PHYSICAL REVIEW D **82,** 096003 (2010)

Nambu–Jona-Lasinio model description of weakly interacting Bose condensate and BEC-BCS crossover in dense QCD-like theories

Lianyi He*

Frankfurt Institute for Advanced Studies and Institute for Theoretical Physics, J. W. Goethe University, 60438 Frankfurt am Main, Germany

(Received 25 July 2010; published 8 November 2010)

QCD-like theories possess a positively definite fermion determinant at finite baryon chemical potential μ_B and the lattice simulation can be successfully performed. While the chiral perturbation theories are sufficient to describe the Bose condensate at low density, to describe the crossover from Bose-Einstein condensation (BEC) to BCS superfluidity at moderate density we should use some fermionic effective model of QCD, such as the Nambu–Jona-Lasinio model. In this paper, using two-color two-flavor QCD as an example, we examine how the Nambu–Jona-Lasinio model describes the weakly interacting Bose condensate at low density and the BEC-BCS crossover at moderate density. Near the quantum phase transition point $\mu_B = m_\pi$ (m_π is the mass of pion/diquark multiplet), the Ginzburg-Landau free energy at the mean-field level can be reduced to the Gross-Pitaevskii free energy describing a weakly repulsive Bose condensate with a diquark-diquark scattering length identical to that predicted by the chiral perturbation theories. The Goldstone mode recovers the Bogoliubov excitation in weakly interacting Bose condensates. The results of in-medium chiral and diquark condensates predicted by chiral perturbation theories are analytically recovered. The BEC-BCS crossover and meson Mott transition at moderate baryon chemical potential as well as the beyond-mean-field corrections are studied. Part of our results can also be applied to real QCD at finite baryon or isospin chemical potential.

DOI: 10.1103/PhysRevD.82.096003 PACS numbers: 11.10.Wx, 12.38.−t, 25.75.Nq

I. INTRODUCTION

A good understanding of quantum chromodynamics (QCD) at finite temperature and baryon density is crucial for us to understand a wide range of physical phenomena. For instance, to understand the evolution of the Universe in the first few seconds, one needs the knowledge of QCD phase transition at temperature $T \sim 200$ MeV and very small baryon density. On the other hand, understanding the physics of neutron stars requires the knowledge of QCD at high baryon density and very low temperature [1]. Lattice simulation of QCD at finite temperature has been successfully performed in the past few decades; however, no successful lattice simulation at high baryon density has been done due to the sign problem [2,3]: The fermion determinant is not positively definite in presence of a nonzero baryon chemical potential μ_B.

We thus look for some special cases which have a positively definite fermion determinant. One case is QCD at finite isospin chemical potential μ_I [4,5], where the ground state changes from a pion condensate to a BCS superfluid with quark-antiquark condensation with increasing isospin density. Another case is the QCD-like theories [6–12] where quarks are in a real or pseudoreal representation of the gauge group, including two-color QCD with quarks in the fundamental representation and QCD with quarks in the adjoint representation. While these cases do not correspond to the real world, they can be simulated on the lattice and may give us some information of real QCD at finite baryon density. For all these special cases, chiral perturbation theories predict a continuous quantum phase transition from the vacuum to the matter phase at baryon or isospin chemical potential equal to the pion mass, in contrast to real QCD where the phase transition takes place at μ_B approximately equal to the nucleon mass. The resulting matter near the quantum phase transition is a dilute Bose condensate of diquarks or pions with weakly repulsive interactions [13]. The equations of state and elementary excitations in such matter have been investigated many years ago by Bogoliubov [14] and Lee, Huang, and Yang [15]. Bose-Einstein condensation (BEC) phenomenon is believed to widely exist in dense matter, such as pions and kaons, can condense in neutron star matter if the electron chemical potential exceeds the effective mass for pions and kaons [16–19]. However, the condensation of pions and kaons in neutron star matter is rather complicated due to the meson-nucleon interactions in dense nuclear medium. On the other hand, at asymptotically high density, perturbative QCD calculations show that the ground state is a weakly coupled BCS superfluid with the condensation of overlapping Cooper pairs [4,5,20–23]. It is interesting that the dense BCS superfluid and the dilute Bose condensate have the same symmetry breaking pattern and thus are continued with one another. In condensed matter physics, this phenomenon was first discussed by Eagles [24] and Leggett [25] and is now called BEC-BCS crossover. It has been successfully realized using ultracold fermionic atoms in the past few years [26].

*lianyi@th.physik.uni-frankfurt.de

While the lattice simulations of two-color QCD at finite baryon chemical potential [27–32] and QCD at finite isospin chemical potential [33–36] have been successfully performed, we still ask for some effective models to link the physics of Bose condensate and the BCS superfluidity. The chiral perturbation theories [4–11,37] as well as the linear sigma models [38], which describe only the physics of Bose condensate, do not meet our purpose. The Nambu–Jona-Lasinio (NJL) model [39] with quarks as elementary blocks, which describes well the mechanism of chiral symmetry breaking and low energy phenomenology of the QCD vacuum, is generally believed to work at low and moderate temperatures and densities [40–42]. Recently, this model has been used to describe the superfluid transition at finite chemical potentials [12,43–56] for the special cases we are interested in this paper. One finds that the critical chemical potential for the superfluid transition predicted by the NJL model is indeed equal to the pion mass [48,52], and the chiral and diquark condensates obtained from the mean-field calculation agree with the results from lattice simulations and chiral perturbation theories near the quantum phase transition [48,52]. The NJL model also predicts a BEC-BCS crossover when the chemical potential increases [50,53–55]. A natural problem arises: how can the fermionic NJL model describe the weakly interacting Bose condensate near the quantum phase transition? In fact, we do not know how the repulsive interactions among diquarks or mesons enter in the pure mean-field calculations [52–54]. In this paper, we will focus on this problem and show that the repulsive interaction is indeed properly included even in the mean-field calculations. This phenomenon is in fact analogous to the BCS description of the molecular condensation in strongly interacting Fermi gases studied by Leggett many years ago [25]. Fermionic models have been used to describe the BEC-BCS crossover in cold Fermi gases by the cold atom community. Recently, it has been shown that we can recover the equation of state of the dilute Bose condensate with correct boson-boson scattering length in the strong coupling limit, including the Lee-Huang-Yang correction by considering the beyond-mean-field corrections [57–59]. In Appendix A, we give a summary of the many-body theoretical approach in cold atoms, which is useful for us to understand the theoretical approach and the results of this paper.

In this paper, using two-color two-flavor QCD as an example and following the theoretical approach of the BEC-BCS crossover in cold Fermi gases [57,58], we examine how the NJL model describes the weakly interacting Bose condensate and the BEC-BCS crossover. Near the quantum phase transition point $\mu_B = m_\pi$, we perform a Ginzburg-Landau expansion of the effective potential at the mean-field level, and show that the Ginzburg-Landau free energy is essentially the Gross-Pitaevskii free energy describing weakly interacting Bose condensates via a proper redefinition of the condensate wave function. As a by-product, we obtain a diquark-diquark scattering length $a_{\rm dd} = m_\pi/(16\pi f_\pi^2)$ (f_π is the pion decay constant) characterizing the repulsive interaction between the diquarks, which recovers the tree-level result predicted by chiral Lagrangian [6–11]. We also show analytically that the Goldstone mode takes the same dispersion as the Bogoliubov excitation in weakly interacting Bose condensates, which gives a diquark-diquark scattering length identical to that in the Gross-Pitaevskii free energy. The mixing between the sigma meson and diquarks plays an important role in recovering the Bogoliubov excitation. The results of in-medium chiral and diquark condensates predicted by chiral perturbation theory are analytically recovered. At high density, we find the superfluid matter undergoes a BEC-BCS crossover at $\mu_B \simeq (m_\sigma/m_\pi)^{1/3} m_\pi \simeq (1.6–2)m_\pi$ with m_σ being the mass of the sigma meson. At $\mu_B \simeq 3m_\pi$, we find that the chiral symmetry is approximated restored and the spectra of pions and sigma meson become nearly degenerated. Well above the chemical potential of chiral symmetry restoration, the degenerate pions and sigma meson undergo a Mott transition, where they become unstable resonances. Because of the spontaneous breaking of baryon number symmetry, mesons can decay into quark pairs in the superfluid medium at nonzero momentum.

The beyond-mean-field corrections are studied. The thermodynamic potential including the Gaussian fluctuations is derived. It is shown that the vacuum state $|\mu_B| < m_\pi$ is thermodynamically consistent in the Gaussian approximation, i.e., all thermodynamic quantities keep vanishing in the regime $|\mu_B| < m_\pi$ even though the beyond-mean-field corrections are included. Near the quantum phase transition point, we expand the fluctuation contribution to the thermodynamic potential in powers of the superfluid order parameter. To leading order, the beyond-mean-field correction is quartic and its effect is to renormalize the diquark-diquark scattering length. The correction to the mean-field result is shown to be proportional to m_π^2/f_π^2. Thus, our theoretical approach provides a new way to calculate the diquark-diquark or meson-meson scattering lengths in the NJL model beyond-mean-field approximation. We also find that we can obtain a correct transition temperature of Bose condensation in the dilute limit, including the beyond-mean-field corrections.

The paper is organized as follows: In Sec. II, we derive the general effective action of the two-color NJL model at finite temperature and density, and determine the model parameters via the vacuum phenomenology. In Sec. III, we investigate the properties of dilute Bose condensate near the quantum phase transition at the mean-field level. In Sec. V, the properties of matter at high density are discussed. Beyond-mean-field corrections are studied in Sec. IV. We summarize in Sec. VI. Natural units are used throughout.

II. NJL MODEL OF TWO-COLOR QCD

Without loss of generality, we study in this paper two-color QCD (the number of colors $N_c = 2$) at finite baryon chemical potential μ_B. For vanishing current quark mass m_0, two-color QCD possesses an enlarged flavor symmetry $SU(2N_f)$ [N_f is the number of flavors], the so-called Pauli-Gursey symmetry which connects quarks and antiquarks [6–11]. For $N_f = 2$, the flavor symmetry SU(4) is spontaneously broken down to Sp(4) driven by a nonzero quark condensate $\langle\bar{q}q\rangle$ and there arise five Goldstone bosons: three pions and two scalar diquarks. For nonvanishing current quark mass, the flavor symmetry is explicitly broken, resulting in five pseudo-Goldstone bosons with a small degenerate mass m_π. At the finite baryon chemical potential μ_B, the flavor symmetry $SU(2N_f)$ is explicitly broken down to $SU_L(N_f)\otimes SU_R(N_f)\otimes U_B(1)$. Further, a nonzero diquark condensate $\langle qq\rangle$ can form at large enough chemical potentials and breaks spontaneously the $U_B(1)$ symmetry. In two-color QCD, the scalar diquarks are in fact the lightest "baryons," and we expect a baryon superfluid phase with $\langle qq\rangle \neq 0$ for $|\mu_B| > m_\pi$.

To construct a NJL model for two-color two-flavor QCD with the above flavor symmetry, we consider a contact current-current interaction $G_c\sum_{a=1}^{3}(\bar{q}\gamma_\mu t_a q)(\bar{q}\gamma^\mu t_a q)$ where t_a ($a = 1, 2, 3$) are the generators of color $SU_c(2)$ and G_c is a phenomenological coupling constant. After the Fierz transformation we can obtain an effective NJL Lagrangian density with scalar mesons and color singlet scalar diquarks [52],

$$\mathcal{L}_{\rm NJL} = \bar{q}(i\gamma^\mu\partial_\mu - m_0)q + G[(\bar{q}q)^2 + (\bar{q}i\gamma_5\boldsymbol{\tau}q)^2 + (\bar{q}i\gamma_5\tau_2 t_2 q_c)(\bar{q}_c i\gamma_5\tau_2 t_2 q)], \tag{1}$$

where $q_c = \mathcal{C}\bar{q}^{\rm T}$ and $\bar{q}_c = q^{\rm T}\mathcal{C}$ are the charge conjugate spinors with $\mathcal{C} = i\gamma_0\gamma_2$ and τ_i ($i = 1, 2, 3$) are the Pauli matrices in the flavor space. The four-fermion coupling constants for the scalar mesons and diquarks are the same, $G = 3G_c/4$ [52], which ensures the enlarged flavor symmetry $SU(2N_f)$ of two-color QCD in the chiral limit $m_0 = 0$. One can show explicitly that there are five Goldstone bosons (three pions and two diquarks) driven by a nonzero quark condensate $\langle\bar{q}q\rangle$. With explicit chiral symmetry broken $m_0 \neq 0$, pions and diquarks are also degenerate, and their mass m_π can be determined via the standard method for the NJL model [40–42].

A. Effective action at finite temperature and density

The partition function of the two-color NJL model ([39]) at finite temperature T and baryon chemical potential μ_B is

$$Z_{\rm NJL} = \int[d\bar{q}][dq]\exp\left[\int dx\left(\mathcal{L}_{\rm NJL} + \frac{\mu_B}{2}\bar{q}\gamma_0 q\right)\right], \tag{2}$$

where we adopt the finite temperature formalism with $\tau = it$, $x = (\tau, \mathbf{r})$, and $\int dx = \int_0^{1/T} d\tau \int d^3\mathbf{r}$. The partition function can be bosonized after introducing the auxiliary boson fields

$$\sigma(x) = -2G\bar{q}(x)q(x), \qquad \boldsymbol{\pi}(x) = -2G\bar{q}(x)i\gamma_5\boldsymbol{\tau}q(x) \tag{3}$$

for mesons and

$$\phi(x) = -2G\bar{q}_c(x)i\gamma_5\tau_2 t_2 q(x) \tag{4}$$

for diquarks. With the help of the Nambu-Gor'kov representation $\bar{\Psi} = (\bar{q}\,\bar{q}_c)$, the partition function can be written as

$$Z_{\rm NJL} = \int[d\bar{\Psi}][d\Psi][d\sigma][d\boldsymbol{\pi}][d\phi^\dagger][d\phi]\exp(-\mathcal{A}_{\rm eff}), \tag{5}$$

where the action $\mathcal{A}_{\rm eff}$ is given by

$$\mathcal{A}_{\rm eff} = \int dx\frac{\sigma^2(x) + \boldsymbol{\pi}^2(x) + |\phi(x)|^2}{4G} - \int dx\int dx'\bar{\Psi}(x)\mathbf{G}^{-1}(x,x')\Psi(x'), \tag{6}$$

with the inverse quark propagator defined as

$$\mathbf{G}^{-1}(x,x') = \begin{pmatrix} \gamma^0\left(-\partial_\tau + \frac{\mu_B}{2}\right) + i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} - \mathcal{M}(x) & -i\gamma_5\phi(x)\tau_2 t_2 \\ -i\gamma_5\phi^\dagger(x)\tau_2 t_2 & \gamma^0\left(-\partial_\tau - \frac{\mu_B}{2}\right) + i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} - \mathcal{M}^{\rm T}(x) \end{pmatrix}\delta(x - x'). \tag{7}$$

Here $\mathcal{M}(x) = m_0 + \sigma(x) + i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x)$. After integrating out the quarks, we can reduce the partition function to $Z_{\rm NJL} = \int[d\sigma][d\boldsymbol{\pi}][d\phi^\dagger][d\phi]\exp\{-\mathcal{S}_{\rm eff}[\sigma,\boldsymbol{\pi},\phi^\dagger,\phi]\}$, where the bosonized effective action $\mathcal{S}_{\rm eff}$ is given by

$$\mathcal{S}_{\rm eff}[\sigma,\boldsymbol{\pi},\phi^\dagger,\phi] = \int dx\frac{\sigma^2(x) + \boldsymbol{\pi}^2(x) + |\phi(x)|^2}{4G} - \frac{1}{2}{\rm Tr}\ln\mathbf{G}^{-1}(x,x'). \tag{8}$$

Here the trace Tr is taken over color, flavor, spin, Nambu-Gor'kov and coordinate (x and x') spaces. The thermodynamic potential density of the system is given by $\Omega(T,\mu_B) = -\lim_{V\to\infty}(T/V)\ln Z_{\rm NJL}$.

B. Evaluating the effective action

The effective action $\mathcal{S}_{\rm eff}$ as well as the thermodynamic potential Ω cannot be evaluated exactly in our 3 + 1 dimensional case. In this work, we firstly consider the saddle point approximation, i.e., the mean-field approximation. Then we investigate the fluctuations around the mean field.

(I) Mean-field approximation. In this approximation, all bosonic auxiliary fields are replaced by their expectation values. To this end, we write $\langle\sigma(x)\rangle = v$, $\langle\phi(x)\rangle = \Delta$ and set $\langle\boldsymbol{\pi}(x)\rangle = 0$. While Δ can be set to be real, we do not do this first in our derivations. We will show in the following that all physical results depend only on $|\Delta|^2$. The zeroth order or mean-field effective action reads

$$\mathcal{S}_{\rm eff}^{(0)} = \frac{V}{T}\left[\frac{v^2+|\Delta|^2}{4G} - \frac{1}{2}\sum_K \mathrm{Tr}\ln\frac{\mathcal{G}^{-1}(K)}{T}\right]. \tag{9}$$

Here and in the following $K = (i\omega_n, \mathbf{k})$ with $\omega_n = (2n+1)\pi T$ being the fermion Matsubara frequency, and $\sum_K = T\sum_n\sum_{\mathbf{k}}$ with $\sum_{\mathbf{k}} = \int d^3\mathbf{k}/(2\pi)^3$. The inverse of the Nambu-Gor'kov quark propagator $\mathcal{G}^{-1}(K)$ is given by

$$\begin{pmatrix} \left(i\omega_n+\frac{\mu_B}{2}\right)\gamma^0 - \boldsymbol{\gamma}\cdot\mathbf{k} - M & -i\gamma_5\Delta\tau_2 t_2 \\ -i\gamma_5\Delta^\dagger\tau_2 t_2 & \left(i\omega_n-\frac{\mu_B}{2}\right)\gamma^0 - \boldsymbol{\gamma}\cdot\mathbf{k} - M \end{pmatrix}, \tag{10}$$

with the effective quark Dirac mass $M = m_0 + v$. The mean-field thermodynamic potential $\Omega_0 = (T/V)\mathcal{S}_{\rm eff}^{(0)}$ can be evaluated as

$$\Omega_0 = \frac{v^2+|\Delta|^2}{4G} - 2N_cN_f\sum_{\mathbf{k}}[\mathcal{W}(E_{\mathbf{k}}^+) + \mathcal{W}(E_{\mathbf{k}}^-)], \tag{11}$$

with the definitions of the function $\mathcal{W}(E) = E/2 + T\ln(1+e^{-E/T})$ and the BCS-like quasiparticle dispersions $E_{\mathbf{k}}^\pm = \sqrt{(E_{\mathbf{k}}\pm\mu_B/2)^2+|\Delta|^2}$ where $E_{\mathbf{k}} = \sqrt{\mathbf{k}^2+M^2}$. The signs $\pm$ correspond to quasiquark and quasi-antiquark excitations, respectively. The integral over the quark momentum $\mathbf{k}$ is divergent at large $|\mathbf{k}|$, and some regularization scheme should be adopted. In this paper, we employ a hard three-momentum cutoff Λ.

The physical values of the variational parameters M (or v) and Δ should be determined by the saddle point condition

$$\frac{\delta\mathcal{S}_{\rm eff}^{(0)}[v,\Delta]}{\delta v} = 0, \qquad \frac{\delta\mathcal{S}_{\rm eff}^{(0)}[v,\Delta]}{\delta\Delta} = 0, \tag{12}$$

which minimizes the mean-field effective action $\mathcal{S}_{\rm eff}^{(0)}$. One can show that the saddle point condition is equivalent to the following Green function relations

$$\begin{aligned}\langle\bar{q}q\rangle &= \sum_K \mathrm{Tr}\mathcal{G}_{11}(K),\\ \langle\bar{q}_c i\gamma_5\tau_2 t_2 q\rangle &= \sum_K \mathrm{Tr}[\mathcal{G}_{12}(K)i\gamma_5\tau_2 t_2],\end{aligned} \tag{13}$$

where the matrix elements of $\mathcal{G}$ are explicitly given by

$$\begin{aligned}\mathcal{G}_{11}(K) &= \frac{i\omega_n+\xi_{\mathbf{k}}^-}{(i\omega_n)^2-(E_{\mathbf{k}}^-)^2}\Lambda_{\mathbf{k}}^+\gamma_0 + \frac{i\omega_n-\xi_{\mathbf{k}}^+}{(i\omega_n)^2-(E_{\mathbf{k}}^+)^2}\Lambda_{\mathbf{k}}^-\gamma_0,\\ \mathcal{G}_{22}(K) &= \frac{i\omega_n-\xi_{\mathbf{k}}^-}{(i\omega_n)^2-(E_{\mathbf{k}}^-)^2}\Lambda_{\mathbf{k}}^-\gamma_0 + \frac{i\omega_n+\xi_{\mathbf{k}}^+}{(i\omega_n)^2-(E_{\mathbf{k}}^+)^2}\Lambda_{\mathbf{k}}^+\gamma_0,\\ \mathcal{G}_{12}(K) &= \frac{-i\Delta\tau_2 t_2}{(i\omega_n)^2-(E_{\mathbf{k}}^-)^2}\Lambda_{\mathbf{k}}^+\gamma_5 + \frac{-i\Delta\tau_2 t_2}{(i\omega_n)^2-(E_{\mathbf{k}}^+)^2}\Lambda_{\mathbf{k}}^-\gamma_5,\\ \mathcal{G}_{21}(K) &= \frac{-i\Delta^\dagger\tau_2 t_2}{(i\omega_n)^2-(E_{\mathbf{k}}^-)^2}\Lambda_{\mathbf{k}}^-\gamma_5 + \frac{-i\Delta^\dagger\tau_2 t_2}{(i\omega_n)^2-(E_{\mathbf{k}}^+)^2}\Lambda_{\mathbf{k}}^+\gamma_5\end{aligned} \tag{14}$$

with the help of the massive energy projectors [60]

$$\Lambda_{\mathbf{k}}^\pm = \frac{1}{2}\left[1 \pm \frac{\gamma_0(\boldsymbol{\gamma}\cdot\mathbf{k}+M)}{E_{\mathbf{k}}}\right]. \tag{15}$$

Here we have defined the notation $\xi_{\mathbf{k}}^\pm = E_{\mathbf{k}} \pm \mu_B/2$ for convenience.

(II) Derivative expansion. Next, we consider the fluctuations around the mean field, corresponding to the bosonic collective excitations. Making the field shifts for the auxiliary fields,

$$\begin{aligned}&\sigma(x)\to v+\sigma(x), \qquad \boldsymbol{\pi}(x)\to 0+\boldsymbol{\pi}(x),\\ &\phi(x)\to\Delta+\phi(x), \qquad \phi^\dagger(x)\to\Delta^\dagger+\phi^\dagger(x),\end{aligned} \tag{16}$$

we can express the total effective action as

$$\begin{aligned}\mathcal{S}_{\rm eff} = \mathcal{S}_{\rm eff}^{(0)} &+ \int dx\left(\frac{\sigma^2+\boldsymbol{\pi}^2+|\phi|^2}{4G}\right.\\ &\left.+\frac{v\sigma+\Delta\phi^\dagger+\Delta^\dagger\phi}{2G}\right)\\ &-\frac{1}{2}\mathrm{Tr}\ln\left[\mathbb{1}+\int dx_1\mathcal{G}(x,x_1)\Sigma(x_1,x')\right].\end{aligned} \tag{17}$$

Here $\mathcal{G}(x,x')$ is the Fourier transformation of $\mathcal{G}(i\omega_n,\mathbf{k})$, and $\Sigma(x,x')$ is defined as

$$\Sigma(x,x') = \begin{pmatrix} -\sigma(x)-i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x) & -i\gamma_5\phi(x)\tau_2 t_2 \\ -i\gamma_5\phi^\dagger(x)\tau_2 t_2 & -\sigma(x)-i\gamma_5\boldsymbol{\tau}^T\cdot\boldsymbol{\pi}(x)\end{pmatrix} \times\delta(x-x'). \tag{18}$$

With the help of the derivative expansion

$$\mathrm{Tr}\ \ln[\mathbb{1}+\mathcal{G}\Sigma] = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{Tr}[\mathcal{G}\Sigma]^n, \tag{19}$$

we can calculate the effective action in powers of the fluctuations $\sigma(x)$, $\boldsymbol{\pi}(x)$, $\phi(x)$, $\phi^\dagger(x)$.

The first order effective action $\mathcal{S}_{\rm eff}^{(1)}$ which includes linear terms of the fluctuations should vanish exactly, since the expectation value of the fluctuations should be exactly zero. In fact, $\mathcal{S}_{\rm eff}^{(1)}$ can be evaluated as

$$\mathcal{S}_{\rm eff}^{(1)} = \int dx\bigg\{\bigg[\frac{v}{2G}+\frac{1}{2}\,{\rm Tr}(\mathcal{G}_{11}+\mathcal{G}_{22})\bigg]\sigma(x) + \frac{1}{2}\,{\rm Tr}[i\gamma_5(\mathcal{G}_{11}\boldsymbol{\tau}+\mathcal{G}_{22}\boldsymbol{\tau}^T)]\cdot\boldsymbol{\pi}(x) + \bigg[\frac{\Delta}{2G}+\frac{1}{2}\,{\rm Tr}(i\gamma_5\tau_2 t_2\mathcal{G}_{12})\bigg]\phi^\dagger(x) + \bigg[\frac{\Delta^\dagger}{2G}+\frac{1}{2}\,{\rm Tr}(i\gamma_5\tau_2 t_2\mathcal{G}_{21})\bigg]\phi(x)\bigg\}. \quad (20)$$

We observe that the coefficients of $\boldsymbol{\pi}(x)$ is automatically zero after taking the trace in Dirac spin space. The coefficients of $\phi(x)$, $\phi^\dagger(x)$ and $\sigma(x)$ vanish once the quark propagator takes the mean-field form and M, Δ take the physical values satisfying the saddle point condition. Thus, in the present approach, the saddle point condition plays a crucial role in having a vanishing linear term in the expansion.

The quadratic term $\mathcal{S}_{\rm eff}^{(2)}$ corresponds to the Gaussian fluctuations. It reads

$$\mathcal{S}_{\rm eff}^{(2)} = \int dx\frac{\sigma^2(x)+\boldsymbol{\pi}^2(x)+|\phi(x)|^2}{4G} + \frac{1}{4}\,{\rm Tr}\bigg[\int dx_1dx_2dx_3\mathcal{G}(x,x_1)\times\Sigma(x_1,x_2)\mathcal{G}(x_2,x_3)\Sigma(x_3,x')\bigg]. \quad (21)$$

For the convenience of our investigation in the following, we will use the form of $\mathcal{S}_{\rm eff}^{(2)}$ in the momentum space. After the Fourier transformation, it can be written as

$$\mathcal{S}_{\rm eff}^{(2)} = \frac{1}{2}\sum_Q\bigg\{\frac{|\sigma(Q)|^2+|\boldsymbol{\pi}(Q)|^2+|\phi(Q)|^2}{2G} + \frac{1}{2}\sum_K{\rm Tr}[\mathcal{G}(K)\Sigma(-Q)\mathcal{G}(K+Q)\Sigma(Q)]\bigg\}, \quad (22)$$

where $Q=(i\nu_m,\mathbf{q})$ with $\nu_m=2m\pi T$ being the boson Matsubara frequency and $\sum_Q = T\sum_m\sum_{\mathbf{q}}$. Here $A(Q)$ is the Fourier transformation of the field $A(x)$, and $\Sigma(Q)$ is defined as [61]

$$\Sigma(Q) = \begin{pmatrix} -\sigma(Q)-i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}(Q) & -i\gamma_5\phi(Q)\tau_2t_2 \\ -i\gamma_5\phi^\dagger(-Q)\tau_2t_2 & -\sigma(Q)-i\gamma_5\boldsymbol{\tau}^T\cdot\boldsymbol{\pi}(Q)\end{pmatrix}. \quad (23)$$

(III) Gaussian fluctuations. After taking the trace in Nambu-Gor'kov space, we find that $\mathcal{S}_{\rm eff}^{(2)}$ can be written in the following bilinear form

$$\mathcal{S}_{\rm eff}^{(2)} = \frac{1}{2}\sum_Q(\phi^\dagger(Q)\phi(-Q)\sigma^\dagger(Q))\mathbf{M}(Q)\begin{pmatrix}\phi(Q)\\ \phi^\dagger(-Q)\\ \sigma(Q)\end{pmatrix} + \frac{1}{2}\sum_Q(\pi_1^\dagger(Q)\pi_2^\dagger(Q)\pi_3^\dagger(Q))\mathbf{N}(Q)\begin{pmatrix}\pi_1(Q)\\ \pi_2(Q)\\ \pi_3(Q)\end{pmatrix}. \quad (24)$$

The matrix $\mathbf{M}$ takes the following nondiagonal form

$$\mathbf{M}(Q) = \begin{pmatrix}\frac{1}{4G}+\Pi_{11}(Q) & \Pi_{12}(Q) & \Pi_{13}(Q)\\ \Pi_{21}(Q) & \frac{1}{4G}+\Pi_{22}(Q) & \Pi_{23}(Q)\\ \Pi_{31}(Q) & \Pi_{32}(Q) & \frac{1}{2G}+\Pi_{33}(Q)\end{pmatrix}. \quad (25)$$

The polarization functions $\Pi_{ij}(Q)$ ($i,j=1,2,3$) are one-loop susceptibilities composed of the matrix elements the Nambu-Gor'kov quark propagator, and can be expressed as

$$\begin{aligned}
\Pi_{11}(Q) &= \frac{1}{2}\sum_K{\rm Tr}[\mathcal{G}_{22}(K)\Gamma\mathcal{G}_{11}(P)\Gamma],\\
\Pi_{22}(Q) &= \frac{1}{2}\sum_K{\rm Tr}[\mathcal{G}_{11}(K)\Gamma\mathcal{G}_{22}(P)\Gamma],\\
\Pi_{12}(Q) &= \frac{1}{2}\sum_K{\rm Tr}[\mathcal{G}_{12}(K)\Gamma\mathcal{G}_{12}(P)\Gamma],\\
\Pi_{21}(Q) &= \frac{1}{2}\sum_K{\rm Tr}[\mathcal{G}_{21}(K)\Gamma\mathcal{G}_{21}(P)\Gamma],\\
\Pi_{33}(Q) &= \frac{1}{2}\sum_K{\rm Tr}[\mathcal{G}_{11}(K)\mathcal{G}_{11}(P)+\mathcal{G}_{22}(K)\mathcal{G}_{22}(P) + \mathcal{G}_{12}(K)\mathcal{G}_{21}(P)+\mathcal{G}_{21}(K)\mathcal{G}_{12}(P)],\\
\Pi_{13}(Q) &= \frac{1}{2}\sum_K{\rm Tr}[\mathcal{G}_{12}(K)\Gamma\mathcal{G}_{11}(P)+\mathcal{G}_{22}(K)\Gamma\mathcal{G}_{12}(P)],\\
\Pi_{31}(Q) &= \frac{1}{2}\sum_K{\rm Tr}[\mathcal{G}_{21}(K)\mathcal{G}_{11}(P)\Gamma+\mathcal{G}_{22}(K)\mathcal{G}_{21}(P)\Gamma],\\
\Pi_{23}(Q) &= \frac{1}{2}\sum_K{\rm Tr}[\mathcal{G}_{11}(K)\Gamma\mathcal{G}_{21}(P)+\mathcal{G}_{21}(K)\Gamma\mathcal{G}_{22}(P)],\\
\Pi_{32}(Q) &= \frac{1}{2}\sum_K{\rm Tr}[\mathcal{G}_{11}(K)\mathcal{G}_{12}(P)\Gamma+\mathcal{G}_{12}(K)\mathcal{G}_{22}(P)\Gamma],
\end{aligned} \quad (26)$$

where $P=K+Q$, $\Gamma=i\gamma_5\tau_2t_2$ and the trace is taken over color, flavor, and spin spaces. Using the fact that $\mathcal{G}_{22}(K,\mu_B)=\mathcal{G}_{11}(K,-\mu_B)$ and $\mathcal{G}_{21}(K,\mu_B)=\mathcal{G}_{12}^\dagger(K,-\mu_B)$, we can easily show that

$$\Pi_{22}(Q)=\Pi_{11}(-Q),\qquad \Pi_{12}(Q)=\Pi_{21}^\dagger(Q),$$
$$\Pi_{13}(Q)=\Pi_{31}^\dagger(Q)=\Pi_{23}^\dagger(-Q)=\Pi_{32}(-Q). \quad (27)$$

Therefore, only five of the polarization functions are independent. At $T=0$, their explicit form is shown in

Appendix B. For general case, we can show that $\Pi_{12} \propto \Delta^2$ and $\Pi_{13} \propto M\Delta$. Thus, in the normal phase where $\Delta = 0$, the matrix $\mathbf{M}$ recovers the diagonal form. The off-diagonal elements Π_{13} and Π_{23} represents the mixing between the sigma meson and diquarks. At large chemical potentials where the chiral symmetry is approximately restored, $M \to m_0$, this mixing can be safely neglected.

On the other hand, the matrix $\mathbf{N}$ of the pion sector is diagonal and proportional to the identity matrix, i.e.,

$$\mathbf{N}_{ij}(Q) = \delta_{ij}\left[\frac{1}{2G} + \Pi_\pi(Q)\right], \qquad i, j = 1, 2, 3. \quad (28)$$

This means pions are eigen mesonic excitations even in the superfluid phase. The polarization function $\Pi_\pi(Q)$ is given by

$$\begin{aligned}\Pi_\pi(Q) = \frac{1}{2}\sum_K \mathrm{Tr}[&\mathcal{G}_{11}(K)i\gamma_5\mathcal{G}_{11}(P)i\gamma_5 \\ &+ \mathcal{G}_{22}(K)i\gamma_5\mathcal{G}_{22}(P)i\gamma_5 - \mathcal{G}_{12}(K)i\gamma_5\mathcal{G}_{21}(P)i\gamma_5 \\ &- \mathcal{G}_{21}(K)i\gamma_5\mathcal{G}_{12}(P)i\gamma_5]. \end{aligned} \quad (29)$$

Its explicit form at $T = 0$ is shown in Appendix B. We find that $\Pi_\pi(Q)$ and $\Pi_{33}(Q)$ is different only to a term proportional to M^2. Thus, at high density where $\langle\bar{q}q\rangle \to 0$, the spectra of pions and sigma meson become nearly degenerate which represents the approximate restoration of chiral symmetry.

(IV) Goldstone's theorem. The $U_B(1)$ baryon number symmetry is spontaneously broken by the nonzero diquark condensate $\langle qq\rangle$ in the superfluid phase, resulting in one Goldstone boson. In our model, this is ensured by the condition $\det\mathbf{M}(Q = 0) = 0$. From the explicit form of the polarization functions shown in Appendix B, we find that this condition holds if and only if the saddle point condition (12) for v and Δ is satisfied. We thus emphasize that in our theoretical framework, the condensates v and Δ should be determined by the saddle point condition, and the beyond-mean-field corrections are possible only through the thermodynamics, i.e., the equations of state.

C. Vacuum and model parameter fixing

For a better understanding our derivation in the following, it is useful to review the vacuum state at $T = \mu_B = 0$. In the vacuum, it is evident that $\Delta = 0$ and the mean-field effective potential Ω_{vac} can be evaluated as

$$\Omega_{\mathrm{vac}}(M) = \frac{(M - m_0)^2}{4G} - 2N_cN_f\sum_{\mathbf{k}} E_{\mathbf{k}}. \quad (30)$$

The physical value of M, denoted by M_*, satisfies the saddle point condition $\partial\Omega_{\mathrm{vac}}/\partial M = 0$ and minimizes Ω_{vac}.

The meson and diquark excitations can be obtained from $\mathcal{S}_{\mathrm{eff}}^{(2)}$, which in the vacuum can be expressed as

$$\begin{aligned}\mathcal{S}_{\mathrm{eff}}^{(2)} = -\frac{1}{2}\int\frac{d^4Q}{(2\pi)^4}\Big[&\sigma(-Q)\mathcal{D}_\sigma^{*-1}(Q)\sigma(Q) \\ &+ \sum_{i=1}^{3}\pi_i(-Q)\mathcal{D}_\pi^{*-1}(Q)\pi_i(Q) \\ &+ \sum_{i=1}^{2}\phi_i(-Q)\mathcal{D}_\phi^{*-1}(Q)\phi_i(Q)\Big], \end{aligned} \quad (31)$$

where ϕ_1, ϕ_2 are the real and imaginary parts of ϕ, respectively. The inverse propagators in vacuum can be expressed in a symmetrical form [41]

$$\begin{aligned}\mathcal{D}_l^{*-1}(Q) &= \frac{1}{2G} + \Pi_l^*(Q), \; l = \sigma, \pi, \phi \\ \Pi_l^*(Q) &= 2iN_cN_f(Q^2 - \epsilon_l^2)I(Q^2) \\ &\quad - 4iN_cN_f\int\frac{d^4K}{(2\pi)^4}\frac{1}{K^2 - M_*^2}, \end{aligned} \quad (32)$$

where $\epsilon_\sigma = 2M_*$, $\epsilon_\pi = \epsilon_\phi = 0$, and the function $I(Q^2)$ is defined as

$$I(Q^2) = \int\frac{d^4K}{(2\pi)^4}\frac{1}{(K_+^2 - M_*^2)(K_-^2 - M_*^2)}, \quad (33)$$

with $K_\pm = K \pm Q/2$. Keeping in mind that M_* satisfies the saddle point condition, we find that the pions and diquarks are Nambu-Goldstone bosons in the chiral limit, corresponding to the symmetry breaking pattern $SU(4) \to Sp(4)$. Using the gap equation of M_*, we find that the masses of mesons and diquarks can be determined by the equation

$$m_l^2 = -\frac{m_0}{M_*}\frac{1}{4iGN_cN_fI(m_l^2)} + \epsilon_l^2. \quad (34)$$

Since the Q^2 dependence of the function $I(Q^2)$ is very weak, we find $m_\pi^2 \sim m_0$ and $m_\sigma^2 \simeq 4M_*^2 + m_\pi^2$.

Since pions and diquarks are deep bound states, their propagators can be well approximated by $\mathcal{D}_\pi^*(Q) \simeq -g_{\pi qq}^2/(Q^2 - m_\pi^2)$ with $g_{\pi qq}^{-2} \simeq -2iN_cN_fI(0)$. The pion decay constant f_π can be determined by the matrix element of the axial current,

$$\begin{aligned}iQ_\mu f_\pi\delta_{ij} &= -\frac{1}{2}\mathrm{Tr}\int\frac{d^4K}{(2\pi)^4} \\ &\quad\times[\gamma_\mu\gamma_5\tau_i\mathcal{G}(K_+)g_{\pi qq}\gamma_5\tau_j\mathcal{G}(K_-)] \\ &= 2N_cN_fg_{\pi qq}M_*Q_\mu I(Q^2)\delta_{ij}. \end{aligned} \quad (35)$$

Here $\mathcal{G}(K) = (\gamma^\mu K_\mu - M_*)^{-1}$. Thus, the pion decay constant can be expressed as

$$f_\pi^2 \approx -2iN_cN_fM_*^2I(0). \quad (36)$$

Finally, together with (34) and (36), we recover the well-known Gell-Mann–Oakes–Renner relation $m_\pi^2f_\pi^2 = -m_0\langle\bar{q}q\rangle_0$.

TABLE I. Model parameters (3-momentum cutoff Λ, coupling constant G, and current quark mass m_0) and related quantities (quark condensate $\langle\bar{u}u\rangle_0$, constituent quark mass M_* and pion mass m_π) for the two-flavor two-color NJL model ([39]). The pion decay constant is fixed to be $f_\pi = 75$ MeV.

Set	Λ [MeV]	$G\Lambda^2$	m_0 [MeV]	$\langle\bar{u}u\rangle_0^{1/3}$ [MeV]	M_* [MeV]	m_π [MeV]
1	657.9	3.105	4.90	−217.4	300	133.6
2	583.6	3.676	5.53	−209.1	400	134.0
3	565.8	4.238	5.43	−210.6	500	134.2
4	565.4	4.776	5.11	−215.1	600	134.4

There are three parameters in our model, the current quark mass m_0, the coupling constant G and the cutoff Λ. In principle they should be determined from the known values of the pion mass m_π, the pion decay constant f_π and the quark condensate $\langle\bar{q}q\rangle_0$. Since two-color QCD does not correspond to our real world, we get the above values from the empirical values $f_\pi \simeq 93$ MeV, $\langle\bar{u}u\rangle_0 \simeq (250\text{ MeV})^3$ in the $N_c = 3$ case, according to the relation f_π^2, $\langle\bar{q}q\rangle_0 \sim N_c$. To obtain the model parameters, we fix the values of the pion decay constant f_π and slightly vary the values of the chiral condensate $\langle\bar{q}q\rangle_0$ and the pion mass m_π. Thus, we can obtain different sets of model parameters corresponding to different values of effective quark mass M_* and hence different values of the sigma meson mass m_σ. Four sets of model parameters are shown in Table. I. As we will show in the following, the physics near the quantum phase transition point $\mu_B = m_\pi$ is not sensitive to different model parameter sets, since the low energy dynamics is dominated by the pseudo-Goldstone bosons (i.e., the diquarks). However, at high density, the physics becomes sensitive to different model parameter sets corresponding to different sigma meson masses. The predictions by the chiral perturbation theories should be recovered in the limit $m_\sigma/m_\pi \to \infty$.

III. DILUTE BOSE CONDENSATE: MEAN-FIELD THEORY

Now we begin to study the properties of two-color matter at finite baryon density. Without loss of generality, we set $\mu_B > 0$. In this section, we study the two-color baryonic matter in the dilute limit, which forms near the quantum phase transition point $\mu_B = m_\pi$. Since the diquark condensate is vanishingly small near the quantum phase transition point, we can make a Ginzburg-Landau expansion for the effective action. As we will see below, this corresponds to the mean-field theory of weakly interacting dilute Bose condensates.

A. Ginzburg-Landau free energy near the quantum phase transition

Since the diquark condensate Δ is vanishingly small near the quantum phase transition, we can derive the Ginzburg-Landau free energy functional $V_{\rm GL}[\Delta(x)]$ at $T = 0$ for the order parameter field $\Delta(x) = \langle\phi(x)\rangle$ in the static and long-wavelength limit. The general form of $V_{\rm GL}[\Delta(x)]$ can be written as

$$V_{\rm GL}[\Delta(x)] = \int dx\Big[\Delta^\dagger(x)\Big(-\delta\frac{\partial^2}{\partial\tau^2} + \kappa\frac{\partial}{\partial\tau} - \gamma\nabla^2\Big)\Delta(x) + \alpha|\Delta(x)|^2 + \frac{1}{2}\beta|\Delta(x)|^4\Big], \quad (37)$$

where the coefficients α, β, γ, δ, κ should be low energy constants which *depend only on the vacuum properties*. The calculation is somewhat similar to the derivation of Ginzburg-Landau free energy of a superconductor from the microscopic BCS theory [63], but for our case there is a difference in that we have another variational parameter, i.e., the effective quark mass M which should be a function of $|\Delta|^2$ determined by the saddle point condition.

(I) The potential terms. In the static and long-wavelength limit, the coefficients α, β of the potential terms can be obtained from the effective action $\mathcal{S}_{\rm eff}$ in the mean-field approximation. At $T = 0$, the mean-field effective action reads $\mathcal{S}_{\rm eff}^{(0)} = \int dx\Omega_0$, where the mean-field thermodynamic potential is given by

$$\Omega_0(|\Delta|^2, M) = \frac{(M - m_0)^2 + |\Delta|^2}{4G} - N_cN_f\sum_{\mathbf{k}}(E_{\mathbf{k}}^+ + E_{\mathbf{k}}^-). \quad (38)$$

The Ginzburg-Landau coefficients α, β can be obtained via a Taylor expansion of Ω_0 in terms of $|\Delta|^2$,

$$\Omega_0 = \Omega_{\rm vac}(M_*) + \alpha|\Delta|^2 + \tfrac{1}{2}\beta|\Delta|^4 + O(|\Delta|^6), \quad (39)$$

where $\Omega_{\rm vac}(M_*)$ is the vacuum contribution which should be subtracted. One should keep in mind that *the effective quark mass M is not a fixed parameter, but a function of* $|\Delta|^2$ via its saddle point condition or gap equation $\partial\Omega_0/\partial M = 0$.

For convenience, we define $y \equiv |\Delta|^2$. The Ginzburg-Landau coefficient α is defined as

$$\alpha = \frac{d\Omega_0(y,M)}{dy}\bigg|_{y=0} = \frac{\partial\Omega_0(y,M)}{\partial y}\bigg|_{y=0} + \frac{\partial\Omega_0(y,M)}{\partial M}\frac{dM}{dy}\bigg|_{y=0} = \frac{\partial\Omega_0(y,M)}{\partial y}\bigg|_{y=0}, \tag{40}$$

where the indirect derivative term vanishes due to the saddle point condition for M. After some simple algebra, we get

$$\alpha = \frac{1}{4G} - N_cN_f\sum_{\mathbf{k}}\frac{E^*_{\mathbf{k}}}{E^{*2}_{\mathbf{k}} - \mu_B^2/4}, \tag{41}$$

where $E^*_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + M_*^2}$. We can make the above expression more meaningful using the pion mass equation in the same three-momentum regularization scheme [41,42],

$$\frac{1}{4G} - N_cN_f\sum_{\mathbf{k}}\frac{E^*_{\mathbf{k}}}{E^{*2}_{\mathbf{k}} - m_\pi^2/4} = 0. \tag{42}$$

We therefore obtain a G-independent result

$$\alpha = \frac{1}{4}N_cN_f(m_\pi^2 - \mu_B^2)\sum_{\mathbf{k}}\frac{E^*_{\mathbf{k}}}{(E^{*2}_{\mathbf{k}} - m_\pi^2/4)(E^{*2}_{\mathbf{k}} - \mu_B^2/4)}. \tag{43}$$

From the fact that $m_\pi \ll 2M_*$ and $\beta > 0$ (see below), we see clearly that a second order quantum phase transition takes place at exactly $\mu_B = m_\pi$. Thus, the Ginzburg-Landau free energy is meaningful only near the quantum phase transition point, i.e., $|\mu_B - m_\pi| \ll m_\pi$, and α can be further simplified as

$$\alpha \simeq (m_\pi^2 - \mu_B^2)\mathcal{J}, \tag{44}$$

where the factor $\mathcal{J}$ is defined as

$$\mathcal{J} = \frac{1}{4}N_cN_f\sum_{\mathbf{k}}\frac{E^*_{\mathbf{k}}}{(E^{*2}_{\mathbf{k}} - m_\pi^2/4)^2}. \tag{45}$$

The coefficient β of the quartic term can be evaluated via the definition

$$\beta = \frac{d^2\Omega_0(y,M)}{dy^2}\bigg|_{y=0} = \frac{\partial^2\Omega_0(y,M)}{\partial y^2}\bigg|_{y=0} + \frac{\partial^2\Omega_0(y,M)}{\partial M\partial y}\frac{dM}{dy}\bigg|_{y=0}. \tag{46}$$

Notice that the last indirect derivative term does not vanish here and *will be important for us to obtain a correct diquark-diquark scattering length*. The derivative dM/dy can be analytically derived from the gap equation for M. From the fact that $\partial\Omega_0/\partial M = 0$, we obtain

$$\frac{\partial}{\partial y}\left(\frac{\partial\Omega_0(y,M)}{\partial M}\right) + \frac{\partial}{\partial M}\left(\frac{\partial\Omega_0(y,M)}{\partial M}\right)\frac{dM}{dy} = 0. \tag{47}$$

Thus, we find

$$\frac{dM}{dy} = -\frac{\partial^2\Omega_0(y,M)}{\partial M\partial y}\left(\frac{\partial^2\Omega_0(y,M)}{\partial M^2}\right)^{-1}. \tag{48}$$

Then the practical expression for β can be written as

$$\beta = \beta_1 + \beta_2, \tag{49}$$

where β_1 is the direct derivative term

$$\beta_1 = \frac{\partial^2\Omega_0(y,M)}{\partial y^2}\bigg|_{y=0}, \tag{50}$$

and β_2 is the indirect term

$$\beta_2 = -\left(\frac{\partial^2\Omega_0(y,M)}{\partial M\partial y}\right)^2\left(\frac{\partial^2\Omega_0(y,M)}{\partial M^2}\right)^{-1}\bigg|_{y=0}. \tag{51}$$

Near the quantum phase transition, all chemical potential dependence can be absorbed into the coefficient α, and we can set $\mu_B = m_\pi$ in β. After a simple algebra, the explicit form of β_1 and β_2 can be evaluated as

$$\beta_1 = \frac{1}{4}N_cN_f\sum_{e=\pm}\sum_{\mathbf{k}}\frac{1}{(E^*_{\mathbf{k}} - em_\pi/2)^3} \tag{52}$$

and

$$\beta_2 = -\left\{\frac{1}{2}N_cN_f\sum_{e=\pm}\sum_{\mathbf{k}}\frac{M_*}{E^*_{\mathbf{k}}}\frac{1}{(E^*_{\mathbf{k}} - em_\pi/2)^2}\right\}^2 \times\left(\frac{m_0}{2GM_*} + 2N_cN_f\sum_{\mathbf{k}}\frac{M_*^2}{E^{*3}_{\mathbf{k}}}\right)^{-1}. \tag{53}$$

The G-dependent term $m_0/(2GM_*)$ in (53) can be approximated as $m_\pi^2f_\pi^2/M_*^2$ using the relation $m_\pi^2f_\pi^2 = -m_0\langle\bar{q}q\rangle$.

(II) The kinetic terms. The kinetic terms in the Ginzburg-Landau free energy can be derived from the inverse of the diquark propagator [63]. In the general case with $\Delta \neq 0$, the diquarks are mixed with the sigma meson. However, approaching the quantum phase transition point, $\Delta \to 0$, the problem is simplified. After the analytical continuation $i\nu_m \to \omega + i0^+$, the inverse of the diquark propagator in the limit $\mu_B \to m_\pi$ can be evaluated as

$$\mathcal{D}_d^{-1}(\omega,\mathbf{q}) = \frac{1}{4G} + \Pi_d(\omega,\mathbf{q}), \tag{54}$$

where the polarization function $\Pi_d(\omega,\mathbf{q})$ is given by

$$\Pi_d(\omega,\mathbf{q}) = N_cN_f\sum_{\mathbf{k}}\frac{E^*_{\mathbf{k}} + E^*_{\mathbf{k}+\mathbf{q}}}{(\omega+\mu_B)^2 - (E^*_{\mathbf{k}} + E^*_{\mathbf{k}+\mathbf{q}})^2}\times\left(1 + \frac{\mathbf{k}\cdot(\mathbf{k}+\mathbf{q}) + M_*^2}{E^*_{\mathbf{k}}E^*_{\mathbf{k}+\mathbf{q}}}\right). \tag{55}$$

In the static and long-wavelength limit ($\omega, |\mathbf{q}| \to 0$), the coefficients κ, δ, γ can be determined by the Taylor expansion $\mathcal{D}_d^{-1}(\omega,\mathbf{q}) = \mathcal{D}_d^{-1}(0,\mathbf{0}) - \delta\omega^2 - \kappa\omega + \gamma\mathbf{q}^2$. Notice that α is identical to $\mathcal{D}_d^{-1}(0,\mathbf{0})$ which is in fact the Thouless criterion for the superfluid transition.

On the other hand, keeping in mind that $\mathcal{D}_d^{-1}(\omega, \mathbf{q})$ can be related to the pion propagator in the vacuum, i.e., $\mathcal{D}_d^{-1}(\omega, \mathbf{q}) = (1/2)\mathcal{D}_\pi^{*-1}(\omega + \mu_B, \mathbf{q})$, in the static and long-wavelength limit and for $\mu_B \to m_\pi \ll 2M_*$ we can well approximate it as [41]

$$\mathcal{D}_d^{-1}(\omega, \mathbf{q}) \simeq -\mathcal{J}[(\omega + \mu_B)^2 - \mathbf{q}^2 - m_\pi^2], \tag{56}$$

where $\mathcal{J}$ is the same factor defined in (44), and one can show that $\mathcal{J} \simeq g_{\pi qq}^{-2}/2$. We thus find that $\delta \simeq \gamma \simeq \mathcal{J}$ which ensures the Lorentz invariance of the vacuum, and $\kappa \simeq 2\mu_B\mathcal{J}$.

B. From Ginzburg-Landau to Gross-Pitaevskii free energy

We now show how the Ginzburg-Landau free energy can be reduced to the theory describing weakly repulsive Bose condensates, i.e., the Gross-Pitaevskii free energy [64,65].

(I) Nonrelativistic version. First, since the Bose condensed matter is indeed dilute, let us consider the nonrelativistic version, where $\omega \ll m_\pi$ and the kinetic term $\propto \partial^2/\partial\tau^2$ is neglected. To this end, we define the nonrelativistic chemical potential μ_d for diquarks, $\mu_d = \mu_B - m_\pi$, and further simplify the coefficient α as

$$\alpha \simeq -\mu_d(2m_\pi\mathcal{J}). \tag{57}$$

Then the Ginzburg-Landau free energy can be reduced to the Gross-Pitaevskii free energy of a dilute repulsive Bose gas, if we define a new condensate wave function $\Psi(x)$ as

$$\Psi(x) = \sqrt{2m_\pi\mathcal{J}}\Delta(x). \tag{58}$$

The resulting Gross-Pitaevskii free energy is given by

$$V_{\rm GP}[\Psi(x)] = \int dx\bigg[\Psi^\dagger(x)\bigg(\frac{\partial}{\partial\tau} - \frac{\boldsymbol{\nabla}^2}{2m_\pi}\bigg)\Psi(x) - \mu_d|\Psi(x)|^2 + \frac{1}{2}g_0|\Psi(x)|^4\bigg], \tag{59}$$

where $g_0 = 4\pi a_{\rm dd}/m_\pi$. The repulsive diquark-diquark interaction is characterized by a positive scattering length $a_{\rm dd}$ defined as

$$a_{\rm dd} = \frac{\beta}{16\pi m_\pi}\mathcal{J}^{-2}. \tag{60}$$

Keep in mind that the scattering length obtained here is *at the mean-field level*. We will discuss the possible beyond-mean-field corrections in Sec. V. Thus, for a dilute medium with density n satisfying $na_{\rm dd}^3 \ll 1$, the system is indeed a weakly interacting Bose condensate [13–15].

(II) Diquark-diquark scattering length. Even though we have shown that the Ginzburg-Landau free energy is indeed a Gross-Pitaevskii version near the quantum phase transition, a key problem is whether the obtained diquark-diquark scattering length $a_{\rm dd}$ is quantitatively correct. A numerical calculation for (60) is straightforward. The obtained values of $a_{\rm dd}$ for the four model parameter sets are shown in Table II. We can also give an analytical expression based on the formula of the pion decay constant in the three-momentum cutoff scheme,

$$f_\pi^2 = N_c M_*^2 \sum_{\mathbf{k}} \frac{1}{E_{\mathbf{k}}^{*3}}. \tag{61}$$

According to the fact that $m_\pi \ll 2M_*$, β and $\mathcal{J}$ can be well approximated as

$$\beta \simeq \frac{f_\pi^2}{M_*^2} - \frac{(2f_\pi^2/M_*)^2}{m_\pi^2 f_\pi^2/M_*^2 + 4f_\pi^2} \simeq \frac{f_\pi^2 m_\pi^2}{4M_*^4}, \qquad \mathcal{J} \simeq \frac{f_\pi^2}{2M_*^2}. \tag{62}$$

Thus, the diquark-diquark scattering length $a_{\rm dd}$ in the limit $m_\pi/(2M_*) \to 0$ is related only to the pion mass and decay constant,

$$a_{\rm dd} = \frac{m_\pi}{16\pi f_\pi^2}. \tag{63}$$

The values of $a_{\rm dd}$ for the four model parameter sets according to the above expression are also listed in Table II. The errors are always about 1% comparing with the exact numerical results, which means that the expression (63) is a good approximation for the diquark-diquark scattering length. The error should come from the finite value of $m_\pi/(2M^*)$. We can obtain a correction in powers of $m_\pi/(2M^*)$ [66], but it is obviously small, and its explicit form is not shown here.

The result $a_{\rm dd} \propto m_\pi$ is universal for the scattering lengths of the pseudo-Goldstone bosons. Even though the SU(4) flavor symmetry is explicitly broken in presence of a nonzero quark mass, a discrete symmetry ϕ_1, $\phi_2 \leftrightarrow \pi_1$, π_2 holds exactly for arbitrary quark mass. This also means that the partition function of two-color QCD has a discrete symmetry $\mu_B \leftrightarrow \mu_I$ [8]. Because of this discrete symmetry of two-color QCD, the analytical expression (63) of $a_{\rm dd}$ (which is in fact the diquark-diquark scattering length in the $B = 2$ channel) should be identical to the pion-pion scattering length at tree level in the $I = 2$ channel which was first obtained by Weinberg many years ago [67].

TABLE II. The values of diquark-diquark scattering length $a_{\rm dd}$ (in units of m_π^{-1}) for different model parameter sets.

Set	1	2	3	4
$a_{\rm dd}$ according to (60) $[m_\pi^{-1}]$	0.0631	0.0635	0.0637	0.0639
$a_{\rm dd}$ according to (63) $[m_\pi^{-1}]$	0.0624	0.0628	0.0630	0.0633

Therefore, the mean-field theory can describe not only the quantum phase transition to a dilute diquark condensate but also the effect of repulsive diquark-diquark interaction.

(III) Equations of state. The mean-field equations of state of the dilute diquark condensate are thus determined by the Gross-Pitaevskii free energy (59). Minimizing $\Psi V_{\rm GP}[(x)]$ with respect to a uniform condensate Ψ, we find the physical minimum is given by

$$|\Psi_0|^2 = \frac{\mu_d}{g_0}, \tag{64}$$

and the baryon density is $n = |\Psi_0|^2$. Using the thermodynamic relations, we therefore get the well-known results for the pressure P, the energy density $\mathcal{E}$ and the chemical potential μ_B in terms of the baryon density n,

$$\begin{aligned} P(n) &= \frac{2\pi a_{\rm dd}}{m_\pi} n^2, \\ \mathcal{E}(n) &= m_\pi n + \frac{2\pi a_{\rm dd}}{m_\pi} n^2, \\ \mu_B(n) &= m_\pi + \frac{4\pi a_{\rm dd}}{m_\pi} n, \end{aligned} \tag{65}$$

which were first obtained by Bogoliubov many years ago [14]. We can examine the above results through a direct numerical calculation with the mean-field thermodynamic potential. The pressure is given by $P = -(\Omega_0 - \Omega_{\rm vac})$ and the baryon density reads $n = -\partial\Omega_0/\partial\mu_B$. In Fig. 1 we show the numerical results for the pressure and the chemical potential as functions of the density for the four model parameter sets. At low enough density, the equations of state are indeed consistent with the results (65) with the scattering length given by (60). It is evident that the results at low density are not sensitive to different model parameter sets, since the physics at low density should be dominated by the pseudo-Goldstone bosons.

In fact, we can derive the equations of state (65) analytically from the mean-field thermodynamic potential Ω_0. For example, the baryon number density reads

$$\begin{aligned} n &= \frac{1}{2} N_c N_f \sum_{\mathbf{k}} \left[\left(1 - \frac{\xi_{\mathbf{k}}^-}{E_{\mathbf{k}}^-}\right) - \left(1 - \frac{\xi_{\mathbf{k}}^+}{E_{\mathbf{k}}^+}\right) \right] \\ &= \frac{1}{2} N_c N_f \sum_{\mathbf{k}} \left[\frac{|\Delta|^2}{E_{\mathbf{k}}^-(E_{\mathbf{k}}^- + \xi_{\mathbf{k}}^-)} - \frac{|\Delta|^2}{E_{\mathbf{k}}^+(E_{\mathbf{k}}^+ + \xi_{\mathbf{k}}^+)} \right]. \end{aligned} \tag{66}$$

Near the quantum phase transition point and to leading order of $|\Delta|^2$, we obtain

$$\begin{aligned} n &\simeq \frac{1}{4} N_c N_f \sum_{\mathbf{k}} \left[\frac{|\Delta|^2}{(E_{\mathbf{k}}^* - m_\pi/2)^2} - \frac{|\Delta|^2}{(E_{\mathbf{k}}^* + m_\pi/2)^2} \right] \\ &= 2 m_\pi \mathcal{J} |\Delta|^2 = |\Psi_0|^2. \end{aligned} \tag{67}$$

Further, since our treatment is only at the mean-field level, the Lee-Huang-Yang corrections [15] which are

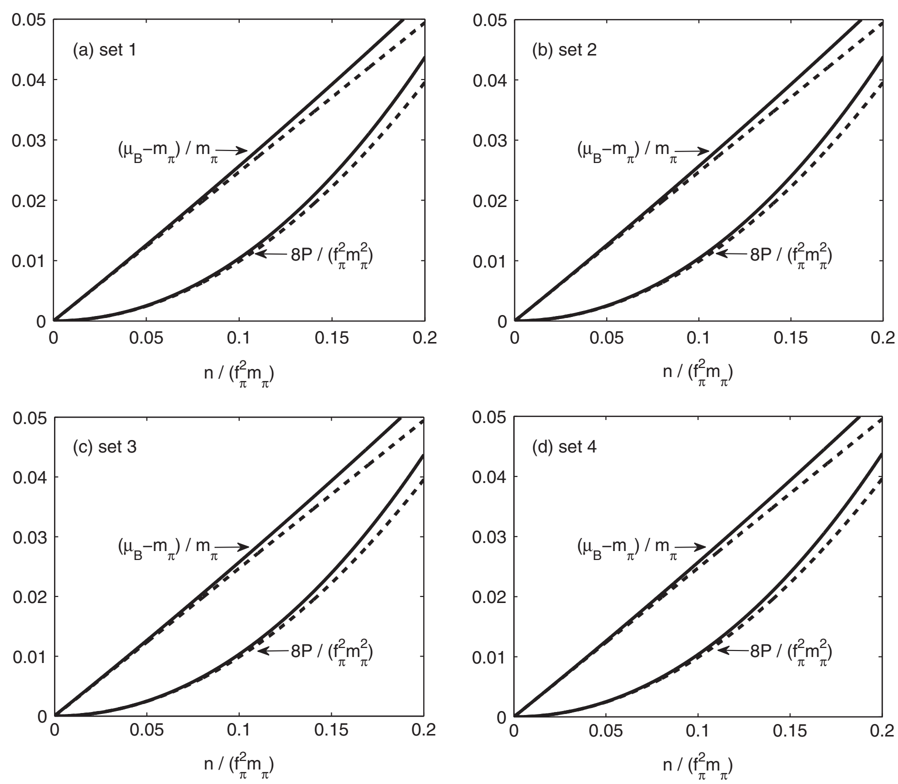

FIG. 1. The baryon chemical potential μ_B and the pressure P as functions of the baryon density n for different model parameter sets. The solid lines correspond to the direct mean-field calculation, and the dashed lines are given by (65).

proportional to $(na_{\rm dd}^3)^{1/2}$ are absent in the equations of state. As we have shown in Appendix A, to obtain such corrections, it is necessary to go beyond the mean field, and a beyond-mean-field correction to the scattering length $a_{\rm dd}$ is also possible [57,58].

(IV) Relativistic version. We can also consider a relativistic version of the Gross-Pitaevskii free energy via defining the condensate wave function

$$\Phi(x) = \sqrt{\mathcal{J}}\Delta(x). \tag{68}$$

In this case, the Ginzburg-Landau free energy is reduced to a relativistic version of the Gross-Pitaevskii free energy,

$$V_{\rm RGP}[\Phi(x)] = \int dx\Big[\Phi^\dagger(x)\Big(-\frac{\partial^2}{\partial\tau^2} + 2\mu_B\frac{\partial}{\partial\tau} - \boldsymbol{\nabla}^2\Big)\Phi(x) + (m_\pi^2 - \mu_B^2)|\Phi(x)|^2 + \frac{\lambda}{2}|\Phi(x)|^4\Big]. \tag{69}$$

The self-interacting coupling $\lambda = \beta\mathcal{J}^{-2}$ is now dimensionless and can be approximated by $\lambda \simeq m_\pi^2/f_\pi^2$. For realistic values of m_π and f_π, we find $\lambda \sim O(1)$. In this sense, the Bose condensate is not weakly interacting, except for the low density limit $na_{\rm dd}^3 \ll 1$. One should keep in mind that this result cannot be applied to high density, since it is valid only near the quantum phase transition point.

C. Bogoliubov excitation in a dilute diquark condensate

An ideal Bose-Einstein condensate is not a superfluid. In presence of weakly repulsive interactions among the bosons, a Goldstone mode which has a linear dispersion in the low energy limit appears, and the condensate becomes a superfluid according to Landau's criterion $\min_{\bf q}[\omega({\bf q})/|{\bf q}|] > 0$. The Goldstone mode which is also called the Bogoliubov mode here should have a dispersion given by [13–15]

$$\omega(\mathbf{q}) = \sqrt{\frac{\mathbf{q}^2}{2m_\pi}\Big(\frac{\mathbf{q}^2}{2m_\pi} + \frac{8\pi a_{\rm dd} n}{m_\pi}\Big)}, \qquad |\mathbf{q}| \ll m_\pi. \tag{70}$$

Since the Gross-Pitaevskii free energy obtained above is at the classical level, to study the bosonic collective excitations we should consider the fluctuations around the mean field [62,68,69]. The propagator of the bosonic collective modes is given by $\mathbf{M}^{-1}(Q)$ and $\mathbf{N}^{-1}(Q)$. The Bogoliubov mode corresponds to the lowest excitation obtained from the equation $\det\mathbf{M}(\omega,\mathbf{q}) = 0$. With the explicit form of the matrix elements of $\mathbf{M}$ in the superfluid phase, we can analytically show that $\det\mathbf{M}(0,\mathbf{0}) = 0$ which ensures the Goldstone's theorem. In fact, for $(\omega,\mathbf{q}) = (0,\mathbf{0})$, we find that $\det\mathbf{M} = (\mathbf{M}_{11}^2 - |\mathbf{M}_{12}|^2)\mathbf{M}_{33} + 2|\mathbf{M}_{13}|^2(|\mathbf{M}_{12}| - \mathbf{M}_{11})$. Using the saddle point condition for Δ, we can show that $\mathbf{M}_{11}(0,\mathbf{0}) = |\mathbf{M}_{12}(0,\mathbf{0})|$ and hence the Goldstone's theorem holds in the superfluid phase. Further, we may obtain an analytical expression of the velocity of the Bogoliubov mode via a Taylor expansion for $\mathbf{M}(\omega,\mathbf{q})$ around $(\omega,\mathbf{q}) = (0,\mathbf{0})$ like those done in [62,68,69]. Such a calculation for our case is more complicated due to the mixing between the sigma meson and diquarks, and it cannot give the full dispersion (70).

On the other hand, since $\Delta \to 0$ near the quantum phase transition point, we can expand the matrix elements of $\mathbf{M}$ in powers of $|\Delta|^2$. The advantage of such an expansion is that it cannot only give the full dispersion (70) but also link the meson properties in the vacuum. Formally, we can write down the following expansions:

$$\begin{aligned}
\mathbf{M}_{11}(\omega,\mathbf{q}) &= \mathcal{D}_d^{-1}(\omega,\mathbf{q}) + |\Delta|^2 A(\omega,\mathbf{q}) + O(|\Delta|^4),\\
\mathbf{M}_{22}(\omega,\mathbf{q}) &= \mathcal{D}_d^{-1}(-\omega,\mathbf{q}) + |\Delta|^2 A(-\omega,\mathbf{q}) + O(|\Delta|^4),\\
\mathbf{M}_{12}(\omega,\mathbf{q}) &= \mathbf{M}_{21}^\dagger(\omega,\mathbf{q}) = \Delta^2 B(\omega,\mathbf{q}) + O(|\Delta|^4),\\
\mathbf{M}_{13}(\omega,\mathbf{q}) &= \mathbf{M}_{31}^\dagger(\omega,\mathbf{q}) = \Delta H(\omega,\mathbf{q}) + O(|\Delta|^3),\\
\mathbf{M}_{23}(\omega,\mathbf{q}) &= \mathbf{M}_{32}^\dagger(\omega,\mathbf{q}) = \Delta^\dagger H(-\omega,\mathbf{q}) + O(|\Delta|^3),\\
\mathbf{M}_{33}(\omega,\mathbf{q}) &= \mathcal{D}_\sigma^{*-1}(\omega,\mathbf{q}) + O(|\Delta|^2).
\end{aligned} \tag{71}$$

Notice that the effective quark mass M is regarded as a function of $|\Delta|^2$ as we have done in deriving the Ginzburg-Landau free energy. Since we are interested in the dispersion in the low energy limit, i.e., ω, $|\mathbf{q}| \ll m_\pi$, we can approximate the coefficients of the leading order terms as their values at $(\omega,\mathbf{q}) = (0,\mathbf{0})$. That is,

$$\begin{aligned}
A(\omega,\mathbf{q}) &\simeq A(-\omega,\mathbf{q}) \simeq A(0,\mathbf{0}) \equiv A_0,\\
B(\omega,\mathbf{q}) &\simeq B(0,\mathbf{0}) \equiv B_0,\\
H(\omega,\mathbf{q}) &\simeq H(-\omega,\mathbf{q}) \simeq H(0,\mathbf{0}) \equiv H_0.
\end{aligned} \tag{72}$$

Further, since $m_\sigma \gg m_\pi$, we can approximate the inverse sigma propagator $\mathcal{D}_\sigma^{*-1}(\omega,\mathbf{q})$ as its value at $(\omega,\mathbf{q}) = (0,\mathbf{0})$. Therefore, the dispersion of the Goldstone mode in the low energy limit can be determined by the following equation:

$$\det\begin{pmatrix} \mathcal{D}_d^{-1}(\omega,\mathbf{q}) + |\Delta|^2 A_0 & \Delta^2 B_0 & \Delta H_0 \\ \Delta^{\dagger 2} B_0 & \mathcal{D}_d^{-1}(-\omega,\mathbf{q}) + |\Delta|^2 A_0 & \Delta^\dagger H_0 \\ \Delta^\dagger H_0 & \Delta H_0 & \mathcal{D}_\sigma^{*-1}(0,\mathbf{0}) \end{pmatrix} = 0. \tag{73}$$

Now we can link the coefficients A_0, B_0, H_0 and $\mathcal{D}_\sigma^{-1}(0,\mathbf{0})$ to the derivatives of the mean-field thermodynamic potential Ω_0 and its Ginzburg-Landau coefficients. Firstly, using the explicit form of $\mathbf{M}_{12}$, we find that

$$|\mathbf{M}_{12}(0,\mathbf{0})| = |\Delta|^2\beta_1 \Rightarrow B_0 = \beta_1. \quad (74)$$

Second, using the fact that

$$\mathbf{M}_{11}(0,\mathbf{0}) - |\mathbf{M}_{12}(0,\mathbf{0})| = \frac{\partial\Omega_0}{\partial|\Delta|^2}, \quad (75)$$

and together with the definition for $A(\omega,\mathbf{q})$,

$$A(\omega,\mathbf{q}) = \frac{d\mathbf{M}_{11}(y,M)}{dy}\bigg|_{y=0} = \frac{\partial\mathbf{M}_{11}(y,M)}{\partial y}\bigg|_{y=0} + \frac{\partial\mathbf{M}_{11}(y,M)}{\partial M}\frac{dM}{dy}\bigg|_{y=0}, \quad (76)$$

we find the following exact relation:

$$A_0 = \beta + B_0 = \beta + \beta_1. \quad (77)$$

On the other hand, we have the following relations for H_0 and $\mathcal{D}_\sigma^{*-1}(0,\mathbf{0})$,

$$H_0 = \frac{\partial^2\Omega_0(y,M)}{\partial M\partial y}\bigg|_{y=0}, \qquad \mathcal{D}_\sigma^{*-1}(0,\mathbf{0}) = \frac{\partial^2\Omega_0(y,M)}{\partial M^2}\bigg|_{y=0}. \quad (78)$$

One can check the above results from the explicit forms of $\mathbf{M}_{13}$ and $\mathbf{M}_{33}$ in Appendix B directly. Thus, we have

$$-\frac{H_0^2}{\mathcal{D}_\sigma^{*-1}(0,\mathbf{0})} = \beta_2. \quad (79)$$

According to the above relations, Eq. (73) can be reduced to

$$3\beta^2|\Delta|^4 + 2\beta|\Delta|^2[\mathcal{D}_d^{-1}(\omega,\mathbf{q}) + \mathcal{D}_d^{-1}(-\omega,\mathbf{q})] + \mathcal{D}_d^{-1}(\omega,\mathbf{q})\mathcal{D}_d^{-1}(-\omega,\mathbf{q}) = 0. \quad (80)$$

It is evident that only the coefficient β appears in the final equation. Further, in the nonrelativistic limit ω, $|\mathbf{q}| \ll m_\pi$ and near the quantum phase transition point, $\mathcal{D}_d^{-1}(\omega,\mathbf{q})$ can be approximated as

$$\mathcal{D}_d^{-1}(\omega,\mathbf{q}) \simeq -2m_\pi\mathcal{J}\left(\omega - \frac{\mathbf{q}^2}{2m_\pi} + \mu_d\right). \quad (81)$$

Together with the mean-field results for the chemical potential $\mu_d = g_0|\Psi_0|^2 = \beta|\Delta|^2/(2m_\pi\mathcal{J})$ and for the baryon density $n = |\Psi_0|^2$, we finally get the Bogoliubov dispersion (70).

We should emphasize that *the mixing between the sigma meson and the diquarks, denoted by the terms* ΔH_0 *and* $\Delta^\dagger H_0$, *plays an important role in recovering the correct Bogoliubov dispersion*. Even though we do get this dispersion, we find the procedure is quite different to the standard theory of weakly interacting Bose gas [13,14,64,65]. There, the elementary excitation is given only by the diquark-diquark sectors, i.e.,

$$\det\begin{pmatrix}\mathbf{M}_{11}(Q) & \mathbf{M}_{12}(Q)\\ \mathbf{M}_{21}(Q) & \mathbf{M}_{22}(Q)\end{pmatrix} = 0 \Rightarrow \det\begin{pmatrix}-\omega + \frac{\mathbf{q}^2}{2m_\pi} - \mu_d + 2g_0|\Psi_0|^2 & g_0|\Psi_0|^2\\ g_0|\Psi_0|^2 & \omega + \frac{\mathbf{q}^2}{2m_\pi} - \mu_d + 2g_0|\Psi_0|^2\end{pmatrix} = 0. \quad (82)$$

But in our case, we cannot get the correct Bogoliubov excitation if we simply set $H_0 = 0$ and consider only the diquark-diquark sector. In fact, this requires $A_0 = 2B_0 = 2\beta$ which is not true in our case.

One can also check how the momentum dependence of A, B, H and $\mathcal{D}_\sigma^{*-1}$ modifies the dispersion. This needs direct numerical solution of the equation $\det\mathbf{M}(\omega,\mathbf{q}) = 0$. We have examined that for $|\mu_B - m_\pi|$ up to $0.01m_\pi$, the numerical result agrees well with the Bogoliubov formula (70). However, at higher density, a significant deviation is observed. This is in fact a signature of BEC-BCS crossover which will be discussed in Sec. IV.

D. In-medium chiral condensate

Up to now we have studied the properties of the dilute Bose condensate induced by a small diquark condensate $\langle qq\rangle$. The chiral condensate $\langle\bar{q}q\rangle$ will be modified in the medium. In such a dilute Bose condensate, we can study the response of the chiral condensate to the baryon density n.

To this end, we expand the effective quark mass M in terms of $y = |\Delta|^2$. We have

$$M - M_* = \frac{dM}{dy}\bigg|_{y=0} y + O(y^2). \quad (83)$$

The expansion coefficient can be approximated as

$$\frac{dM}{dy}\bigg|_{y=0} \simeq -\frac{2f_\pi^2/M_*}{m_\pi^2 f_\pi^2/M_*^2 + 4f_\pi^2} = -\frac{1}{2M_*}\left[1 + O\left(\frac{m_\pi^2}{4M_*^2}\right)\right]. \quad (84)$$

Using the definition of the effective quark mass, $M = m_0 - 2G\langle\bar{q}q\rangle$, we find that

$$\frac{\langle\bar{q}q\rangle_n}{\langle\bar{q}q\rangle_0} = 1 - \frac{|\Delta|^2}{4G\langle\bar{q}q\rangle_0 M_*} \simeq 1 - \frac{|\Delta|^2}{2M_*^2}. \tag{85}$$

Since the baryon number density reads $n = |\Psi_0|^2 = 2m_\pi \mathcal{J}|\Delta|^2$, using the fact that $\mathcal{J} \simeq f_\pi^2/(2M_*^2)$, we obtain to leading order

$$\frac{\langle\bar{q}q\rangle_n}{\langle\bar{q}q\rangle_0} \simeq 1 - \frac{n}{2f_\pi^2 m_\pi}. \tag{86}$$

This formula is in fact a two-color analogue of the density dependence of the chiral condensate in the $N_c = 3$ case, where we have [70,71]

$$\frac{\langle\bar{q}q\rangle_n}{\langle\bar{q}q\rangle_0} \simeq 1 - \frac{\Sigma_{\pi\mathrm{N}}}{f_\pi^2 m_\pi^2} n, \tag{87}$$

with $\Sigma_{\pi\mathrm{N}}$ being the pion-nucleon sigma term. In Fig. 2, we show the numerical results via solving the mean-field gap equations. One finds that the chiral condensate has a perfect linear behavior at low density. For large value of M_* (and hence the sigma meson mass m_σ), the linear behavior persists even at higher density.

In fact, the Eq. (86) can be obtained in a model independent way. Applying the Hellmann-Feynman theorem to a dilute diquark gas with energy density $\mathcal{E}(n)$ given by (65), we can obtain (86) directly. According to the Hellmann-Feynman theorem, we have

$$2m_0(\langle\bar{q}q\rangle_n - \langle\bar{q}q\rangle_0) = m_0 \frac{d\mathcal{E}}{dm_0}. \tag{88}$$

The derivative $d\mathcal{E}/dm_0$ can be evaluated via the chain rule $d\mathcal{E}/dm_0 = (d\mathcal{E}/dm_\pi)(dm_\pi/dm_0)$. Together with the Gell-Mann–Oakes–Renner relation $m_\pi^2 f_\pi^2 = -m_0\langle\bar{q}q\rangle_0$ and the fact that $da_{\mathrm{dd}}/dm_\pi \simeq a_{\mathrm{dd}}/m_\pi$, we can obtain to leading order Eq. (86). Beyond the leading order, we find the correction of order $O(n^2)$ vanishes. Thus, the next-to-leading order correction should be $O(n^{5/2})$ coming from the Lee-Huang-Yang correction to the equation of state [72].

Finally, we can show analytically that the "chiral rotation" behavior [4–11] predicted by the chiral perturbation theories is valid in the NJL model near the quantum phase transition. In the chiral perturbation theories, the chemical potential dependence of the chiral and diquark condensates can be analytically expressed as

$$\frac{\langle\bar{q}q\rangle_{\mu_B}}{\langle\bar{q}q\rangle_0} = \frac{m_\pi^2}{\mu_B^2}, \qquad \frac{\langle qq\rangle_{\mu_B}}{\langle\bar{q}q\rangle_0} = \sqrt{1 - \frac{m_\pi^4}{\mu_B^4}}. \tag{89}$$

Near the phase transition point, we can expand the above formula in powers of $\mu_d = \mu_B - m_\pi$. To leading order, we have

$$\frac{\langle\bar{q}q\rangle_{\mu_B}}{\langle\bar{q}q\rangle_0} \simeq 1 - \frac{2\mu_d}{m_\pi}, \qquad \frac{\langle qq\rangle_{\mu_B}}{\langle\bar{q}q\rangle_0} \simeq 2\sqrt{\frac{\mu_d}{m_\pi}}. \tag{90}$$

Using the mean-field result (64) for the chemical potential μ_d, one can easily check that the above relations are also valid in our NJL model.

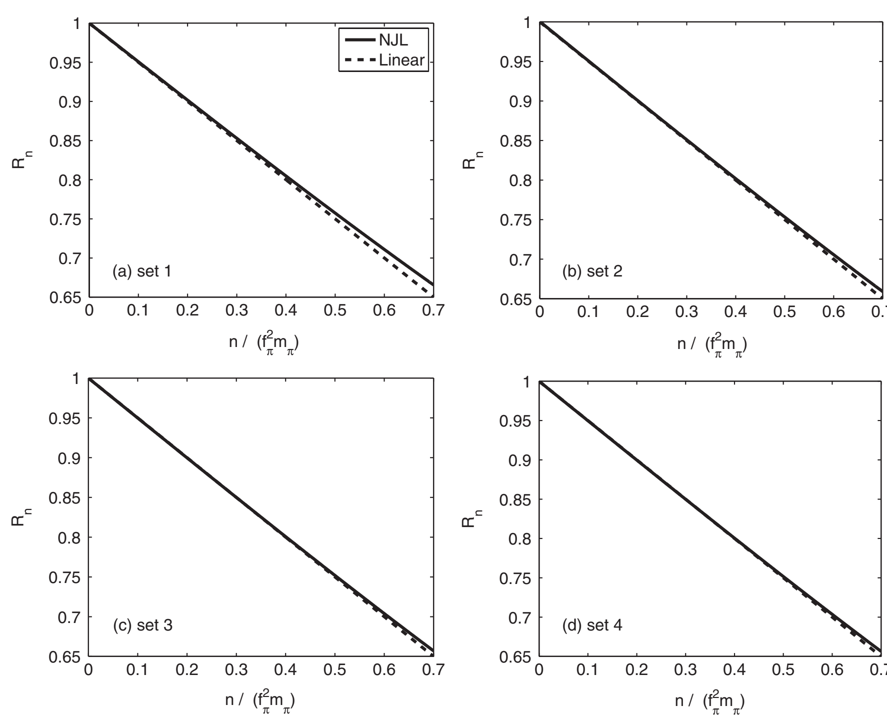

FIG. 2. The ratio $R_n = \langle\bar{q}q\rangle_n/\langle\bar{q}q\rangle_0$ as a function of $n/(f_\pi^2 m_\pi)$ for different model parameter sets. The dashed line is the linear behavior given by (86).

E. Chiral limit

In the above studies we focused on the "physical point" where $m_0 \neq 0$. In the final part of this section, we briefly discuss the chiral limit with $m_0 = 0$.

We may naively expect that the results at $m_0 \neq 0$ can be directly generalized to the chiral limit via setting $m_\pi = 0$. The ground state is a noninteracting Bose condensate of massless diquarks, since $m_\pi = 0$ and $a_{\rm dd} = 0$. However, this cannot be true since many divergences develop due to the vanishing pion mass. In fact, the conclusion of second order phase transition is not correct since the Ginzburg-Landau coefficient β vanishes. Instead, the superfluid phase transition is of strongly first order in the chiral limit [48,52].

In the chiral limit, the effective action in the vacuum should depend only on the combination $\sigma^2 + \boldsymbol{\pi}^2 + |\phi|^2$ due to the exact flavor symmetry $SU(4) \simeq SO(6)$. The vacuum is chosen to be associated with a nonzero chiral condensate $\langle\sigma\rangle$ without loss of generality. At zero and at finite chemical potential, the thermodynamic potential $\Omega_0(M, |\Delta|)$ has two minima locating at $(M, |\Delta|) = (a, 0)$ and $(M, |\Delta|) = (0, b)$. At zero chemical potential, these two minima are degenerate due to the exact flavor symmetry. However, at nonzero chemical potential (even arbitrarily small), the minimum $(0, b)$ has the lowest free energy. Analytically, we can show that $b \rightarrow M_*$ at $\mu_B = 0^+$. This means the superfluid phase transition in the chiral limit is of strongly first order, and takes place at arbitrarily small chemical potential. Since the effective quark mass M keeps vanishing in the superfluid phase, a low density Bose condensate does not exist in the chiral limit.

IV. MATTER AT HIGH DENSITY: BEC-BCS CROSSOVER AND MOTT TRANSITION

The investigations in Sec. III are restricted near the quantum phase transition point $\mu_B = m_\pi$. Generally the state of matter at high density should not be a relativistic Bose condensate described by (69). In fact, perturbative QCD calculations show that the matter is a weakly coupled BCS superfluid at asymptotic density [20–23]. In this section, we will discuss the evolution of the superfluid matter as the baryon density increases from the NJL model point of view. While some results presented in the following have been published elsewhere [52–55], we will still show them for the sake of completeness.

A. Chiral and diquark condensates

The numerical results for the chiral condensate $\langle\bar{q}q\rangle$ and diquark condensate $\langle qq\rangle$ are shown in Fig. 3. As a comparison, we also show the analytical result (89) predicted by chiral perturbation theories. While the behavior of the chiral condensate is in good agreement with the chiral perturbation theories, the diquark condensate deviates significantly from the result (89) for small values of M_*.

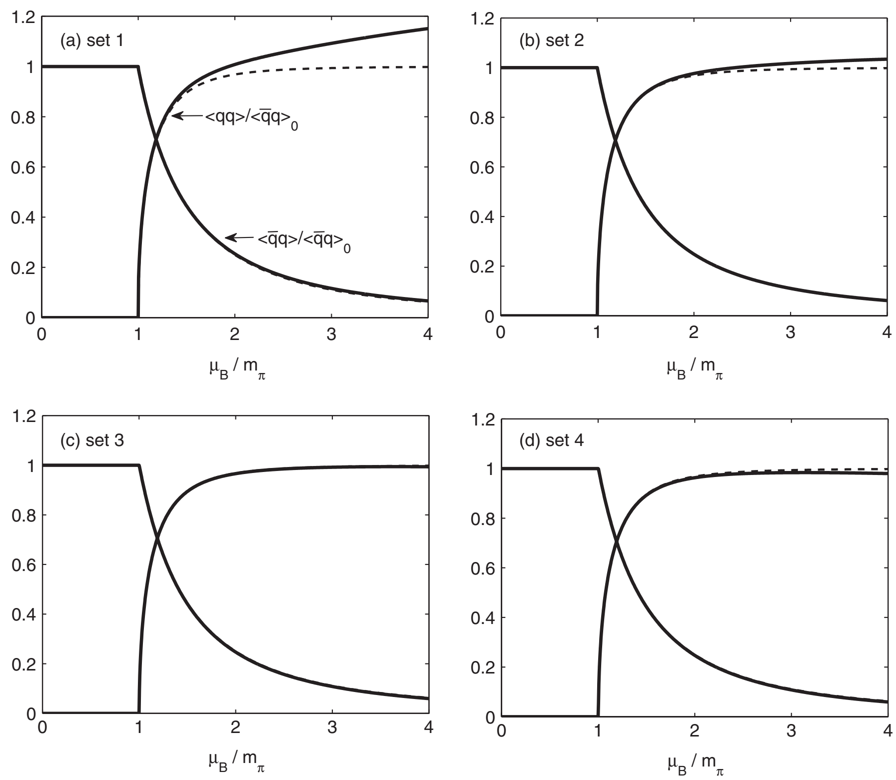

FIG. 3. The chiral and diquark condensates (in units of $\langle\bar{q}q\rangle_0$) as functions of the baryon chemical potential (in units of m_π) for different model parameter sets.

This deviation can be understood from the fact that the chiral perturbation theories correspond to the nonlinear sigma model limit $m_\sigma \to \infty$. For finite value of m_σ, one should consider the O(6) linear sigma model [55]

$$\mathcal{L}_{\rm LSM} = \frac{1}{2}(\partial_\mu \boldsymbol{\varphi})^2 - \frac{1}{2}m^2\boldsymbol{\varphi}^2 + \frac{1}{4}\lambda\boldsymbol{\varphi}^4 - H\sigma, \qquad (91)$$

where $\boldsymbol{\varphi} = (\sigma, \boldsymbol{\pi}, \phi_1, \phi_2)$ and $m^2 < 0$. The model parameters m^2, λ, H can be determined from the vacuum phenomenology. In this model, we can show that the chiral and diquark condensates are given by [48,55]

$$\frac{\langle\bar{q}q\rangle_{\mu_B}}{\langle\bar{q}q\rangle_0} = \frac{m_\pi^2}{\mu_B^2}, \qquad \frac{\langle qq\rangle_{\mu_B}}{\langle\bar{q}q\rangle_0} = \sqrt{1 - \frac{m_\pi^4}{\mu_B^4} + 2\frac{\mu_B^2 - m_\pi^2}{m_\sigma^2 - m_\pi^2}}. \qquad (92)$$

In the nonlinear sigma model limit $m_\sigma \to \infty$, the above results are indeed reduced to the result (89) predicted by chiral perturbation theories. However, for finite values of m_σ, the results can be significantly different from (89) at large chemical potential.

B. BEC-BCS crossover

While the Ginzburg-Landau free energy can be reduced to the Gross-Pitaevskii free energy near the quantum phase transition point, it is not the case at arbitrary μ_B.

When μ_B increases, we find that the fermionic excitation spectra $E_{\mathbf{k}}^{\pm}$ undergo a characteristic change. Near the quantum phase transition $\mu_B = m_\pi$ they are nearly degenerate since $m_\pi \ll 2M_*$ and their minima are located at $|\mathbf{k}| = 0$. However, at very large μ_B the minimum of $E_{\mathbf{k}}^-$ moves to $|\mathbf{k}| \simeq \mu_B/2$ since $M \to m_0$. Meanwhile the excitation energy of the antifermion excitations become much larger than that of the fermion excitations and can be neglected. This characteristic change of the fermionic excitation spectra takes place when the minimum of the lowest band excitation $E_{\mathbf{k}}^-$ moves from $|\mathbf{k}| = 0$ to $|\mathbf{k}| \neq 0$ [53,54,62,68,69,73–79], i.e., $\mu_B/2 = M(\mu_B)$ [80]. A schematic plot of this characteristic change is shown in Fig. 4. The equation $\mu_B/2 = M(\mu_B)$ defines the so-called crossover point $\mu_B = \mu_0$ which can be numerically determined by the mean-field gap equations. The numerical results of the crossover chemical potential μ_0 for the four model parameter sets are shown in Table III. For reasonable parameter sets, the crossover chemical potential is in the range $(1.6–2)m_\pi$.

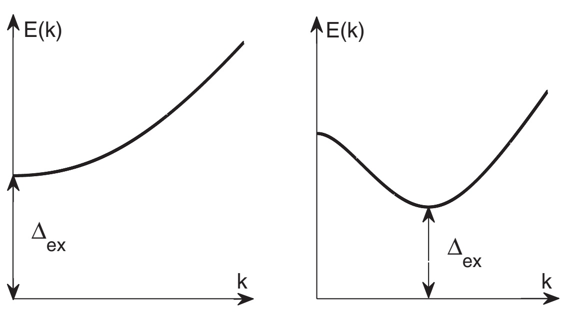

FIG. 4. A schematic plot of the fermionic excitation spectrum in the BEC state (left) and the BCS state (right).

TABLE III. The crossover chemical potential μ_0 (in units of m_π) for different model parameter sets.

Set	1	2	3	4
Crossover chemical potential μ_0 $[m_\pi]$	1.65	1.81	1.95	2.07

In fact, an analytical expression for μ_0 can be achieved according to the fact that the chiral rotation behavior $\langle\bar{q}q\rangle_{\mu_B}/\langle\bar{q}q\rangle_0 \simeq m_\pi^2/\mu_B^2$ is still valid in the NJL model at large chemical potentials as shown in Fig. 3. We obtain [53]

$$\frac{\mu_0}{2} \simeq \frac{m_\pi^2}{\mu_0^2}M_* \Rightarrow \mu_0 \simeq (2M_*m_\pi^2)^{1/3}. \qquad (93)$$

Using the fact that $m_\sigma \simeq 2M_*$, we find that μ_0 can be expressed as

$$\frac{\mu_0}{m_\pi} \simeq \left(\frac{m_\sigma}{m_\pi}\right)^{1/3}. \qquad (94)$$

Thus, in the nonlinear sigma model limit $m_\sigma/m_\pi \to \infty$, there should be no BEC-BCS crossover. On the other hand, this means the physical prediction power of the chiral perturbation theories is restricted near the quantum phase transition point.

The fermionic excitation gap $\Delta_{\rm ex}$ (as shown in Fig. 4), defined as the minimum of the fermionic excitation energy, i.e., $\Delta_{\rm ex} = \min_{\mathbf{k}}\{E_{\mathbf{k}}^-, E_{\mathbf{k}}^+\}$, can be evaluated as

$$\Delta_{\rm ex} = \begin{cases} \sqrt{(M - \frac{\mu_B}{2})^2 + |\Delta|^2} & \mu_B < \mu_0 \\ |\Delta| & \mu_B > \mu_0 \end{cases}. \qquad (95)$$

It is evident that the fermionic excitation gap is equal to the superfluid order parameter only in the BCS regime. This is similar to the BEC-BCS crossover in nonrelativistic systems [62], and we find that the corresponding fermion chemical potential μ can be defined as $\mu = \mu_B/2 - M$. The numerical results of the fermionic excitation gap $\Delta_{\rm ex}$ for different model parameter sets are shown in Fig. 5. We find that for a wide range of the baryon chemical potential, it is of order $O(M_*)$. The fermionic excitation gap is equal to the pairing gap $|\Delta|$ only at the BCS side of the crossover, and exhibits a minimum at the quantum phase transition point.

On the other hand, the momentum distributions of quarks (denoted by $n(\mathbf{k})$) and antiquarks (denoted by $\bar{n}(\mathbf{k})$) can be evaluated using the quark Green function $\mathcal{G}_{11}(K)$. We obtain

$$n(\mathbf{k}) = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}^-}{E_{\mathbf{k}}^-}\right), \qquad \text{for quarks,}$$
$$\bar{n}(\mathbf{k}) = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}^+}{E_{\mathbf{k}}^+}\right), \qquad \text{for antiquarks.} \qquad (96)$$

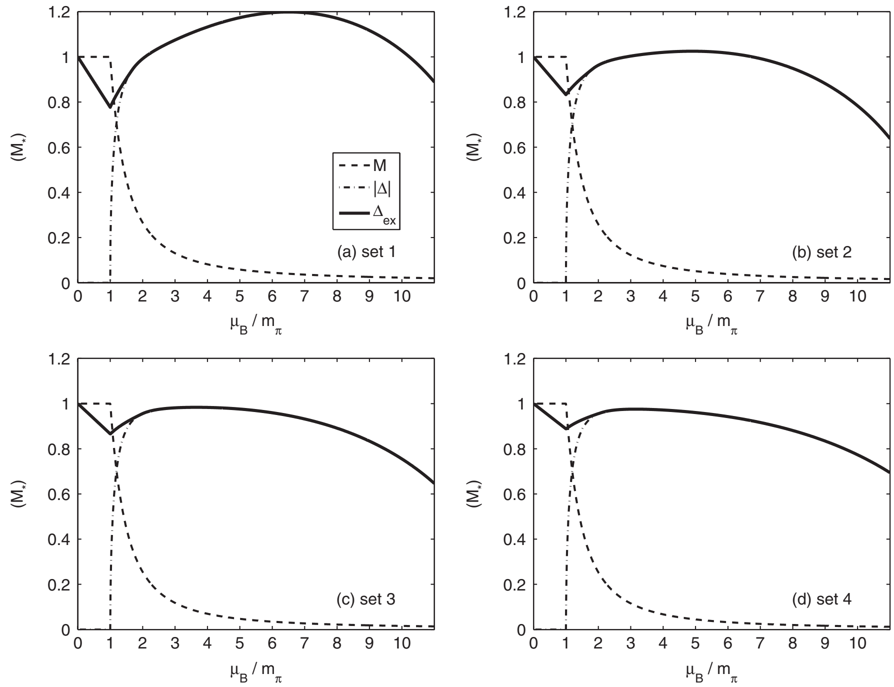

FIG. 5. The fermionic excitation gap $\Delta_{\rm ex}$ (in units of M_*) as a function of the baryon chemical potential (in units of m_π) for different model parameter sets. The effective quark mass M and the pairing gap $|\Delta|$ are also shown by dashed and dash-dotted lines, respectively.

The numerical results for $n(\mathbf{k})$ and $\bar{n}(\mathbf{k})$ (for model parameter set 1) are shown in Fig. 6. Near the quantum phase transition point, the quark momentum distribution $n(\mathbf{k})$ is a very smooth function in the whole momentum space. In the opposite limit, i.e., at large chemical potentials, it approaches unity at $|\mathbf{k}| = 0$ and decreases rapidly around the effective "Fermi surface" at $|\mathbf{k}| \simeq |\mu|$. For the antiquarks, we find that the momentum distribution $\bar{n}(\mathbf{k})$ exhibits a nonmonotonous behavior: it is suppressed at both low and high densities and is visible only at moderate chemical potentials. However, even at very large chemical potentials, e.g., $\mu_B = 10m_\pi$, the momentum distribution $n(\mathbf{k})$ does not approach the standard BCS behavior, which means the dense matter is not a weakly coupled BCS superfluid for a wide range of the baryon chemical potential. In Fig. 7, we show the ratio $|\Delta|/\mu$ up to $\mu_B \simeq 10m_\pi$. It is clear that the ratio is not small even at large chemical potentials. At $\mu_B = 10m_\pi$, it is about 0.5, which means the dense matter is still a strongly coupled BCS superfluid.

The Goldstone mode also undergoes a characteristic change in the BEC-BCS crossover. Near the quantum phase transition point, i.e., in the dilute limit, the Goldstone mode recovers the Bogoliubov excitation of weakly interacting Bose condensates. In the opposite limit, we expect the Goldstone mode approaches the Anderson-Bogoliubov mode of a weakly coupled BCS superfluid, which takes a dispersion $\omega(\mathbf{q}) = |\mathbf{q}|/\sqrt{3}$ up to the two-particle continuum $\omega \simeq 2|\Delta|$. In fact, at large chemical

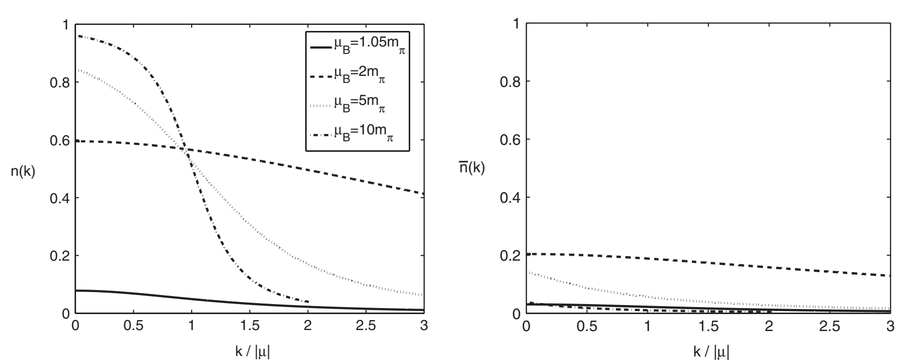

FIG. 6. The momentum distributions for quarks (upper panel) and antiquarks (lower panel) for various values of μ_B. The momentum is scaled by $|\mu| = |\mu_B/2 - M|$.

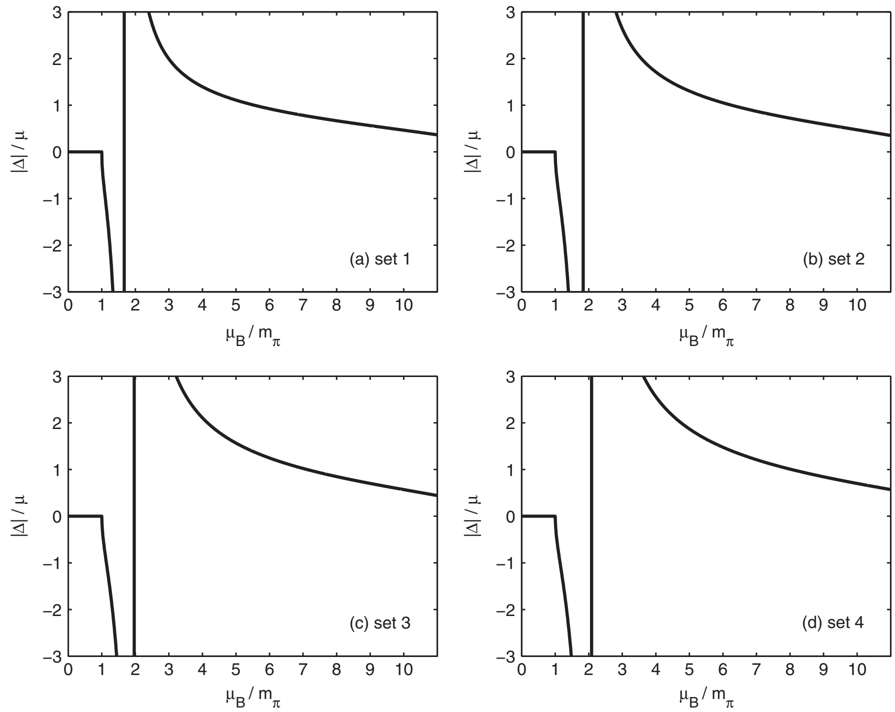

FIG. 7. The ratio of the pairing gap $|\Delta|$ to the effective fermionic chemical potential $\mu = \mu_B/2 - M$ as a function of the baryon chemical potential (in units of m_π) for different model parameter sets. The divergent point corresponds to $\mu_B = \mu_0$, i.e., the BEC-BCS crossover point.

potentials, we can safely neglect the mixing between the sigma meson and diquarks. The Goldstone boson dispersion is thus determined by the equation

$$\det\begin{pmatrix} \mathbf{M}_{11}(Q) & \mathbf{M}_{12}(Q) \\ \mathbf{M}_{21}(Q) & \mathbf{M}_{22}(Q) \end{pmatrix} = 0. \tag{97}$$

The problem is totally the same as that has been investigated in [68,69]. Therefore, at very large chemical potentials where $|\Delta|/\mu$ becomes small enough, the Goldstone mode recovers the Anderson-Bogoliubov mode of a weakly coupled BCS superfluid.

Finally, we should emphasize that the existence of a smooth crossover from the Bose condensate to the BCS superfluid depends on whether there exists a deconfinement phase transition at finite μ_B [31,32,81] and where it takes place. Recent lattice calculation predicts a deconfinement crossover which occurs at a baryon chemical potential larger than that of the BEC-BCS crossover [32].

C. Chiral restoration and meson Mott transition

As in real QCD with two quark flavors, we expect the chiral symmetry is restored and the spectra of sigma meson and pions become degenerate at high density [82]. For the two-flavor case and with vanishing m_0, the residue $SU_L(2) \otimes SU_R(2) \otimes U_B(1)$ symmetry group at $\mu_B \neq 0$ is spontaneously broken down to $Sp_L(2) \otimes Sp_R(2)$ in the superfluid medium with nonzero $\langle\bar{q}q\rangle$ and $\langle qq\rangle$, resulting in one Goldstone boson. For small nonzero m_0, we expect the spectra of sigma meson and pions become approximately degenerate when the in-medium chiral condensate $\langle\bar{q}q\rangle$ becomes small enough.

In fact, according to the result $\langle\bar{q}q\rangle_n/\langle\bar{q}q\rangle_0 \simeq 1 - n/(2f_\pi^2 m_\pi)$ at low density, we can roughly expect that the chiral symmetry is approximately restored at $n \sim 2f_\pi^2 m_\pi$. From the chemical potential dependence of the chiral condensate $\langle\bar{q}q\rangle$ shown in Fig. 3, we find that it becomes smaller and smaller as the density increases. As a result, we should have nearly degenerate spectra for the sigma meson and pions. To show this we need the explicit form of the matrix $\mathbf{M}(Q)$ and $\mathbf{N}(Q)$ given in Appendix B. Since $\mathbf{M}_{13}$, $\mathbf{M}_{32} \propto M\Delta$, at high density where $\langle\bar{q}q\rangle \to 0$, they can be safely neglected and the sigma meson decouples from the diquarks. The propagator of the sigma meson is then given by $\mathbf{M}_{33}^{-1}(Q)$. From the explicit form of the polarization functions $\Pi_\sigma(Q) = \Pi_{33}(Q)$ and $\Pi_\pi(Q)$, we can see that the inverse propagators of the sigma meson and pions differ from each other in a term proportional to M^2. Thus, at high density their spectra are nearly degenerate, and their masses are given by the equation

$$1\text{–}2G\Pi_\pi(\omega, \mathbf{0}) = 0. \tag{98}$$

Using the mean-field gap equation for Δ, we find the solution is $\omega = \mu_B$, which means the meson masses are equal to μ_B at large chemical potentials. In Fig. 8, we show the chemical potential dependence of the meson and diquark mass spectra determined at zero momentum.

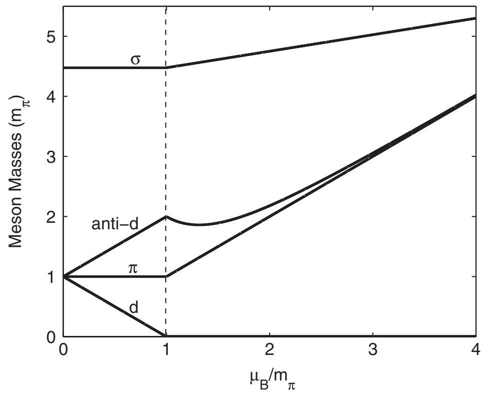

FIG. 8. The mass spectra of mesons and diquarks (in units of m_π) as functions of the baryon chemical potential (in units of m_π) for model parameter set 1. For other model parameter sets, the mass of the heaviest mode is changed but others are almost the same.

We find from the meson spectra that the chiral symmetry is approximately restored at $\mu_B \simeq 3m_\pi$, corresponding to $n \simeq 3.5 f_\pi^2 m_\pi$. It is interesting that near the quantum phase transition point $\mu_B = m_\pi$ the mixing between the sigma meson and diquarks is very strong and makes the sigma meson lost its way. Since it is continuous with the antidiquark mode in the normal phase, it is also called the "antidiquark" mode in the superfluid phase [52,83]. The "sigma meson," which is continuous with the sigma meson in the normal phase, is in fact the Higgs mode of the BCS superfluid with a mass 2Δ at high density.

Even though the deconfinement transition or crossover which corresponds to the gauge field sector cannot be described in the NJL model, we can on the other hand study the meson Mott transition associated with the chiral restoration [84–86]. The meson Mott transition is defined as the point where the meson energy becomes larger than the two-particle continuum $\omega_{\bar{q}q}$ for the decay process $\pi \to \bar{q}q$ at zero momentum, which means the mesons are no longer bound states. The two-particle continuum $\omega_{\bar{q}q}$ is different at the BEC and the BCS sides. From the explicit form of $\Pi_\pi(Q)$, we find that

$$\omega_{\bar{q}q} = \begin{cases} \sqrt{(M - \frac{\mu_B}{2})^2 + |\Delta|^2} + \sqrt{(M + \frac{\mu_B}{2})^2 + |\Delta|^2} & \mu_B < \mu_0 \\ |\Delta| + \sqrt{(M + \frac{\mu_B}{2})^2 + |\Delta|^2} & \mu_B > \mu_0 \end{cases}. \tag{99}$$

Thus, the pions and the sigma meson will undergo a Mott transition when their masses become larger than the two-particle continuum $\omega_{\bar{q}q}$, i.e., $\mu_B > \omega_{\bar{q}q}$. Using the mean-field results for Δ and M, we can calculate the two-particle continuum $\omega_{\bar{q}q}$ as a function of μ_B, which is shown in Fig. 9. We find that the Mott transition does occur at a chemical potential $\mu_B = \mu_{\rm M1}$ which is sensitive to the value of M_*. The values of $\mu_{\rm M1}$ for the four model parameter sets are shown in Table IV. For reasonable model

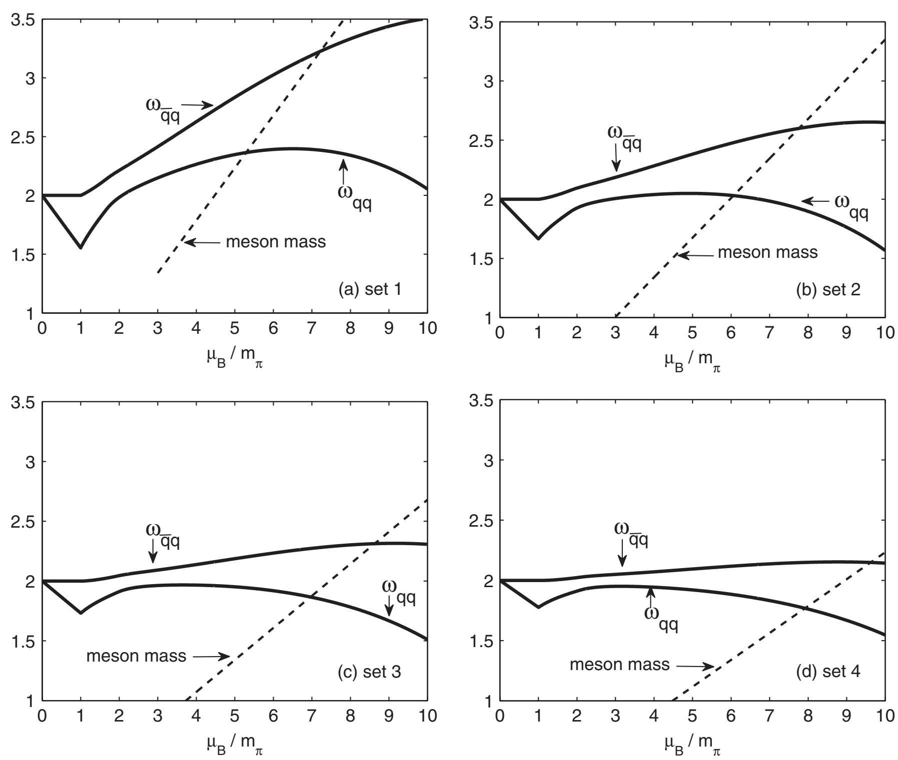

FIG. 9. The two-particle continua $\omega_{\bar{q}q}$ and ω_{qq} (in units of M_*) as functions of the baryon chemical potential (in units of m_π) for different model parameter sets. The degenerate mass of pions and sigma meson is shown by dashed line.

parameter sets, the value of μ_{M1} is in the range (7–10)m_π. Above this chemical potential, the mesons are no longer stable bound states and can decay into quark-antiquark pairs even at zero momentum. We note that the Mott transition takes place well above the chiral restoration, in contrast to the pure finite temperature case where the mesons are dissociated once the chiral symmetry is restored [84,85].

On the other hand, we find from the explicit forms of the meson propagators in Appendix B that the decay process $\pi \to qq$ is also possible at $\mathbf{q} \neq 0$ (even though $|\mathbf{q}|$ is small) due to the presence of superfluidity. Thus, we have another unusual Mott transition in the superfluid phase. Notice that this process is not in contradiction to the baryon number conservation law, since the $U_B(1)$ baryon number symmetry is spontaneously broken in the superfluid phase. Quantitatively, this transition occurs when the meson mass becomes larger than the two-particle continuum ω_{qq} for the decay process $\pi \to qq$ at $\mathbf{q} = 0^+$. In this case, we have

$$\omega_{qq} = \begin{cases} 2\sqrt{\left(M - \frac{\mu_B}{2}\right)^2 + |\Delta|^2} & \mu_B < \mu_0 \\ 2|\Delta| & \mu_B > \mu_0 \end{cases}. \tag{100}$$

The two-particle continuum ω_{qq} is also shown in Fig. 9. We find that the unusual Mott transition does occur at another chemical potential $\mu_B = \mu_{M2}$ which is also sensitive to the value of M_*. The values of μ_{M2} for the four model parameter sets are also shown in Table IV. For reasonable model parameter sets, this value is in the range (5–8)m_π. This process can also occur in the 2SC phase of quark matter in the $N_c = 3$ case [87]. In the 2SC phase, the symmetry breaking pattern is $SU_c(3) \otimes U_B(1) \to SU_c(2) \otimes \tilde{U}_B(1)$ where the generator of the residue baryon number symmetry $\tilde{U}_B(1)$ is $\tilde{B} = B - 2T_8/\sqrt{3} = \mathrm{diag}(0, 0, 1)$ corresponding to the unpaired blue quarks. Thus the baryon number symmetry for the paired red and green quarks are broken and our results can be applied. To show this explicitly, we write down the explicit form of the polarization function for pions in the 2SC phase [87]

$$\Pi_\pi^{2SC}(Q) = \Pi_\pi^{2-\mathrm{color}}(Q) + \sum_K \mathrm{Tr}[\mathcal{G}_0(K) i\gamma_5 \mathcal{G}_0(P) i\gamma_5], \tag{101}$$

where $\mathcal{G}_0(K)$ is the propagator for the unpaired blue quarks. Here $\Pi_\pi^{2-\mathrm{color}}(Q)$ is given by (29) (the effective quarks mass M and the pairing gap Δ should be given by the $N_c = 3$ case of course) and corresponds to the contribution from the paired red and green sectors. The second term is the contribution from the unpaired blue quarks. Therefore, the unusual decay process is only available for the paired quarks.

TABLE IV. The chemical potentials μ_{M1} and μ_{M2} (in units of m_π) for different model parameter sets.

Set	1	2	3	4
μ_{M1} $[m_\pi]$	7.22	7.76	8.63	9.62
μ_{M2} $[m_\pi]$	5.29	6.06	6.96	7.92

V. BEYOND-MEAN-FIELD CORRECTIONS

The investigations in Sec. III and IV are restricted in the mean-field approximation, even though the bosonic collective excitations are studied. In this section, we will include the Gaussian fluctuations in the thermodynamic potential, and thus really go beyond the mean field. The scheme of going beyond the mean field is somewhat like those done in the study of finite temperature thermodynamics of the NJL model [88,89]; however, in this paper we will focus on the beyond-mean-field corrections at zero temperature, i.e., the pure quantum fluctuations. We will first derive the thermodynamic potential beyond the mean field which is valid at arbitrary chemical potential and temperature, and then briefly discuss the beyond-mean-field corrections near the quantum phase transition. The numerical calculations are deferred for future studies.

A. Thermodynamic potential beyond the mean field

In the Gaussian approximation, the partition function can be expressed as

$$Z_{\mathrm{NJL}} \simeq \exp(-\mathcal{S}_{\mathrm{eff}}^{(0)}) \int [d\sigma][d\boldsymbol{\pi}][d\phi^\dagger][d\phi] \exp(-\mathcal{S}_{\mathrm{eff}}^{(2)}). \tag{102}$$

Integrating out the Gaussian fluctuations, we can express the total thermodynamic potential as

$$\Omega(T, \mu_B) = \Omega_0(T, \mu_B) + \Omega_{\mathrm{fl}}(T, \mu_B), \tag{103}$$

where the contribution from the Gaussian fluctuations can be written as

$$\Omega_{\mathrm{fl}} = \frac{1}{2}\sum_Q [\ln\det\mathbf{M}(Q) + \ln\det\mathbf{N}(Q)]. \tag{104}$$

However, there is a problem with the above expression, since it is actually ill-defined: the sum over the boson Matsubara frequency is divergent and we need appropriate convergent factors to make it meaningful. In the simpler case without superfluidity, the convergent factor is simply given by $e^{i\nu_m 0^+}$ [88,89]. In our case, the situation is somewhat different due to the introduction of the Nambu-Gor'kov spinors. Keep in mind that in the equal time limit, there are additional factors $e^{i\omega_n 0^+}$ for $\mathcal{G}_{11}(K)$ and $e^{-i\omega_n 0^+}$ for $\mathcal{G}_{22}(K)$. Therefore, to get the proper convergent factors for Ω_{fl}, we should keep these factors when we make the sum over the fermion Matsubara frequency ω_n in evaluating the polarization functions $\Pi_{ij}(Q)$ and $\Pi_\pi(Q)$.

The problem in the expression of $\Omega_{\rm fl}$ is thus from the opposite convergent factors for $\mathbf{M}_{11}$ and $\mathbf{M}_{22}$. From the above arguments, we find that there is a factor $e^{i\nu_m 0^+}$ for $\mathbf{M}_{11}$ and $e^{-i\nu_m 0^+}$ for $\mathbf{M}_{22}$. Keep in mind that the Matsubara sum $\sum_m$ is converted to a standard contour integral ($i\nu_m \to z$). The convergence for $z \to +\infty$ is automatically guaranteed by the Bose distribution function $b(z) = 1/(e^{\beta z} - 1)$, we thus should treat only the problem for $z \to -\infty$. To this end, we write the first term of $\Omega_{\rm fl}$ as

$$\sum_Q \ln\det\mathbf{M}(Q) = \sum_Q \bigg[\ln\mathbf{M}_{11} e^{i\nu_m 0^+} + \ln\mathbf{M}_{22} e^{-i\nu_m 0^+} + \ln\bigg(\frac{\det\mathbf{M}}{\mathbf{M}_{11}\mathbf{M}_{22}}\bigg) e^{i\nu_m 0^+} \bigg]. \quad (105)$$

Using the fact that $\mathbf{M}_{22}(Q) = \mathbf{M}_{11}(-Q)$, we obtain

$$\sum_Q \ln\det\mathbf{M}(Q) = \sum_Q \ln\bigg[\frac{\mathbf{M}_{11}(Q)}{\mathbf{M}_{22}(Q)} \det\mathbf{M}(Q)\bigg] e^{i\nu_m 0^+}. \quad (106)$$

Therefore, the well-defined form of $\Omega_{\rm fl}$ is given by the above formula together with the other term $\sum_Q \ln\det\mathbf{N}(Q)$ associated with a factor $e^{i\nu_m 0^+}$.

The Matsubara sum can be written as the contour integral via the theorem $\sum_m g(i\nu_m) = \oint_{\rm C} dz/(2\pi i) b(z) g(z)$, where C runs on either side of the imaginary z axis, enclosing it counterclockwise. Distorting the contour to run above and below the real axis, we obtain

$$\Omega_{\rm fl} = \sum_{\mathbf{q}} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} b(\omega)[\delta_{\rm M}(\omega, \mathbf{q}) + \delta_{11}(\omega, \mathbf{q}) - \delta_{22}(\omega, \mathbf{q}) + 3\delta_\pi(\omega, \mathbf{q})], \quad (107)$$

where the scattering phases are defined as

$$\begin{aligned} \delta_{\rm M}(\omega, \mathbf{q}) &= {\rm Im}\ln\det\mathbf{M}(\omega + i0^+, \mathbf{q}), \\ \delta_{11}(\omega, \mathbf{q}) &= {\rm Im}\ln\mathbf{M}_{11}(\omega + i0^+, \mathbf{q}), \\ \delta_{22}(\omega, \mathbf{q}) &= {\rm Im}\ln\mathbf{M}_{22}(\omega + i0^+, \mathbf{q}), \\ \delta_\pi(\omega, \mathbf{q}) &= {\rm Im}\ln[(2G)^{-1} + \Pi_\pi(\omega + i0^+, \mathbf{q})]. \end{aligned} \quad (108)$$

Keep in mind the pressure of the vacuum should be zero, the physical thermodynamic potential at finite temperature and chemical potential should be defined as

$$\Omega_{\rm phy}(T, \mu_B) = \Omega(T, \mu_B) - \Omega(0, 0). \quad (109)$$

B. Thermodynamic consistency of the vacuum

As we have shown in the mean-field theory, at $T = 0$, the vacuum state is restricted in the region $|\mu_B| < m_\pi$. In this region, all thermodynamic quantities should keep zero, no matter how large the value of μ_B is. While this should be an obvious physical conclusion, it is important to check whether our beyond-mean-field theory satisfies this condition.

Notice that the physical thermodynamic potential is defined as $\Omega_{\rm phy}(\mu_B) = \Omega(\mu_B) - \Omega(0)$, we therefore should prove that the thermodynamic potential $\Omega(\mu_B)$ keeps a constant in the region $|\mu_B| < m_\pi$. For the mean-field part Ω_0, the proof is quite easy. Because of the fact that $M_* > m_\pi/2$, the solution for M is always given by $M = M_*$. Thus Ω_0 keeps its value at $\mu_B = 0$ in the region $|\mu_B| < m_\pi$.

Now we turn to the complicated part $\Omega_{\rm fl}$. Since $\Delta = 0$, all the off-diagonal elements of $\mathbf{M}$ vanishes, and $\Omega_{\rm fl}$ is reduced to

$$\Omega_{\rm fl} = \frac{1}{2}\sum_Q \ln\bigg[\frac{1}{2G} + \Pi_\sigma(Q)\bigg] e^{i\nu_m 0^+} + \frac{3}{2}\sum_Q \ln\bigg[\frac{1}{2G} + \Pi_\pi(Q)\bigg] e^{i\nu_m 0^+} + \sum_Q \ln\bigg[\frac{1}{4G} + \Pi_d(Q)\bigg] e^{i\nu_m 0^+}, \quad (110)$$

where $\Pi_\sigma(Q) = \Pi_{33}(Q)$, and we should set $\Delta = 0$ and $M = M_*$ in evaluating the polarization functions. First, we can easily show that the contributions from the sigma meson and pions do not have explicit μ_B dependence and thus keep the same values as those at $\mu_B = 0$. In fact, since the effective quark mass M keeps its vacuum value M_* guaranteed by the mean-field part, all the μ_B dependence in $\Pi_{\sigma,\pi}(Q)$ is included in the Fermi distribution functions $f(E \pm \mu_B/2)$. Since $M_* > \mu_B/2$, they vanish automatically at $T = 0$. In fact, from the explicit expressions for $\Pi_{\sigma,\pi}(Q)$ in Appendix B, we can check that there is no μ_B independence in $\Pi_{\sigma,\pi}(Q)$.

The diquark contribution, however, has an explicit μ_B dependence through the combination $i\nu_m + \mu_B$ in the polarization function $\Pi_d(Q)$. The diquark contribution (at $T = 0$) can be written as

$$\begin{aligned} \Omega_d &= -\sum_{\mathbf{q}} \int_{-\infty}^{0} \frac{d\omega}{\pi} \delta_d(\omega, \mathbf{q}), \\ \delta_d(\omega, \mathbf{q}) &= {\rm Im}\ln[(4G)^{-1} + \Pi_d(\omega + i0^+, \mathbf{q})]. \end{aligned} \quad (111)$$

Making a shift $\omega \to \omega - \mu_B$, and noticing that fact $\Pi_d(\omega - \mu_B, \mathbf{q}) = \Pi_\pi(\omega, \mathbf{q})/2$, we obtain

$$\Omega_d = -\sum_{\mathbf{q}} \int_{-\infty}^{-\mu_B} \frac{d\omega}{\pi} \delta_\pi(\omega, \mathbf{q}). \quad (112)$$

To show the above quantity is in fact μ_B independent, we separate it into a pole part and a continuum part. There is a well-defined two-particle continuum $E_c(\mathbf{q})$ for pions at arbitrary momentum $\mathbf{q}$,

$$E_c(\mathbf{q}) = \min_{\mathbf{k}}(E^*_{\mathbf{k}} + E^*_{\mathbf{k}+\mathbf{q}}). \quad (113)$$

The pion propagator has two symmetric poles $\pm\omega_\pi(\mathbf{q})$ when $\mathbf{q}$ satisfies $\omega_\pi(\mathbf{q}) < E_c(\mathbf{q})$. Thus in the region $|\omega| < E_c(\mathbf{q})$, the scattering phase δ_π can be analytically evaluated as

$$\delta_\pi(\omega, \mathbf{q}) = \pi[\Theta(-\omega - \omega_\pi(\mathbf{q})) - \Theta(\omega - \omega_\pi(\mathbf{q}))]. \tag{114}$$

Since $E_c(\mathbf{q}) > \omega_\pi(\mathbf{q}) > m_\pi > \mu_B$, the thermodynamic potential Ω_d can be separated as

$$\Omega_d = \sum_{\mathbf{q}}[\omega_\pi(\mathbf{q}) - E_c(\mathbf{q})] - \sum_{\mathbf{q}}\int_{-\infty}^{-E_c(\mathbf{q})}\frac{d\omega}{\pi}\delta_\pi(\omega, \mathbf{q}), \tag{115}$$

which is indeed μ_B independent. Notice that in the first term the integral over $\mathbf{q}$ is restricted in the region $|\mathbf{q}| < q_c$ where q_c is defined as $\omega_\pi(q_c) = E_c(q_c)$.

In conclusion, we have shown that the thermodynamic potential Ω in the Gaussian approximation keeps a constant in the vacuum state, i.e., at $|\mu_B| < m_\pi$ and at $T = 0$. All other thermodynamic quantities such as the baryon number density keep zero in the vacuum. The subtraction term $\Omega(0, 0)$ in the Gaussian approximation can be expressed as

$$\Omega(0,0) = \Omega_{\mathrm{vac}}(M_*) + \frac{5}{2}\sum_{\mathbf{q}}[\omega_\pi(\mathbf{q}) - E_c(\mathbf{q})] - \sum_{\mathbf{q}}\int_{-\infty}^{-E_c(\mathbf{q})}\frac{d\omega}{2\pi}[\delta_\sigma(\omega, \mathbf{q}) + 5\delta_\pi(\omega, \mathbf{q})]. \tag{116}$$

C. Quantum corrections near the phase transition

Now we consider the beyond-mean-field corrections near the quantum phase transition point $\mu_B = m_\pi$. Notice that the effective quark mass M and the diquark condensate Δ are determined at the mean-field level, and the beyond-mean-field corrections are possible only through the equations of state.

Formally, the Gaussian contribution to the thermodynamic potential Ω_{fl} is a function of μ_B, M and $y = |\Delta|^2$, i.e., $\Omega_{\mathrm{fl}} = \Omega_{\mathrm{fl}}(\mu_B, y, M)$. In the superfluid phase, the total baryon density including the Gaussian contribution can be evaluated as

$$n(\mu_B) = n_0(\mu_B) + n_{\mathrm{fl}}(\mu_B), \tag{117}$$

where the mean-field part is simply given by $n_0(\mu_B) = -\partial\Omega_0/\partial\mu_B$ and the Gaussian contribution can be expressed as

$$n_{\mathrm{fl}}(\mu_B) = -\frac{\partial\Omega_{\mathrm{fl}}}{\partial\mu_B} - \frac{\partial\Omega_{\mathrm{fl}}}{\partial y}\frac{dy}{d\mu_B} - \frac{\partial\Omega_{\mathrm{fl}}}{\partial M}\frac{dM}{d\mu_B}. \tag{118}$$

The physical values of M and $|\Delta|^2$ should be determined by their mean-field gap equations. In fact, from the gap equations $\partial\Omega_0/\partial M = 0$ and $\partial\Omega_0/\partial y = 0$, we obtain

$$\begin{aligned}&\frac{\partial^2\Omega_0}{\partial\mu_B\partial M} + \frac{\partial^2\Omega_0}{\partial y\partial M}\frac{dy}{d\mu_B} + \frac{\partial^2\Omega_0}{\partial M^2}\frac{dM}{d\mu_B} = 0,\\ &\frac{\partial^2\Omega_0}{\partial\mu_B\partial y} + \frac{\partial^2\Omega_0}{\partial y^2}\frac{dy}{d\mu_B} + \frac{\partial^2\Omega_0}{\partial M\partial y}\frac{dM}{d\mu_B} = 0.\end{aligned} \tag{119}$$

Thus, we can obtain the derivatives $dM/d\mu_B$ and $dy/d\mu_B$ analytically. Finally, $n_{\mathrm{fl}}(\mu_B)$ is a continuous function of μ_B guaranteed by the properties of second order phase transition, and we have $n_{\mathrm{fl}}(m_\pi) = 0$.

Next we focus on the beyond-mean-field corrections near the quantum phase transition. Since the diquark condensate Δ is vanishingly small, we can expand the Gaussian part Ω_{fl} in powers of $|\Delta|^2$. Notice that μ_B and M can be evaluated as functions of $|\Delta|^2$ from the Ginzburg-Landau potential and mean-field gap equations. Thus, to order $O(|\Delta|^2)$, the expansion takes the form

$$\Omega_{\mathrm{fl}} \simeq \eta|\Delta|^2, \tag{120}$$

where the expansion coefficient η is defined as

$$\eta = \left(\frac{\partial\Omega_{\mathrm{fl}}}{\partial y} + \frac{\partial\Omega_{\mathrm{fl}}}{\partial\mu_B}\frac{d\mu_B}{dy} + \frac{\partial\Omega_{\mathrm{fl}}}{\partial M}\frac{dM}{dy}\right)\Bigg|_{\mu_B=m_\pi, y=0, M=M_*}. \tag{121}$$

Using the definition of n_{fl}, we find that η can be related to n_{fl} by

$$\eta = n_{\mathrm{fl}}(m_\pi)\frac{d\mu_B}{dy}\Bigg|_{y=0}. \tag{122}$$

Thus, the coefficient η vanishes, and the leading order of the expansion should be $O(|\Delta|^4)$.

As shown above, to leading order, the expansion of Ω_{fl} can be formally expressed as

$$\Omega_{\mathrm{fl}} \simeq -\frac{\zeta}{2}\beta|\Delta|^4. \tag{123}$$

The method to derive the exact expression of the numerical factor ζ is shown in Appendix C. Notice that the factor ζ is in fact μ_B independent, thus the total baryon density to leading order is

$$n = n_0 + \zeta\beta|\Delta|^2\frac{d|\Delta|^2}{d\mu_B}\Bigg|_{\mu_B=m_\pi}. \tag{124}$$

Near the quantum phase transition point, the mean-field contribution is $n_0 = |\Psi_0|^2 = 2m_\pi\mathcal{J}|\Delta|^2$ from the Gross-Pitaevskii free energy. The last term can be evaluated using the analytical result

$$|\Psi_0|^2 = \frac{\mu_d}{g_0} \Rightarrow |\Delta|^2 = \frac{2m_\pi\mathcal{J}}{\beta}\mu_d, \tag{125}$$

which is in fact the solution of the mean-field gap equations. Therefore, to leading order, the total baryon density reads

$$n = (1+\zeta)2m_\pi\mathcal{J}|\Delta|^2. \tag{126}$$

On the other hand, the total pressure P can be expressed as

$$P = (1+\zeta)\frac{\beta}{2}|\Delta|^4. \tag{127}$$

Thus we find that the leading order quantum corrections are totally included in the numerical factor ζ. Setting $\zeta = 0$, we recover the mean-field results obtained in Sec. III.

Including the quantum fluctuations, the equations of state shown in (65) are modified to be

$$P(n) = \frac{1}{1+\zeta}\frac{2\pi a_{\rm dd}}{m_\pi}n^2, \qquad \mu_B(n) = m_\pi + \frac{1}{1+\zeta}\frac{4\pi a_{\rm dd}}{m_\pi}n. \tag{128}$$

This means, to leading order, the effect of quantum fluctuations is giving a correction to the diquark-diquark scattering length. The renormalized scattering length is

$$a'_{\rm dd} = \frac{a_{\rm dd}}{1+\zeta}. \tag{129}$$

Generally, we have $\zeta > 0$ and the renormalized scattering length is smaller than the mean-field result.

An exact calculation of the numerical factor ζ can be performed using the method shown in Appendix C. However, this needs huge numerical power and we defer it to future work. In this paper we will give an analytical estimation of ζ based on the fact that the quantum fluctuations are dominated by the gapless Goldstone mode. To this end, we approximate the Gaussian contribution $\Omega_{\rm fl}$ as

$$\Omega_{\rm fl} \simeq \frac{1}{2}\sum_Q \ln[\mathcal{D}_{\rm d}^{-1}(Q)\mathcal{D}_d^{-1}(-Q) + 3\beta^2|\Delta|^4 + 2\beta|\Delta|^2(\mathcal{D}_d^{-1}(Q) + \mathcal{D}_d^{-1}(-Q))], \tag{130}$$

where $\mathcal{D}_d^{-1}(Q)$ is given by (54) and can be approximated by (56). Subtracting the value of $\Omega_{\rm fl}$ at $\mu_B = m_\pi$ with $\Delta = 0$, and using the result $\mu_B = m_\pi + g_0|\Psi_0|^2$ from the Gross-Pitaevskii equation, we find that ζ can be evaluated as

$$\zeta = \frac{\beta}{\mathcal{J}^2}(I_1 + I_2) \simeq \frac{m_\pi^2}{f_\pi^2}(I_1 + I_2), \tag{131}$$

where the numerical factors I_1 and I_2 are given by

$$I_1 = \frac{1}{2}\sum_m\sum_{\mathbf{X}}\frac{Z_m^2 + \mathbf{X}^2}{(Z_m^2 - \mathbf{X}^2)^2 - 4Z_m^2}, \qquad I_2 = 4\sum_m\sum_{\mathbf{X}}\frac{(3Z_m^2 - \mathbf{X}^2)^2}{[(Z_m^2 - \mathbf{X}^2)^2 - 4Z_m^2]^2}. \tag{132}$$

Here the dimensionless notations Z_m and $\mathbf{X}$ are defined as $Z_m = i\nu_m/m_\pi$ and $\mathbf{X} = \mathbf{q}/m_\pi$ respectively. Notice that the integral over $\mathbf{X}$ is divergent and hence such an estimation has no prediction power due to the fact that the NJL model is nonrenormalizable. However, regardless of the numerical factor $I_1 + I_2$, we find that $\zeta \propto m_\pi^2/f_\pi^2$. Thus, the correction should be small in the nonlinear sigma model limit $m_\pi \ll 2M_*$.

D. Transition temperature

While the effect of the Gaussian fluctuations at zero temperature is to give a small correction to the diquark-diquark scattering length and the equations of state, it can be significant at finite temperature. In fact, as the temperature approaches the critical value of superfluidity, the Gaussian fluctuations should dominate. In this part, we will show that to get a correct critical temperature in terms of the baryon density n, we must go beyond the mean field. The situation is analogous to the Nozieres–Schmitt-Rink treatment of molecular condensation in strongly interacting Fermi gases [75–78].

The transition temperature T_c is determined by the Thouless criterion $\mathcal{D}_d^{-1}(0, \mathbf{0}) = 0$ which can be shown to be consistent with the saddle point condition $\delta\mathcal{S}_{\rm eff}/\delta\phi|_{\phi=0} = 0$. Its explicit form is a BCS-type gap equation

$$\frac{1}{4G} = N_cN_f\sum_{e=\pm}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1-2f(\xi_{\mathbf{k}}^e)}{2\xi_{\mathbf{k}}^e}. \tag{133}$$

Meanwhile, the dynamic quark mass M satisfies the mean-field gap equation

$$\frac{M-m_0}{2GM} = N_cN_f\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1-f(\xi_{\mathbf{k}}^-)-f(\xi_{\mathbf{k}}^+)}{E_{\mathbf{k}}}. \tag{134}$$

To obtain the transition temperature as a function of n, we need the so-called number equation given by $n = -\partial\Omega/\partial\mu_B$, which includes both the mean-field contribution $n_0(\mu_B, T) = 2N_f\sum_{\mathbf{k}}[f(\xi_{\mathbf{k}}^-) - f(\xi_{\mathbf{k}}^+)]$ and the Gaussian contribution $n_{\rm fl}(\mu_B, T) = -\partial\Omega_{\rm fl}/\partial\mu_B$. At the transition temperature where $\Delta = 0$, $\Omega_{\rm fl}$ can be expressed as

$$\Omega_{\rm fl} = \int\frac{d^3\mathbf{q}}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}b(\omega)\times[2\delta_d(\omega,\mathbf{q}) + \delta_\sigma(\omega,\mathbf{q}) + 3\delta_\pi(\omega,\mathbf{q})], \tag{135}$$

where the scattering phases are defined as $\delta_d(\omega,\mathbf{q}) = \mathrm{Im}\ln[1/(4G) + \Pi_d(\omega + i0^+, \mathbf{q})]$ for the diquarks, $\delta_\sigma(\omega,\mathbf{q}) = \mathrm{Im}\ln[1/(2G) + \Pi_\sigma(\omega + i0^+, \mathbf{q})]$ for the sigma meson, and $\delta_\pi(\omega,\mathbf{q}) = \mathrm{Im}\ln[1/(2G) + \Pi_\pi(\omega + i0^+, \mathbf{q})]$ for the pions. Obviously, the polarization functions should take their forms at finite temperature in the normal phase.

The transition temperature T_c at arbitrary baryon number density n can be determined numerically via solving simultaneously the gap and number equations. However, in the dilute limit $n \to 0$ which we are interested in this section, analytical result can be achieved. Keep in mind that $T_c \to 0$ when $n \to 0$, we find that the Fermi distribution functions $f(\xi_{\mathbf{k}}^\pm)$ vanish exponentially (since $M_* - m_\pi/2 \gg T_c$) and we obtain $\mu_B = m_\pi$ and $M = M_*$ from the gap Eqs. (133) and (134), respectively. Meanwhile the mean-field contribution of the density n_0 can be neglected and the total density n is thus dominated by the Gaussian part $n_{\rm fl}$. When $T \to 0$ we can show that $\Pi_\sigma(\omega,\mathbf{q})$ and $\Pi_\pi(\omega,\mathbf{q})$ are independent of μ_B, and the number equation is reduced to

$$n = -\sum_{\mathbf{q}} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} b(\omega) \frac{\partial \delta_d(\omega, \mathbf{q})}{\partial \mu_B}. \tag{136}$$

Since $T_c \to 0$, the inverse diquark propagator can be reduced to $\mathcal{D}_d^{-1}(\omega, \mathbf{q})$ in (54). Thus the scattering phase δ_d can be well approximated by $\delta_d(\omega, \mathbf{q}) = \pi[\Theta(\mu_B - \epsilon_{\mathbf{q}} - \omega) - \Theta(\omega - \mu_B - \epsilon_{\mathbf{q}})]$ with $\epsilon_{\mathbf{q}} = \sqrt{\mathbf{q}^2 + m_\pi^2}$. Therefore, the number equation can be further reduced to the well-known equation for ideal Bose-Einstein condensation,

$$n = \sum_{\mathbf{q}} [b(\epsilon_{\mathbf{q}} - \mu_B) - b(\epsilon_{\mathbf{q}} + \mu_B)]|_{\mu_B = m_\pi}. \tag{137}$$

Since the above equation is valid only in the low density limit $n \to 0$, the critical temperature is thus given by the nonrelativistic result

$$T_c = \frac{2\pi}{m_\pi} \left[\frac{n}{\xi(3/2)} \right]^{2/3}. \tag{138}$$

At finite density but $na_{\rm dd}^3 \ll 1$, there exists a correction to T_c which is proportional to $n^{1/3} a_{\rm dd}$ [13]. Such a correction is hard to handle analytically in our model since we should consider simultaneously the corrections to M and μ_B, as well as the contribution from the sigma meson and pions.

VI. SUMMARY

In summary, we have examined the NJL model description of weakly interacting Bose condensate and BEC-BCS crossover in QCD-like theories at finite baryon density. Our main conclusions are as follows:

(1) Near the quantum phase transition point $\mu_B = m_\pi$, we have performed a Ginzburg-Landau expansion of the effective potential. At the mean-field level, the Ginzburg-Landau free energy is essentially the Gross-Pitaevskii free energy describing weakly repulsive Bose condensates after a proper redefinition of the condensate wave function. The obtained diquark-diquark scattering length reads $a_{\rm dd} = m_\pi/(16\pi f_\pi^2)$, which recovers the tree-level result predicted by chiral Lagrangian.

(2) We have analytically shown that the Goldstone mode near the quantum phase transition point takes the same dispersion as the Bogoliubov excitation in weakly interacting Bose condensates, which gives a diquark-diquark scattering length identical to that in the Gross-Pitaevskii free energy. The mixing between the sigma meson and the diquarks plays an important role in recovering the Bogoliubov dispersion.

(3) The results of baryon number density and in-medium chiral and diquark condensates predicted by chiral perturbation theory are analytically recovered near the quantum phase transition point in the NJL model.

(4) At high density, the superfluid matter undergoes a BEC-BCS crossover at $\mu_B \simeq (m_\sigma/m_\pi)^{1/3} m_\pi \simeq (1.6\text{–}2)m_\pi$. At $\mu_B \simeq 3m_\pi$, the chiral symmetry is approximated restored and the spectra of pions and sigma meson become nearly degenerate. Well above the chemical potential of chiral symmetry restoration, the degenerate pions and sigma meson undergo a Mott transition, where they become unstable resonances. Because of the spontaneous breaking of baryon number symmetry, mesons can decay into quark pairs in the superfluid medium at nonzero momentum.

(5) The general theoretical framework of the thermodynamics beyond the mean field is established. It is shown that the vacuum state in the region $|\mu_B| < m_\pi$ is thermodynamically consistent in the Gaussian approximation, i.e., all thermodynamic quantities keep vanishing for $|\mu_B| < m_\pi$ even though the Gaussian fluctuations are included.

(6) Near the quantum phase transition point, we find that the effect of the leading order beyond-mean-field correction is to renormalize the diquark-diquark scattering length. The correction to the mean-field result is estimated to be proportional to m_π^2/f_π^2. Our theoretical approach provides a new way to calculate the diquark-diquark or meson-meson scattering lengths in the NJL model beyond-mean-field approximation. We also find that we can obtain a correct transition temperature of Bose condensation in the dilute limit once the beyond-mean-field corrections are included.

Our studies can be generalized to describe pion condensation at finite isospin chemical potential μ_I [4,5] and kaon condensation at finite strangeness chemical potential $\mu_{\rm S}$ [90]. In the NJL model, pion condensation is shown to occur at $|\mu_I| = m_\pi$ when $|\mu_{\rm S}| < m_{\rm K} - m_\pi/2$, and kaon is shown to condense at $|\mu_{\rm S}| = m_{\rm K}$ when $\mu_I \to 0$ [46,48]. The generalization to pion condensation is straightforward. The obtained Ginzburg-Landau and Gross-Pitaevskii free energies are the same as those derived in this paper, if we replace $\mu_B \to \mu_I$. The results are valid both for $N_c = 2$ and $N_c = 3$ cases. At the mean-field level, the results for diquark condensation at $N_c = 2$ and pion condensation at $N_c = 3$ are formally identical in the NJL model. Significant difference may appear if we consider the beyond-mean-field corrections, since for the $N_c = 3$ case the scalar diquarks are not pseudo-Goldstone bosons. A calculation of the pion-pion scattering length in the $I = 2$ channel can be performed within our theoretical framework. The calculations of kaon condensation and kaon-kaon scattering length are also possible, but somewhat complicated due to the large mass difference between the light and strange quarks.

We are also interested in how the beyond-mean-field corrections modify the superfluid equations of state. As we learn from the knowledge of BEC-BCS crossover in cold Fermi gases, the superfluid equations of state can be strongly modified in the crossover regime [57,58], corresponding to the moderate baryon density in our case. This issue is also important to the color-superconducting quark matter [91–93] at moderate density, i.e., for quark chemical potential around 400 MeV where the pairing gap can be of order $O(100\ \text{MeV})$. The numerical works are in progress.

ACKNOWLEDGMENTS

The work is supported by the Alexander von Humboldt Foundation. L. He would like to thank T. Brauner for valuable discussions and critical reading of the manuscript, and D. Rischke for encouragement during the work.

APPENDIX A: FERMIONIC MODEL DESCRIPTION OF DILUTE BOSE CONDENSATE

In this appendix, we briefly review the theory of molecular Bose condensation in two-component Fermi gases in the strong coupling limit. While there exist many theoretical approaches [94–96] to deal with this problem, we employ the field theoretical approach [57,58] parallel to that used in this paper.

The Lagrangian density of the system can be written as

$$\mathcal{L} = \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger\left(i\partial_t + \frac{\nabla^2}{2m} + \mu\right)\psi_\sigma + g\psi_\uparrow^\dagger\psi_\downarrow^\dagger\psi_\downarrow\psi_\uparrow, \quad \text{(A1)}$$

where $\psi_{\uparrow,\downarrow}$ denote the two-component (nonrelativistic) fermion fields with equal masses m and chemical potentials μ. The gas is assumed to be dilute, and the coupling constant g can be related to the s-wave fermion-fermion scattering length a_s as

$$\frac{m}{4\pi a_s} = -\frac{1}{g(\Lambda)} + \sum_{|\mathbf{k}|<\Lambda}\frac{1}{2\epsilon_\mathbf{k}}, \quad \text{(A2)}$$

where $\epsilon_\mathbf{k} = \mathbf{k}^2/(2m)$. In the dilute limit, we can take the limit $\Lambda \to \infty$ in the final result.

Performing the Hubbard-Stratonovich transformation with the auxiliary boson field $\phi(x) = g\psi_\downarrow(x)\psi_\uparrow(x)$, and defining the Nambu-Gor'kov representation $\Psi^\dagger = (\psi_\uparrow^\dagger, \psi_\downarrow)$, we can evaluate the partition function of the system as $Z = \int[d\Psi^\dagger][d\Psi][d\phi^\dagger][d\phi]\exp(-\mathcal{A}_{\text{eff}})$, where

$$\mathcal{A}_{\text{eff}} = \int dx \frac{|\phi(x)|^2}{g} - \int dx \int dx' \Psi^\dagger(x)\mathbf{G}^{-1}(x,x')\Psi(x), \quad \text{(A3)}$$

and the inverse fermion propagator $\mathbf{G}^{-1}$ is given by

$$\begin{pmatrix} -\partial_\tau + \frac{\nabla^2}{2m} + \mu & \phi(x) \\ \phi^\dagger(x) & -\partial_\tau - \frac{\nabla^2}{2m} - \mu \end{pmatrix}\delta(x-x'). \quad \text{(A4)}$$

Then integrating out the fermionic degree of freedom, we get $Z = \int[d\phi^\dagger][d\phi]\exp(-\mathcal{S}_{\text{eff}})$ where the bosonized effective action reads

$$\mathcal{S}_{\text{eff}}[\phi^\dagger, \phi] = \int dx \frac{|\phi(x)|^2}{g} - \text{Tr}\ln\mathbf{G}^{-1}(x,x'). \quad \text{(A5)}$$

1. Mean-field theory

First, we consider the mean-field theory where the auxiliary boson field $\phi(x)$ is replaced by its expectation value $\langle\phi(x)\rangle = \Delta(x)$. In the strong coupling limit $a_s \to 0^+$, the fermion chemical potential μ approaches $-E_b/2$ with $E_b = 1/(ma_s^2)$ being the molecular binding energy. Since the pairing gap $|\Delta| \ll |\mu|$, we can expand the effective action in powers of $|\Delta|$, which resulting in a Ginzburg-Landau free energy functional

$$V_{\text{GL}}[\Delta(x)] = \int dx\left[\Delta^\dagger(x)\left(\kappa\frac{\partial}{\partial\tau} - \gamma\nabla^2\right)\Delta(x) + \alpha|\Delta(x)|^2 + \frac{1}{2}\beta|\Delta(x)|^4\right]. \quad \text{(A6)}$$

The coefficients α, β of the potential terms can be obtained from the mean-field thermodynamic potential $\Omega_0 = (T/V)\mathcal{S}_{\text{eff}}[\Delta^\dagger, \Delta]$ which can be evaluated as

$$\Omega_0 = -\frac{m}{4\pi a_s}|\Delta|^2 - \sum_\mathbf{k}\left(E_\mathbf{k} - \xi_\mathbf{k} - \frac{|\Delta|^2}{2\epsilon_\mathbf{k}}\right), \quad \text{(A7)}$$

where $\xi_\mathbf{k} = \epsilon_\mathbf{k} - \mu$ and $E_\mathbf{k} = \sqrt{\xi_\mathbf{k}^2 + |\Delta|^2}$. After a simple algebra, the coefficients α and β can be evaluated as

$$\alpha = \frac{m}{4\pi}\left(\sqrt{-2m\mu} - \frac{1}{a_s}\right), \qquad \beta = \frac{m^3}{8\pi}\frac{1}{(-2m\mu)^{3/2}}. \quad \text{(A8)}$$

From the expression of α, we find that a quantum phase transition from vacuum to Bose condensation takes place at $\mu = -1/(2ma_s^2) = -E_b/2$. Thus, near the phase transition, α can be simplified as

$$\alpha \simeq -\frac{m^2 a_s}{8\pi}\mu_b, \quad \text{(A9)}$$

where $\mu_b = 2\mu + E_b$ is the boson chemical potential. Further, setting $\mu = -E_b/2$, β can be simplified as

$$\beta \simeq -\frac{m^3 a_s^3}{16\pi}. \quad \text{(A10)}$$

The coefficients γ, κ of the kinetic terms can be obtained from the inverse boson propagator $\mathcal{D}^{-1}(Q)$ with $\Delta = 0$. It can be evaluated as

$$\mathcal{D}^{-1}(Q) = -\frac{m}{4\pi a_s} + \sum_\mathbf{k}\left(\frac{1}{i\nu_m - \xi_\mathbf{k} - \xi_{\mathbf{k}+\mathbf{q}}} + \frac{1}{2\epsilon_\mathbf{k}}\right). \quad \text{(A11)}$$

In the strong coupling limit, it can be well approximated as [94]

$$\mathcal{D}^{-1}(Q) \simeq -\frac{m^2 a_s}{8\pi}\left(i\nu_m - \frac{\mathbf{q}^2}{4m}\right). \quad \text{(A12)}$$

In summary, if we define the new condensate wave function $\Psi(x)$ by

$$\Psi(x) = \sqrt{\frac{m^2 a_s}{8\pi}}\Delta(x), \tag{A13}$$

the Ginzburg-Landau free energy can be reduced to the Gross-Pitaevskii free energy of dilute Bose gases,

$$V_{\rm GP}[\Psi(x)] = \int dx\left[\Psi^\dagger(x)\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m_b}\right)\Psi(x) - \mu_b|\Psi(x)|^2 + \frac{1}{2}\frac{4\pi a_{\rm bb}}{m_b}|\Psi(x)|^4\right], \tag{A14}$$

where $m_b = 2m$ is the boson mass and $a_{\rm bb} = 2a_s$ is the boson-boson scattering length. Since $a_s \to 0^+$, the interactions among the composite bosons are repulsive and weak.

2. Beyond-mean-field corrections

To study the beyond-mean-field corrections, we consider the fluctuations around the mean field. Making the field shift $\phi(x) \to \Delta + \phi(x)$, we can expand the effective action $\mathcal{S}_{\rm eff}$ in powers of the fluctuations. The zeroth order term $\mathcal{S}_{\rm eff}^{(0)}$ is just the mean-field result, and the linear terms vanish automatically guaranteed by the saddle point condition for Δ. The quadratic terms, corresponding to Gaussian fluctuations, can be evaluated as

$$\mathcal{S}_{\rm eff}^{(2)} = \frac{1}{2}\sum_Q (\phi^\dagger(Q)\phi(-Q))\mathbf{M}(Q)\begin{pmatrix}\phi(Q)\\ \phi^\dagger(-Q)\end{pmatrix}, \tag{A15}$$

where the inverse boson propagator $\mathbf{M}$ is given by

$$\mathbf{M}_{11}(Q) = \mathbf{M}_{22}(-Q) = \frac{1}{g} + \sum_K \mathcal{G}_{22}(K)\mathcal{G}_{11}(K+Q) = \frac{1}{g} + \sum_{\mathbf{k}}\left(\frac{u_{\mathbf{k}}^2 u_{\mathbf{k}+\mathbf{q}}^2}{i\nu_m - E_{\mathbf{k}} - E_{\mathbf{k}+\mathbf{q}}} - \frac{\upsilon_{\mathbf{k}}^2 \upsilon_{\mathbf{k}+\mathbf{q}}^2}{i\nu_m + E_{\mathbf{k}} + E_{\mathbf{k}+\mathbf{q}}}\right) \tag{A16}$$

and

$$\mathbf{M}_{12}(Q) = \mathbf{M}_{21}(Q) = \sum_K \mathcal{G}_{12}(K)\mathcal{G}_{21}(K+Q) = \sum_{\mathbf{k}}\left(\frac{u_{\mathbf{k}}\upsilon_{\mathbf{k}}u_{\mathbf{k}+\mathbf{q}}\upsilon_{\mathbf{k}+\mathbf{q}}}{i\nu_m + E_{\mathbf{k}} + E_{\mathbf{k}+\mathbf{q}}} - \frac{u_{\mathbf{k}}\upsilon_{\mathbf{k}}u_{\mathbf{k}+\mathbf{q}}\upsilon_{\mathbf{k}+\mathbf{q}}}{i\nu_m - E_{\mathbf{k}} - E_{\mathbf{k}+\mathbf{q}}}\right). \tag{A17}$$

Here the fermion Green function $\mathcal{G}$ is defined as $\mathcal{G}^{-1} = \mathbf{G}^{-1}[\Delta]$ and the BCS distribution functions $\upsilon_{\mathbf{k}}^2 = (1 - \xi_{\mathbf{k}}/E_{\mathbf{k}})/2$ and $u_{\mathbf{k}}^2 = 1 - \upsilon_{\mathbf{k}}^2$ are used.

In the strong coupling limit where $|\Delta|/|\mu| \ll 1$, the matrix elements of $\mathbf{M}$ can be analytically evaluated. We have [57]

$$\mathbf{M}(Q) \simeq \frac{m^2 a_s}{8\pi}\times\begin{pmatrix}-\omega + \frac{\mathbf{q}^2}{2m_b} - \mu_b + 2g_0|\Psi_0|^2 & g_0|\Psi_0|^2\\ g_0|\Psi_0|^2 & \omega + \frac{\mathbf{q}^2}{2m_b} - \mu_b + 2g_0|\Psi_0|^2\end{pmatrix}, \tag{A18}$$

where $g_0 = 4\pi a_{\rm bb}/m_b$ and $|\Psi_0|^2$ is the minimum of the Gross-Pitaevskii free energy. Together with the mean-field result for the boson density $n_b = |\Psi_0|^2$, we can show that the Goldstone mode takes a dispersion relation given by

$$\omega(\mathbf{q}) = \sqrt{\frac{\mathbf{q}^2}{2m_b}\left(\frac{\mathbf{q}^2}{2m_b} + \frac{8\pi a_{\rm bb}n_b}{m_b}\right)}, \tag{A19}$$

which is just the Bogoliubov excitation in a dilute Bose condensate.

To evaluate the thermodynamic potential beyond the mean field, we express the partition function in the Gaussian approximation as

$$Z \simeq \exp(-\mathcal{S}_{\rm eff}^{(0)})\int[d\phi^\dagger][d\phi]\exp(-\mathcal{S}_{\rm eff}^{(2)}). \tag{A20}$$

Integrating out the Gaussian fluctuations, the total thermodynamic potential can be expressed as

$$\Omega(\mu) = \Omega_0(\mu) + \Omega_{\rm fl}(\mu), \tag{A21}$$

where the contribution from the Gaussian fluctuations can be evaluated as [58]

$$\Omega_{\rm fl} = \frac{1}{2}\sum_Q \ln\left[\frac{\mathbf{M}_{11}(Q)}{\mathbf{M}_{22}(Q)}\det\mathbf{M}(Q)\right]e^{i\nu_m 0^+}. \tag{A22}$$

Near the quantum phase transition point $\mu = -E_b/2$, we can expand $\Omega_{\rm fl}$ in powers of $|\Delta|^2$. Because of the properties of second order phase transition, the terms of order $O(|\Delta|^2)$ vanish. To leading order, the result is [58]

$$\Omega_{\rm fl} \simeq -\frac{\zeta}{256\pi}\left(\frac{2m}{-\mu}\right)^{3/2}|\Delta|^4 \simeq -\zeta\frac{m^3 a_s^3}{32\pi}|\Delta|^4, \tag{A23}$$

where the numerical factor $\zeta = 2.61$. From the Gross-Pitaevskii free energy, we find that the pressure P in the mean-field approximation can be expressed as

$$P = \frac{m^3 a_s^3}{32\pi}|\Delta|^4. \tag{A24}$$

Thus, to leading order, the beyond-mean-field corrections renormalize the boson-boson scattering length $a_{\rm bb}$. The new renormalized scattering length reads

$$a_{\rm bb} = \frac{2a_s}{1+\zeta} \simeq 0.55a_s. \tag{A25}$$

Notice that this result is quite close to the exact result for the four body problem of $0.6a_s$ [97]. This means the quantum fluctuations are almost correctly included in the present theoretical approach.

Further, going beyond the leading order we find that we can fit $\Omega_{\rm fl}$ to the functional form [58]

$$\Omega_{\rm fl} = \frac{E_b}{2a_s^3}(c_1\tilde{\mu}_b^2 + c_2\tilde{\mu}_b^{5/2} + \cdots), \tag{A26}$$

where $\tilde{\mu}_b = \mu_b/E_b$ and the dimensionless factors c_1, c_2 can be numerically determined. Solving for the molecular chemical potential μ_b one obtains

$$\mu_b = \frac{4\pi a_{\rm bb} n_b}{m_b}\left[1 + \xi\frac{32}{3\sqrt{\pi}}(n_b a_{\rm bb}^3)^{1/2} + \cdots\right], \tag{A27}$$

with the coefficient $\xi = 0.94$ [58] which is 6% smaller than the Lee-Huang-Yang result $\xi = 1$ [15].

APPENDIX B: THE ONE-LOOP SUSCEPTIBILITIES

In this appendix, we evaluate the explicit forms of the one-loop susceptibilities $\Pi_{ij}(Q)$ $(i,\ j = 1, 2, 3)$ and $\Pi_\pi(Q)$. At arbitrary temperature, their expressions are rather huge. However, at $T = 0$, they can be written in rather compact forms. For convenience, we define $\Delta = |\Delta|e^{i\theta}$ in this appendix.

(I) Diquark sector. First, the polarization functions $\Pi_{11}(Q)$ and $\Pi_{12}(Q)$ can be evaluated as

$$\begin{aligned}
\Pi_{11}(Q) = N_cN_f\sum_{\mathbf{k}}\Bigg[&\left(\frac{(u_{\mathbf{k}}^-)^2(u_{\mathbf{p}}^-)^2}{i\nu_m - E_{\mathbf{k}}^- - E_{\mathbf{p}}^-} - \frac{(\upsilon_{\mathbf{k}}^-)^2(\upsilon_{\mathbf{p}}^-)^2}{i\nu_m + E_{\mathbf{k}}^- + E_{\mathbf{p}}^-} - \frac{(u_{\mathbf{k}}^+)^2(u_{\mathbf{p}}^+)^2}{i\nu_m + E_{\mathbf{k}}^+ + E_{\mathbf{p}}^+} + \frac{(\upsilon_{\mathbf{k}}^+)^2(\upsilon_{\mathbf{p}}^+)^2}{i\nu_m - E_{\mathbf{k}}^+ - E_{\mathbf{p}}^+}\right)\mathcal{T}_+ \\
&+ \left(\frac{(u_{\mathbf{k}}^-)^2(\upsilon_{\mathbf{p}}^+)^2}{i\nu_m - E_{\mathbf{k}}^- - E_{\mathbf{p}}^+} - \frac{(\upsilon_{\mathbf{k}}^-)^2(u_{\mathbf{p}}^+)^2}{i\nu_m + E_{\mathbf{k}}^- + E_{\mathbf{p}}^+} - \frac{(u_{\mathbf{k}}^+)^2(\upsilon_{\mathbf{p}}^-)^2}{i\nu_m + E_{\mathbf{k}}^+ + E_{\mathbf{p}}^-} + \frac{(\upsilon_{\mathbf{k}}^+)^2(u_{\mathbf{p}}^-)^2}{i\nu_m - E_{\mathbf{k}}^+ - E_{\mathbf{p}}^-}\right)\mathcal{T}_-\Bigg], \\
\Pi_{12}(Q) = N_cN_f\sum_{\mathbf{k}}\Bigg[&\left(\frac{u_{\mathbf{k}}^-\upsilon_{\mathbf{k}}^-u_{\mathbf{p}}^-\upsilon_{\mathbf{p}}^-}{i\nu_m + E_{\mathbf{k}}^- + E_{\mathbf{p}}^-} - \frac{u_{\mathbf{k}}^-\upsilon_{\mathbf{k}}^-u_{\mathbf{p}}^-\upsilon_{\mathbf{p}}^-}{i\nu_m - E_{\mathbf{k}}^- - E_{\mathbf{p}}^-} + \frac{u_{\mathbf{k}}^+\upsilon_{\mathbf{k}}^+u_{\mathbf{p}}^+\upsilon_{\mathbf{p}}^+}{i\nu_m + E_{\mathbf{k}}^+ + E_{\mathbf{p}}^+} - \frac{u_{\mathbf{k}}^+\upsilon_{\mathbf{k}}^+u_{\mathbf{p}}^+\upsilon_{\mathbf{p}}^+}{i\nu_m - E_{\mathbf{k}}^+ - E_{\mathbf{p}}^+}\right)\mathcal{T}_+ \\
&+ \left(\frac{u_{\mathbf{k}}^-\upsilon_{\mathbf{k}}^-u_{\mathbf{p}}^+\upsilon_{\mathbf{p}}^+}{i\nu_m + E_{\mathbf{k}}^- + E_{\mathbf{p}}^+} - \frac{u_{\mathbf{k}}^-\upsilon_{\mathbf{k}}^-u_{\mathbf{p}}^+\upsilon_{\mathbf{p}}^+}{i\nu_m - E_{\mathbf{k}}^- - E_{\mathbf{p}}^+} + \frac{u_{\mathbf{k}}^+\upsilon_{\mathbf{k}}^+u_{\mathbf{p}}^-\upsilon_{\mathbf{p}}^-}{i\nu_m + E_{\mathbf{k}}^+ + E_{\mathbf{p}}^-} - \frac{u_{\mathbf{k}}^+\upsilon_{\mathbf{k}}^+u_{\mathbf{p}}^-\upsilon_{\mathbf{p}}^-}{i\nu_m - E_{\mathbf{k}}^+ - E_{\mathbf{p}}^-}\right)\mathcal{T}_-\Bigg]e^{2i\theta},
\end{aligned} \tag{B1}$$

where $\mathbf{p} = \mathbf{k} + \mathbf{q}$. Here $\mathcal{T}_\pm$ are factors arising from the trace in spin space,

$$\mathcal{T}_\pm = \frac{1}{2} \pm \frac{\mathbf{k}\cdot\mathbf{p} + M^2}{2E_{\mathbf{k}}E_{\mathbf{p}}}, \tag{B2}$$

and $u_{\mathbf{k}}^\pm$, $\upsilon_{\mathbf{k}}^\pm$ are the BCS distribution functions defined as

$$(u_{\mathbf{k}}^\pm)^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}^\pm}{E_{\mathbf{k}}^\pm}\right), \qquad (\upsilon_{\mathbf{k}}^\pm)^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}^\pm}{E_{\mathbf{k}}^\pm}\right). \tag{B3}$$

At $Q = 0$, we find that

$$\Pi_{12}(0) = \Delta^2\frac{1}{4}N_cN_f\sum_{\mathbf{k}}\left[\frac{1}{(E_{\mathbf{k}}^-)^3} + \frac{1}{(E_{\mathbf{k}}^+)^3}\right]. \tag{B4}$$

Thus, near the quantum phase transition point, we have $\Pi_{12}(0) = \Delta^2\beta_1 + O(|\Delta|^4)$. On the other hand, a simple algebra shows that

$$\frac{1}{4G} + \Pi_{11}(0) - |\Pi_{12}(0)| = \frac{\partial\Omega_0}{\partial|\Delta|^2}. \tag{B5}$$

Therefore, the mean-field gap equation for Δ ensures the Goldstone's theorem in the superfluid phase.

(II) Diquark-sigma mixing terms. The term Π_{13} standing for the mixing between the sigma meson and the diquarks reads

$$\begin{aligned}
\Pi_{13}(Q) = N_cN_f\sum_{\mathbf{k}}\Bigg[&\left(\frac{u_{\mathbf{k}}^+\upsilon_{\mathbf{k}}^+(\upsilon_{\mathbf{p}}^+)^2 + u_{\mathbf{p}}^+\upsilon_{\mathbf{p}}^+(\upsilon_{\mathbf{k}}^+)^2}{i\nu_m - E_{\mathbf{k}}^+ - E_{\mathbf{p}}^+} + \frac{u_{\mathbf{k}}^+\upsilon_{\mathbf{k}}^+(u_{\mathbf{p}}^+)^2 + u_{\mathbf{p}}^+\upsilon_{\mathbf{p}}^+(u_{\mathbf{k}}^+)^2}{i\nu_m + E_{\mathbf{k}}^+ + E_{\mathbf{p}}^+} - \frac{u_{\mathbf{k}}^-\upsilon_{\mathbf{k}}^-(u_{\mathbf{p}}^-)^2 + u_{\mathbf{p}}^-\upsilon_{\mathbf{p}}^-(u_{\mathbf{k}}^-)^2}{i\nu_m - E_{\mathbf{k}}^- - E_{\mathbf{p}}^-}\right. \\
&\left.- \frac{u_{\mathbf{k}}^-\upsilon_{\mathbf{k}}^-(\upsilon_{\mathbf{p}}^-)^2 + u_{\mathbf{p}}^-\upsilon_{\mathbf{p}}^-(\upsilon_{\mathbf{k}}^-)^2}{i\nu_m + E_{\mathbf{k}}^- + E_{\mathbf{p}}^-}\right)\mathcal{I}_+ + \left(\frac{u_{\mathbf{k}}^+\upsilon_{\mathbf{k}}^+(u_{\mathbf{p}}^-)^2 + u_{\mathbf{p}}^-\upsilon_{\mathbf{p}}^-(\upsilon_{\mathbf{k}}^+)^2}{i\nu_m - E_{\mathbf{k}}^+ - E_{\mathbf{p}}^-} + \frac{u_{\mathbf{k}}^+\upsilon_{\mathbf{k}}^+(\upsilon_{\mathbf{p}}^-)^2 + u_{\mathbf{p}}^-\upsilon_{\mathbf{p}}^-(u_{\mathbf{k}}^+)^2}{i\nu_m + E_{\mathbf{k}}^+ + E_{\mathbf{p}}^-}\right. \\
&\left.- \frac{u_{\mathbf{k}}^-\upsilon_{\mathbf{k}}^-(\upsilon_{\mathbf{p}}^+)^2 + u_{\mathbf{p}}^+\upsilon_{\mathbf{p}}^+(u_{\mathbf{k}}^-)^2}{i\nu_m - E_{\mathbf{k}}^- - E_{\mathbf{p}}^+} - \frac{u_{\mathbf{k}}^-\upsilon_{\mathbf{k}}^-(u_{\mathbf{p}}^+)^2 + u_{\mathbf{p}}^+\upsilon_{\mathbf{p}}^+(\upsilon_{\mathbf{k}}^-)^2}{i\nu_m + E_{\mathbf{k}}^- + E_{\mathbf{p}}^+}\right)\mathcal{I}_-\Bigg]e^{i\theta},
\end{aligned} \tag{B6}$$

where the factors $\mathcal{I}_\pm$ are defined as

$$\mathcal{I}_\pm = \frac{M}{2}\left(\frac{1}{E_{\mathbf{k}}} \pm \frac{1}{E_{\mathbf{p}}}\right). \quad \text{(B7)}$$

One can easily find that $\Pi_{13} \sim M\Delta$, thus it vanishes when Δ or M approaches zero. At $Q = 0$, we have

$$\Pi_{13}(0) = \Delta \frac{1}{2} N_c N_f \sum_{\mathbf{k}} \frac{M}{E_{\mathbf{k}}}\left[\frac{\xi_{\mathbf{k}}^-}{(E_{\mathbf{k}}^-)^3} + \frac{\xi_{\mathbf{k}}^+}{(E_{\mathbf{k}}^+)^3}\right]. \quad \text{(B8)}$$

Thus the quantity H_0 defined in (71) can be evaluated as

$$H_0 = \frac{1}{2} N_c N_f \sum_{e=\pm} \sum_{\mathbf{k}} \frac{M_*}{E_{\mathbf{k}}^*} \frac{1}{(E_{\mathbf{k}}^* - e m_\pi/2)^2} = \frac{\partial^2 \Omega_0(y, M)}{\partial M \partial y}\bigg|_{y=0}. \quad \text{(B9)}$$

(III) Sigma meson and pions. The polarization function Π_{33} which stands for the sigma meson can be evaluated as

$$\begin{aligned}\Pi_{33}(Q) = N_c N_f \sum_{\mathbf{k}} \bigg[&(\upsilon_{\mathbf{k}}^- u_{\mathbf{p}}^- + u_{\mathbf{k}}^- \upsilon_{\mathbf{p}}^-)^2 \left(\frac{1}{i\nu_m - E_{\mathbf{k}}^- - E_{\mathbf{p}}^-} - \frac{1}{i\nu_m + E_{\mathbf{k}}^- + E_{\mathbf{p}}^-}\right)\mathcal{T}'_- + (\upsilon_{\mathbf{k}}^+ u_{\mathbf{p}}^+ + u_{\mathbf{k}}^+ \upsilon_{\mathbf{p}}^+)^2 \bigg(\frac{1}{i\nu_m - E_{\mathbf{k}}^+ - E_{\mathbf{p}}^+} \\ &- \frac{1}{i\nu_m + E_{\mathbf{k}}^+ + E_{\mathbf{p}}^+}\bigg)\mathcal{T}'_- + (\upsilon_{\mathbf{k}}^+ \upsilon_{\mathbf{p}}^- + u_{\mathbf{k}}^+ u_{\mathbf{p}}^-)^2 \left(\frac{1}{i\nu_m - E_{\mathbf{k}}^+ - E_{\mathbf{p}}^-} - \frac{1}{i\nu_m + E_{\mathbf{k}}^+ + E_{\mathbf{p}}^-}\right)\mathcal{T}'_+ \\ &+ (\upsilon_{\mathbf{k}}^- \upsilon_{\mathbf{p}}^+ + u_{\mathbf{k}}^- u_{\mathbf{p}}^+)^2 \left(\frac{1}{i\nu_m - E_{\mathbf{k}}^- - E_{\mathbf{p}}^+} - \frac{1}{i\nu_m + E_{\mathbf{k}}^- + E_{\mathbf{p}}^+}\right)\mathcal{T}'_+ \bigg], \end{aligned} \quad \text{(B10)}$$

where the factors $\mathcal{T}'_\pm$ are defined as

$$\mathcal{T}'_\pm = \frac{1}{2} \pm \frac{\mathbf{k}\cdot\mathbf{p} - M^2}{2E_{\mathbf{k}}E_{\mathbf{p}}}. \quad \text{(B11)}$$

At $Q = 0$ and for $\Delta = 0$, we find that

$$\mathbf{M}_{33}(0) = \frac{1}{2G} - 2N_c N_f \sum_{\mathbf{k}} \frac{1}{E_{\mathbf{k}}^*} + 2N_c N_f \sum_{\mathbf{k}} \frac{M_*^2}{E_{\mathbf{k}}^{*3}} = \frac{\partial^2 \Omega_0(y, M)}{\partial M^2}\bigg|_{y=0}. \quad \text{(B12)}$$

Finally, the polarization function $\Pi_\pi(Q)$ for pions can be obtained by replacing $\mathcal{T}'_\pm \to \mathcal{T}_\pm$. Thus, when $M \to 0$, the sigma meson and pions become degenerate and chiral symmetry is restored.

APPENDIX C: EXPANSION OF Ω_{fl} IN TERMS OF $|\Delta|^2$

In this appendix, we derive the expression of the Taylor expansion of Ω_{fl} in terms of $|\Delta|^2 \equiv y$. As we have shown in Sec. V, the leading order term should be $O(|\Delta|^4)$. Thus, we need to evaluate the numerical factor ζ. A key problem here is that the effective quark mass M and the chemical potential μ_B are both functions of $|\Delta|^2$ determined at the mean-field level.

First, we expand the matrix elements of $\mathbf{M}$ and $\mathbf{N}$ in terms of y. Any of these elements denoted by F is a function of μ_B, M and y. Our method of expansion is as follows. We firstly expand $F(\mu_B, M, y)$ in terms of y formally with μ_B and M being fixed parameters, i.e.,

$$F(\mu_B, M, y) = F_0(\mu_B, M) + F_1(\mu_B, M)y + F_2(\mu_B, M)y^2 + O(y^3), \quad \text{(C1)}$$

where $F_0(\mu_B, M) \equiv F(\mu_B, M, 0)$ and the expansion coefficients are defined as

$$F_1(\mu_B, M) = \frac{\partial F(\mu_B, M, y)}{\partial y}\bigg|_{y=0}, \qquad F_2(\mu_B, M) = \frac{1}{2}\frac{\partial^2 F(\mu_B, M, y)}{\partial y^2}\bigg|_{y=0}. \quad \text{(C2)}$$

We then expand the coefficients $F_i(\mu_B, M)$ ($i = 0, 1, 2$) at $(\mu_B, M) = (m_\pi, M_*)$, using the fact that

$$M \simeq M_* - \frac{1}{2M_*}y, \qquad \mu_B \simeq m_\pi + \frac{\beta}{2m_\pi \mathcal{J}}y. \quad \text{(C3)}$$

Doing this we formally obtain

$$F_i(\mu_B, M) = F_i(m_\pi, M_*) + F_i^1(m_\pi, M_*)y + F_i^2(m_\pi, M_*)y^2 + O(y^3). \quad \text{(C4)}$$

Finally, up to order $O(y^2)$, we have

$$\begin{aligned} F(\mu_B, M, y) = F_0(m_\pi, M_*) &+ [F_0^1(m_\pi, M_*) + F_1(m_\pi, M_*)]y + [F_0^2(m_\pi, M_*) \\ &+ F_1^1(m_\pi, M_*) + F_2(m_\pi, M_*)]y^2. \end{aligned} \quad \text{(C5)}$$

Using this method, we can expand the matrix elements of $\mathbf{M}$ and $\mathbf{N}$ formally as follows:

$$\begin{aligned}
\mathbf{M}_{11}(Q) &= \mathcal{D}_d^{*-1}(Q) + X_1(Q)|\Delta|^2 + Y_1(Q)|\Delta|^4 + O(|\Delta|^6),\\
\mathbf{M}_{12}(Q) &= \Delta^2 Z(Q) + O(|\Delta|^4),\\
\mathbf{M}_{13}(Q) &= \Delta W(Q) + O(|\Delta|^3),\\
\mathbf{M}_{33}(Q) &= \mathcal{D}_\sigma^{*-1}(Q) + X_2(Q)|\Delta|^2 + Y_2(Q)|\Delta|^4 + O(|\Delta|^6),\\
\mathbf{N}_{11}(Q) &= \mathcal{D}_\pi^{*-1}(Q) + X_3(Q)|\Delta|^2 + Y_3(Q)|\Delta|^4 + O(|\Delta|^6).
\end{aligned} \tag{C6}$$

Here $\mathcal{D}_d^{*-1}(Q)$ is defined as $\mathcal{D}_d^{-1}(Q;\mu_B = m_\pi)$.

Meanwhile, the thermodynamic potential $\Omega_{\rm fl}$ can be expressed as

$$\begin{aligned}
\Omega_{\rm fl} &= \frac{1}{2}\sum_Q\left\{\ln\left[\frac{\mathbf{M}_{11}(Q)}{\mathbf{M}_{22}(Q)}\det\mathbf{M}(Q)\right] + \ln\det\mathbf{N}(Q)\right\}e^{i\nu_m 0^+}\\
&\quad - \frac{1}{2}\sum_Q[2\ln\mathcal{D}_d^{*-1}(Q) + \ln\mathcal{D}_\sigma^{*-1}(Q)\\
&\quad + 3\mathcal{D}_\pi^{*-1}(Q)]e^{i\nu_m 0^+}.
\end{aligned} \tag{C7}$$

Using the expansion (C6), we find that the factor ζ is given by

$$\begin{aligned}
\beta\zeta &= \sum_Q\left\{\frac{X_2(Q)}{\mathcal{D}_\sigma^{*-1}(Q)} + \frac{X_1(Q)}{\mathcal{D}_d^{*-1}(Q)} + \frac{X_1(-Q)}{\mathcal{D}_d^{*-1}(-Q)} + \frac{W^2(Q)}{\mathcal{D}_d^{*-1}(Q)\mathcal{D}_\sigma^{*-1}(Q)} + \frac{W^2(-Q)}{\mathcal{D}_d^{*-1}(-Q)\mathcal{D}_\sigma^{*-1}(Q)}\right\}^2\\
&\quad - \sum_Q\left\{\frac{Y_2(Q)}{\mathcal{D}_\sigma^{*-1}(Q)} + \frac{Y_1(Q)}{\mathcal{D}_d^{*-1}(Q)} + \frac{Y_1(-Q)}{\mathcal{D}_d^{*-1}(-Q)} + \frac{X_1(Q)X_1(-Q) - Z^2(Q)}{\mathcal{D}_d^{*-1}(Q)\mathcal{D}_d^{*-1}(-Q)}\right\}\\
&\quad + \sum_Q\frac{W^2(-Q)X_1(Q) + W^2(Q)X_1(-Q) - 2W(Q)W(-Q)Z(Q)}{\mathcal{D}_d^{*-1}(Q)\mathcal{D}_d^{*-1}(-Q)\mathcal{D}_\sigma^{*-1}(Q)} + \sum_Q\left\{\left[\frac{X_3(Q)}{\mathcal{D}_\pi^{*-1}(Q)}\right]^2 - \frac{Y_3(Q)}{\mathcal{D}_\pi^{*-1}(Q)}\right\}.
\end{aligned} \tag{C8}$$

It is obvious that $Z(Q) = B(Q)$ and $W(Q) = H(Q)$ where $B(Q)$ and $H(Q)$ are defined in (71). On the other hand, since

$$\frac{\beta}{2m_\pi\mathcal{J}} \simeq \frac{m_\pi}{2M_*}\frac{1}{2M_*}, \tag{C9}$$

to leading order of $O(m_\pi/2M_*)$, we can set $\mu_B = m_\pi$ in all equations and identify $X_1(Q) = A(Q)$ defined in (71).

[1] N. Itoh, Prog. Theor. Phys. **44**, 291 (1970); E. Witten, Phys. Rev. D **30**, 272 (1984).
[2] F. Karsch, Lect. Notes Phys. **583**, 209 (2002).
[3] S. Muroya, A. Nakamura, C. Nonaka, and T. Takaishi, Prog. Theor. Phys. **110**, 615 (2003).
[4] D. T. Son and M. A. Stephanov, Phys. Rev. Lett. **86**, 592 (2001).
[5] D. T. Son and M. A. Stephanov, Phys. At. Nucl. **64**, 834 (2001).
[6] J. B. Kogut, M. A. Stephanov, and D. Toublan, Phys. Lett. B **464**, 183 (1999).
[7] J. B. Kogut, M. A. Stephanov, D. Toublan, J. J. M. Verbaarschot, and A. Zhitnitsky, Nucl. Phys. **B582**, 477 (2000).
[8] K. Splittorff, D. T. Son, and M. A. Stephanov, Phys. Rev. D **64**, 016003 (2001).
[9] J. T. Lenaghan, F. Sannino, and K. Splittorff, Phys. Rev. D **65**, 054002 (2002).
[10] K. Splittorff, D. Toublan, and J. J. M. Verbaarschot, Nucl. Phys. **B620**, 290 (2002).
[11] K. Splittorff, D. Toublan, and J. J. M. Verbaarschot, Nucl. Phys. **B639**, 524 (2002).
[12] T. Zhang, T. Brauner, and D. H. Rischke, J. High Energy Phys. 06 (2010) 64.
[13] For review, see J. O. Andersen, Rev. Mod. Phys. **76**, 599 (2004).
[14] N. N. Bogoliubov, J. Phys. USSR **11**, 23 (1947).
[15] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).
[16] R. F. Sawyer, Phys. Rev. Lett. **29**, 382 (1972).
[17] D. J. Scalapino, Phys. Rev. Lett. **29**, 386 (1972).
[18] G. Baym, Phys. Rev. Lett. **30**, 1340 (1973).
[19] D. K. Campbell, R. F. Dashen, and J. T. Manassah, Phys. Rev. D **12**, 979 (1975); **12**, 1010 (1975).
[20] D. T. Son, Phys. Rev. D **59**, 094019 (1999).
[21] T. Schäfer and F. Wilczek, Phys. Rev. D **60**, 114033 (1999).
[22] D. Pisarski and D. H. Rischke, Phys. Rev. D **61**, 074017 (2000); **61**, 051501 (2000).

[23] T. Kanazawa, T. Wettig, and N. Yamamoto, J. High Energy Phys. 08 (2009) 003.
[24] D. M. Eagles, Phys. Rev. **186**, 456 (1969).
[25] A. J. Leggett, in *Modern Trends in the Theory of Condensed Matter*, edited by A. Pekalski and R. Przystawa (Springer-Verlag, Berlin, 1980).
[26] M. Greiner *et al.*, Nature (London) **426**, 537 (2003); S. Jochim *et al.*, Science **302**, 2101 (2003); M. W. Zwierlein *et al.*, Nature (London) **435**, 1047 (2005).
[27] S. Hands, I. Montvay, S. Morrison, M. Oevers, L. Scorzato, and J. Skullerud, Eur. Phys. J. C **17**, 285 (2000).
[28] S. Hands, I. Montvay, L. Scorzato, and J. Skullerud, Eur. Phys. J. C **22**, 451 (2001).
[29] J. B. Kogut, D. K. Sinclair, S. J. Hands, and S. E. Morrison, Phys. Rev. D **64**, 094505 (2001).
[30] J. B. Kogut, D. Toublan, and D. K. Sinclair, Phys. Lett. B **514**, 77 (2001).
[31] S. Hands, S. Kim, and J. Skullerud, Eur. Phys. J. C **48**, 193 (2006).
[32] S. Hands, S. Kim, and J. Skullerud, Phys. Rev. D **81**, 091502(R) (2010).
[33] J. B. Kogut and D. K. Sinclair, Phys. Rev. D **66**, 034505 (2002).
[34] J. B. Kogut and D. K. Sinclair, Phys. Rev. D **66**, 014508 (2002).
[35] J. B. Kogut and D. K. Sinclair, Phys. Rev. D **70**, 094501 (2004).
[36] P. Forcrand, M. A. Stephanov, and U. Wenger, Proc. Sci., LAT2007 (2007) 237.
[37] M. Loewe and C. Villavicencio, Phys. Rev. D **67**, 074034 (2003).
[38] J. O. Andersen, Phys. Rev. D **75**, 065011 (2007).
[39] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).
[40] U. Vogl and W. Weise, Prog. Part. Nucl. Phys. **27**, 195 (1991).
[41] S. P. Klevansky, Rev. Mod. Phys. **64**, 649 (1992).
[42] T. Hatsuda and T. Kunihiro, Phys. Rep. **247**, 221 (1994).
[43] D. Toublan and J. B. Kogut, Phys. Lett. B **564**, 212 (2003).
[44] M. Frank, M. Buballa, and M. Oertel, Phys. Lett. B **562**, 221 (2003).
[45] A. Barducci, R. Casalbuoni, G. Pettini, and L. Ravagli, Phys. Rev. D **69**, 096004 (2004).
[46] A. Barducci, R. Casalbuoni, G. Pettini, and L. Ravagli, Phys. Rev. D **71**, 016011 (2005).
[47] L. He and P. Zhuang, Phys. Lett. B **615**, 93 (2005).
[48] L. He, M. Jin, and P. Zhuang, Phys. Rev. D **71**, 116001 (2005).
[49] H. J. Warringa, D. Boer, and J. O. Andersen, Phys. Rev. D **72**, 014015 (2005).
[50] L. He, M. Jin, and P. Zhuang, Phys. Rev. D **74**, 036005 (2006).
[51] Z. Zhang and Y. -X. Liu, Phys. Rev. C **75**, 064910 (2007).
[52] C. Ratti and W. Weise, Phys. Rev. D **70**, 054013 (2004).
[53] G. Sun, L. He, and P. Zhuang, Phys. Rev. D **75**, 096004 (2007).
[54] T. Brauner, K. Fukushima, and Y. Hidaka, Phys. Rev. D **80**, 074035 (2009).
[55] J. O. Andersen and T. Brauner, Phys. Rev. D **81**, 096004 (2010).
[56] J. Xiong, M. Jin, and J. Li, J. Phys. G **36**, 125005 (2009).
[57] H. Hu, X.-J. Liu, and P. D. Drummond, Europhys. Lett. **74**, 574 (2006); Nature Phys. **3**, 469 (2007).
[58] R. B. Diener, R. Sensarma, and M. Randeria, Phys. Rev. A **77**, 023626 (2008).
[59] X. Leyronas and R. Combescot, Phys. Rev. Lett. **99**, 170402 (2007).
[60] M. Huang, P. Zhuang, and W. Chao, Phys. Rev. D **65**, 076012 (2002).
[61] To be consistent with the convention used in [57,58,62], here $\phi^\dagger(Q)$ is defined as $[\phi(Q)]^\dagger$.
[62] J. R. Engelbrecht, M. Randeria, and C. A. R. Sá de Melo, Phys. Rev. B **55**, 15153 (1997).
[63] N. Nagaosa, *Quantum Field Theory in Condensed Matter Physics* (Springer, Heidelberg, Germany, 1999).
[64] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Oxford University Press, Oxford, England, 2003).
[65] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, Cambridge, England, 2002).
[66] H. J. Schulze, J. Phys. G **21**, 185 (1995).
[67] S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966).
[68] L. He and P. Zhuang, Phys. Rev. D **75**, 096003 (2007).
[69] T. Brauner, Phys. Rev. D **77**, 096006 (2008).
[70] T. D. Cohen, R. J. Furnstahl, and D. K. Griegel, Phys. Rev. C **45**, 1881 (1992).
[71] T. D. Cohen, R. J. Furnstahl, D. K. Griegel, and X. Jin, Prog. Part. Nucl. Phys. **35**, 221 (1995).
[72] L. He, Y. Jiang, and P. Zhuang, Phys. Rev. C **79**, 045205 (2009).
[73] D. T. Son and M. A. Stephanov, Phys. Rev. A **74**, 013614 (2006).
[74] M. Kitazawa, D. H. Rischke, and I. A. Shovkovy, Phys. Lett. B **663**, 228 (2008).
[75] P. Nozières and S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985).
[76] C. A. R. Sá de Melo, M. Randeria, and J. R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
[77] Y. Nishida and H. Abuki, Phys. Rev. D **72**, 096004 (2005).
[78] H. Abuki, Nucl. Phys. **A791**, 117 (2007).
[79] L. He and P. Zhuang, Phys. Rev. D **76**, 056003 (2007).
[80] In nonrelativistic systems, the BCS and BEC states are distinguished by $\mu > 0$ and $\mu < 0$, respectively, where μ is the fermion chemical potential.
[81] D. Toublan and A. R. Zhitnitsky, Phys. Rev. D **73**, 034009 (2006).
[82] This is only possible for our two-flavor case. For $N_f > 2$, the chiral symmetry is spontaneously broken at high density [23].
[83] For a careful explanation of the meson spectra in chiral perturbation theory, see T. Brauner, Mod. Phys. Lett. A **21**, 559 (2006).
[84] D. Blaschke, F. Reinholz, G. Ropke, and D. Kremp, Phys. Lett. **151B**, 439 (1985).
[85] T. Hatsuda and T. Kunihiro, Phys. Rev. Lett. **55**, 158 (1985); Phys. Lett. B **185**, 304 (1987).
[86] The mesons we called here are the collective excitations which have the same quantum numbers as the pions and

the sigma meson in the vacuum. They are also called "pions" and "sigma meson" in the chiral restored regime.

[87] For the studies of meson properties in color-superconducting quark matter, see D. Ebert, K. G. Klimenko, and V. L. Yudichev, Phys. Rev. C **72**, 015201 (2005); D. Zablocki, D. Blaschke, and R. Anglani, AIP Conf. Proc. **1038**, 159 (2008). However, in these papers the meson properties are investigated only at zero momentum.

[88] J. Hüfner, S. P. Klevansky, P. Zhuang, and H. Voss, Ann. Phys. (N.Y.) **234**, 225 (1994).

[89] P. Zhuang, J. Hüfner, and S. P. Klevansky, Nucl. Phys. **A576**, 525 (1994).

[90] J. B. Kogut and D. Toublan, Phys. Rev. D **64**, 034007 (2001).

[91] R. Rapp, T. Schaefer, E. V. Shuryak, and M. Velkovsky, Phys. Rev. Lett. **81**, 53 (1998).

[92] M. Alford, K. Rajagopal, and F. Wilczek, Phys. Lett. B **422**, 247 (1998).

[93] M. Alford, K. Rajagopal, T. Schäfer, and A. Schmitt, Rev. Mod. Phys. **80**, 1455 (2008); K. Rajagopal and F. Wilczek, arXiv:hep-ph/0011333; D. K. Hong, Acta Phys. Pol. B **32**, 1253 (2001); M. Alford, Annu. Rev. Nucl. Part. Sci. **51**, 131 (2001); T. Schäfer, arXiv:hep-ph/0304281; D. H. Rischke, Prog. Part. Nucl. Phys. **52**, 197 (2004); M. Buballa, Phys. Rep. **407**, 205 (2005); H. -C. Ren, arXiv: hep-ph/0404074; M. Huang, Int. J. Mod. Phys. E **14**, 675 (2005); I. A. Shovkovy, Found. Phys. **35**, 1309 (2005); Q. Wang, Progress in Physics **30**, 173 (2010).

[94] A. Perali, P. Pieri, G. C. Strinati, and C. Castellani, Phys. Rev. B **66**, 024510 (2002); P. Pieri, L. Pisani, and G. C. Strinati, Phys. Rev. B **70**, 094508 (2004).

[95] R. Haussmann, W. Rantner, S. Cerrito, and W. Zwerger, Phys. Rev. A **75**, 023610 (2007).

[96] Q. Chen, J. Stajic, S. Tan, and K. Levin, Phys. Rep. **412**, 1 (2005).

[97] D. S. Petrov, C. Salomon, and G. V. Shlyapnikov, Phys. Rev. Lett. **93**, 090404 (2004).