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Towards a quantum theory of the chiral magnetic effect

V. D. Orlovsky and V. I. Shevchenko

Institute of Theoretical and Experimental Physics, B .Cheremushkinskaya 25, 117218 Moscow, Russia

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We discuss three possible ways of addressing quantum physics behind chiral magnetic effect and electric charge fluctuation patterns in heavy-ion collisions. The first one makes use of P-parity violation probed by local order parameters, the second considers the chiral magnetic effect in the quantum measurement theory framework, and the third way is to study a product of two P-odd contributions. In the latter approach, the relevant form factor is constructed and computed for a weak magnetic field in the confinement region and for free quarks in a strong field. It is shown that the effect is negligible in the former case. We also discuss the saturation effect—the charge fluctuation asymmetry for free fermions reaches a constant value at asymptotically large fields.

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I. INTRODUCTION

One of the main theoretical challenges of modern quantum chromodynamics (QCD) is to build a detailed theoretical picture of strong interaction physics relevant for heavy-ion collisions. Currently running experimental programs have already brought lots of exciting results. Despite tremendous progress in understanding, a rich pattern of observed effects is still waiting to be placed into a coherent theoretical picture based on QCD.

In the course of studies of hadronic matter at large temperatures and/or densities one can make use of the scale separation allowing us to neglect effects of weak and electromagnetic interactions in most cases. A possible interesting exception is pointed out in [1,2]. When relativistic ions undergo noncentral collision, a strong magnetic field is generated in the collision region. This field rapidly changes with time and its typical magnitude is estimated [3] as $\sqrt{eB} = 10^{2\pm1}$ MeV, i.e., of the order of dynamical QCD scale. Correspondingly, any studies of strongly interacting matter in heavy-ion collisions have to take the effects of this Abelian magnetic field into account. Of particular interest in this respect is the so-called chiral magnetic effect (CME). The physics behind it can be explained in several different but complementary ways [1–26]. Let us consider nonzero density of one flavor of free massless quarks in an external magnetic field. Suppose there are unequal chemical potentials for left- and right-handed quarks: $\mu_L \neq \mu_R$. When it can be shown that a nonzero *classical* electric current flows along the magnetic field (see [9] and references therein, see also [27] for another perspective):

$$\mathbf{j} = \frac{e^2}{2\pi^2}\mu_5\mathbf{B}, \tag{1}$$

where $2\mu_5 = \mu_R - \mu_L$. The physical reason for this chiral charge excess to electric charge current conversion is quark magnetic moment interaction with the magnetic field (which is of a different sign for positively and negatively charged quarks) together with the correlation of spin and momentum for chiral fermions. Both sides of (1) have of course the same transformation properties under P- and CP-parity conjugation. Many different aspects of CME have been extensively discussed in the literature and there is no doubt that CME is a robust theoretical effect. However, it is not a simple task to apply this clear physical picture to real processes described by nonperturbative QCD. One of the most important questions on this way is about the physical origin of the chiral chemical potential μ_5, which is absent in the fundamental QCD Lagrangian. In the original picture [8], the appearance of effective $\mu_5 \neq 0$ is a nonperturbative QCD effect, caused by the interaction of quarks with topologically nontrivial gluon field configurations *above* the phase transition. The physical explanation goes as follows. As is well known, the topological charge in the QCD vacuum fluctuates as described by Veneziano-Witten formula [28,29]

$$\chi = \int d^4x \langle G\tilde{G}(x) G\tilde{G}(0)\rangle \propto f^2_{\eta'} m^2_{\eta'}, \tag{2}$$

where $f_{\eta'}$ is meson decay constant and $m_{\eta'}$ is η' meson mass. The nonperturbative parameter in the right-hand side scales as $\Lambda^4_{\rm QCD}$ which means that the topological charge fluctuates over Euclidean 4-volumes of typical size determined by the nonperturbative QCD scale. It is worth stressing that these fluctuations are *quantum*, i.e., the states of the different topological charge are to be summed over for whatever Euclidean 4-volume V and one always has

$$\int_V d^4x \langle G\tilde{G}(x)\rangle = 0. \tag{3}$$

In other words, (3) is valid because the integrand in the left-hand side vanishes identically, at each point. There is no special space-time fluctuation pattern in the problem other than the correlator (2) (and higher ones).

The situation however may change at nonzero temperature/density. Since the Euclidean $\mathbb{O}(4)$ invariance of the

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vacuum is broken in this case, one can think of different fluctuation patterns in spatial and in temporal directions. Moreover, since in real collision experiments external conditions are time dependent they can play a dual role of the background and of a measuring device. In other words the meaning of averaging in (3) changes: one has to integrate only over those field excitations which are present at a given Minkowski 3-volume for a given time period and the problem becomes essentially nonstationary in this sense. One can say that the average over fields $\langle .. \rangle$ becomes V dependent. Such a quantity—physically corresponding to a "single event"—can, in principle, be nonvanishing. Of course it is natural to expect that random character of fluctuations leads to zero result for (3) after averaging over many events.

The CME is often considered as a reasonable explanation of outgoing particles electric charge asymmetry observed at the relativistic heavy ion collider [30–41] in $\sqrt{s_{NN}} = 200$ GeV Au + Au and Cu + Cu collisions. The latter effect can be described as follows. For a noncentral collision one can fix the reaction plane by two vectors: beam momentum and impact parameter (without a loss of generality this is always chosen as a 12 plane in the present paper and no adjustment angle Ψ_{RP} is introduced). Thus, the angular momentum of the beams (and the corresponding magnetic field) is oriented along axis 3. The azimuthal angle $\phi \in [0, 2\pi)$ is defined in plane 23. With this notation, in any particular event one studies charged particles distribution in ϕ using the following conventional parametrization:

$$\frac{dN_\pm}{d\phi} \propto 1 + 2v_{1,\pm} \cos\phi + 2v_{2,\pm} \cos 2\phi + 2a_\pm \sin\phi + \ldots . \quad (4)$$

The coefficients $v_{1,\pm}$ and $v_{2,\pm}$ account for the so-called directed and elliptic flow. They are believed to be universal for positively and negatively charged particles with good accuracy. The coefficients a_+ and a_- describe charge flow along the third axis, i.e., normal to the reaction plane. This P-parity forbidden correlation between a polar vector (electric current) and the axial one (angular momentum) is considered as a signature of P-parity violation in a given event with $a_\pm \neq 0$. On the other hand, the random nature of the process dictates $\langle a_+ \rangle_e = \langle a_- \rangle_e = 0$ (there the averaging over events is taken).

Trying to construct a theory of the phenomenon one has first to choose adequate language. Since at the end, the heavy-ion collision is a scattering problem, the ultimate framework would be the S-matrix and inelastic scattering amplitudes' formalism with two colliding ions as incoming particles. Because of its extreme complexity this way seems to be totally hopeless. Instead, one uses some effective theories like hydrodynamics to predict distribution of outgoing particles. In the particular problem of a charge fluctuation' asymmetry, the crucial point distinguishing different theoretical models is whether the currents of interest are treated as classical or as quantum. In the former case one makes use of the expression (1) as classical equation. The quantum nature of the problem here is hidden in a theory for μ_5 and corresponding correlators and fluctuations for this effective chiral chemical potential. In the later case one is to consider quantum averages like $\langle \Omega | j_\mu | \Omega \rangle$, $\langle \Omega | j_\mu j_\nu | \Omega \rangle$, etc. and to understand (1) as operator relation. However, if one takes the diagonal matrix element of (1) in the vacuum the answer is of course trivial: $\langle 0 | \mathbf{j} | 0 \rangle = 0$ even for a nonzero external magnetic field.

We discuss three basic complementary ways to address the quantum nature of CME in this paper:

(1) To make use of P-parity violation probed by local order parameters
(2) To consider CME in quantum measurement theory framework
(3) To study a product of two P-odd contributions

We discuss all these approaches in the present paper and start with the first one in the next section which is phenomenologically the simplest.

II. P-PARITY VIOLATION PROBED BY LOCAL ORDER PARAMETERS

As is well known, quantum field theoretical averages of local operators have typically the following leading contribution:

$$\langle \Omega | \mathcal{O}(x) | \Omega \rangle \propto c \cdot \Lambda^{d_{\mathcal{O}}}, \quad (5)$$

where Λ is an ultraviolet cutoff and the numerical constant c is generally nonvanishing if $c = 0$ is not protected by some symmetry. Therefore, the crucial step in the discussed problem is to model transition from a local microscopic current j_μ to a nonlocal macroscopic one J_μ. It is done by taking the matrix elements of the current j_μ over the medium degrees of freedom $|\Phi\rangle$ from a full state vector $|\Omega\rangle = |\Phi\rangle \otimes |\phi\rangle$:

$$j_\mu(x) = \bar\psi \gamma_\mu \psi(x) \leftrightarrow J_\mu \propto \langle \Phi | \int dx \rho_V(x) j_\mu(x) | \Phi \rangle. \quad (6)$$

Here the function $\rho_V(x)$ defines the measure of integration over the "physically infinitesimal volume," as is usual in condensed matter physics.

The second important ingredient is the existence of the medium itself. For phenomenological purposes it is not important what particular kind of microscopic description for the medium is chosen. What does matter is Lorentz symmetry breaking following from the existence of a distinguished frame—the medium rest frame. In the simplest cases of a uniform medium characterized by nonzero temperature/density it is usually parametrized by a unit vector u_μ—the medium four velocity. The general answer for the induced current in uniform electromagnetic field reads

$$\langle \phi | J_\mu | \phi \rangle = c^{(+)} u^\nu F_{\mu\nu} + c^{(-)} u^\nu \tilde{F}_{\mu\nu}, \tag{7}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ and $c^{(+)}$ ($c^{(-)}$) are parity-even (parity-odd) functions. They may depend on invariant combinations like F^2, $F\tilde{F}$, etc. but, on the other hand, the same functions encode microscopic properties of the medium. In particular, for P-even medium $c^{(-)} = 0$, at least at the lowest order in external field.

We say about local parity violation in a state $|\Omega\rangle$ when a local parity-odd operator $\mathcal{O}(x) = -P\mathcal{O}(x)P^\dagger$ has nonzero expectation value in this state

$$\langle \Omega | \mathcal{O}(x) | \Omega \rangle \neq 0, \tag{8}$$

for example, $\langle \psi^\dagger \gamma_5 \psi \rangle \neq 0$. The condition of locality here is important. Operationally it means that the operators and their products are defined at the scale $a \sim \Lambda^{-1}$ where Λ is ultraviolet cutoff. For nonlocal averages, on the other hand, it is not a problem to have nonzero P-odd matrix element, e.g., $\langle j_0(x) j_3(y) \rangle$. The medium, characterized by a finite coherence length, brings physical meaning to this nonlocality. An expression analogous to (7) now reads

$$\langle \phi | J_\mu \partial J^5 | \phi \rangle = \bar{c}^{(-)} u^\nu F_{\mu\nu} + \bar{c}^{(+)} u^\nu \tilde{F}_{\mu\nu}, \tag{9}$$

where $\bar{c}^{(+)}$ ($\bar{c}^{(-)}$) are another parity-even (parity-odd) functions, (here and in the following text $\partial J^5 = \partial^\mu J^5_\mu$). Again for the P-even medium $\bar{c}^{(-)}$ should vanish at the lowest order (notice that the P-odd invariant $F\tilde{F}$ vanishes for the particular case of the uniform magnetic field).

To feel the physical meaning of (9) let us imagine the radial distribution of velocities $\mathbf{v}$ of the matter in a uniform magnetic field $\mathbf{B}$. If the divergence ∂J^5 is also uniform in the (''fireball'') volume, the charge density is to be of a different sign above and below the reaction plane:

$$\langle \phi | J_0 \partial J^5 | \phi \rangle \propto \mathbf{v} \cdot \mathbf{B}. \tag{10}$$

In the medium rest frame characterized by $u_\mu = (1, 0, 0, 0)$ for the uniform magnetic background, the electric current $\mathbf{J}$ flows along the magnetic field $\mathbf{B}$.

It seems quite natural to interpret (9) in the following way: as soon as the concept of a medium can be applied to the discussed problem one can easily construct classical nonzero local P-odd parameters without specifying any particular ''chiral microscopy.'' The medium (manifested by existence of the selected frame) is crucial in two aspects: first, it allows us to consider meaningful local objects and not badly divergent quantities like (5) and second, by Lorentz-invariance breaking it provides invariant meaning for the electric and magnetic fields, thus making possible correlations between local (in macroscopic sense) operators of different parities. We also see here the importance of the uniformity condition: if ∂J^5 is short-correlated, there is no net effect. This brings us back to the question about dynamical scales hierarchy.

There is a deep question behind the above consideration: if the (microscopic) current nonconservation is anomalous [e.g. in (9)]—how is this fact encoded in equations for macroscopic, effective currents? We leave aside the discussion of this important point and refer an interested reader to [42] where this question is addressed in a hydrodynamic setup.

From a heavy ions collision point of view the P-parity violating average (9) is not an observable by itself. The reason is physically clear: instead of measuring components of vector (electric) current and axial (chiral) current and studying their correlation, only the quantities of the former kind are being measured (in the form of final particles electric charge distribution). As for the latter quantities related to chiral charge—it is assumed that the quark-gluon medium created after the collision of two heavy ions plays a role in the measuring device performing an effective measurement of the topological charge in the corresponding space-time region. It should be mentioned that this is a rather strong assumption: to say that one part of some quantum system can measure (in classical sense) the state of another part of the same system means in fact to address some scenario for decoherence of the subsystems and information loss. To model this effect one has to adopt the language of quantum measurement theory. This is done in the next section.

III. CME IN QUANTUM MEASUREMENT THEORY FRAMEWORK

It is possible to understand (1) as a correlation between the preferred direction of an outgoing electric charge distribution asymmetry and the magnetic field in a particular event. The sign of this P-parity–odd asymmetry is fixed by the sign of effective μ_5 in this event (and of course varies randomly from event to event due to the topological neutrality of the QCD vacuum). The quantitative theory would require information about the distribution function of effective μ_5.

A simple quantum-mechanical analogy can be useful to illustrate the point. In one-dimensional bound state problem with P-parity–even potential $V(x) = V(-x)$ one has $\langle x \rangle = \int x dx |\psi_0(x,t)|^2 = 0$ where $\psi_0(x,t)$ is the ground state P-parity–even wave function. On the other hand, performing particle position measurements on an ensemble of N identical systems all in the ground state one gets a sequence of positive and negative numbers $x_1, x_2, \ldots, x_N$ (with some uncertainties determined by the measuring device properties). Quantum mechanics does not predict the result of a single measurement, but guarantees $\langle x \rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N x_i = 0$. For each measurement with the outcome $x_i \neq 0$ one can say that P invariance is broken in this particular experiment, ''event-by-event.'' In this simple case ''breaking'' is clearly of statistical origin and has nothing to do with dynamics—i.e., properties of the potential $V(x)$. Therefore, it is common in quantum mechanics not to use such terminology and compute instead nonzero P-parity–even observables, such as, e.g., $\langle x^2 \rangle =$

$\int x^2 dx |\psi_0(x,t)|^2 \neq 0$, characterizing the pattern of quantum fluctuations. What is, however, important is the textbook average over events/average over probability density equivalence.

By way of another simple analogy consider a system of massless quantum fields subject to boundary conditions at a typical distance scale L characterized by a unit 3-vector $\mathbf{n}$. To be concrete, one can think of electromagnetic Casimir vacuum between parallel plates at distance L with $\mathbf{n}$ being normal to the planes. Let this vector smoothly fluctuate in random directions with typical frequency ω, which is assumed to be much smaller than c/L. One studies the quantum average of the energy-momentum tensor for the fields, $\langle T_{\mu\nu}(x)\rangle$. Since the problem is quasi-stationary, the general answer is given by

$$\langle T_{\mu\nu}(x)\rangle = a(x) g_{\mu\nu} + b(x) n_\mu n_\nu + \mathcal{O}(\omega L/c). \tag{11}$$

On the other hand, performing an average over time period $T \gg \omega^{-1}$ one should have

$$\frac{1}{T}\int_0^T dt \langle T_{\mu\nu}(x)\rangle = \bar{a}(x) g_{\mu\nu} \tag{12}$$

since no memory has remained about the particular direction the vector $\mathbf{n}$ is pointing to. Thus, experiments with the detector time resolution $\omega T \ll 1$ will observe $\mathbb{O}(3)$ violating local answer (11) while those over long time scales $\omega T \gg 1$ will see $\mathbb{O}(3)$ respecting answer (12). It is of crucial importance that some physical process with the typical life time scale comparable or larger than the plasma life time does exist and it is responsible for the creation of P- and CP-odd domains in the dense and hot matter in Minkowski space-time. It seems to be a rather subtle point in this case how a relation between Euclidean expression (2) and Minkowskian dynamics should look. In any case, the existence of scale separation between the process dynamical scales and the measuring device ones is necessary for the whole picture to make sense.

Since a detailed picture of the discussed microscopic quantum/classical interplay is beyond us, our attitude here is purely phenomenological. We define the effective η-dependent current $J_\mu(x,\eta)$ as

$$J_\mu(x,\eta) = \langle \Omega_\eta | j_\mu(x) | \Omega_\eta \rangle, \tag{13}$$

where electric current $j_\mu(x) = \bar{\psi}(x) Q \gamma_\mu \psi(x)$ with a quarks charge matrix $Q = \mathrm{diag}(2/3, -1/3, -1/3)$. The state $|\Omega_\eta\rangle$ is characterized by

$$\langle \Omega_\eta | \int_V d^4 y \partial^\mu j^5_\mu(y) | \Omega_\eta \rangle = \eta, \tag{14}$$

where $j^5_\mu = \bar{\psi}\tau\gamma_\mu\gamma^5\psi$ and τ is a matrix in flavor space fixing the axial current flavor quantum numbers. It is physically obvious that $J_\mu(x,\eta)$ must be an odd function in η and

$$\int_{-\infty}^{\infty} d\eta J_\mu(x,\eta) = 0. \tag{15}$$

Since by assumption each event is characterized by some value of η, positive or negative with equal probability, this corresponds to "averaging to zero" over many events.

To proceed further it is convenient to use the formalism of partial partition functions:

$$Z = \int D\Phi \exp(-S[\Phi]) \prod_i \int d\eta_i \tilde{\delta}(\eta_i - O_i[\Phi]), \tag{16}$$

where $S[\Phi]$ is the standard Euclidean QCD action, Φ stays for dynamical quark and gluon fields A, $\bar{\psi}$, ψ, and $O_i[\Phi]$ is a gauge-invariant operator made of these fields. We approximate the real detector with finite resolution by the choice of the "detector function" $\tilde{\delta}(x)$ in the Gaussian form:

$$\tilde{\delta}(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \exp(-\lambda^2 l^2/2 + i\lambda\eta), \tag{17}$$

so that $\int_{-\infty}^{\infty} d\eta \tilde{\delta}(\eta) = 1$.

We are interested in a value of the electric current (13). For an exactly conserved axial current $\partial j^5 = 0$ one would have $\langle \Omega | j_\mu(x) \cdot \partial j^5(y) | \Omega \rangle = 0$. Because of the (electromagnetic) anomaly, however, the result reads

$$i\int dx e^{iq(x-y)} \langle \Omega | j_\mu(x) \cdot \partial j^5(y) | \Omega \rangle = -\mathrm{Tr}_f[Q^2\tau]\mathrm{Tr}_c \mathbf{1} \cdot \left(\frac{1}{2\pi^2}\right) \cdot q_\nu \tilde{F}_{\mu\nu}, \tag{18}$$

where the traces Tr_f and Tr_c run over flavor and color indices, respectively, and are normalized according to $\mathrm{Tr}_f \mathbf{1} = N_f$ and $\mathrm{Tr}_c \mathbf{1} = N_c$.

The anomalous divergence of the axial current is given by the following general expression:

$$\partial j^5 = -\mathrm{Tr}_f[Q^2\tau]\mathrm{Tr}_c \mathbf{1} \cdot \left(\frac{1}{8\pi^2}\right) \cdot F_{\mu\nu}\tilde{F}^{\mu\nu} - \mathrm{Tr}_f[\tau] \cdot \left(\frac{1}{8\pi^2}\right) \cdot \mathrm{Tr}_c G_{\mu\nu}\tilde{G}^{\mu\nu} \tag{19}$$

(notice that for uniform magnetic field $F_{\mu\nu}\tilde{F}^{\mu\nu} = 0$). Since for singlet current $\tau = \mathbf{1}$, computing

$$J_\mu(\eta, x) = \frac{1}{Z}\int D\Phi j_\mu(x) \tilde{\delta}(\eta - n_V) \exp(-S[\Phi]), \tag{20}$$

where

$$n_V = \int_V d^4 y \partial j^5 = -\frac{N_f}{8\pi^2}\int_V d^4 y \, \mathrm{Tr}_c G_{\mu\nu}\tilde{G}^{\mu\nu} \tag{21}$$

at the leading order of the cluster expansion

$$\langle A \exp B \rangle \approx \langle AB \rangle \exp(\langle B^2 \rangle /2) \tag{22}$$

valid for $\langle A \rangle = 0$ and $\langle B \rangle = 0$, one gets in this approximation:

$$J_\mu(x,\eta) = -\frac{\eta e^{-\eta^2/2L^2}}{\sqrt{2\pi^5 L^6}} \cdot \left[\int \frac{d^4q}{(2\pi)^4} e^{iqx} f_V(q) iq^\nu\right] \cdot \tilde{F}_{\mu\nu}. \tag{23}$$

Here $L^2 = l^2 + \langle n_V^2\rangle$ and the form factor is given by $f_V(q) = \int_V d^4y \exp(-iqy)$. In the infinite volume limit $\chi = \lim_{V\to\infty}\langle n_V^2\rangle/VN_f^2$ where χ is the standard topological susceptibility.

The expression (23) deserves a few comments. First, the right-hand side of (23) is an odd function of η as it should be, and at small η the current is linear in η. If the point x is far from $y \in V$ the current vanishes due to form-factor $f_V(q)$, i.e., the current flows only in the interaction volume V. On the other hand, if $x \in V$ and V is large enough to neglect surface terms, the current also vanishes as it should for any finite-volume effect. The volume scaling $\langle n_V^2\rangle \sim V$ for the phase with finite correlation length is another manifestation of the same fact.

It is worth mentioning that the maximal current is reached at $\eta \sim L$ and decreases as $J^{\max} \propto B/\tau L^2$ (where $\tau \sim V^{1/4}$). This result seems to be counterintuitive. Indeed, a naive picture would suggest that stronger fluctuations of the topological charge $\langle n_V^2\rangle$ are to correspond to stronger currents $J_\mu(x,\eta)$. This, in fact, is not the case. A rough physical explanation follows from (18): since the product of j_μ and ∂j^5 is fixed by electromagnetic anomaly (i.e. by the magnitude of an external Abelian field $F_{\mu\nu}$) large ∂j^5 corresponds to small j_μ and vice versa. Let us recall that according to the lattice data [43] the magnitude of topological charge fluctuations experience a rather sharp drop above the deconfinement transition. According to the above it means the effective *enhancement* of maximal possible electric current fluctuations. Of course, at too small $\langle n_V^2\rangle$ the Gaussian approximation (neglect of higher order correlators) we have used breaks down.

It is seen that the discussed effect is a result of the subtle interplay between strong and electromagnetic anomalies (see related remarks in [9]). While the later one is responsible for the correlation between vector and axial currents, the former anomaly provides nonconservation of an axial charge due to topological nonperturbative gluon fluctuations. The question about μ_5 distribution addressed in the introduction is translated here into the question about η distribution for experimental events.

IV. CHARGE FLUCTUATIONS ASYMMETRY AND POLARIZATION OPERATOR

Perhaps the most logically consistent way is to study transition matrix elements of (1) between states of opposite P-parity. This corresponds to

$$\langle\Omega|j_i j_k|\Omega\rangle \to \sum_A \langle\Omega|j_i|A\rangle\langle A|j_k|\Omega\rangle, \tag{24}$$

where the states $|\Omega\rangle$ and $|A\rangle$ have opposite P-parities and $\langle A|\Omega\rangle = 0$. Of course, the expression (24) is nothing but the electromagnetic polarization operator in the state $|\Omega\rangle$ saturated by particular states in spectral expansion.

This line of reasoning has been addressed in the literature before. Local averages like $\langle j_\mu^2(x)\rangle$ were computed in pioneering studies of CME on the lattice [44,45] and many interesting patterns were found. Later nonlocal averages $\langle j_\mu(x) j_\nu(y)\rangle$ are computed [46,47]. We find it worth recalling once again that since the typical correlators we are interested in are given by dimension six operators, their local matrix elements are strongly UV-singular

$$\langle j_\mu^2(x)\rangle_F \propto \Lambda^6 + F^2\Lambda^2 + \text{UV-finite}, \tag{25}$$

where Λ is an UV cutoff and F-external field strength. Even subtracted average $\langle j_\mu^2(x)\rangle_F - \langle j_\mu^2(x)\rangle_0$ is divergent. In actual numerical calculations on the lattice [44,45] this $F^2\Lambda^2$ term is not seen. This could be a consequence of the specific structure of UV cuts used in [44,45]. Anyway, this problem presents an analytical challenge for any attempt to describe CME in terms of local matrix elements. To our view this is a clear signal about the intrinsic nonlocal nature of the discussed phenomenon.

The polarization operator in the CME context is studied in [23]. There are two main differences between our approach and that of the cited paper. First, the regular contribution (given by the polarization operator in magnetic field) and CME contribution (proportional to μ_5) are separated from the beginning in [23] [in some sense, quantum currents are superimposed on top of the classical current (1)]. We follow another logic and consider the polarization operator as the only source of asymmetric charge fluctuations, but extract a particular form factor from it, which corresponds to negative parity intermediate states. Second, the expression for charge fluctuations observable as a functional depending on a polarization operator is different in our paper from that of [23]. We will make more comments on that below.

In this section we discuss a P-even observable which is a product of two P-odd contributions. The role of the former is played by the current correlator $\langle j_\mu j_\nu\rangle$. It seems physically clear that this object should contain some information about charge distribution (4). The exact form of this correspondence is, however, far from trivial. One could think of several ways to relate these quantities. Before presenting our approach to this problem let us mention other methods used in the literature. First, we notice that the current in the ϕ direction is given by

$$\mathbf{e}_z j_3 + \mathbf{e}_y j_2 = \sqrt{j_3^2 + j_2^2}(\mathbf{e}_z \sin\phi + \mathbf{e}_y \cos\phi), \tag{26}$$

and the corresponding charge difference from (4) is

$$\left\langle \int \frac{d(N_+ - N_-)}{d\phi} d\phi \int \frac{d(N'_+ - N'_-)}{d\phi'} d\phi' \right\rangle_e, \tag{27}$$

where by the brackets $\langle\ldots\rangle_e$ we denote the average over events. One has $\langle(a_+ - a_-)^2\rangle_e \propto \langle j_3^2 + j_2^2\rangle$, where the current product is assumed to be local. This is very close (but different) to the definition used in [44]. It is natural to expect that positive-definite $\langle (a_+ - a_-)^2 \rangle_e$ should be nonzero even without any magnetic field.

Another relation is suggested in [23]. It is written in terms of event average of the cosine, where $\alpha, \beta = +, -$ and $N_\pm$ is the total number of outgoing particles of a given charge:

$$\langle \cos(\phi_\alpha + \phi'_\beta) \rangle_e \propto \frac{\alpha\beta}{N_\alpha N_\beta}(j_2^2 - j_3^2), \tag{28}$$

where, up to some background terms

$$\langle \cos(\phi_\alpha + \phi'_\beta) \rangle_e = \langle v_{1,\alpha} v_{1,\beta} \rangle_e - \langle a_\alpha a_\beta \rangle_e. \tag{29}$$

Assuming charge independence of $v_{1,\alpha}$ and equal numbers of particle species $N_+ = N_- = N$ one gets $\langle (a_+ - a_-)^2 \rangle_e \propto \langle j_3^2 - j_2^2 \rangle$ if one neglects the $v_{1,\alpha}$ term with respect to the a_α term. In fact, the leading term, which is always contained in the j_3 component, coincides for both expressions, while the procedure of taking into account fluctuations in the reaction plane is different.

In this paper we use an alternative signature provided by charge density fluctuations and not spatial components of the currents. An attractive feature of this quantity is that it is well defined even in the static limit. To this end consider the electric charge in some spatial volume V at temperature T:

$$eQ_V = e\int_V d\mathbf{x}\, j_0(x). \tag{30}$$

Since we work in the zero density approximation the quantum average of this object vanishes:

$$\langle Q_V \rangle = 0. \tag{31}$$

This is not the case for its square:

$$\langle Q_V^2 \rangle = -\hat{\kappa}\int_V d\mathbf{x}\int_V d\mathbf{x}'\,\Pi_{44}(x, x'), \tag{32}$$

where $\Pi_{44}(x, x')$ is Euclidean polarization operator in the constant external field $F_{\mu\nu}$ and at temperature T Wick-rotated from the standard Minkowski expression $\Pi_{00}^{(M)}(x, x')$:

$$\Pi_{\mu\nu}^{(M)}(x, x') = i\langle T\{j_\mu(x) j_\nu(x')\}\rangle_{F,T}, \tag{33}$$

with $j_\mu = \bar{\psi} Q \gamma_\mu \psi$; $\Pi_{\mu\nu}^{(M)} \leftrightarrow \Pi_{\mu\nu}^{(E)}$, notice the sign convention (32) corresponding to positive-definite $\langle Q_V^2 \rangle$ in the static limit. In the standard way we denote

$$\Pi_{\mu\nu}(q) = \int d^4x e^{-iq(x-x')}\Pi_{\mu\nu}(x, x'), \tag{34}$$

with $\mu, \nu = 1, 2, 3, 4$, and $\mathbf{q} = (q_1, q_2, q_3)$, $q_\perp = (q_1, q_2)$.

The operator $\hat{\kappa}$ in (32) accounts for the temporal profile of the process. In terms of momentum space components, (32) takes the following form:

$$\langle Q_V^2 \rangle = -\int \frac{dq_4}{2\pi}\kappa(q_4)\int \frac{d\mathbf{q}}{(2\pi)^3}|F_V(\mathbf{q})|^2 \Pi_{44}(\mathbf{q}, q_4), \tag{35}$$

where the form factor $F_V(\mathbf{q}) = \int_V d\mathbf{x}\exp(i\mathbf{q}\mathbf{x})$ keeps information about the spatial profile of the volume V, while the temporal factor $\kappa(q_4) = \int d\tau g(\tau)\exp(iq_4\tau)$ encodes a temporal (in the Euclidean sense) profile. For the finite temperature case we consider here the standard Matsubara replacements $q_4 \to \omega_n = 2\pi n T$ and $(2\pi)^{-1} \times \int dq_4 \to T\sum_n$ that are to be performed. The choice $g(\tau) = T$ we will adopt in the rest of the paper physically corresponds to the static limit where only the lowest Matsubara frequency $n = 0$ contributes:

$$\langle Q_V^2 \rangle_{\rm st} = -T\int \frac{d\mathbf{q}}{(2\pi)^3}|F_V(\mathbf{q})|^2 \Pi_{44}(\mathbf{q}, 0). \tag{36}$$

Using the expressions from the Appendix it can be checked that in the thermodynamic limit $V \to \infty$ without an external field one reproduces the standard Stefan-Boltzmann answer for elementary fermions $\lim_{V\to\infty}\langle e^2 Q_V^2 \rangle_{\rm st}/V = e^2T^3/3$. In case of quarks one should of course understand eB as $q_f eB$ and introduce additional trace over flavors with the factor $N_c Q^2$: $\Pi_{44}^{eB,T} \to N_c \sum_f q_f^2 \Pi_{44}^{q_f eB,T}$. For the sake of brevity we will use the simple notation as for elementary fermions of the unit electric charge having in mind the necessity to make the replacement discussed above in the final answers.

In the limiting case of no background $B = 0$, $T = 0$ one has $\Pi_{44}(\mathbf{q}, q_4) = \mathbf{q}^2\Pi(q^2)$ and, at the leading order, for large 4-volumes V_4:

$$\langle Q_V^2 \rangle_{B=0,T=0} \propto \Pi'(0) \cdot V_4^{-1/2}, \tag{37}$$

where the condition of gauge invariance $\Pi(0) = 0$ has been taken into account and the volume $V_4 = R^3 \times t$ is assumed to be uniform: $R \sim t$. Thus, the expression (32) is UV safe and vacuum charge fluctuations in a given space-time region are a purely finite-volume effect.

We can now come back to the definition (32) and rewrite the coordinate integration in cylinder coordinates with axis 1 as the polar axis and angle ϕ defined in the 23 plane. This is the same notation as in (4); notice that in the standard setup the azimuthal angle is usually defined in plane 12. This allows us to represent the form factor $F_V(\mathbf{q})$ as

$$F_V(\mathbf{q}) = \int dx_1 e^{iq_1x_1}\int_0 \rho d\rho \int_0^{2\pi} d\phi e^{i\bar{q}\rho}, \tag{38}$$

where $\bar{q}\rho = q_2x_2 + q_3x_3 = q_2\rho\cos\phi + q_3\rho\sin\phi$ and the structure of the integration upper limit is determined by the chosen model for spatial distribution (the sharp boundary, smoothed boundary, Gaussian shape, exponential shape etc.). The $\sin\phi$ mode in the Fourier expansion of (38) is multiplied by the following coefficient:

$$c_1 = (1/\pi)\int_0^{2\pi} d\phi \sin\phi e^{i\bar{q}\rho} = \frac{2iq_3}{\hat{q}}J_1(\hat{q}\rho), \tag{39}$$

where $\hat{q} = \sqrt{q_2^2 + q_3^2}$. Thus, we have for the expansion of (36) in harmonics:

$$\langle Q_V^2 \rangle = \ldots + \int_0^{2\pi} d\phi \sin\phi \int_0^{2\pi} d\phi' \sin\phi' \langle (q_V^a)^2 \rangle + \ldots, \tag{40}$$

where $\langle (q_V^a)^2 \rangle$ is given by the same expression (36) with the change $F_V(\mathbf{q}) \to f_V(q_1, q_2, q_3)$ where

$$f_V(q_1, q_2, q_3) = \frac{2iq_3}{\hat{q}} \int dx_1 e^{iq_1 x_1} \int_0 \rho J_1(\hat{q}\rho) d\rho. \tag{41}$$

In the same way $\langle (q_V^{v_1})^2 \rangle$ corresponds to the exchange $q_3 \leftrightarrow q_2$ and $\sin\phi \leftrightarrow \cos\phi$. Making use of (4), (27), and (40) we obtain the following relation for the asymmetry:

$$\langle q_V^2 \rangle = \langle (q_V^a)^2 \rangle - \langle (q_V^{v_1})^2 \rangle = - \sum_{\alpha,\beta=\pm} \alpha\beta \cos(\phi_\alpha + \phi'_\beta), \tag{42}$$

$$\begin{aligned} \langle q_V^2 \rangle &= N^2 \cdot (\langle (a_+ - a_-)^2 \rangle_e - \langle (v_{1,+} - v_{1,-})^2 \rangle_e) \\ &= T \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{q_3^2 - q_2^2}{q_3^2 + q_2^2} \\ &\quad \times \left| \int dx_1 e^{iq_1 x_1} \int_0 \rho J_1(\hat{q}\rho) d\rho \right|^2 \Pi_{44}(\mathbf{q}, 0). \end{aligned} \tag{43}$$

It is obvious that the above expression has to be proportional to the magnetic field since there are no other $O(3)$-violating factors in the problem. The effect we are looking for corresponds to the *strong enhancement* of (43) in an external magnetic field and hence, from experimental point of view, strong dependence of (43) on centrality. It is to be stressed that the multiplicity factor N^2 is by itself strongly centrality dependent. This dependence is kinematical and has nothing to do with the magnetic field dependence of $\langle q_V^2 \rangle$. Only the latter lies at the heart of CME.

V. GENERAL STRUCTURE OF POLARIZATION OPERATOR

In this section we analyze the general structure of the polarization operator in the background of the nonzero temperature and magnetic field. As is clear from the above discussion, this is a necessary prerequisite before one can compute the charge fluctuation asymmetry (43).

First, let us make a few general comments about the space-time dependence of the current-current correlator. In the confinement phase (i.e. at sufficiently low temperatures) the polarization operator is saturated by the lowest lying meson states allowed by quantum numbers. For a correlator of vector currents it is of course ρ meson at the zero field/temperature. The situation changes, however, at the nonzero external field since quantum numbers can be exchanged between currents and the background in this case. For example, absorbing one quantum of the electromagnetic field from the background ρ meson can be converted to pion (the $\langle VVP \rangle$ correlation). At large distances the lightest resonance will always dominate, so (for the weak magnetic field) one expects the general structure of Euclidean polarization operator of the following form, motivated by perturbation theory in the external field B:

$$\langle j(x)j(x') \rangle \propto e^{-m_\rho |x-x'|} + C(B) \cdot e^{-m_\pi |x-x'|}, \tag{44}$$

with $C(B) \propto B^2$. This interesting effect of different parity states mixing in an external field is similar to the one observed a long time ago in [48] at finite temperature. In that case the existence of a thermal bath of pions makes it possible to convert the vector current to the pseudoscalar one by absorbing one pion from the bath (the $\langle VPP \rangle$ correlation) and the amplitude of this process is proportional to T in the chiral limit. Of course this physical analogy between finite B and finite T cases is not precise, for example, the tensor structure is absolutely different.

Thus, the long-distance correlations are saturated by the lightest degrees of freedom (i.e. pions in the confinement phase). On the other hand, in the deconfinement phase at strong fields, if the Larmor radius is much smaller than $\Lambda_{\mathrm{QCD}}^{-1}$ no quarks can propagate in the transverse direction at all:

$$\langle j(x)j(x') \rangle \propto e^{-eB(x-x')_\perp^2/2}. \tag{45}$$

Large-N_c suppressed transverse correlations are possible only due to the gluon degrees of freedom.

We confine our attention in what follows to a particular case of a purely magnetic constant Abelian background field $F_{\mu\nu}$ in the thermal bath rest frame at nonzero temperature T. We have chosen $F_{12} = -F_{21} = B$, i.e., the magnetic field is directed along the third axis. The temperature effects break Lorentz invariance and the physical answers depend on a 4-vector u_μ which represents four velocity of the thermal bath. It is normalized as $u_\mu u^\mu = 1$. In the present paper we take zero chemical potential $\mu = 0$. It is to be noticed that many general conclusions concerning the structure of polarization operator stay intact for $\mu \neq 0$ since the latter is associated with the same four vector u_μ given by $u_\mu = (1, 0, 0, 0)$ in the medium rest frame.

The polarization operator (34) is a rank two tensor depending on two polar vectors q_μ and u_μ and antisymmetric tensor $F_{\mu\nu}$. The general decomposition of (34) in terms of independent tensors was extensively studied in the literature starting from [49,50], see [51] for a recent exposition and [52] for a useful collection of references. Generally, one is to deal with $4 \times 4 = 16$ independent tensor structures, built by multiplying the four independent base vectors q_μ, u_μ, $q^\alpha F_{\alpha\mu}$, $q^\alpha F_\alpha^\beta F_{\beta\mu}$. It can be shown, however, that general requirements of being transversal

$$q^\mu \Pi_{\mu\nu}(q) = q^\nu \Pi_{\mu\nu}(q) = 0 \tag{46}$$

and Bose symmetric $\Pi_{\mu\nu}(q) = \Pi_{\nu\mu}(-q)$ together with generalized Furry's theorem [49]

$$\Pi_{\mu\nu}(q, u, F) = \Pi_{\mu\nu}(q, -u, -F) \tag{47}$$

reduce the number of independent tensor structures to six. Two of them are field independent, the other two depend on $F_{\mu\nu}$ linearly, and the last two—quadratically (notice that our numeration of the tensors is different from the one adopted in [49]). Their explicit form reads

$$\begin{aligned}
\Psi^{(1)}_{\mu\nu} &= q^2\delta_{\mu\nu} - q_\mu q_\nu,\\
\Psi^{(2)}_{\mu\nu} &= (q^2u_\mu - q_\mu(uq))(q^2u_\nu - q_\nu(uq)),\\
\Psi^{(3)}_{\mu\nu} &= (uq)(q_\mu F_{\nu\rho}q^\rho - q_\nu F_{\mu\rho}q^\rho + q^2F_{\mu\nu}),\\
\Psi^{(4)}_{\mu\nu} &= (u_\mu F_{\nu\rho}q^\rho - u_\nu F_{\mu\rho}q^\rho + (uq)F_{\mu\nu}),\\
\Psi^{(5)}_{\mu\nu} &= F_{\mu\rho}q^\rho F_{\nu\sigma}q^\sigma,\\
\Psi^{(6)}_{\mu\nu} &= (q^2\delta_{\mu\rho} - q_\mu q_\rho)F^\rho_\alpha F^{\alpha\sigma}(q^2\delta_{\sigma\nu} - q_\sigma q_\nu).
\end{aligned} \tag{48}$$

The coefficient functions of the decomposition

$$\Pi_{\mu\nu}(q, u, F) = \sum_{i=1}^{6} \pi^{(i)} \cdot \Psi^{(i)}_{\mu\nu} \tag{49}$$

depend on q^2, mixed invariants $(uq)^2$, $(qF)^2$, $(uF)^2$, $(qFu)^2$, pure field invariants F^2, $F\tilde{F}$, and also the temperature T and particle data, encoded in matrices Q and M. The expression (49) allows us to discuss the current correlations' asymmetries in an invariant way in any theory where the expression for polarization operator can be obtained.

Having these general prerequisites let us come back to the analysis of the correlation patterns. For our choice $F_{12} = B$ the invariants $(uF)^2$, $(qFu)^2$, and $F\tilde{F}$ are equal to zero. In what follows we will be especially interested in a particular type of contribution to $\Pi_{\mu\nu}(q)$ proportional to the tensor structure $\Psi^{(7)}_{\mu\nu}$ given by the product of two axial vectors

$$\Psi^{(7)}_{\mu\nu} = \tilde{F}_{\mu\rho}q^\rho \tilde{F}_{\nu\sigma}q^\sigma. \tag{50}$$

It is not independent and one easily checks that $\Psi^{(7)}_{\mu\nu}$ can be expressed as a linear combination of (48)

$$q^2\Psi^{(7)}_{\mu\nu} = (q^2F^2/2 - (qF)^2)\Psi^{(1)}_{\mu\nu} + q^2\Psi^{(5)}_{\mu\nu} + \Psi^{(6)}_{\mu\nu}. \tag{51}$$

Let us consider the tensor structure of the polarization operator in more detail. First, since we are interested only in diagonal 11, 22, 33, 44, and also 34 components in this paper, we have no contributions from $\pi^{(3)}$ and $\pi^{(4)}$ because the tensors $\Psi^{(3)}_{\mu\nu}$ and $\Psi^{(4)}_{\mu\nu}$ are antisymmetric and also vanish for $\mu = 3$, $\nu = 4$ in the chosen background field. Second, we notice that for μ, ν equal to 3 or 4, one has identically $\Psi^{(5)}_{\mu\nu} = 0$. Adopting conventional notation: $q_\perp = (q_1, q_2)$, $q_\parallel = (q_3, q_4)$ we can rewrite (49) using (51) as

$$\Pi_\parallel(q) = \pi^{(Q)} \cdot \Psi^{(1)}_\parallel + \pi^{(T)} \cdot \Psi^{(2)}_\parallel + \tilde{\pi}^{(F)} \cdot \Psi^{(7)}_\parallel, \tag{52}$$

where the new invariant functions are given by

$$\begin{aligned}
\pi^{(Q)} &= \pi^{(1)} - (q^2F^2/2 - (qF)^2)\pi^{(6)},\\
\pi^{(T)} &= \pi^{(2)}, \qquad \tilde{\pi}^{(F)} = q^2\pi^{(6)}.
\end{aligned} \tag{53}$$

As for the diagonal correlators in the 12 plane, one has

$$\Pi_\perp(q) = \pi^{(Q)} \cdot \Psi^{(1)}_\perp + \pi^{(T)} \cdot \Psi^{(2)}_\perp + \pi^{(F)} \cdot \Psi^{(5)}_\perp, \tag{54}$$

where $\pi^{(Q)}$ and $\pi^{(T)}$ are defined by the same expressions (53) while the $\pi^{(F)}$ form factor reads

$$\pi^{(F)} = \pi^{(5)} - q^2\pi^{(6)}. \tag{55}$$

It is seen that the correlators of our interest can be decomposed into just three independent structures. The first, $\pi^{(Q)}$, corresponds to purely quantum fluctuations. It has a nonzero limit at both $B \to 0$ and $T \to 0$, which coincides in this case with the textbook expression for polarization operator. The second structure, $\pi^{(T)}$, is responsible for thermal fluctuations. It vanishes at $T \to 0$. It is worth mentioning that both functions $\pi^{(Q)}$ and $\pi^{(T)}$ depend on the temperature and external field (since the pattern of both quantum and thermal fluctuations is sensitive to the external conditions) and our notation corresponds, rather, to the limiting form of these functions.

We notice that the terms proportional to $\pi^{(Q)}$ and $\pi^{(T)}$ are identical in (52) and (54) up to an obvious change of notation $\parallel \leftrightarrows \perp$. This is to be expected since quantum and thermal fluctuation are $\mathbb{O}(3)$ isotropic. The only nonisotropic terms (and the most interesting for us here) are the last terms $\tilde{\pi}^{(F)}$ in (52) and $\pi^{(F)}$ in (54). The former one takes into account charge (and also the current component j_3) fluctuations induced by the external magnetic field. The P-parity structure of this term is given by

$$\begin{aligned}
\delta_B\langle j_3 j_3\rangle &= \tilde{\pi}^{(F)} \times \tilde{F}_{3\rho}p^\rho \times \tilde{F}_{3\sigma}p^\sigma,\\
\text{P-even} &= \text{P-even} \times \text{axial} \times \text{axial}.
\end{aligned}$$

It is to be compared with the thermal contribution proportional to $\Psi^{(2)}_\parallel$

$$\begin{aligned}
\delta_T\langle j_3 j_3\rangle &= \pi^{(T)} \times p_3(\text{up}) \times p_3(\text{up}),\\
\text{P-even} &= \text{P-even} \times \text{vector} \times \text{vector}.
\end{aligned}$$

This directly corresponds to our discussion in the introduction: in the latter case the thermal fluctuations are distributed isotropically in the thermal bath rest frame, while in the former one there are electric currents fluctuating along the magnetic field. The magnitude of these fluctuations is measured by the function $\tilde{\pi}^{(F)}$, and no physical principle forces it to vanish either below or abovethe critical temperature. Physically, $\tilde{\pi}^{(F)}$ corresponds to P-odd intermediate states in the polarization operator.

The function $\pi^{(F)}$ entering (54) is a sum of two terms according to (55). This also is to be expected. Charged particles flowing in the plane perpendicular to the magnetic field are deflected by the Lorentz force, and this *diamagnetic* effect is taken into account by the form factor $\pi^{(5)}$. It is absent in $\Pi_{\parallel}$. But the particle's spin interacts with the field by means of a $\sigma_{\alpha\beta}\mathbf{F}^{\alpha\beta}$ term in $\Pi_{\parallel}$ as well as in $\Pi_{\perp}$, which results in the factor $q^2\pi^{(6)}$ in both expressions (52) and (54). It is worth noting that according to our general logic the electric charge asymmetry is computed for the full expression for Π_{44}, not just from some part of it, proportional to $\tilde{\pi}^{(F)}$. Thus, it is legitimate to speak about the CME interpretation of the answer (43) only in the limiting case when $\tilde{\pi}^{(F)}$ provides the dominant contribution. We discuss that in more detail below.

VI. MODEL EXAMPLES

We analyze in this section two limiting cases where one can construct $\tilde{\pi}^{(F)}$ in an explicit way. The first one corresponds to weak magnetic fields in the confinement phase. In this case the intermediate states are hadron resonances of negative P-parity (see a closely related discussion in [53]). However, to select explicitly physical states making dominant contribution is far from trivial and the answer strongly depends on kinematics. We confine ourselves in this paper to the simplest case keeping only three neutral 0^{-+} intermediate states: π^0, η, η'. Technically, it is more convenient to consider from the very beginning matrix elements of vector currents between vacuum and these states in external fields. Making use of the definition of the off-shell vector-vector-axial form factor $\mathcal{F}_\pi \equiv \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q^2, q_1^2, q_2^2)$ (see, e.g., [54]) with $q = q_1 + q_2$,

$$\int dx \int dy e^{iq_1x+iq_2y}\langle 0|\mathrm{Tr}\{j_\mu(x)j_\nu(y)\}|\pi^0(q)\rangle = \epsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta \mathcal{F}_\pi(q^2, q_1^2, q_2^2), \quad (56)$$

one gets at the leading order in the constant external field:

$$\langle 0|j_\mu(-q)|\pi^0(q)\rangle_F = ieq^\rho \tilde{F}_{\rho\mu}\mathcal{F}_\pi(q^2, q^2, 0). \quad (57)$$

The expressions for η and η' contributions are completely analogous with the replacement of $\mathcal{F}_\pi$ by $\mathcal{F}_\eta$ and $\mathcal{F}_{\eta'}$.

Thus, the q^2 dependence of the polarization operator in an external field is determined in this approximation by the form factors $\mathcal{F}_\phi(q^2, q^2, 0)$ with one on-shell leg (corresponding to external field vertex). These form factors are essentially nonperturbative QCD objects. Let us recall that on-shell [i.e. at the point $\mathcal{F}_\phi(m_\phi^2, 0, 0)$] they are fixed by the triangle anomaly, for example, for pion:

$$\mathcal{F}_\pi(m_\pi^2, 0, 0) = -\frac{N_c}{12\pi^2 F_\pi}. \quad (58)$$

Another important case is the large $q^2 \to \infty$ limit where one has (for chiral fermions) $\mathcal{F}_\phi(q^2, q^2, 0) \to \chi_F F_\pi/3$ where χ_F is a QCD quark condensate magnetic susceptibility, defined by $\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F = e_q\chi_F\langle\bar{q}q\rangle F_{\mu\nu}$. Different approximation schemes valid at intermediate momenta are discussed in the literature (see, e.g., [55]).

Having written the field-dependent matrix element (57) one is able to express the invariant function $\tilde{\pi}^{(F)}(q^2)$ as follows:

$$\tilde{\pi}^{(F)}(q^2) = \sum_{\phi=\pi,\eta,\eta'} \frac{|\mathcal{F}_\phi(q^2, q^2, 0)|^2}{q^2 - m_\phi^2}. \quad (59)$$

From the point of view of expression (43) the dominant contribution to asymmetry is this phase that comes from the lightest degree of freedom, i.e., massless in the chiral limit pion (to be more precise, we assume the limit $m_\pi R \ll 1$). Choosing for concreteness the Gaussian boundary condition [i.e. introducing the factors $\exp(-q_i^2R^2/2)$ into (43)] one obtains

$$\langle q_V^2\rangle = \gamma\left(\frac{eB}{F_\pi}\right)^2 TR^3, \quad (60)$$

where the numerical factor $\gamma = 1.6 \times 10^{-4}$ is of course specific for this boundary choice. Certainly the result trivially follows from dimensional considerations. We see $\langle q_V^2\rangle \ll 1$ for a phenomenologically reasonable choice of parameters. Contributions of mass gapped states bring additional suppression (and, in particular, break $\sim R^3$ scaling).

As the second example we consider free fermions in the strong field limit. This regime would correspond to the deconfinement phase where proper dynamical degrees of freedom are quarks and gluons with perturbatively weak interaction between each other. To compute the polarization operator under external conditions in perturbation theory one usually makes use of the Schwinger proper-time technique and there is extensive literature on the subject [56–60] where different kinds of external backgrounds were studied. The polarization operator in a constant magnetic field and at nonzero temperature was calculated in [61] in an imaginary time formalism. Our aim here is to put these results in a charge fluctuations asymmetry perspective. For the reader's convenience we reproduce the explicit one-loop expressions for the polarization operator $\Pi_{\parallel}$ given by [61] in the Appendix of the present paper.

It is convenient to present the Euclidean polarization operator in the following form:

$$\Pi_{\mu\nu}(q_\perp, q_3, n) = \sum A_{\mu\nu}(q)e^{-\phi(q)} + Q_{\mu\nu}(q), \quad (61)$$

where the sum includes integration over proper times and summation over Matsubara frequencies [see expression (A3) in the Appendix], the functions $A_{\mu\nu}[q]$ polynomially depend on momenta components q, and the universal Euclidean phase $\phi(q)$ is given by expression (A4). The contact terms $Q_{\mu\nu}(q)$ have no sensitivity to infrared parameters (like temperature or external field) and provide the correct limit of $\Pi_{\mu\nu}$ at vanishing background.

One can notice that $\tilde{\pi}^{(F)}$ can be simply related to the polarization operator components. Namely, solving the system of three linear equations (52) for the choices $(\mu\nu) = 44$, 33, and 34 one finds all three invariant form factors, including $\tilde{\pi}^{(F)}$:

$$B^2\tilde{\pi}^{(F)} = -\frac{q_3q_4\Pi_{44} + (q_\perp^2 + q_3^2)\Pi_{34}}{q_\perp^2 q_3 q_4}, \tag{62}$$

where $q_\perp^2 = q_1^2 + q_2^2$ and $q_4 \equiv \omega_n = 2\pi T n$.

Thus, the chiral magnetic form factor is a nontrivial linear combination of Π_{34} and Π_{44}. First, we are to check that at $B \to 0$ the right-hand side of (62) vanishes. This is obvious at zero temperature since in this case there is the only tensor structure given by $\Psi^{(1)}_{\mu\nu}$ and

$$q_3q_4\Psi^{(1)}_{44} + (q_\perp^2 + q_3^2)\Psi^{(1)}_{34} = 0. \tag{63}$$

For temperature dependent parts it is rather nontrivial, the proof that this is indeed the case can be found in the Appendix.

The explicit expression for $\tilde{\pi}^{(F)}$ looks especially simple in the small T regime. It reads

$$\tilde{\pi}^{(F)} = -\frac{1}{(4\pi)^2}\frac{1}{eB}\int_\epsilon^\infty du \int_{-1}^{+1} dv((1-v^2)\coth\bar{u} + f_\perp(\bar{u}, v))\exp(-\phi^{(0)}), \tag{64}$$

where $\bar{u} = ueB$ and the functions $\phi^{(0)}$ and $f_\perp(\bar{u}, v)$ are given in the Appendix. Notice that such a form factor was discussed in a different context in [50].

In the weak field limit one has

$$\lim_{B\to 0}\tilde{\pi}^{(F)} = \frac{1}{6\pi^2}\int_{-1}^{1} dv \frac{(1-v^2)(3-v^2)}{(4m^2 + (1-v^2)q^2)^2}. \tag{65}$$

In the strong field limit (still at small T) the situation becomes more interesting—the form factor $\tilde{\pi}^{(F)}$ provides the dominant contribution to the polarization operator:

$$\Pi_{44} \to q_3^2(eB)^2\tilde{\pi}^{(F)} \to -\frac{eB}{4\pi^2}e^{-q_\perp^2/(2|eB|)}\int_{-1}^{1} dv \frac{(1-v^2)q_3^2}{4m^2 + (1-v^2)q_3^2} \tag{66}$$

up to the terms $\mathcal{O}(q_\perp^2/eB)$. One can say that all asymmetry of charge fluctuations is due to a CME-like form factor in this limit.

We see another interesting effect. In the chiral limit (66) does not depend on q_3 at all, while the dependence on $q_\perp$ is suppressed by the field B. On the other hand, the essence of the asymmetry of interest is just different dependence of the polarization operator on different components of momentum. Since the polarization operator itself linearly rises with B for the strong field it is not *a priori* clear which effect is to win. Detailed calculations show that in fact they balance each other and the asymmetry (43) is not asymptotically rising with B—there is an effect of saturation. It is reasonable to separate different regimes depending on ratios between basic parameters such as B, m, T, and R where the latter one stays for the typical 3-dimensional size of the volume V_3. For two light flavors one can safely neglect quark masses m. Three other parameters are in the ballpark of 100 MeV (for a large fireball one can think of phenomenologically realistic $eBR^2 = 5 \div 10$). Without the intention to cook up numerical factors but just to get feeling of the numbers, plugging (66) into (43) we get

$$\langle q_V^2\rangle = \gamma' \cdot RT, \tag{67}$$

where again the numerical factor $\gamma' = 4.1 \times 10^{-2}$ corresponds to the Gaussian boundary shape. Thus, for asymptotically large B one reaches the "kinematical limit" for the asymmetry in our picture, despite the fact that numerically it is still very small.

VII. CONCLUSIONS

We have discussed three possible ways to study quantum physics behind the chiral magnetic effect and electric charge fluctuation asymmetry observed in heavy-ion collisions. For all approaches the importance of scale separation is stressed—there should be a hierarchy of dynamical scales characterizing the life of the quark-gluon phase after the collision and intrinsic QCD scales (perhaps field/temperature shifted) characterizing the non-Abelian topological charge fluctuation pattern. The physical essence of CME as we tried to present it here is that the quark-gluon medium plays the role of a measuring device with respect to the topological QCD vacuum with the final particle's electric charge asymmetry as an outcome. This is most clearly illustrated by the expression (23).

The third approach we have considered is somewhat different because it provides nonzero results even for free fermions in the magnetic field, i.e., without any "topological origin." We believe that this can be considered as a particular case of CME as well. Just the nonzero matrix element of the vector current between the vacuum and J^{-+} states in external magnetic field leads to an asymmetric charge/current pattern as if there is a fluctuating vector current collinear to $\mathbf{B}$. Of course the detailed picture depends on the actual quantum dynamics of these J^{-+} degrees of freedom, and we have shown that, indeed, it is strongly suppressed in the confinement phase. Nevertheless we find it legitimate to interpret this dynamics using the same CME-like language since, namely, this anomaly-driven vector-axial correlation is at the heart of the effect, while the concrete way of life of the axial degrees of freedom (distribution function for μ_5 in the standard CME analysis) is of secondary importance.

We have left without attention all aspects of temperature dynamics in this paper. Despite the fact that no drastic qualitative effects are expected it is interesting to study the asymmetry in the whole parameter space spanned by (B, T, m, R). This could clearly have phenomenological applications to heavy-ion collision physics whose understanding is the main challenge for modern QCD.

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APPENDIX

For the reader's convenience we present explicit expressions for the one-loop polarization operator as computed in [61]. It reads

$$\Pi_{\mu\nu}(\mathbf{q}, q_4) = -T\int\frac{d\mathbf{p}}{(2\pi)^3}\sum_{l=-\infty}^{\infty} \times \mathrm{Tr}\{\gamma_\mu S_l(\mathbf{p})\gamma_\nu S_{l-n}(\mathbf{p}-\mathbf{q})\} + Q_{\mu\nu}(q), \tag{A1}$$

where $S_l(\mathbf{p})$ is a fermion propagator in an external constant magnetic field and $Q_{\mu\nu}(q)$ is the "contact term" needed to cancel the ultraviolet divergencies. It has no dependence on soft backgrounds like temperature or external field. The sum goes over fermionic Matsubara frequencies $\hat{\omega}_l = (2l+1)\pi T$, while the bosonic one is $q_4 = \omega_n = 2n\pi T$.

Thus, the general structure of Euclidean polarization operator is given by

$$\Pi_{\mu\nu}(q_\perp, q_3, n) = \sum A_{\mu\nu}(q)e^{-\phi(q)} + Q_{\mu\nu}(q). \tag{A2}$$

The sum is given by the following expression:

$$\sum = \frac{T}{4\pi}\frac{eB}{\sqrt{\pi}}\int_\epsilon^\infty du\sqrt{u}\int_{-1}^1 dv\sum_{l=-\infty}^{\infty}. \tag{A3}$$

The universal phase $\phi(q)$ has the form

$$\begin{aligned}\phi(q) &= \phi^{(0)}(q) + uW_l^2 \\ &= \frac{q_\perp^2}{eB}\frac{\cosh\bar{u} - \cosh\bar{u}v}{2\sinh\bar{u}} \\ &\quad + u\left[m^2 + W_l^2 + \frac{1-v^2}{4}(q_4^2 + q_3^2)\right],\end{aligned} \tag{A4}$$

where $W_l = \hat{\omega}_l - \frac{1-v}{2}\omega_n$ and $\bar{u} = ueB$. The contact term is given by

$$Q_{\mu\nu} = \frac{1}{12\pi^2}\int_\epsilon^\infty\frac{du}{u}e^{-um^2}(q^2\delta_{\mu\nu} - q_\mu q_\nu). \tag{A5}$$

The function $A_{\mu\nu}(q)$ polynomially depends on momenta components q and for 3, 4 components reads

$$A_{44}(q) = \coth\bar{u}\left(\frac{1}{u} - 2W_l^2 + q_4 vW_l - \frac{1-v^2}{2}q_3^2\right) + \frac{q_\perp^2}{2}f_\perp(\bar{u}, v), \tag{A6}$$

$$A_{34}(q) = q_3\left[vW_l + \frac{1-v^2}{2}q_4\right]\coth\bar{u}, \tag{A7}$$

$$A_{33}(q) = -\coth\bar{u}\left[q_4^2\frac{1-v^2}{2} + q_4 vW_l\right] + \frac{q_\perp^2}{2}f_\perp(\bar{u}, v), \tag{A8}$$

where

$$f_\perp(\bar{u}, v) = \frac{v\coth\bar{u}\sinh\bar{u}v - \cosh\bar{u}v}{\sinh\bar{u}}. \tag{A9}$$

To get the zero temperature limit of the above expressions, the Poisson summation formula is useful

$$\sum_{l=-\infty}^{\infty} e^{-a(l-z)^2} = \left(\frac{\pi}{a}\right)^{1/2}\sum_{k=-\infty}^{\infty} e^{-(\pi^2k^2/a)-2\pi izk}. \tag{A10}$$

In particular one gets

$$\lim_{T\to 0} T\sum_{l=-\infty}^{\infty} e^{-uW_l^2} = \frac{1}{2\sqrt{u\pi}}. \tag{A11}$$

Another necessary demonstration of self-consistency is proof of vanishing of $B^2\tilde{\pi}^{(F)}$ defined by (62) at $B = 0$ for any T. One has, at $B \to 0$,

$$\begin{aligned}q_3q_4A_{44} + \mathbf{q}^2A_{34} &= \frac{q_3}{u}\left(q_4\left(\frac{1}{u} - 2W_l^2\right) + vW_lq^2\right) \\ &= \frac{q_3}{2u^2}\frac{d}{dv}(W_le^{-u(W_l^2+[(1-v^2)/4]q^2)}).\end{aligned} \tag{A12}$$

It is easy to check that the latter expression gives zero result when integrated from −1 to 1.

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