}(k_1, k_2, k_3) = \left(\frac{P_Z(k_1)}{2}\frac{P_Z(k_2)}{2}f_{Z,s_1 s_2}^{s_3} + 2\text{ perms}\right). \tag{12}$$

In the following discussion, we use the general expression (3) as the bispectra of the tensor and vector perturbations without specifying the type of the non-Gaussianity.

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[1]One can easily include the scalar mode into our notation by considering $s = 0$ initial stochastic field. For such a case, $\xi^0 = \Phi$ and $\tfrac{1}{2}f_{S,00}^0 = f_{NL}$ where the index S is used for the scalar mode.

[2]For the scalar mode, $P_S(k)/2 = P_\Phi(k)$.

III. CMB BISPECTRUM IN THE ALL-SKY ANALYSIS

In this section, we derive the formulae of the CMB bispectra sourced from tensor and vector perturbations on the full sky. The primordial perturbations in the scalar, vector, and tensor sectors introduced in the previous section are transferred through the primordial plasma to the CMB epoch and observed in the CMB temperature and polarization fluctuations. In the all-sky analysis, the CMB spin-0 temperature field I and spin-2 polarization fields Q, U, are expanded by spin-weighted spherical harmonics [26,34]. Following the usual manner, we convert the $Q \pm iU$ fields into spin-0 E and B fields by using the "spin raising operator" and "spin lowering operator" as Eqs. (B2) and (B3). Their radiative transfer functions are shown in Appendix B.

A. Scalar mode case

First, we give a brief review of the CMB bispectrum sourced from scalar perturbation. For the scalar case, the CMB bispectrum can be written as [2]

$$\langle a^{(S)}_{X,\ell_1 m_1} a^{(S)}_{X,\ell_2 m_2} a^{(S)}_{X,\ell_3 m_3} \rangle = \mathcal{G}^{m_1 m_2 m_3}_{\ell_1 \ell_2 \ell_3} b^{(S)}_{X,\ell_1 \ell_2 \ell_3}, \quad (13)$$

where the index (S) means that a source of the CMB fluctuation is the scalar perturbation and the index X denotes the temperature (I), the E-mode polarization (E), and the B-mode polarization (B).[3] Here, $\mathcal{G}^{m_1 m_2 m_3}_{\ell_1 \ell_2 \ell_3}$ is the Gaunt integral given by

$$\mathcal{G}^{m_1 m_2 m_3}_{\ell_1 \ell_2 \ell_3} \equiv \int d\Omega_y Y_{\ell_1 m_1}(\Omega_y) Y_{\ell_2 m_2}(\Omega_y) Y_{\ell_3 m_3}(\Omega_y), \quad (14)$$

and $b^{(S)}_{X,\ell_1 \ell_2 \ell_3}$ is the scalar reduced bispectrum formulated as

$$b^{(S)}_{X,\ell_1 \ell_2 \ell_3} = \int_0^\infty y^2 dy \left[\prod_{i=1}^3 \frac{2}{\pi} \int_0^\infty k_i^2 dk_i j_{\ell_i}(k_i y) \mathcal{T}^{(S)}_{X,\ell_i}(k_i) \right] \\ \times F^{000}(k_1, k_2, k_3). \quad (15)$$

Here $\mathcal{T}^{(S)}_{X,\ell_i}(k_i)$ is the time-integrated transfer function of the scalar perturbation as shown in Eqs. (B16) and (B17), and $j_\ell(x)$ is the spherical Bessel function.

B. Tensor and vector-mode case

Let us follow the above formulation for the tensor and vector cases. Tensor and vector 1-point functions are explicitly given as Eqs. (B15) and (B18)–(B23) in Appendix B. From those equations, one may wonder why tensor and vector 1-point functions depend on the spin-weighted spherical harmonics $_sY_{lm}(\Omega_k)$, although I,

E, and B modes are spin-0 fields. This dependence arises as a consequence of calculating the transfer function in the arbitrary direction of the wave vector $\mathbf{k}$. As discussed in detail in Appendix B, the transfer function for the arbitrary $\mathbf{k}$ is written with the Wigner D matrix [33,35,37] under the rotational transformation of $\mathbf{k}$ from a particular direction (e.g., z direction) to an arbitrary direction. This D matrix can be transcribed into $_sY_{lm}(\Omega_k)$ as Eq. (B13). As we will show in the following discussion, because of this spin-weighted spherical harmonics $_sY_{lm}(\Omega_k)$, the CMB bispectra sourced from the tensor and vector modes on each angular momentum, ℓ, depends on the sum of the reduced bispectrum over all angular momenta.[4]

For example, let us consider the CMB temperature fluctuation sourced from the tensor perturbation which has the spin-2 spherical harmonics as

$$a^{(T)}_{I,\ell m} \supset {}_{-2}Y_{\ell m}(\Omega_k) \xi^{+2}(\mathbf{k}), \qquad {}_{+2}Y_{\ell m}(\Omega_k) \xi^{-2}(\mathbf{k}). \quad (17)$$

Here we consider that the bispectrum of tensor-temperature fluctuations can be sourced from the non-Gaussianity of the primordial tensor perturbations which is characterized by the primordial bispectrum given by Eq. (3) and hence we can easily find

$$\langle a^{(T)}_{I,\ell_1 m_1} a^{(T)}_{I,\ell_2 m_2} a^{(T)}_{I,\ell_3 m_3} \rangle \supset \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3). \quad (18)$$

By making use of the expansion of 3D Dirac delta function given by

$$\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) = \int \frac{d^3 \mathbf{y}}{(2\pi)^3} e^{i(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \cdot \mathbf{y}} \\ = \int_0^\infty y^2 dy \int d\Omega_y \prod_{i=1}^3 2 \sum_{\ell_i' m_i'} i^{\ell_i'} j_{\ell_i'}(k_i y) \\ \times Y^*_{\ell_i' m_i'}(\Omega_y) Y_{\ell_i' m_i'}(\Omega_{k_i}), \quad (19)$$

and Eq. (17), a part of the bispectrum of tensor-temperature fluctuations can be expressed as

$$\langle a^{(T)}_{I,\ell_1 m_1} a^{(T)}_{I,\ell_2 m_2} a^{(T)}_{I,\ell_3 m_3} \rangle \supset \sum_{\substack{\ell_1' \ell_2' \ell_3' \\ m_1' m_2' m_3'}} \mathcal{G}^{m_1' m_2' m_3'}_{\ell_1' \ell_2' \ell_3'} \prod_{i=1}^3 \\ \times \int d\Omega_{k_i} Y_{\ell_i' m_i'}(\Omega_{k_i}) {}_{-2}Y^*_{\ell_i m_i}(\Omega_{k_i}). \quad (20)$$

Although the bispectrum of the scalar-temperature fluctuation (and E-mode polarization induced from the scalar-type perturbation) is derived in the same manner, the

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[3]Of course, scalar perturbation contributes only to the E-mode polarization and not to the B-mode one. In this paper, we use this index also for the tensor and vector modes which contribute not only to the E mode but also the B mode.

[4]This complexity does not occur for the CMB 2-point power spectra sourced even from the tensor and vector modes and as is well known all CMB power spectra can be described as

$$\langle a^{(Z)*}_{X',\ell' m'} a^{(Z)}_{X,\ell m} \rangle = C^{(Z)}_{X'X,\ell} \delta_{\ell'\ell} \delta_{m'm}. \quad (16)$$

orthogonality of the spin-0 spherical harmonic functions gives us the quite simple expression of Eq. (13) whose form is the Gaunt integral multiplied by the scalar reduced bispectrum. However, as seen in the above expression in the tensor case (also the vector case) the CMB bispectra on each ℓ depends on the sum of the reduced bispectrum over all angular momenta ℓ' as Eq. (21) in contrast to the scalar case such as Eq. (13), due to the nonorthogonality of the θ_k dependence between $Y_{\ell'm'}$ and $_sY_{\ell m}$ ($s = \pm 1$ or ± 2).

One may think that this complexity would be evaded once the plane wave could be expanded using spin-weighted spherical harmonics, rather than Eq. (19). In this paper, however, instead of pursuing this possibility we will use the flat-sky approximation to evade this difficulty as we shall show below.

Thus, the bispectrum formulae of the CMB fluctuations sourced from tensor and vector modes are, respectively, given by

$$
\langle a_{X,\ell_1 m_1}^{(Z)} a_{X,\ell_2 m_2}^{(Z)} a_{X,\ell_3 m_3}^{(Z)} \rangle = \sum_{\ell_1',\ell_2',\ell_3'} i^{\ell_1'+\ell_2'+\ell_3'} \mathcal{G}_{\ell_1'\ell_2'\ell_3'}^{m_1 m_2 m_3} \int_0^\infty y^2 dy \bigg[\prod_{i=1}^3 \frac{2}{\pi} (-i)^{\ell_i} \int_0^\infty k_i^2 dk_i j_{\ell_i'}(k_i y) \mathcal{T}_{X,\ell_i}^{(Z)}(k_i) \bigg]
$$
$$
\times \sum_{s_1 s_2 s_3} \mathrm{sgn}(s_1)^{s_1+x_1} \mathrm{sgn}(s_2)^{s_2+x_2} \mathrm{sgn}(s_3)^{s_3+x_3} F^{s_1 s_2 s_3}(k_1, k_2, k_3) \mathcal{Y}_{\ell\ell'm}^{(Z)}(s), \qquad (21)
$$

with

$$
\mathcal{Y}_{\ell\ell'm}^{(Z)}(s) \equiv \prod_{i=1}^3 \int d\Omega_{k_i} Y_{\ell'_i m_i}(\Omega_{k_i})_{-s_i} Y_{\ell_i m_i}^*(\Omega_{k_i}). \quad (22)
$$

Here $\mathcal{T}_{X,\ell}^{(Z)}(k_i)$ is the time-integrated transfer function generated from vector ($Z = V$) or tensor ($Z = T$) perturbation as described in Eqs. (B18)–(B23), and x is the index: $x \equiv 0$ for $X = I, E$ and $x \equiv 1$ for $X = B$.

As we have mentioned before, in the tensor and vector cases, due to the nonorthogonality of the θ_k dependence between $_sY_{\ell m}$ and $Y_{\ell'm'}$, the CMB bispectra on each ℓ depend on the sum of the reduced bispectrum over all angular momenta ℓ' as Eq. (21) in contrast to the scalar case such as Eq. (13). For this complexity, the numerical calculations of the tensor and vector bispectra take much longer time than that of the scalar one. However, this problem can be evaded by using the flat-sky approximation as shown in the next section.

IV. CMB BISPECTRA IN THE FLAT-SKY ANALYSIS

Here, we explain the formulation of the CMB bispectrum sourced from tensor and vector perturbations by using the flat-sky approximation as mentioned in Refs. [6,16,34,38]. The flat-sky approximation uses the (2D) plane wave expansion of the CMB fluctuation instead of the spherical harmonics one, and it is valid if we restrict observed direction $\hat{n}$ only close to the z axis. As confirmed in Ref. [34], the flat-sky power spectra of E- and B-mode polarizations sourced from the primordial tensor perturbations are in good agreement with the all-sky ones for $\ell \gtrsim 40$. In Ref. [6], the validity of the flat-sky analysis is also shown in the calculation of the temperature bispectra generated from the Sachs-Wolfe term by evaluating the convergence of the modified Bessel function. In addition, in Ref. [16], the consistency between the flat-sky result and all-sky one in the calculation of the scalar-temperature power spectrum and bispectrum are discussed

Based on these studies, we have also compared the all-sky power spectra with the flat-sky ones for the I, E, B modes from the tensor and vector perturbations and found their consistencies at $\ell \gtrsim 40$. We have also compared all-sky and flat-sky temperature bispectra induced from scalar-type perturbations, and confirmed that the flat-sky approximation is also applicable in the calculation of the bispectrum for the angular scales where the flat-sky power spectrum is a good approximation of the all-sky power spectrum. From these considerations, even if we cannot compare the all-sky bispectra with the flat-sky ones in the tensor and vector modes due to the difficulties discussed in the previous section, we can regard the flat-sky bispectra from the tensor and vector perturbations as good approximations for $\ell \gtrsim 40$.

A. Scalar bispectra in the flat-sky analysis

As described in Refs. [1,2], in the flat-sky approximation the scalar bispectrum Eq. (13) is modified as

$$
\langle a_X^{(S)}(\ell_1) a_X^{(S)}(\ell_2) a_X^{(S)}(\ell_3) \rangle
$$
$$
= (2\pi)^2 \delta^{(2)}(\ell_1 + \ell_2 + \ell_3) b_X^{(S)}(\ell_1, \ell_2, \ell_3). \quad (23)
$$

Since $\mathcal{G}_{\ell_1\ell_2\ell_3}^{m_1 m_2 m_3} \approx (2\pi)^2 \delta^{(2)}(\ell_1 + \ell_2 + \ell_3)$, Eq. (23) indicates $b_{X,\ell_1\ell_2\ell_3}^{(S)} \approx b_X^{(S)}(\ell_1, \ell_2, \ell_3)$. A detailed derivation of 1-point functions $a_X^{(Z)}(\ell)$ is presented in Appendix C. The scalar reduced bispectra are formulated, by using Eqs. (C6) and (C7), as

$$
b_X^{(S)}(\ell_1, \ell_2, \ell_3)
$$
$$
= \int_{-\infty}^\infty y^2 dy \bigg[\prod_{i=1}^3 \int_0^{\tau_0} d\tau_i \int_{\ell_i/D_i}^\infty \frac{dk_i}{2\pi} g_X^{(S)}(\ell_i, k_i, \tau_i, y) \bigg]
$$
$$
\times F^{000}(k_1, k_2, k_3), \quad (24)
$$

where $D_i \equiv \tau_0 - \tau_i$ and the scalar g functions are described as

$$g_I^{(S)}(\ell, k, \tau, y) = S_I^{(S)}(k, \tau)\frac{k}{\sqrt{k^2 - (\ell/D)^2}}\frac{2}{D^2}$$

$$\times \cos[\sqrt{k^2 - (\ell/D)^2}(y - D)], \quad (25)$$

$$g_E^{(S)}(\ell, k, \tau, y) = S_P^{(S)}(k, \tau)\frac{k}{\sqrt{k^2 - (\ell/D)^2}}\left(\frac{\ell}{kD}\right)^2\frac{2}{D^2}$$

$$\times \cos[\sqrt{k^2 - (\ell/D)^2}(y - D)]. \quad (26)$$

Here $S_I^{(S)}(k, \tau)$ and $S_P^{(S)}(k, \tau)$ are the scalar-type source functions of the temperature and polarization fluctuations as mentioned in Appendix A.

B. Tensor and vector bispectra in the flat-sky analysis

Let us consider the tensor-temperature bispectrum in the flat-sky analysis. By using Eq. (C4), a component of the flat-sky bispectrum of tensor-temperature mode is written as

$$\langle a_I^{(T)}(\ell_1)a_I^{(T)}(\ell_2)a_I^{(T)}(\ell_3)\rangle = \left[\prod_{i=1}^{3}\int_0^{\tau_0}d\tau_i\int_{-\infty}^{\infty}\frac{dk_{iz}}{2\pi}S_I^{(T)}\left(k_i = \sqrt{k_{iz}^2 + (\ell_i/D_i)^2}, \tau_i\right)\frac{\ell_i^2}{(k_{iz}D_i)^2 + \ell_i^2}\frac{1}{D_i^2}e^{-ik_{iz}D_i}\right]$$

$$\times \sum_{s_1s_2s_3=\pm2}F^{s_1s_2s_3}\left(\sqrt{k_{1z}^2 + (\ell_1/D_1)^2}, \sqrt{k_{2z}^2 + (\ell_2/D_2)^2}, \sqrt{k_{3z}^2 + (\ell_3/D_3)^2}\right)$$

$$\times (2\pi)^3\delta^{(2)}\left(\frac{\ell_1}{D_1} + \frac{\ell_2}{D_2} + \frac{\ell_3}{D_3}\right)\delta(k_{1z} + k_{2z} + k_{3z}). \quad (27)$$

By using the expansion of the 1D Dirac delta function and the approximation of 2D Dirac delta function as

$$\delta(k_{1z} + k_{2z} + k_{3z}) = \int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{i(k_{1z}+k_{2z}+k_{3z})y}, \quad (28)$$

$$\delta^{(2)}\left(\frac{\ell_1}{D_1} + \frac{\ell_2}{D_2} + \frac{\ell_3}{D_3}\right) = D_1^2\delta^{(2)}(\ell_1 + \ell_2 + \ell_3 + \frac{D_1 - D_2}{D_2}\ell_2 + \frac{D_1 - D_3}{D_3}\ell_3) \approx D_1^2\delta^{(2)}(\ell_1 + \ell_2 + \ell_3), \quad (29)$$

we can derive the simple form of the tensor-temperature bispectrum in the flat limit as

$$\langle a_I^{(T)}(\ell_1)a_I^{(T)}(\ell_2)a_I^{(T)}(\ell_3)\rangle \approx (2\pi)^2\delta^{(2)}(\ell_1 + \ell_2 + \ell_3)\int_{-\infty}^{\infty}y^2dy\left[\prod_{i=1}^{3}\int_0^{\tau_0}d\tau_i\int_{\ell_i/D_i}^{\infty}\frac{dk_i}{2\pi}g_I^{(T)}(\ell_i, k_i, \tau_i, y)\right]$$

$$\times \sum_{s_1,s_2,s_3=\pm2}F^{s_1s_2s_3}(k_1, k_2, k_3). \quad (30)$$

The approximation of Eq. (29) is valid because the bispectra are suppressed when the triangle in the ℓ space does not close as discussed in Ref. [6]. In Eq. (30), we use the approximation $(D_1/y)^2 \approx 1$, which is valid because the integrand has large value for $D_1 \sim y \sim \tau_0$. Similar to the discussion in the previous section, as the other tensor bispectra and the vector bispectra can be derived in the same manner, these bispectra can be written by the same form as the scalar bispectra of Eq. (23) which can be written as a 2D Dirac delta function multiplied by the reduced bispectra;

$$\langle a_X^{(Z)}(\ell_1)a_X^{(Z)}(\ell_2)a_X^{(Z)}(\ell_3)\rangle = (2\pi)^2\delta^{(2)}(\ell_1 + \ell_2 + \ell_3)b_X^{(Z)}(\ell_1, \ell_2, \ell_3), \quad (31)$$

where the tensor or vector reduced bispectrum is expressed as

$$b_X^{(Z)}(\ell_1, \ell_2, \ell_3) = \int_{-\infty}^{\infty}y^2dy\left[\prod_{i=1}^{3}\int_0^{\tau_0}d\tau_i\int_{\ell_i/D_i}^{\infty}\frac{dk_i}{2\pi}g_X^{(Z)}(\ell_i, k_i, \tau_i, y)\right]$$

$$\times \sum_{s_1,s_2,s_3}\text{sgn}(s_1)^{x_1}\text{sgn}(s_2)^{x_2}\text{sgn}(s_3)^{x_3}F^{s_1s_2s_3}(k_1, k_2, k_3). \quad (32)$$

Tensor g functions are written as

$$g_I^{(T)}(\ell, k, \tau, y) = S_I^{(T)}(k, \tau) \frac{k}{\sqrt{k^2 - (\ell/D)^2}} \left(\frac{\ell}{kD}\right)^2 \frac{2}{D^2}$$
$$\times \cos[\sqrt{k^2 - (\ell/D)^2}(y - D)], \qquad (33)$$

$$g_E^{(T)}(\ell, k, \tau, y) = S_P^{(T)}(k, \tau) \frac{k}{\sqrt{k^2 - (\ell/D)^2}} \left[2 - \left(\frac{\ell}{kD}\right)^2\right] \frac{2}{D^2}$$
$$\times \cos[\sqrt{k^2 - (\ell/D)^2}(y - D)], \qquad (34)$$

$$g_B^{(T)}(\ell, k, \tau, y) = -S_P^{(T)}(k, \tau) \frac{4}{D^2} \sin[\sqrt{k^2 - (\ell/D)^2}(y - D)].$$
$$\qquad (35)$$

Vector g functions are also described as

$$g_I^{(V)}(\ell, k, \tau, y) = iS_I^{(V)}(k, \tau) \frac{\ell}{\sqrt{(kD)^2 - \ell^2}} \frac{2}{D^2}$$
$$\times \cos[\sqrt{k^2 - (\ell/D)^2}(y - D)], \qquad (36)$$

$$g_E^{(V)}(\ell, k, \tau, y) = -iS_P^{(V)}(k, \tau) \left(\frac{\ell}{kD}\right) \frac{2}{D^2}$$
$$\times \sin[\sqrt{k^2 - (\ell/D)^2}(y - D)], \qquad (37)$$

$$g_B^{(V)}(\ell, k, \tau, y) = -iS_P^{(V)}(k, \tau) \frac{\ell}{\sqrt{(kD)^2 - \ell^2}} \frac{2}{D^2}$$
$$\times \cos[\sqrt{k^2 - (\ell/D)^2}(y - D)]. \qquad (38)$$

Here $S_I^{(Z)}(k, \tau)$ and $S_P^{(Z)}(k, \tau)$ are the Z-type source functions of the temperature and polarization fluctuations as mentioned in Appendix A.

By comparing Eq. (31) to Eq. (23), we find that the tensor and vector bispectra are formulated in the same form as the scalar one. It is because the helicity dependence, which brings nontrivial couplings between angular momenta in the reduced bispectra, vanishes in the CMB 1-point functions induced from the tensor and vector perturbations due to the absence of the contribution of azimuthal angle from **k** to $\hat{n}$ in the transfer functions as discussed in Appendix C. Hence, unlike the all-sky analysis, the sum of the reduced bispectrum is not needed in calculating the tensor and vector bispectra in the flat-sky limit and one can calculate the tensor and vector bispectra with the same computational cost taken in the scalar case. This corresponds to the restoration of the orthogonality of θ_k between $Y_{\ell'm'}$ and $_sY_{\ell m}$ ($s = \pm 1$ or ± 2) for $\ell \gg 1$, namely, $\mathcal{Y}_{\ell \ell' m}^{(Z)}(s) \to \delta_{\ell_1 \ell_1'} \delta_{\ell_2 \ell_2'} \delta_{\ell_3 \ell_3'}$ (there is no dependence on m_1, m_2 and m_3) due to $_sY_{\ell m} \to Y_{\ell m}$ for $\ell \gg 1$. In other words, it means that because the degeneracy factor of m equal to $2\ell + 1$ becomes so large in the large ℓ limit,

the spin eigenstate of $s = \pm 1$ or ± 2 and that of $s = 0$ are almost indistinguishable.

In addition, interestingly, from Eq. (32) and (12), we find that B-mode bispectra ($x = 1$) from tensor and vector perturbations vanish if each $f_{Z,s_1 s_2}^{s_3}$ is identical value. This situation corresponds to the parity conservation at the non-linear level. Therefore, if one detects the finite value of the B-mode bispectrum in the squeezed limit, it may offer further evidence of the cosmological parity violation.

V. SUMMARY AND DISCUSSION

In this paper, we derive the complete set of CMB temperature and polarization bispectra generated from non-Gaussianity in the tensor and vector-mode perturbations both in the all- and flat-sky analyses. For the primordial non-Gaussianity in the tensor and vector sectors, we consider the more general type such as Eq. (3), which contains the squeezed type given by Eqs. (8) and (9) and the equilateral type.

Note that the formulation presented can be easily extended in a straightforward manner to the other cases, such as a case in which the nonlinear tensor perturbation is excited by the linear-order scalar-tensor couplings. As an example of this, we can consider the scalar-graviton interaction during inflation shown by Ref. [39]. Through such interaction, the non-Gaussianity of the primordial fluctuations can be generated as a scalar-scalar-tensor type, namely, $\langle a^{(S)} a^{(S)} a^{(T)} \rangle$. Although, in the standard slow-roll inflation, such type of non-Gaussianity is expected to be suppressed by the slow-roll parameter, it seems interesting that one investigates such type of non-Gaussianity through the future CMB observations in the sense of the confirmation of the standard inflation scenario, by using our formulation. Furthermore, the 3-point cross correlations between CMB intensity and polarizations, such as $\langle a_I a_I a_E \rangle$, and higher-order correlations than the 3-point one can be easily formulated in the same manner [33,40].

In the formulation of all-sky bispectra, we find that those formulae take complicated forms compared to the scalar one due to the helicity dependence which is represented by the azimuthal angle dependence between the wave vector of photon and the unit vector specifying the line of sight direction in the photon propagation. However, in the formulation of flat-sky bispectra, we find that the above difficulty is solved for the absence of the above azimuthal dependence. In addition, we also show that if the bispectra of B-mode polarization are a nonzero value, it may become evidence of the cosmological parity violation in the non-linear sector.

ACKNOWLEDGMENTS

APPENDIX A: RADIATION TRANSFER FUNCTIONS

Here we show the radiation transfer functions of temperature mode Δ_I and two polarization modes Δ_Q, Δ_U. Transfer functions induced by the scalar and tensor modes in a particular basis in which the wave vector of photons **k** is parallel to z axis $\hat{z}$ are formulated in Refs. [34,35,41] by the line of sight integral method. For the vector case, the method of calculation can be obtained in Ref. [36]. Based on the Stokes parameters as defined in Ref. [16], these are expressed as follows:

$$\Delta_I^{(S)}(\tau_0, \mathbf{k}, \Omega_{k,n}) = \Phi(\mathbf{k}) \int_0^{\tau_0} d\tau S_I^{(S)}(k, \tau) e^{-i\mu_{k,n}x}, \quad (A1)$$

$$\Delta_Q^{(S)}(\tau_0, \mathbf{k}, \Omega_{k,n})$$
$$= (1 - \mu_{k,n}^2)\Phi(\mathbf{k}) \int_0^{\tau_0} d\tau S_P^{(S)}(k, \tau) e^{-i\mu_{k,n}x}, \quad (A2)$$

$$\Delta_I^{(T)}(\tau_0, \mathbf{k}, \Omega_{k,n})$$
$$= (1 - \mu_{k,n}^2)(e^{2i\phi_{k,n}}\xi^{+2}(\mathbf{k}) + e^{-2i\phi_{k,n}}\xi^{-2}(\mathbf{k}))$$
$$\times \int_0^{\tau_0} d\tau S_I^{(T)}(k, \tau) e^{-i\mu_{k,n}x}, \quad (A3)$$

$$(\Delta_Q^{(T)} \pm i\Delta_U^{(T)})(\tau_0, \mathbf{k}, \Omega_{k,n})$$
$$= [(1 \mp \mu_{k,n})^2 e^{2i\phi_{k,n}}\xi^{+2}(\mathbf{k}) + (1 \pm \mu_{k,n})^2 e^{-2i\phi_{k,n}}\xi^{-2}(\mathbf{k})]$$
$$\times \int_0^{\tau_0} d\tau S_P^{(T)}(k, \tau) e^{-i\mu_{k,n}x}, \quad (A4)$$

$$\Delta_I^{(V)}(\tau_0, \mathbf{k}, \Omega_{k,n})$$
$$= -i\sqrt{1 - \mu_{k,n}^2}(\xi^{+1}(\mathbf{k})e^{i\phi_{k,n}} + \xi^{-1}(\mathbf{k})e^{-i\phi_{k,n}})$$
$$\times \int_0^{\tau_0} d\tau S_I^{(V)}(k, \tau) e^{-i\mu_{k,n}x}, \quad (A5)$$

$$(\Delta_Q^{(V)} \pm i\Delta_U^{(V)})(\tau_0, \mathbf{k}, \Omega_{k,n})$$
$$= \sqrt{1 - \mu_{k,n}^2}[\mp(1 \mp \mu_{k,n})\xi^{+1}(\mathbf{k})e^{i\phi_{k,n}} \pm (1 \pm \mu_{k,n})$$
$$\times \xi^{-1}(\mathbf{k})e^{-i\phi_{k,n}}] \int_0^{\tau_0} d\tau S_P^{(V)}(k, \tau) e^{-i\mu_{k,n}x}, \quad (A6)$$

where $\Omega_{k,n}(\equiv (\theta_{k,n}, \phi_{k,n}))$ denotes the orientation of the line of sight direction $\hat{n}$ in a particular basis in which $\mathbf{k}||\hat{z}$, $\mu_{k,n} \equiv \cos\theta_{k,n}$, and $S_I^{(Z)}(k, \tau)$ and $S_P^{(Z)}(k, \tau)$ are the Z-type source functions of the temperature and polarization fluctuations [34,42,43].

In order to estimate the 1-point function $a_{\ell m}$, one must construct the transfer functions for the arbitrary **k**. In other words, we want to obtain the transfer functions expressed by the arbitrary **k** (whose direction is denoted by Ω_k) and $\hat{n}$ (denoted by Ω_n) instead of $\Omega_{k,n}$. To achieve this we introduce the rotational matrix

$$S(\Omega_k) \equiv \begin{pmatrix} \cos\theta_k\cos\phi_k & -\sin\phi_k & \sin\theta_k\cos\phi_k \\ \cos\theta_k\sin\phi_k & \cos\phi_k & \sin\theta_k\sin\phi_k \\ -\sin\theta_k & 0 & \cos\theta_k \end{pmatrix}, \quad (A7)$$

which expresses the basis rotation that transforms $\hat{z} \parallel \mathbf{k}$ to the arbitrary $\hat{z}$. Then the relation between Ω_k, Ω_n, and $\Omega_{k,n}$ is written

$$\begin{pmatrix} \sin\theta_n\cos\phi_n \\ \sin\theta_n\sin\phi_n \\ \cos\phi_n \end{pmatrix} = S(\Omega_k) \begin{pmatrix} \sin\theta_{k,n}\cos\phi_{k,n} \\ \sin\theta_{k,n}\sin\phi_{k,n} \\ \cos\phi_{k,n} \end{pmatrix}. \quad (A8)$$

In the temperature modes, only by changing $\Omega_{k,n}$ to Ω_k and Ω_n with the relation (A8), the transfer functions for the arbitrary **k** can be obtained. In the E, B modes, in addition to this treatment, one must consider the mixing between Δ_Q and Δ_U under the transformation $S(\Omega_k)$ as described in Ref. [34].

This effect is expressed as

$$(\Delta_Q' \pm i\Delta_U')(\tau_0, \mathbf{k}, \Omega_n) = e^{\mp 2i\psi}(\Delta_Q \pm i\Delta_U)(\tau_0, \mathbf{k}, \Omega_{k,n}), \quad (A9)$$

with the mixing angle ψ. The angle ψ represents the rotation angle between $\hat{\theta}_{k,n}$ and $\hat{\theta}_n$, where $\hat{\theta}_{k,n}$ and $\hat{\theta}_n$ are the unit vectors orthogonal to $\hat{n}$ in a particular basis in which $\mathbf{k} \parallel \hat{z}$ and a general basis, respectively.

In the flat-sky analysis, i.e., $\theta_n \to 0$, by using Eqs. (A1)–(A6) and (A8) and by using the limit of ψ as $\psi \to \phi_n - \phi_k + \pi$, the transfer functions for the arbitrary **k** are derived as

$$\Delta_I^{(S)}(\tau_0, \mathbf{k}, \Omega_n) \to \Phi(\mathbf{k}) \int_0^{\tau_0} d\tau S_I^{(S)}(k, \tau) e^{-i\mathbf{k}\cdot\hat{n}D}, \quad (A10)$$

$$(\Delta_Q^{(S)} \pm i\Delta_U^{(S)})(\tau_0, \mathbf{k}, \Omega_n)$$
$$\to e^{\mp 2i(\phi_n - \phi_k)}\sin^2\theta_k\Phi(\mathbf{k}) \int_0^{\tau_0} d\tau S_P^{(S)}(k, \tau) e^{-i\mathbf{k}\cdot\hat{n}D}, \quad (A11)$$

$$\Delta_I^{(T)}(\tau_0, \mathbf{k}, \Omega_n)$$

$$\rightarrow (1 - \mu_k^2)(\xi^{+2} + \xi^{-2})(\mathbf{k}) \int_0^{\tau_0} d\tau S_I^{(T)}(k, \tau) e^{-i\mathbf{k}\cdot\hat{n}D},$$

$$(A12)$$

$$(\Delta_Q^{(T)} \pm i\Delta_U^{(T)})(\tau_0, \mathbf{k}, \Omega_n)$$

$$\rightarrow e^{\mp 2i(\phi_n - \phi_k)}[(1 + \mu_k^2)(\xi^{+2} + \xi^{-2})(\mathbf{k})$$

$$\mp 2\mu_k(\xi^{+2} - \xi^{-2})(\mathbf{k})] \int_0^{\tau_0} d\tau S_P^{(T)}(k, \tau) e^{-i\mathbf{k}\cdot\hat{n}D}, \quad (A13)$$

$$\Delta_I^{(V)}(\tau_0, \mathbf{k}, \Omega_n)$$

$$\rightarrow i\sin\theta_k(\xi^{+1} + \xi^{-1})(\mathbf{k}) \int_0^{\tau_0} d\tau S_I^{(V)}(k, \tau) e^{-i\mathbf{k}\cdot\hat{n}D},$$

$$(A14)$$

$$(\Delta_Q^{(V)} \pm i\Delta_U^{(V)})(\tau_0, \mathbf{k}, \Omega_n)$$

$$\rightarrow e^{\mp 2i(\phi_n - \phi_k)} \sin\theta_k[-\cos\theta_k(\xi^{+1} + \xi^{-1})(\mathbf{k})$$

$$\pm (\xi^{+1} - \xi^{-1})(\mathbf{k})] \int_0^{\tau_0} d\tau S_P^{(V)}(k, \tau) e^{-i\mathbf{k}\cdot\hat{n}D}. \quad (A15)$$

It is important to note that the ϕ_k dependence which are inherent in the vector and tensor perturbations vanishes in the flat-sky approximation, besides a trivial ϕ_k dependence due to a spin-2 nature of the Stokes Q and U parameters. One may explicitly see that $\phi_{k,n}$ dependence vanishes in the transfer functions when taking $\theta_n \rightarrow 0$ because the S matrix rotates the basis with the new z axis always being on the x–z plane in a particular basis in which $\mathbf{k} \parallel \hat{z}$. This approximation means that for $\theta_n \ll 1$, it is valid to calculate the CMB fluctuation on the basis of vector and tensor perturbations fixed as $\theta_n = 0$, namely, $\phi_{k,n} = \pi$.

APPENDIX B: 1-POINT FUNCTION IN THE ALL-SKY ANALYSIS

Here we formulate the all-mode 1-point functions $a_{\ell m}$ in the all-sky analysis based on the derivation in Ref. [35]. One-point functions of the I, E, B modes are generated from Δ_I, Δ_Q, Δ_U as

$$a_{I,\ell m} = \int d\Omega_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Delta_I(\tau_0, \mathbf{k}, \Omega_n) Y_{\ell m}^*(\Omega_n), \quad (B1)$$

$$a_{E,\ell m} = -\frac{1}{2} \int d\Omega_n \int \frac{d^3\mathbf{k}}{(2\pi)^3}$$

$$\times [(\Delta_Q + i\Delta_U)(\tau_0, \mathbf{k}, \Omega_n)_2 Y_{\ell m}^*(\Omega_n)$$

$$+ (\Delta_Q - i\Delta_U)(\tau_0, \mathbf{k}, \Omega_n)_{-2} Y_{\ell m}^*(\Omega_n)]$$

$$= -\frac{1}{2}\left[\frac{(\ell - 2)!}{(\ell + 2)!}\right]^{1/2} \int d\Omega_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} [\bar{\eth}^2(\Delta_Q + i\Delta_U)$$

$$+ \eth^2(\Delta_Q - i\Delta_U)](\tau_0, \mathbf{k}, \Omega_n) Y_{\ell m}^*(\Omega_n), \quad (B2)$$

$$a_{B,\ell m} = \frac{i}{2} \int d\Omega_n \int \frac{d^3\mathbf{k}}{(2\pi)^3}$$

$$\times [(\Delta_Q + i\Delta_U)(\tau_0, \mathbf{k}, \Omega_n)_2 Y_{\ell m}^*(\Omega_n)$$

$$- (\Delta_Q - i\Delta_U)(\tau_0, \mathbf{k}, \Omega_n)_{-2} Y_{\ell m}^*(\Omega_n)]$$

$$= \frac{i}{2}\left[\frac{(\ell - 2)!}{(\ell + 2)!}\right]^{1/2} \int d\Omega_n \int \frac{d^3\mathbf{k}}{(2\pi)^3}$$

$$\times [\bar{\eth}^2(\Delta_Q + i\Delta_U) - \eth^2(\Delta_Q - i\Delta_U)]$$

$$\times (\tau_0, \mathbf{k}, \Omega_n) Y_{\ell m}^*(\Omega_n). \quad (B3)$$

Here we expand with the spin raising (lowering) operators $\eth$ ($\bar{\eth}$) as introduced in Refs. [34,37,44] and $Y_{\ell m}(\Omega_n)$ for being easily understanding that E, B modes are spin-0 fields. $\eth$ and $\bar{\eth}$ act the spin-s function $_s f(\theta_n, \phi_n)$ as

$$\eth_s f(\theta_n, \phi_n) = -\sin^s \theta_n [\partial_{\theta_n} + i\csc\theta_n \partial_{\phi_n}]$$

$$\times \sin^{-s}\theta_{ns} f(\theta_n, \phi_n), \quad (B4)$$

$$\bar{\eth}_s f(\theta_n, \phi_n) = -\sin^{-s}\theta_n [\partial_{\theta_n} - i\csc\theta_n \partial_{\phi_n}]$$

$$\times \sin^s\theta_{ns} f(\theta_n, \phi_n). \quad (B5)$$

From here, we derive the 1-point function of tensor-temperature mode as an example. As mentioned in Sec. III, This is calculated by using Wigner D matrix $D_{mm'}^{(\ell)}$, which is the unitary irreducible matrix of rank $2\ell + 1$ that forms a representation of the rotational group. The property of this matrix and the relation with spin-weighted spherical harmonics are explained in Refs. [33,35,37]. By using Eq. (B1), the relation between the $Y_{\ell m}$ and D matrix, and the relation corresponding to Eq. (A8) as

$$Y_{\ell m}^*(\Omega_n) = \sum_{m'} D_{mm'}^{(\ell)}(S(\Omega_k)) Y_{\ell m'}^*(\Omega_{k,n}), \quad (B6)$$

$$d\Omega_n = d\Omega_{k,n}, \quad (B7)$$

the 1-point function of tensor-temperature mode is written as

$$a_{I,\ell m}^{(T)} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\sum_{m'} D_{mm'}^{(\ell)}(S(\Omega_k))\right.$$

$$\left. \times \int d\Omega_{k,n} Y_{\ell m'}^*(\Omega_{k,n}) \Delta_I^{(T)}(\tau_0, \mathbf{k}, \Omega_{k,n})\right]. \quad (B8)$$

Next, with the mathematical relations as

$$Y_{\ell m'}^*(\Omega_{k,n}) = \left[\frac{2l + 1}{4\pi} \frac{(\ell - m')!}{(\ell + m')!}\right]^{1/2} P_\ell^{m'}(\mu_{k,n}) e^{-im'\phi_{k,n}}, \quad (B9)$$

$$P_\ell^{-2}(\mu_{k,n}) = \frac{(\ell - 2)!}{(\ell + 2)!} P_\ell^2(\mu_{k,n}), \quad (B10)$$

$$\int_{-1}^{1} d\mu_{k,n}(1 - \mu_{k,n}^2)P_\ell^2(\mu_{k,n})e^{-i\mu_{k,n}x}$$

$$= -2(-i)^\ell \frac{(\ell + 2)!}{(\ell - 2)!} \frac{j_\ell(x)}{x^2}, \tag{B11}$$

the integration for $\Omega_{k,n}$ can be performed to obtain

$$a_{I,\ell m}^{(T)} = -4\pi(-i)^\ell \left[\frac{(\ell + 2)!}{(\ell - 2)!}\right]^{1/2}\left[\frac{2\ell + 1}{4\pi}\right]^{1/2} \int \frac{d^3\mathbf{k}}{(2\pi)^3}$$

$$\times [D_{m2}^{(\ell)}(S(\Omega_k))\xi^{+2}(\mathbf{k}) + D_{m,-2}^{(\ell)}(S(\Omega_k))\xi^{-2}(\mathbf{k})]$$

$$\times \int_0^{\tau_0} d\tau S_I^{(T)}(k, \tau)\frac{j_\ell(x)}{x^2} \tag{B12}$$

Because D matrix is written by the spin-weighted spherical harmonics as

$$D_{ms}^{(\ell)}(S(\Omega_k)) = \left[\frac{4\pi}{2\ell + 1}\right]^{1/2}(-1)^s {}_{-s}Y_{\ell m}^*(\Omega_k), \tag{B13}$$

we obtain the final form, namely,

$$a_{I,\ell m}^{(T)} = -4\pi(-i)^\ell \left[\frac{(\ell + 2)!}{(\ell - 2)!}\right]^{1/2} \int \frac{d^3\mathbf{k}}{(2\pi)^3}$$

$$\times [{}_{-2}Y_{\ell m}^*(\Omega_k)\xi^{+2}(\mathbf{k}) + {}_{2}Y_{\ell m}^*(\Omega_k)\xi^{-2}(\mathbf{k})]$$

$$\times \int_0^{\tau_0} d\tau S_I^{(T)}(k, \tau)\frac{j_\ell(x)}{x^2}. \tag{B14}$$

For the other modes, we can derive in the same manner with Eqs. (B1)–(B3), (A1), (A2), and (A4)–(A6).

As a result, all-sky 1-point functions can be formulated:

$$a_{X,\ell m}^{(Z)}$$

$$= 4\pi(-i)^\ell \int \frac{d^3\mathbf{k}}{(2\pi)^3}$$

$$\times \begin{cases} Y_{\ell m}^*(\Omega_k)\Phi(\mathbf{k})\mathcal{T}_{X,\ell}^{(Z)}(k) & \text{(for } Z = S) \\ \sum_s \text{sgn}(s)^{s+x} {}_{-s}Y_{\ell m}^*(\Omega_k)\xi^s(\mathbf{k})\mathcal{T}_{X,\ell}^{(Z)}(k) & \text{(for } Z = T, V) \end{cases},$$

$$\tag{B15}$$

where $x = 0$ for $X = I$, E, $x = 1$ for $X = B$, time-integrated transfer functions $\mathcal{T}_{X,\ell}^{(Z)}(k)$ are expressed as

$$\mathcal{T}_{I,\ell}^{(S)}(k) = \int_0^{\tau_0} d\tau S_I^{(S)}(k, \tau)j_\ell(x), \tag{B16}$$

$$\mathcal{T}_{E,\ell}^{(S)}(k) = \left[\frac{(\ell - 2)!}{(\ell + 2)!}\right]^{1/2} \int_0^{\tau_0} d\tau S_P^{(S)}(k, \tau)\hat{\mathcal{E}}^{(S)}(x)j_\ell(x), \tag{B17}$$

$$\mathcal{T}_{I,\ell}^{(T)}(k) = -\left[\frac{(\ell + 2)!}{(\ell - 2)!}\right]^{1/2} \int_0^{\tau_0} d\tau S_I^{(T)}(k, \tau)\frac{j_\ell(x)}{x^2}, \tag{B18}$$

$$\mathcal{T}_{E,\ell}^{(T)}(k) = -\int_0^{\tau_0} d\tau S_P^{(T)}(k, \tau)\hat{\mathcal{E}}^{(T)}(x)\frac{j_\ell(x)}{x^2}, \tag{B19}$$

$$\mathcal{T}_{B,\ell}^{(T)}(k) = \int_0^{\tau_0} d\tau S_P^{(T)}(k, \tau)\hat{\mathcal{B}}^{(T)}(x)\frac{j_\ell(x)}{x^2}, \tag{B20}$$

$$\mathcal{T}_{I,\ell}^{(V)}(k) = -\left[\frac{(\ell + 1)!}{(\ell - 1)!}\right]^{1/2} \int_0^{\tau_0} d\tau S_I^{(V)}(k, \tau)\frac{j_\ell(x)}{x}, \tag{B21}$$

$$\mathcal{T}_{E,\ell}^{(V)}(k) = \left[\frac{(\ell + 1)!}{(\ell - 1)!}\frac{(\ell - 2)!}{(\ell + 2)!}\right]^{1/2}$$

$$\times \int_0^{\tau_0} d\tau S_P^{(V)}(k, \tau)\hat{\mathcal{E}}^{(V)}(x)\frac{j_\ell(x)}{x}, \tag{B22}$$

$$\mathcal{T}_{B,\ell}^{(V)}(k) = \left[\frac{(\ell + 1)!}{(\ell - 1)!}\frac{(\ell - 2)!}{(\ell + 2)!}\right]^{1/2}$$

$$\times \int_0^{\tau_0} d\tau S_P^{(V)}(k, \tau)\hat{\mathcal{B}}^{(V)}(x)\frac{j_\ell(x)}{x}, \tag{B23}$$

and the operators $\mathcal{E}$, $\mathcal{B}$ are defined as

$$\hat{\mathcal{E}}^{(S)}(x) \equiv (1 + \partial_x^2)^2 x^2,$$

$$\hat{\mathcal{E}}^{(T)}(x) \equiv -12 + x^2(1 - \partial_x^2) - 8x\partial_x,$$

$$\hat{\mathcal{B}}^{(T)}(x) \equiv 8x + 2x^2\partial_x, \tag{B24}$$

$$\hat{\mathcal{E}}^{(V)}(x) \equiv 4x + (12 + x^2)\partial_x + 8x\partial_x^2 + x^2\partial_x^3,$$

$$\hat{\mathcal{B}}^{(V)}(x) \equiv x^2 + 4x\partial_x + x^2\partial_x^2.$$

Note that in the all-sky analysis, due to the dependence of transfer functions on $\phi_{k,n}$, 1-point functions depend on the helicity state through the spin spherical harmonics.

APPENDIX C: 1-POINT FUNCTION IN THE FLAT-SKY ANALYSIS

In this section, we formulate the all-mode 1-point functions $a_{\ell m}$ in the flat-sky analysis. In this limit, 1-point functions in the all-sky analysis described as Eqs. (B1)–(B3) are modified by using the plane wave as

$$a_{I,\ell m} \rightarrow \int d^2\Theta \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Delta_I(\tau_0, \mathbf{k}, \Omega_n)e^{-i\ell\cdot\Theta} \equiv a_I(\ell), \tag{C1}$$

$$a_{E,\ell m} \rightarrow \frac{1}{2} \int d^2\Theta \int \frac{d^3\mathbf{k}}{(2\pi)^3}[(\Delta_Q + i\Delta_U)e^{-2i(\phi_\ell - \phi_n)}$$

$$+ (\Delta_Q - i\Delta_U)e^{2i(\phi_\ell - \phi_n)}](\tau_0, \mathbf{k}, \Omega_n)e^{-i\ell\cdot\Theta}$$

$$\equiv a_E(\ell), \tag{C2}$$

$$a_{B,\ell m} \rightarrow \frac{i}{2} \int d^2\Theta \int \frac{d^3\mathbf{k}}{(2\pi)^3}[-(\Delta_Q + i\Delta_U)e^{-2i(\phi_\ell - \phi_n)}$$

$$+ (\Delta_Q - i\Delta_U)e^{2i(\phi_\ell - \phi_n)}](\tau_0, \mathbf{k}, \Omega_n)e^{-i\ell\cdot\Theta}$$

$$\equiv a_B(\ell), \tag{C3}$$

where Θ is the 2D vector projecting $\hat{n}$ to the flat-sky plane expressed as $\Theta = (\Theta \cos\phi_n, \Theta \sin\phi_n)$. For example, in order to obtain the 1-point function of the tensor-temperature mode, we substitute Eq. (A12) into Eq. (C1) and calculate as follows:

$$a_I^{(T)}(\ell) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}(1 - \mu_k^2)(\xi^{+2} + \xi^{-2})(\mathbf{k}) \int_0^{\tau_0} d\tau \int d^2\mathbf{\Theta} e^{-i(\mathbf{k}^\parallel D + \ell)\cdot\mathbf{\Theta}} S_I^{(T)}(k, \tau) e^{-ik_z D}$$

$$= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sin^2\theta_k (\xi^{+2} + \xi^{-2})(\mathbf{k}) \int_0^{\tau_0} d\tau (2\pi)^2 \delta^{(2)}(\mathbf{k}^\parallel D + \ell) S_I^{(T)}(k, \tau) e^{-ik_z D}$$

$$= \int_0^{\tau_0} d\tau \int_{-\infty}^{\infty} \frac{dk_z}{2\pi}(\xi^{+2} + \xi^{-2})(\mathbf{k}^\parallel = -\ell/D, k_z) \frac{\ell^2}{(k_z D)^2 + \ell^2} S_I^{(T)}(k = \sqrt{k_z^2 + (\ell/D)^2}, \tau)\frac{1}{D^2} e^{-ik_z D}, \quad (C4)$$

where $D = \tau_0 - \tau$ is the conformal distance and we have decomposed $\mathbf{k}$ into two-dimensional vector parallel to the flat sky and that orthogonal to it, $\mathbf{k} = (\mathbf{k}^\parallel, k_z)$. In order to obtain the last equation, we use following relations which are satisfied under $\mathbf{k}^\parallel = -\ell/D$ as

$$k = \sqrt{k_z^2 + \left(\frac{\ell}{D}\right)^2},$$

$$\sin\theta_k = \frac{\ell}{kD} = \frac{\ell}{\sqrt{(k_z D)^2 + \ell^2}},$$

$$\cos\theta_k = \text{sgn}(k_z)\sqrt{1 - \left(\frac{\ell}{kD}\right)^2},$$

$$\phi_k = \phi_\ell + \pi. \quad (C5)$$

One-point functions of the other modes are calculated in the same manner by using Eqs. (A10), (A11), (A13)–(A15), and (C1)–(C3) as

$$a_I^{(S)}(\ell) = \int_0^{\tau_0} d\tau \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \Phi(\mathbf{k}^\parallel = -\ell/D, k_z)$$
$$\times S_I^{(S)}(k = \sqrt{k_z^2 + (\ell/D)^2}, \tau)\frac{1}{D^2} e^{-ik_z D}, \quad (C6)$$

$$a_E^{(S)}(\ell) = \int_0^{\tau_0} d\tau \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \Phi(\mathbf{k}^\parallel = -\ell/D, k_z)$$
$$\times \frac{\ell^2}{(k_z D)^2 + \ell^2} S_P^{(S)}(k = \sqrt{k_z^2 + (\ell/D)^2}, \tau)\frac{1}{D^2} e^{-ik_z D}, \quad (C7)$$

$$a_E^{(T)}(\ell) = \int_0^{\tau_0} d\tau \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} (\xi^{+2} + \xi^{-2})(\mathbf{k}^\parallel = -\ell/D, k_z)$$
$$\times \left(2 - \frac{\ell^2}{(k_z D)^2 + \ell^2}\right) S_P^{(T)}(k = \sqrt{k_z^2 + (\ell/D)^2}, \tau)$$
$$\times \frac{1}{D^2} e^{-ik_z D}, \quad (C8)$$

$$a_B^{(T)}(\ell) = i \int_0^{\tau_0} d\tau \int_{-\infty}^{\infty} dk_z (\xi^{+2} - \xi^{-2})(\mathbf{k}^\parallel = -\ell/D, k_z)$$
$$\times 2\,\text{sgn}(k_z)\sqrt{1 - \frac{\ell^2}{(k_z D)^2 + \ell^2}}$$
$$\times S_P^{(T)}(k = \sqrt{k_z^2 + (\ell/D)^2}, \tau)\frac{1}{D^2} e^{-ik_z D}, \quad (C9)$$

$$a_I^{(V)}(\ell) = i \int_0^{\tau_0} d\tau \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} (\xi^{+1} + \xi^{-1})(\mathbf{k}^\parallel = -\ell/D, k_z)$$
$$\times \frac{\ell}{\sqrt{(k_z D)^2 + \ell^2}} S_I^{(T)}(k = \sqrt{k_z^2 + (\ell/D)^2}, \tau)$$
$$\times \frac{1}{D^2} e^{-ik_z D}, \quad (C10)$$

$$a_E^{(V)}(\ell) = - \int_0^{\tau_0} d\tau \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} (\xi^{+1} + \xi^{-1})(\mathbf{k}^\parallel = -\ell/D, k_z)$$
$$\times \text{sgn}(k_z)\frac{\ell}{\sqrt{(k_z D)^2 + \ell^2}}\sqrt{1 - \frac{\ell^2}{(k_z D)^2 + \ell^2}}$$
$$\times S_P^{(V)}(k = \sqrt{k_z^2 + (\ell/D)^2}, \tau)\frac{1}{D^2} e^{-ik_z D}, \quad (C11)$$

$$a_B^{(V)}(\ell) = -i \int_0^{\tau_0} d\tau \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} (\xi^{+1} - \xi^{-1})(\mathbf{k}^\parallel = -\ell/D, k_z)$$
$$\times \frac{\ell}{\sqrt{(k_z D)^2 + \ell^2}} S_P^{(V)}(k = \sqrt{k_z^2 + (\ell/D)^2}, \tau)$$
$$\times \frac{1}{D^2} e^{-ik_z D}. \quad (C12)$$

Note that the helicity dependence vanishes in flat-sky 1-point functions unlike in the all-sky ones as shown in Appendix B. It is due to the absence of $\phi_{k,n}$ dependence in the flat-sky transfer functions as explained in Appendix A.

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