PHYSICAL REVIEW D **82**, 096004 (2010)

Exact one-loop evolution invariants in the standard model

P. F. Harrison* and R. Krishnan†
Department of Physics, University of Warwick, Coventry CV4 7AL, United Kingdom

W. G. Scott‡
Rutherford Appleton Laboratory, Chilton, Didcot, Oxon OX11 0QX, United Kingdom
(Received 16 June 2010; published 8 November 2010)

Guided by considerations of flavor symmetry, we construct a set of exact standard model renormalization group evolution invariants which link quark masses and mixing parameters. We examine their phenomenological implications and infer a simple combination of Yukawa coupling matrices which plays a unique role in the standard model, suggesting a possible new insight into the observed spectrum of quark masses. Our evolution invariants are readily generalized to the leptons in the case of Dirac neutrinos, but do not appear to be relevant for either quarks or leptons in the minimal supersymmetric standard model.

DOI: 10.1103/PhysRevD.82.096004 PACS numbers: 11.10.Hi, 12.15.Ff, 14.65.−q

I. INTRODUCTION

Recently, there has been interest in evolution invariants [1–4], combinations of observables which do not evolve under the renormalization group (RG). Applications have thus far been primarily focused beyond the standard model (SM) [1,2], although approximate evolution invariants of the SM have also been identified [3,4]. Any empirical relations among evolution invariants are more likely to be fundamental than relations valid at a particular scale between observables which evolve differently with energy.

The RG evolution equations (RGEs) of the Yukawa couplings are compactly written as matrix equations [5], since the problem is intrinsically flavor symmetric—all flavors are treated equivalently. Conventional flavor observables, such as the quark and lepton masses (proportional to the eigenvalues of the Yukawa coupling matrices) or their mixing angles, break the flavor symmetry so that their RG equations are more complicated [4,6]. This complexity has meant that in most cases, only quantities that are invariant in certain approximations have been found, e.g. assuming no fermion mixing [1], assuming only two generations of fermions [3], or neglecting the contributions of light quark masses [4]. Motivated by our earlier work on flavor-symmetric variables [7], we introduce a set of flavor-symmetric observables whose one-loop RG equations in the SM are especially simple. These lead straightforwardly without approximation to SM evolution invariants which, for the first time, are exact (at this order). This new approach might find further application beyond the SM. For illustration, we consider primarily the quarks, but our considerations are equally valid for the leptons in the case that neutrinos are Dirac particles, in which case more invariants follow.

We define the Hermitian squares of the Yukawa coupling matrices for charge $+\frac{2}{3}$ (U) and charge $-\frac{1}{3}$ (D) quarks respectively,

$$\mathcal{U} = U^\dagger U, \qquad \mathcal{D} = D^\dagger D \tag{1}$$

and introduce a complete set of ten flavor-symmetric invariants (each is invariant under independent $S3$ permutations of the (u, c, t) and/or the (d, s, b) flavor labels):

$$\begin{aligned}
\mathcal{T}_{+0} &= \mathrm{Tr}(\mathcal{U}) & \mathcal{T}_{0+} &= \mathrm{Tr}(\mathcal{D}) \\
\mathcal{T}_{-0} &= \mathrm{Tr}(\mathcal{U}^{-1}) & \mathcal{T}_{0-} &= \mathrm{Tr}(\mathcal{D}^{-1}) \\
\mathcal{T}_{++} &= \mathrm{Tr}(\mathcal{U}\mathcal{D}) & \mathcal{T}_{+-} &= \mathrm{Tr}(\mathcal{U}\mathcal{D}^{-1}) \\
\mathcal{T}_{-+} &= \mathrm{Tr}(\mathcal{U}^{-1}\mathcal{D}) & \mathcal{T}_{--} &= \mathrm{Tr}(\mathcal{U}^{-1}\mathcal{D}^{-1}) \\
\mathcal{D}_{\mathcal{U}} &= \mathrm{Det}(\mathcal{U}) & \mathcal{D}_{\mathcal{D}} &= \mathrm{Det}(\mathcal{D}).
\end{aligned} \tag{2}$$

The set is complete in the sense that the ten variables are fully determined by the physical masses and mixings, and are, in turn, sufficient to fully determine them (up to discrete permutations of the flavor labels). A further ten analogous variables can be similarly constructed using Hermitian squares of Yukawa matrices for the neutrinos ($\mathcal{N}$) and the charged leptons ($\mathcal{L}$).

II. SM EVOLUTION

We start with the one-loop RG equations for the quark Yukawa coupling matrices in the SM [5]:

$$U^{-1}\frac{dU}{dt} = \gamma_u + \frac{3}{2}(\mathcal{U} - \mathcal{D}), \tag{3}$$

$$D^{-1}\frac{dD}{dt} = \gamma_d + \frac{3}{2}(\mathcal{D} - \mathcal{U}) \tag{4}$$

where $t = \frac{1}{16\pi^2}\ln(\mu/\mu_0)$ for renormalization scale μ, and

$$\gamma_u = T - G_U; \qquad \gamma_d = T - G_D, \tag{5}$$

with:

*p.f.harrison@warwick.ac.uk
†k.rama@warwick.ac.uk
‡w.g.scott@rl.ac.uk

$$T = \mathrm{Tr}(3\mathcal{U} + 3\mathcal{D} + \mathcal{N} + \mathcal{L}), \tag{6}$$

$$G_U = \frac{17}{12}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2, \tag{7}$$

$$G_D = \frac{5}{12}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2 \tag{8}$$

(the g_i are the gauge couplings [8]). For $\mathcal{U}$ and $\mathcal{D}$, the Hermitian-squared matrices of Eq. (1), we get

$$\frac{d\mathcal{U}}{dt} = 2\gamma_u\mathcal{U} + 3\mathcal{U}^2 - \frac{3}{2}\{\mathcal{U}, \mathcal{D}\}, \tag{9}$$

$$\frac{d\mathcal{D}}{dt} = 2\gamma_d\mathcal{D} + 3\mathcal{D}^2 - \frac{3}{2}\{\mathcal{U}, \mathcal{D}\}. \tag{10}$$

Differentiating Eqs. (2) and using Eqs. (9) and (10), we obtain the separate evolution equations of our ten flavor-symmetric observables:

$$\frac{d\mathcal{T}_{+0}}{dt} = 2\gamma_u\mathcal{T}_{+0} + 3(\mathcal{T}_{+0}^2 - 2\mathcal{T}_{-0}\mathcal{D}_{\mathcal{U}} - \mathcal{T}_{++}) \tag{11a}$$

$$\frac{d\mathcal{T}_{0+}}{dt} = 2\gamma_d\mathcal{T}_{0+} + 3(\mathcal{T}_{0+}^2 - 2\mathcal{T}_{0-}\mathcal{D}_{\mathcal{D}} - \mathcal{T}_{++}) \tag{11b}$$

$$\frac{d\mathcal{T}_{-0}}{dt} = -2\gamma_u\mathcal{T}_{-0} - 9 + 3\mathcal{T}_{-+} \tag{11c}$$

$$\frac{d\mathcal{T}_{0-}}{dt} = -2\gamma_d\mathcal{T}_{0-} - 9 + 3\mathcal{T}_{+-} \tag{11d}$$

$$\frac{d\mathcal{T}_{++}}{dt} = 2(\gamma_u + \gamma_d)\mathcal{T}_{++} \tag{11e}$$

$$\frac{d\mathcal{T}_{--}}{dt} = -2(\gamma_u + \gamma_d)\mathcal{T}_{--} \tag{11f}$$

$$\frac{d\mathcal{T}_{+-}}{dt} = 2(\gamma_u - \gamma_d + 3\mathcal{T}_{+0})\mathcal{T}_{+-} - 6\mathcal{T}_{+0} + 6\mathcal{D}_{\mathcal{U}}(\mathcal{T}_{--} - \mathcal{T}_{-0}\mathcal{T}_{0-}) \tag{11g}$$

$$\frac{d\mathcal{T}_{-+}}{dt} = 2(-\gamma_u + \gamma_d + 3\mathcal{T}_{0+})\mathcal{T}_{-+} - 6\mathcal{T}_{0+} + 6\mathcal{D}_{\mathcal{D}}(\mathcal{T}_{--} - \mathcal{T}_{-0}\mathcal{T}_{0-}) \tag{11h}$$

$$\frac{d\mathcal{D}_{\mathcal{U}}}{dt} = 3\mathcal{D}_{\mathcal{U}}[2\gamma_u + (\mathcal{T}_{+0} - \mathcal{T}_{0+})] \tag{11i}$$

$$\frac{d\mathcal{D}_{\mathcal{D}}}{dt} = 3\mathcal{D}_{\mathcal{D}}[2\gamma_d - (\mathcal{T}_{+0} - \mathcal{T}_{0+})]. \tag{11j}$$

We make the following observations:

(1) Most of the variables's evolutions have two parts:

(i) a part proportional to the variable itself, whose coefficient depends at most on γ_u, γ_d, $\mathcal{T}_{+0}$ and $\mathcal{T}_{0+}$. We call this the ''pure'' part.

(ii) a part which depends more generally on the other variables—the ''mixed'' part.

(2) The four variables $\mathcal{D}_{\mathcal{U}}$, $\mathcal{D}_{\mathcal{D}}$, $\mathcal{T}_{++}$ and $\mathcal{T}_{--}$ have only pure parts (this is also the case for Jarlskog's determinant [9], which was the main result of Ref. [10]). This feature seems to be peculiar to the SM—we will rely on it in the next stage of our derivation.

III. SM EVOLUTION INVARIANTS

Exploiting the opportunity to cancel the terms involving $\mathcal{T}_{+0}$ and $\mathcal{T}_{0+}$ in Eqs. (11i) and (11j), we note that the quantity $\mathrm{Det}(\mathcal{U}\mathcal{D}) = (\mathcal{D}_{\mathcal{U}}\mathcal{D}_{\mathcal{D}})$ has a pure evolution with exactly a factor 3 times the coefficient which appears in Eqs. (11e) and (11f):

$$\frac{d}{dt}\ln\mathrm{Det}(\mathcal{U}\mathcal{D}) = 6(\gamma_u + \gamma_d). \tag{12}$$

We may thus form two independent combinations which are exact evolution invariants at one-loop order [11]:

$$I_{TD}^{q} \equiv \frac{\mathcal{T}_{++}}{(\mathcal{D}_{\mathcal{U}}\mathcal{D}_{\mathcal{D}})^{1/3}} \equiv \frac{\mathrm{Tr}(\mathcal{U}\mathcal{D})}{\mathrm{Det}^{1/3}(\mathcal{U}\mathcal{D})}; \quad \frac{dI_{TD}^q}{dt} = 0, \tag{13}$$

$$I_{PD}^{q} \equiv \mathcal{T}_{--}(\mathcal{D}_{\mathcal{U}}\mathcal{D}_{\mathcal{D}})^{1/3} \equiv \mathrm{Tr}(\mathcal{U}\mathcal{D})^{-1}\mathrm{Det}^{1/3}(\mathcal{U}\mathcal{D}); \qquad \frac{dI_{PD}^q}{dt} = 0. \tag{14}$$

The pure evolutions expressed by Eqs. (11e), (11f), and (12), and the two resulting RG invariants, Eqs. (13) and (14), are the key results of this paper. I_{TD}^q and I_{PD}^q appear to be the only exact RG invariants that can be constructed from the quark Yukawa coupling matrices alone in the SM case. We have not succeeded in finding similar exact RG invariants involving only Yukawa couplings in the minimal supersymmetric standard model (MSSM) or the 2 Higgs doublet model (2HDM).

We can construct entirely analogous evolution invariants using $\mathcal{N}$ and $\mathcal{L}$, the (Hermitian squares of the) Yukawa coupling matrices for the leptons (in the Dirac neutrino case). The RGEs of $\mathcal{N}$ and $\mathcal{L}$ are analogous to those in Eqs. (9) and (10) with γ_ν and γ_ℓ defined as in Eq. (5) with the same value of T (Eq. (6)) and the gauge contributions, Eqs. (7) and (8), modified to $G_N = \frac{3}{4}g_1^2 + \frac{9}{4}g_2^2$ and $G_L = \frac{15}{4}g_1^2 + \frac{9}{4}g_2^2$. The leptonic analogue of the pure evolution rate $2(\gamma_u + \gamma_d)$, Eqs. (11e), (11f), and (12), is just $2(\gamma_\nu + \gamma_\ell)$, being the pure evolution rate of $\mathrm{Tr}(\mathcal{N}\mathcal{L})$ and $\mathrm{Det}^{1/3}(\mathcal{N}\mathcal{L})$. Thus two more invariants follow, which we call I_{TD}^ℓ and I_{PD}^ℓ respectively, having definitions in terms of $\mathcal{N}$ and $\mathcal{L}$ analogous to those in Eqs. (13) and (14).

For completeness, we present here other exact one-loop evolution invariants of the SM. The T-dependence cancels in the ratio of any corresponding pair of purely evolving quark and lepton observables, leaving only a dependence on gauge couplings, g_i ($i = 1\ldots3$). The one-loop RGEs for the g_i in the SM (at high energies) are [12]

$$\frac{dg_1}{dt} = \frac{41}{6}g_1^3, \qquad \frac{dg_2}{dt} = -\frac{19}{6}g_2^3, \qquad \frac{dg_3}{dt} = -7g_3^3. \tag{15}$$

Thus, e.g. using Eq. (12), together with its leptonic analogue and Eq. (15), we have that

$$\mathcal{I}^{ql}_{\rm prod} \equiv \frac{\mathrm{Det}(\mathcal{U}\mathcal{D})}{\mathrm{Det}(\mathcal{N}\mathcal{L})} g_1^{-(96/41)} g_3^{-(96/7)} \tag{16}$$

is also an exact one-loop evolution invariant.

We note that by combining Eqs. (11i) and (11j), to form the pure-evolving $\mathrm{Det}(\mathcal{U}\mathcal{D})$, we have effectively removed one independent evolution equation from the complete set, Eqs. (11). Thus, we may add the (independent) Jarlskog commutator [9] which also has a pure RGE [10]

$$\frac{d}{dt}\ln(\mathrm{Det}[\mathcal{U},\mathcal{D}]) = 3[2(\gamma_u + \gamma_d) + \mathrm{Tr}(\mathcal{U}) + \mathrm{Tr}(\mathcal{D})] \tag{17}$$

and likewise for the leptons. Noting the definition of T, Eq. (6), and using Eqs. (12) and (17), their leptonic analogues, and Eq. (15), we find another RG invariant

$$\mathcal{I}^{ql}_{\rm comm} \equiv \frac{\mathrm{Det}^3[\mathcal{U},\mathcal{D}]\mathrm{Det}[\mathcal{N},\mathcal{L}]}{\mathrm{Det}^3(\mathcal{U}\mathcal{D})\mathrm{Det}^{5/4}(\mathcal{N}\mathcal{L})} g_1^{-(81/82)} g_2^{(81/38)}. \tag{18}$$

Using Eqs. (15), two more RG invariants can be constructed from gauge couplings alone:

$$\mathcal{I}^{g}_{12} \equiv \frac{6}{41}g_1^{-2} + \frac{6}{19}g_2^{-2}, \tag{19}$$

$$\mathcal{I}^{g}_{13} \equiv \frac{6}{41}g_1^{-2} + \frac{1}{7}g_3^{-2}. \tag{20}$$

Finally, we note the SM RGE of the Higgs vacuum expectation value, v [13]:

$$\frac{dv}{dt} = v\left(-T + \frac{3}{4}g_1^2 + \frac{9}{4}g_2^2\right). \tag{21}$$

Since its product with any Yukawa coupling gives a mass term, we have that if we use mass matrices directly, rather than Yukawa matrices, the T and g_2 dependences of the γ_i, Eq. (5), are exactly cancelled leaving only the dependences on g_1 and g_3. Thus, using v together with purely evolving quantities, and the gauge couplings, allows the construction of other RG invariants, e.g.

$$\mathcal{I}^{q}_{DV} \equiv \mathrm{Det}^{1/3}(\mathcal{U}\mathcal{D})v^4 g_1^{4/41} g_3^{-(32/7)}. \tag{22}$$

Of course, only one of these invariants involving v is independent of the set already defined.

IV. EVALUATION

In constructing our RG invariants, we have used only four of the variables defined in Eq. (2), namely $\mathcal{D}_\mathcal{U}$, $\mathcal{D}_\mathcal{D}$, $\mathcal{T}_{++}$ and $\mathcal{T}_{--}$. While $\mathrm{Det}(\mathcal{U}\mathcal{D}) = \mathcal{D}_\mathcal{U}\mathcal{D}_\mathcal{D}$ is simply the product of all six eigenvalues, variables of the form $\mathrm{Tr}(\mathcal{U}^n\mathcal{D}^m)$ depend also on the mixing matrix elements. It is easy to show that such quantities are simple mass moment transforms [14] of the "P matrix" [15] of transition probabilities $|V_{\alpha i}|^2$. Writing $u = m_u^2/v^2$, etc., with analogous expressions for the charge $-\frac{1}{3}$ quarks

$$\begin{aligned}\mathrm{Tr}(\mathcal{U}^n\mathcal{D}^m) &= (u^n, \quad c^n, \quad t^n)\cdot\begin{pmatrix}|V_{ud}|^2|V_{us}|^2|V_{ub}|^2\\ |V_{cd}|^2|V_{cs}|^2|V_{cb}|^2\\ |V_{td}|^2|V_{ts}|^2|V_{tb}|^2\end{pmatrix}\cdot\begin{pmatrix}d^m\\ s^m\\ b^m\end{pmatrix}\\ &= \sum_{\alpha i} m_\alpha^{2n} m_i^{2m}|V_{\alpha i}|^2/v^{2(m+n)} \quad \forall\ m, n\end{aligned} \tag{23}$$

(with $\alpha = u$, c, t and $i = d$, s, b) in which terms, the flavor-symmetry property is manifest. We may now expand our new RG invariants explicitly. From Eq. (13)

$$\begin{aligned}\mathcal{I}^q_{TD} &= \sum_{\alpha i}\frac{m_\alpha^2 m_i^2|V_{\alpha i}|^2}{(m_u m_c m_t m_d m_s m_b)^{2/3}}\\ &= \sum_{\alpha\neq\beta\neq\gamma, i\neq j\neq k}\left(\frac{m_\alpha^2}{m_\beta m_\gamma}\frac{m_i^2}{m_j m_k}\right)^{2/3}|V_{\alpha i}|^2.\end{aligned} \tag{24}$$

From Eq. (14)

$$\begin{aligned}\mathcal{I}^q_{PD} &= (m_u m_c m_t m_d m_s m_b)^{2/3}\sum_{\alpha i} m_\alpha^{-2} m_i^{-2}|V_{\alpha i}|^2\\ &= \sum_{\alpha\neq\beta\neq\gamma, i\neq j\neq k}\left(\frac{m_\beta m_\gamma}{m_\alpha^2}\frac{m_j m_k}{m_i^2}\right)^{2/3}|V_{\alpha i}|^2.\end{aligned} \tag{25}$$

Analogous formulae are obtained for the leptonic RG invariants, $\mathcal{I}^\ell_{TD}$ and $\mathcal{I}^\ell_{PD}$.

From Eq. (16)

$$\mathcal{I}^{ql}_{\rm prod} = \frac{m_u m_c m_t m_d m_s m_b}{m_1 m_2 m_3 m_e m_\mu m_\tau} g_1^{-(96/41)} g_3^{-(96/7)}, \tag{26}$$

while from Eq. (18)

$$\begin{aligned}\mathcal{I}^{ql}_{\rm comm} &= J_q^3 f^3(u) f^3(d)\times J_\ell f(\nu) f(\ell)\\ &\quad\times(y_1 y_2 y_3 y_e y_\mu y_\tau)^{-(1/4)} g_1^{-(81/82)} g_2^{81/38},\end{aligned} \tag{27}$$

with $f(u) = (m_t^2 - m_c^2)(m_c^2 - m_u^2)(m_t^2 - m_u^2)/(m_t^2 m_c^2 m_u^2)$ and similar definitions for the charge $-\frac{1}{3}$ quarks, and the leptons. The y_ν and y_ℓ are the eigenvalues of $\mathcal{N}$ and $\mathcal{L}$.

For brevity, we limit the following discussion to $\mathcal{I}^q_{TD}$ and $\mathcal{I}^q_{PD}$, the RG invariants constructed only from quark Yukawa matrices. Using the experimental values of the quark masses [16], and the Wolfenstein parameters [17], λ, A, ρ and η for the Cabibbo-Kobayashi-Maskawa (CKM) matrix, we find both invariants to be of the order of 10^8, as shown in Fig. 1, with their ratio $(\mathcal{I}^q_{PD}/\mathcal{I}^q_{TD}) = 0.7^{+1.1}_{-0.4}$, consistent with unity. The strongly hierarchical quark masses and the small CKM mixing angles mean that each of them is dominated by a single leading term. We find at next-to-leading order in small quantities (small mass ratios and λ^2)

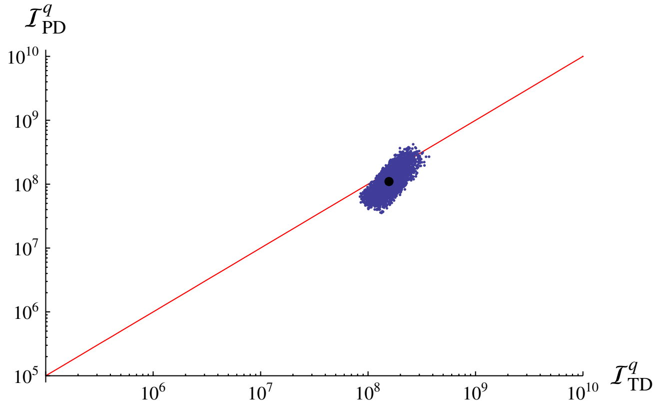

FIG. 1 (color online). The black point shows the values of the RG invariants I^q_{TD} and I^q_{PD} found using quark masses from [16] and measured values of the CKM mixings (all renormalized to M_Z). The cluster of points indicates the range allowed by experimental and theoretical uncertainties. The straight line shows the hypothesis $I^q_{TD} = I^q_{PD}$ suggested by the data.

$$I^q_{TD} \approx \left(\frac{m_t}{m_u}\frac{m_t}{m_c}\frac{m_b}{m_d}\frac{m_b}{m_s}\right)^{2/3}\left(1+\lambda^2\left(\frac{m_c}{m_t}\frac{m_s}{m_b}\right)^2\right), \quad (28)$$

$$I^q_{PD} \approx \left(\frac{m_t}{m_u}\frac{m_c}{m_u}\frac{m_b}{m_d}\frac{m_s}{m_d}\right)^{2/3}(1-\lambda^2). \quad (29)$$

Since for I^q_{TD}, Eq. (28), the leading term is several orders of magnitude larger than the next-to-leading term, we conclude that the combination $(m_t^2 m_b^2/m_u m_c m_d m_s)^{2/3}$ is itself invariant to a very good approximation. At next-to-leading order, the $\mathcal{O}(1)$ invariant ratio is

$$\frac{I^q_{PD}}{I^q_{TD}} \approx (m_c^2 m_s^2/m_t m_u m_b m_d)^{2/3}(1-\lambda^2). \quad (30)$$

It is well known that from the weak scale to the GUT scale, the various quark masses evolve by typically 55–65% [16]. The different mass ratios, on the other hand, vary at a slower rate, e.g. m_b/m_s changes by $\sim$16% and m_s/m_d by $\sim$1.8%. As a check on our analysis, we have numerically solved Eqs. (3) and (4) together with the RG equations for the gauge couplings, Eq. (15), and verified that our RG invariants do not evolve at all. We have similarly verified that the leading terms of our RG invariants given in Eqs. (28) and (29) change by 0.05% or less.

V. INTERPRETATION

While the Yukawa coupling matrices $\mathcal{U}$ and $\mathcal{D}$ separately have the mixed and coupled evolutions given by Eqs. (9) and (10), it is an interesting feature, apparently peculiar to the SM, that the eigenvalues, λ_i, of the *product* matrix $\mathcal{U}\mathcal{D}$ [18] have pure evolutions with common rate, leaving the eigenvalue *ratios* RG-invariant. This follows since $\mathcal{T}_{++} = \mathrm{Tr}(\mathcal{U}\mathcal{D})$ and $\mathrm{Det}(\mathcal{U}\mathcal{D})$ with pure RGEs given in Eqs. (11e) and (12), are simply the order-one and order-three coefficients in the eigenvalue equation of the matrix $\mathcal{U}\mathcal{D}$, while $\mathcal{T}_{--} = \mathrm{Tr}(\mathcal{U}\mathcal{D})^{-1}$, with pure evolution given by Eq. (11f), is simply $P(\mathcal{U}\mathcal{D})/\mathrm{Det}(\mathcal{U}\mathcal{D})$ where

$$P(\mathcal{U}\mathcal{D}) \equiv \frac{1}{2}[\mathrm{Tr}^2(\mathcal{U}\mathcal{D}) - \mathrm{Tr}(\mathcal{U}\mathcal{D})^2] \quad (31)$$

$(=\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)$ is the corresponding order-two coefficient. From Eqs. (11e), (11f), and (12), we thus see that each of the coefficients in the eigenvalue equation of $\mathcal{U}\mathcal{D}$ has a pure RGE with an evolution rate which is simply given by the order of the coefficient times the same basic rate, $2(\gamma_u + \gamma_d)$. Since the three eigenvalues of $\mathcal{U}\mathcal{D}$ are all order-one in terms of these coefficients via the formula for the roots of a cubic, it follows that they also have pure RGEs with common evolution rate $2(\gamma_u + \gamma_d)$. We thus conclude that the ratios of the eigenvalues of $\mathcal{U}\mathcal{D}$, λ_i/λ_j $(i \neq j)$, are also each RG invariants (although clearly they are not individually flavor-symmetric).

While it is an undoubted mystery why the two independent invariants, $I^q_{TD} \approx (\lambda_3^2/\lambda_1\lambda_2)^{1/3}$ and $I^q_{PD} \approx (\lambda_2\lambda_3/\lambda_1^2)^{1/3}$, should be so large $(\mathcal{O}(10^8))$, it is also a puzzle why they should be so nearly equal to each other—the proximity to unity of their observed ratio, $(I^q_{PD}/I^q_{TD}) \simeq 0.7^{+1.1}_{-0.4}$ (see Fig. 1), represents a significant fine-tuning of SM parameters. Moreover, it is interesting to observe that if this ratio were exactly unity, then the spectrum of the product matrix $\mathcal{U}\mathcal{D}$ would be geometric, i.e.

$$I^q_{TD} = I^q_{PD}(= I,\ \text{say}) \Rightarrow \lambda_3/\lambda_2 = \lambda_2/\lambda_1 \approx I, \quad (32)$$

relations which are then valid at all scales. Indeed, one might reasonably postulate that nature requires the spectrum of the matrix $\mathcal{U}\mathcal{D}$ to be exactly geometric, $(I^q_{PD}/I^q_{TD}) \equiv 1$, at some (presumably high) energy scale, the data being fully consistent with this. Of course, the separate spectra of the $\mathcal{U}$ and $\mathcal{D}$ matrices have long been known [19] to be approximately geometric: $m_c^2/(m_u m_t) \sim \mathcal{O}(1)$, $m_s^2/(m_d m_b) \sim \mathcal{O}(1)$. However, such separate relations are not RG-invariant and are therefore *a priori* less interesting and generally more difficult to test experimentally.

We consider briefly why the SM admits RG invariants constructed from only the Yukawa couplings. It can be seen from Eqs. (9) and (10) that the mixed parts of the evolution equations for the Yukawa coupling matrices $\mathcal{U}$ and $\mathcal{D}$ have balanced positive and negative coefficients. These are exploited in the evolution of the product $\mathcal{U}\mathcal{D}$ where these terms cancel on taking the trace of simple powers. The existence of balanced coefficients in the SM can be traced back to the use of the conjugate Higgs for the Yukawa couplings of the charge $\frac{2}{3}$ quarks, by contrast with the MSSM and the 2HDM, which use independent Higgs fields in each charge sector, resulting in mixed evolutions [10] with coefficients all having the same sign so that no such cancellation is possible.

VI. SUMMARY

We have recast the SM RG equations using flavor-symmetric weak-basis invariant functions of the Yukawa coupling matrices, leading to the identification of exact one-loop RG invariants in the SM. We have identified two such invariants involving quark Yukawas alone, and two similar ones for leptons in the case of Dirac neutrinos. The SM seems at least somewhat unusual in allowing such RG invariants—we have not been able to find any in the MSSM or 2HDM. Despite the fact that the evolutions of $\mathcal{U}$ and $\mathcal{D}$ are coupled and mixed, the weak-basis invariants of their product matrix $\mathcal{U}\mathcal{D}$ have pure evolutions with a rate simply proportional to their order so that its eigenvalue ratios are RG-invariant, and are furthermore experimentally observed to be consistent with a geometric spectrum.

ACKNOWLEDGMENTS

This work was supported by the UK Science and Technology Facilities Council (STFC). Two of us (P. F. H. and R. K.) acknowledge the hospitality of the Centre for Fundamental Physics (CfFP) at the Rutherford Appleton Laboratory. We also acknowledge useful comments by Steve King and Roman Zwicky.

[1] D. A. Demir, J. High Energy Phys. 11 (2005) 003.
[2] S. Chang and T. K. Kuo, Phys. Rev. D **66**, 111302 (2002).
[3] L.-x. Liu, arXiv:0910.1326v2.
[4] S. Luo and Z.-z. Xing, J. Phys. G **37**, 075 018 (2010).
[5] M. E. Machacek and M. T. Vaughn, Nucl. Phys. **B236**, 221 (1984); P. F. Harrison and W. G. Scott, Phys. Lett. B **333**, 471 (1994).
[6] K. S. Babu, Z. Phys. C **35**, 69 (1987); K. Sasaki, Z. Phys. C **32**, 149 (1986).
[7] P. F. Harrison and W. G. Scott, Phys. Lett. B **628**, 93 (2005); P. F. Harrison, D. R. J. Roythorne, and W. G. Scott, *ibid.* **657**, 210 (2007).
[8] Throughout, we use the $U(1)$ gauge coupling normalization of [6].
[9] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985); Z. Phys. C **29**, 491 (1985).
[10] G. G. Athanasiu, S. Dimopoulos, and F. J. Gilman, Phys. Rev. Lett. **57**, 1982 (1986).
[11] Our notation for $\mathcal{I}^q_{TD}$ and $\mathcal{I}^q_{PD}$ is based on which of the coefficients of the eigenvalue equation of the matrix $\mathcal{U}\mathcal{D}$ the RG invariant may be constructed from, e.g. $\mathcal{I}^q_{PD} = P(\mathcal{U}\mathcal{D})/\mathrm{Det}^{2/3}(\mathcal{U}\mathcal{D})$, where $P(\mathcal{U}\mathcal{D})$ is defined in Eq. (31).
[12] M. E. Machacek and M. T. Vaughn, Nucl. Phys. **B222**, 83 (1983).
[13] Yu. F. Pirogov and O. V. Zenin, Eur. Phys. J. C **10**, 629 (1999).
[14] P. F. Harrison, D. R. J. Roythorne, and W. G. Scott, arXiv:0805.3440
[15] P. F. Harrison, W. G. Scott, and T. J. Weiler, Phys. Lett. B **641**, 372 (2006).
[16] Z.-z. Xing, H. Zhang, and S. Zhou, Phys. Rev. D **77**, 113016 (2008).
[17] L. Wolfenstein, Phys. Rev. Lett. **51**, 1945 (1983).
[18] There is no significance to the choice of the product order $\mathcal{U}\mathcal{D}$ over $\mathcal{D}\mathcal{U}$, since the eigenvalues are the same in each case. All our results are equally applicable to both cases.
[19] A. Davidson, T. Schwartz, and K. C. Wali, J. Phys. G **24**, L55 (1998).