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Spectral representation for baryon to meson transition distribution amplitudes

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We construct a spectral representation for the baron to meson transition distribution amplitudes (TDAs), i.e. matrix elements involving three quark correlators which arise in the description of baryon to meson transitions within the factorization approach to hard exclusive reactions. We generalize for these quantities the notion of double distributions introduced in the context of generalized parton distributions. We propose the generalization of Radyushkin's factorized ansatz for the case of baryon to meson TDAs. Our construction opens the way to modeling of baryon to meson TDAs in their complete domain of definition and quantitative estimates of cross sections for various hard exclusive reactions.

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I. INTRODUCTION

The concept of generalized parton distributions (GPDs) [1–4], which in the simplest (leading twist) case are nondiagonal matrix elements of quark-antiquark or gluon-gluon nonlocal operators on the light cone, has recently been extended [5,6] to baryon to meson (and baryon to photon) transition distribution amplitudes (TDAs), nondiagonal matrix elements of three quark operators between two hadronic states of different baryon number (or between a baryon state and a photon). Nucleon to meson TDAs are conceptually much related to meson-nucleon generalized distribution amplitudes [7,8] since they involve the same nonlocal operators [9–12]. These objects are useful for the description of exclusive processes characterized by a baryonic exchange such as backward electroproduction of mesons [13–15] or proton-antiproton hard exclusive annihilation processes [16]. Nucleon to meson TDAs are also considered to be a useful tool to quantify the pion cloud in baryons [17].

Up to now TDAs between the states of unequal baryon number lacked any suitable phenomenological parametrization in the whole domain of their definition, as, for example, in the framework of the quark model developed in [18]. The complete parametrization should properly take into account the fundamental requirement of Lorentz covariance which is manifest as the polynomiality property of the Mellin moments in the relevant light-cone momentum fraction on the complete domain of their definition. For the case of the GPDs an elegant way to fulfill this requirement consists in employing the spectral representation. The corresponding spectral properties were established with the help of the α-representation techniques [19,20]. Radyushkin's factorized ansatz based on the double distribution representation for GPDs [21–24] became the basis for various successful phenomenological GPD models (see [25–29]).

In this paper we address the problem of construction of a spectral representation of baryon to meson transition distribution amplitudes. We introduce the notion of quadruple distributions and generalize Radyushkin's factorized ansatz for this issue. This allows the modeling of baryon to meson TDAs in the complete domain of their definition and quantitative rate estimates in various hard exclusive reactions.

Similarly as the nucleon to meson TDAs factorize in backward meson electroproduction, nucleon to photon TDAs may factorize in backward virtual Compton scattering [30]. The main part of the analysis performed in our paper can be directly applied to the nucleon to photon TDAs. But the modelling of the quadruple distribution has to account for the anomalous nature of a photon. The studies of the anomalous photon structure functions [31] and of the photon GPDs [32] show that taking it into account is a nontrivial task which deserves separate studies.

II. BASIC DEFINITIONS AND KINEMATICS

Nucleon to meson transition distribution amplitudes also called in the literature as skewed DAs [5] and superskewed parton distributions [6] which extend the concept of usual generalized parton distributions arise e.g. in the description of meson electroproduction on the nucleon target [13–15]. For definiteness below we consider the case of nucleon to pion transition distribution amplitudes (πN TDAs for brevity) although our analysis is general enough to be applied to other baryon-meson and also baryon to photon TDAs. πN TDAs arise in the description of backward pion electroproduction

$$\gamma^*(q) + N(p_1) \rightarrow N'(p_2) + \pi(p_\pi), \qquad (1)$$

in the generalized Bjorken regime ($-q^2$ is large and $q^2/(2p_1 \cdot q)$ kept fixed $-q^2 \gg -u$). The factorization theorem was argued for the process Eq. (1) in [5,6] (see Fig. 1). The appropriate kinematics is described as follows [15]:

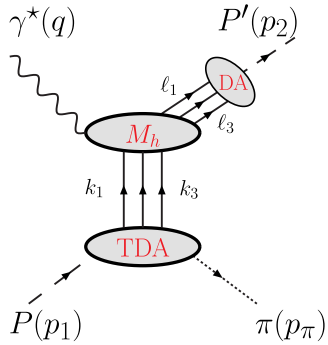

FIG. 1 (color online). The factorization of the process $\gamma^* + P \to P' + \pi$. The lower blob is the pion-nucleon transition distribution amplitude, M_h denotes the hard subprocess amplitude, DA is the nucleon distribution amplitude.

$$P = \frac{1}{2}(p_1 + p_\pi); \qquad \Delta = p_\pi - p_1; \\ u = \Delta^2; \qquad \xi = -\frac{\Delta \cdot n}{2P \cdot n}, \tag{2}$$

where u denotes the transfer momentum squared between the meson and the nucleon target and ξ is the skewness parameter. n and p are the usual light-cone vectors occurring in the Sudakov decomposition of momenta ($n^2 = p^2 = 0$, $n \cdot p = 1$). The light-cone decomposition of the particular vector v^μ is given by $v^\mu = v^+ p^\mu + v^- n^\mu + v_T^\mu$.

The definition of πN TDAs can be symbolically written as [5,6]

$$\int \Big[\prod_{i=1}^{3} \frac{dz_i^-}{2\pi}\Big] e^{ix_1(P\cdot z_1)+ix_2(P\cdot z_2)+ix_3(P\cdot z_3)} \\ \times \langle \pi(P+\Delta/2)| \epsilon_{abc} \psi_{j_1}^a(z_1) \psi_{j_2}^b(z_2) \psi_{j_3}^c(z_3) \\ \times |N(P-\Delta/2)\rangle|_{z_i^+ = z_i^\perp = 0} \\ \sim \delta(2\xi - x_1 - x_2 - x_3) H_{j_1 j_2 j_3}(x_1, x_2, x_3, \xi, u). \tag{3}$$

Here $j_{1,2,3}$ stand for spin-flavor indices and a, b, c are color indices. The decomposition of the Fourier transform (3) of the matrix element of the three-local light-cone quark operator involves a set of independent spin-flavor structures multiplied by corresponding invariant functions: πN TDAs.

It is worth to mention that in order to preserve gauge invariance one has to insert the path-ordered gluonic exponentials $[z_i; z_0]$ along the straight line connecting an arbitrary initial point $z_0 n$ and a final one $z_i n$:

$$\langle \pi | \epsilon^{abc} \psi_{j_1}^{a'}(z_1)[z_1; z_0]_{a',a} \psi_{j_2}^{b'}(z_2)[z_2; z_0]_{b'b} \psi_{j_3}^{c'}(z_3) \\ \times [z_3; z_0]_{c'c} |N\rangle. \tag{4}$$

Throughout this paper we adopt the light-cone gauge $A^+ = 0$, so that the gauge link is equal to unity. Thus we do not show it explicitly in the definition (3).

For the case of proton to π^0 transition the decomposition of (3) over the independent spinor structures at the leading twist involves 8 independent terms. It reads[1] [15]:

$$4\mathcal{F}(\langle \pi^0(p_\pi)| \epsilon_{abc} u_\alpha^a(z_1 n) u_\beta^b(z_2 n) d_\gamma^c(z_3 n)|P(p_1, s_1)\rangle) \\ = \delta(2\xi - x_1 - x_2 - x_3) i \frac{f_N}{f_\pi} [V_1^{p\pi^0}(\not p C)_{\alpha\beta}(N^+)_\gamma + A_1^{p\pi^0}(\not p \gamma^5 C)_{\alpha\beta}(\gamma^5 N^+)_\gamma + T_1^{p\pi^0}(\sigma_{p\mu} C)_{\alpha\beta}(\gamma^\mu N^+)_\gamma \\ + M^{-1} V_2^{p\pi^0}(\not p C)_{\alpha\beta}(\Delta_T N^+)_\gamma + M^{-1} A_2^{p\pi^0}(\not p \gamma^5 C)_{\alpha\beta}(\gamma^5 \Delta_T N^+)_\gamma + M^{-1} T_2^{p\pi^0}(\sigma_{p\Delta_T} C)_{\alpha\beta}(N^+)_\gamma \\ + M^{-1} T_3^{p\pi^0}(\sigma_{p\mu} C)_{\alpha\beta}(\sigma^{\mu\Delta_T} N^+)_\gamma + M^{-2} T_4^{p\pi^0}(\sigma_{p\Delta_T} C)_{\alpha\beta}(\Delta_T N^+)_\gamma]. \tag{5}$$

Here $\not p$ is the usual Dirac slash notation ($\not p = p_\mu \gamma^\mu$), $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ with $\sigma^{p\mu} = p_\nu \sigma^{\nu\mu}$, C is the charge conjugation matrix and N^+ is the large component of the nucleon spinor ($N = (\not p \not n + \not n \not p)N = N^- + N^+$ with $N^+ \sim \sqrt{p_1^+}$ and $N^- \sim \sqrt{1/p_1^+}$). M stands for the nucleon mass, f_π is the pion decay constant ($f_\pi = 131$ MeV) and f_N is a constant with the dimension of energy squared. All the 8 $p \to \pi^0$ TDAs V_i, A_i and T_i are dimensionless.

In this paper we concentrate on the dependence of the invariant functions V_i, A_i, T_i multiplying the independent spin-flavor structures in (5) on the longitudinal momentum fractions x_1, x_2, x_2 and skewness parameter ξ. Let us stress that our subsequent analysis is completely general: all invariant functions can be treated at the same footing. For simplicity in what follows we employ the same notation for all the invariant functions

$$H(x_1, x_2, x_3, \xi, u) \equiv \{V_i, A_i, T_i\}(x_1, x_2, x_3, \xi, u). \tag{6}$$

A basic feature of model building cleverness is to fulfill fundamental requirements of field theory, such as general Lorentz covariance. In particular this requirement leads to the so-called polynomiality property of the Mellin moments in light-cone momentum fractions x_1, x_2, x_3 of πN TDAs:

$$\int dx_1 dx_2 dx_3 \delta(2\xi - x_1 - x_2 - x_3) x_1^{n_1} x_2^{n_2} x_3^{n_3} \\ \times H(x_1, x_2, x_3, \xi, u) \sim \Big(i\frac{d}{dz_1^-}\Big)^{n_1} \Big(i\frac{d}{dz_2^-}\Big)^{n_2} \Big(i\frac{d}{dz_3^-}\Big)^{n_3} \\ \times [\langle \pi(P+\Delta/2)| \psi(z_1)\psi(z_2)\psi(z_3)|N(P-\Delta/2)\rangle]|_{z_i=0}. \tag{7}$$

[1]We make use of the notation $\mathcal{F}(\cdot) = (P\cdot n)^3 \times \int [\prod_{i=1}^3 \frac{dz_i}{2\pi}] e^{ix_1(P\cdot z_1)+ix_2(P\cdot z_2)+ix_3(P\cdot z_3)}(\cdot)$

Indeed the x_1, x_2, x_3- Mellin moments of πN TDA are the form factors of the local twist-3 three quark operators between nucleon and pion states. This leads to the appearance of polynomials in ξ at the right hand side of (7).[2]

III. SUPPORT PROPERTIES OF πN TDAS

A. ERBL-like and DGLAP-like domains for πN TDAs

In order to specify the support properties of πN TDAs let us first consider the case of the GPDs [see Fig. 2(a)]. Let x_1 and x_2 be the fractions (defined with respect to average nucleon momentum $P = \frac{p_1+p_2}{2}$) of the light-cone momentum carried by quark and antiquark inside nucleon ($x_1 + x_2 = 2\xi$). In the so-called Efremov-Radyushkin-Brodsky-Lepage (ERBL) region both x_1 and x_2 are positive. The variable x is usually defined as

$$x = \frac{x_1 - x_2}{2}. \quad (8)$$

In the ERBL region $x_1, x_2 \in [0, 2\xi]$ and thus $x \in [-\xi, \xi]$. In the so-called Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) region either x_1 is positive $x_1 \in [2\xi, 1 + \xi]$ and x_2 is negative $x_2 \in [-1 + \xi, 0]$ or vice versa (x_1 is negative $x_1 \in [-1 + \xi, 0]$ and x_2 positive $x_2 \in [2\xi, 1 + \xi]$). These two DGLAP domains result in $x \in [\xi, 1]$ and $x \in [-1, -\xi]$, respectively.

Now let us turn to the case of πN TDAs. Let x_1, x_2 and x_3 satisfying the constraint $x_1 + x_2 + x_3 = 2\xi$, with $\xi \geq 0$ be the light-cone momentum fractions carried by three quarks. As usual the light-cone momentum fractions are defined with respect to the average hadron momentum $P = \frac{p_1+p_\pi}{2}$. The convenient way to depict the support properties of πN TDAs is to employ barycentric coordinates (Mandelstam plane).

First of all we identify the analogous of the ERBL domain, in which three longitudinal momentum fraction carried by three quarks are positive. In the barycentric coordinates the ERBL-like region corresponds to the interior of the equilateral triangle with the height 2ξ (see Fig. 3). It is natural to assume that the DGLAP-like domains are bounded by the lines

$$\begin{aligned} &x_1 = -1 + \xi; \qquad x_1 = 0; \qquad x_1 = 1 + \xi; \\ &x_2 = -1 + \xi; \qquad x_2 = 0; \qquad x_2 = 1 + \xi; \\ &x_3 = -1 + \xi; \qquad x_3 = 0; \qquad x_3 = 1 + \xi. \end{aligned} \quad (9)$$

We are guided by the following requirements.

[2]Naive counting gives $n_1 + n_2 + n_3$ for the order of this polynomial. However, the problem of determination of the highest possible power of ξ in (7) still lacks some analysis. This is a rather important question since it would allow to make the conclusion on the necessity of adding of D-term like contributions [33] to the spectral representation of πN TDAs (see discussion in Sec. VII).

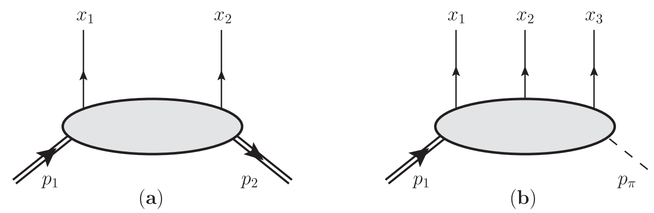

FIG. 2. Longitudinal momentum flow in the ERBL regime for GPDs (a) and πN TDAs (b).

(i) The complete domain of definition of πN TDA should be symmetric in x_1, x_2, x_3.
(ii) In the limiting case $\xi = 1$ this domain should reduce to the ERBL-like domain on which the nucleon DA is defined. In the barycentric coordinates the domain of definition of the nucleon DA is equilateral triangle.
(iii) For any x_i set to zero we should recover the usual domain of definition of GPDs for the two remaining variables.

Three small equilateral triangles correspond to DGLAP-like type I domains, where only one longitudinal momentum fractions is positive while two others are negative. Three trapezoid domains correspond to DGLAP-like type II, where two longitudinal momentum fractions are positive and one is negative.

The support properties (9) are invariant under the permutation of the longitudinal momentum fractions x_i. In the limit $\xi \to 1$ the support of πN TDA is reduced to the ERBL-like domain (the equilateral triangle) (see Fig. 4) and coincide with that of the nucleon distribution amplitude (DA). In fact this is natural since $\xi = 1$ corresponds to the soft pion limit in which πN TDA reduces to the corresponding nucleon DA [15].

In the limiting case $\xi \to 0$ the support of πN TDA in the barycentric coordinates is given by the regular hexagon.

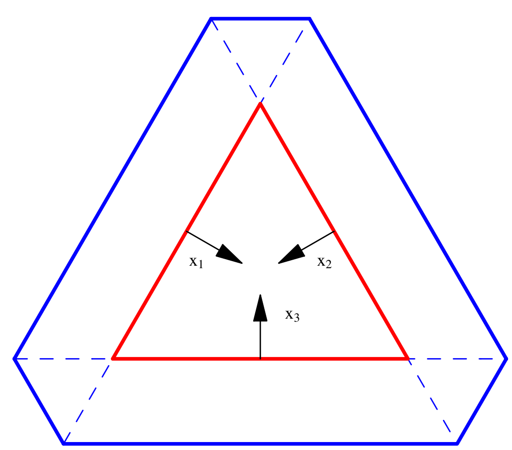

FIG. 3 (color online). Physical domains for πN TDAs in the barycentric coordinates.

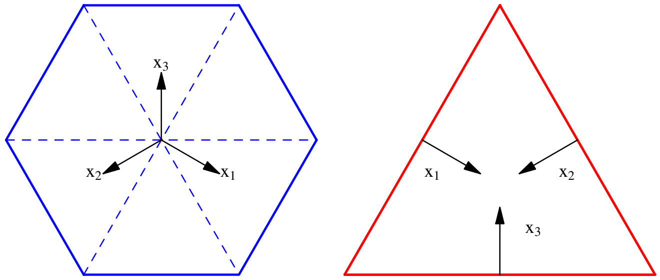

FIG. 4 (color online). Physical domains for πN TDAs in the barycentric coordinates. Two limiting cases: $\xi = 0$ (left) and $\xi = 1$ (right).

B. Quark-diquark coordinates

In order to describe πN TDA instead of x_1, x_2, x_3 which satisfy

$$x_1 + x_2 + x_3 = 2\xi, \tag{10}$$

it is convenient to introduce the so-called quark-diquark coordinates. Let us stress that we do not imply any dynamical meaning to the notion of "diquark." There are three different possible choices depending on which quarks are supposed to form a "diquark system":

$$\begin{aligned} v_1 &= \frac{x_2 - x_3}{2}; & w_1 &= \frac{x_1 - x_2 - x_3}{2}; \\ v_2 &= \frac{x_3 - x_1}{2}; & w_2 &= \frac{x_2 - x_3 - x_1}{2}; \\ v_3 &= \frac{x_1 - x_2}{2}; & w_3 &= \frac{x_3 - x_1 - x_2}{2}. \end{aligned} \tag{11}$$

We suggest to introduce the notations ξ'_1, ξ'_2 and ξ'_3 for the fraction of the longitudinal momentum carried by the diquark:

$$\frac{x_2 + x_3}{2} = \frac{\xi - w_1}{2} \equiv \xi'_1; \qquad \frac{x_1 + x_3}{2} = \frac{\xi - w_2}{2} \equiv \xi'_2; \qquad \frac{x_1 + x_2}{2} = \frac{\xi - w_3}{2} \equiv \xi'_3. \tag{12}$$

The variables x_1, x_2, x_3 are expressed through the new variables (11) as follows:

$$\begin{aligned} x_1 &= \xi + w_1; & x_2 &= v_1 + \xi'_1; & x_3 &= -v_1 + \xi'_1; \\ x_1 &= -v_2 + \xi'_2; & x_2 &= \xi + w_2; & x_3 &= v_2 + \xi'_2; \\ x_1 &= v_3 + \xi'_3; & x_2 &= -v_3 + \xi'_3; & x_3 &= \xi + w_3. \end{aligned} \tag{13}$$

C. ERBL-like and DGLAP-like domains for πN TDA in quark-diquark coordinates

Let us consider how the ERBL-like and DGLAP-like domains for πN TDA look like in quark-diquark coordinates. Throughout the rest of this section we employ the particular choice of quark-diquark coordinates (11):

$$v \equiv v_3 = \frac{x_1 - x_2}{2}; \qquad w \equiv w_3 = \frac{x_3 - x_1 - x_2}{2}; \qquad \xi' \equiv \xi'_3 = \frac{\xi - w_3}{2}. \tag{14}$$

The generalization for the alternative cases is straightforward.

The ERBL-like and DGLAP-like domains for πN TDA in quark-diquark coordinates (14). are depicted on Fig. 5. In these coordinates the ERBL-like region corresponds to the central isosceles triangular domain. It is bounded by the lines

$$v = -\xi' \;\; (x_1 = 0); \quad v = \xi' \;\; (x_2 = 0); \quad w = -\xi \;\; (x_3 = 0). \tag{15}$$

DGLAP-like type I regions correspond to three smaller isosceles triangular domains. Finally, three trapezoid domains correspond to DGLAP-like type II region.

For $w \in [-1, -\xi]$ DGLAP-like region is bounded by

$$\begin{aligned} v &= 1 + \xi - \xi' \quad (x_1 = 1 + \xi) \quad \text{and} \\ v &= -1 - \xi + \xi' \quad (x_2 = 1 + \xi). \end{aligned} \tag{16}$$

For $w \in [-\xi, \xi]$ DGLAP-like region is bounded by

$$\begin{aligned} &v = -1 + \xi - \xi' \quad (x_1 = -1 + \xi); \qquad v = -\xi'; \\ &v = \xi'; \qquad v = 1 - \xi + \xi' \quad (x_2 = -1 + \xi). \end{aligned} \tag{17}$$

For $w \in [\xi, 1]$ DGLAP-like region is bounded by

$$\begin{aligned} &v = -1 + \xi - \xi' \quad (x_1 = -1 + \xi); \qquad v = -\xi'; \\ &v = \xi'; \qquad v = 1 - \xi + \xi' \quad (x_2 = -1 + \xi). \end{aligned} \tag{18}$$

One can easily check that for $\xi \geq 0$ the following inequalities are valid:

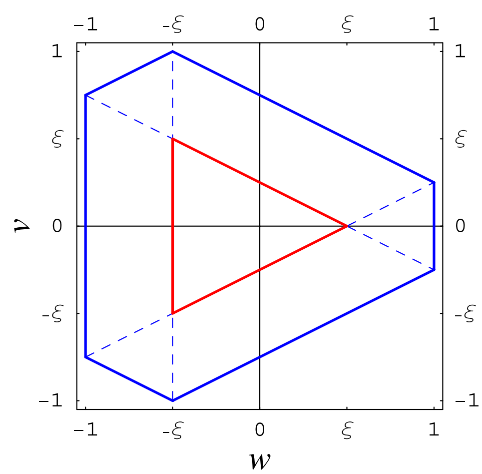

FIG. 5 (color online). ERBL-like and DGLAP-like domains for πN TDA in quark-diquark coordinates (14). Three lines: $w = -\xi$ and $v = \pm\xi'$ form the isosceles triangle which corresponds to ERBL-like region. Three smaller isosceles triangles correspond to DGLAP-like type I region. Three trapezoid domains correspond to DGLAP-like type II region.

$$\begin{cases} w < -\xi \\ \xi' > \xi \end{cases} ; \quad \text{and} \quad \begin{cases} w > -\xi \\ \xi' < \xi \end{cases} . \tag{19}$$

$\xi' = \xi$ occurs on the line $w = -\xi$. Thus the whole domain of definition of πN TDA in quark-diquark coordinates depicted on Fig. 5 can parameterized as follows:

$$-1 \le w \le 1; \qquad -1 + |\xi - \xi'| \le v \le 1 - |\xi - \xi'|. \tag{20}$$

Let us briefly summarize our result.

(i) $w \in [-1; -\xi]$ with $v \in [\xi'; 1 - \xi' + \xi]$ or $v \in [-1 + \xi' - \xi; -\xi']$ correspond to DGLAP-like type I domains.

(ii) $w \in [-1; -\xi]$ and $v \in [-\xi'; \xi']$ corresponds to DGLAP-like type II domain.

(iii) $w \in [-\xi; \xi]$ with $v \in [-\xi'; \xi']$ corresponds to ERBL-like domain.

(iv) $w \in [-\xi; \xi]$ with $v \in [\xi'; 1 - \xi + \xi']$ or $v \in [-1 + \xi - \xi'; -\xi']$ correspond to DGLAP-like type II domain.

(v) $w \in [\xi; 1]$ with $v \in [-\xi'; 1 - \xi + \xi']$ or $v \in [-1 + \xi - \xi'; \xi']$ correspond to DGLAP-like type II domain.

(vi) $w \in [\xi; 1]$ with $v \in [\xi'; -\xi']$ correspond to DGLAP-like type I domain.

The Mellin moments of πN TDAs in x_1, x_2, x_3 computed with the weight

$$\int dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 2\xi) \tag{21}$$

are the quantities of major theoretical importance. In the quark-diquark coordinates (14) the corresponding integrals can be rewritten as

$$\begin{aligned} &\int_{-1+\xi}^{1+\xi} dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 2\xi) \\ &\quad \times x_1^{n_1} x_2^{n_2} x_3^{n_3} H(x_1, x_2, x_3 = 2\xi - x_1 - x_2) \\ &\qquad = \int_{-1}^{1} dw \int_{-1+|\xi-\xi'|}^{1-|\xi-\xi'|} dv (v + \xi')^{n_1} (-v + \xi')^{n_2} \\ &\qquad \times (w + \xi)^{n_3} H(w, v, \xi). \end{aligned} \tag{22}$$

IV. SPECTRAL REPRESENTATION FOR πN TDAS FROM THE SUPPORT PROPERTIES AND THE POLYNOMIALITY CONDITION

The double distribution representation [21–24] was found to be an elegant way to incorporate both the polynomiality property of the Mellin moments and the support properties of GPDs. In the framework of this representation the GPD H is given as a one dimensional section of the double distribution (DD) $f(\alpha, \beta)$:

$$H(x, \xi) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \delta(x - \beta - \alpha\xi) f(\beta, \alpha). \tag{23}$$

The spectral representation (23) was originally recovered in the diagrammatical analysis employing the α-representation techniques [19,20]. The spectral conditions $|\beta| \le 1$ and $|\alpha| \le 1 - |\beta|$ ensure the support property of GPD $|x| \le 1$ for any $|\xi| \le 1$.

The polynomiality property of the Mellin moments in x which resides on the fundamental field theoretic requirements (Lorentz covariance) is ensured by the fact that the x dependence of GPD in (23) is introduced solely through the integration path. In [34] it was pointed out that the relation between GPDs and DDs is the particular case of the Radon transform. It is worth to mention that the polynomiality property is well known in the framework of the Radon transform theory as the Cavalieri conditions [35].

Now we propose to invert the logic. From the pure mathematical point of view representing GPD as the Radon transform of a certain spectral density is the most natural way to ensure polynomiality property. Postulating the polynomiality property of GPD and the support property $|x| \le 1$ one can put down the spectral representation (23) and unambiguously recover the spectral conditions $|\beta| \le 1$ and $|\alpha| \le 1 - |\beta|$. Let us stress that this does not provide the alternative derivation of (23) since there is no way to show independently the support property $|x| \le 1$ of GPD. However we think that this line of argumentation justifies the use of the Radon transform (23) which is a rather general representation for a function satisfying the polynomiality condition with the restricted support in x as the building block for the spectral representation of multipartonic generalizations of GPDs and, in particular, for πN TDAs. In order to derive the form of the spectral representation for πN TDA let us first consider the simple example of ordinary GPDs.

A. Test ground: Spectral representation for GPDs

We are going to treat the example of usual GPDs in a slightly unusual way, which we find more suitable for further generalization. Let us introduce the light-cone momentum fractions x_1 and x_2 of the average hadron momentum carried by the quark and antiquark, respectively. The variables x_1 and x_2 satisfy the condition $x_1 + x_2 = 2\xi$. The support property in x_1, x_2 is known to be given by

$$-1 + \xi \le x_1 \le 1 + \xi; \qquad -1 + \xi \le x_2 \le 1 + \xi. \tag{24}$$

In order to write down the spectral representation for GPD we introduce two sets of spectral parameters $\beta_{1,2}$, $\alpha_{1,2}$. The momentum fractions $x_{1,2}$ are supposed to have the following decomposition in terms of spectral parameters:

$$x_1 = \xi + \beta_1 + \alpha_1\xi; \qquad x_2 = \xi + \beta_2 + \alpha_2\xi. \tag{25}$$

The condition $x_1 + x_2 = 2\xi$ can be taken into account by introducing two δ-functions $\delta(\beta_1 + \beta_2)\delta(\alpha_1 + \alpha_2)$. This allows us to write down the following spectral representation for GPD $H(x_1, x_2 = 2\xi - x_1, \xi)$:

$$H(x_1, x_2 = 2\xi - x_1, \xi) = \int_{\Omega_1} d\beta_1 d\alpha_1 \int_{\Omega_2} d\beta_2 d\alpha_2 \delta(x_1 - \xi - \beta_1 - \alpha_1\xi) \times \delta(\beta_1 + \beta_2)\delta(\alpha_1 + \alpha_2)F(\beta_1, \beta_2, \alpha_1, \alpha_2). \quad (26)$$

Here by $\Omega_{1,2}$ we denote the usual domains in the parameter space:

$$\Omega_{1,2} = \{|\beta_{1,2}| \le 1; |\alpha_{1,2}| \le 1 - |\beta_{1,2}|\}; \quad (27)$$

and $F(\beta_1, \beta_2, \alpha_1, \alpha_2)$ is a certain quadruple distribution.

The important advantage of the spectral representation (26) is that it is symmetric under the interchange of the longitudinal momentum fractions x_1 and x_2. Note that the spectral conditions (27) ensure the support properties (24) both in x_1 and x_2. The (n_1, n_2)-th Mellin moments in x_1, x_2 of $H(x_1, x_2 = 2\xi - x_1, \xi)$ are polynomial of ξ of order $n_1 + n_2$:

$$\begin{aligned}\int_{-1+\xi}^{1+\xi} dx_1 \int_{-1+\xi}^{1+\xi} dx_2 \delta(2\xi - x_1 - x_2) &\times x_1^{n_1} x_2^{n_2} H(x_1, x_2 = 2\xi - x_1, \xi) \\ &= \int_{\Omega_1} d\beta_1 d\alpha_1 \int_{\Omega_2} d\beta_2 d\alpha_2 (\xi + \beta_1 + \alpha_1\xi)^{n_1} \\ &\times (\xi + \beta_2 + \alpha_2\xi)^{n_2} \delta(\beta_1 + \beta_2)\delta(\alpha_1 + \alpha_2) \\ &\times F(\beta_1, \beta_2, \alpha_1, \alpha_2) \\ &= P_{n_1+n_2}(\xi). \end{aligned} \quad (28)$$

Now we are about to show that the spectral representation (26) is equivalent to the usual Radyushkin's representation (23) for GPDs in terms of double rather than quadruple distributions. For this issue we can lift the two superfluous integrations employing the two delta functions. In order to perform this in the astute way let us introduce the natural spectral variables $\alpha_\pm$, $\beta_\pm$:

$$\alpha_\pm = \frac{\alpha_1 \pm \alpha_2}{2}; \qquad \beta_\pm = \frac{\beta_1 \pm \beta_2}{2}. \quad (29)$$

It is also useful to perform the related change of the variables in the (x_1, x_2) space in the initial spectral representation (26). The corresponding natural variables are

$$x_- = \frac{x_1 - x_2}{2} = \alpha_- + \beta_-\xi; \quad \text{and} \quad x_+ = \frac{x_1 + x_2}{2} = \xi + \alpha_+ + \beta_+\xi. \quad (30)$$

Thus instead of using (26) we switch to the natural variables and consider

$$H(x_1, x_2 = 2\xi - x_1, \xi) = \frac{1}{2}\int_{\Omega_1} d\beta_1 d\alpha_1 \int_{\Omega_2} d\beta_2 d\alpha_2 \delta(x_- - \beta_- - \alpha_-\xi) \times \delta(\beta_+)\delta(\alpha_+)F(\beta_1, \beta_2, \alpha_1, \alpha_2). \quad (31)$$

The appropriate definition of the integration domain in (31) after the change of the variables (29) require special attention. In particular,

$$\int_{-1}^{1} d\beta_1 \int_{-1}^{1} d\beta_2 \ldots = 2\int_{-1}^{1} d\beta_- \int_{-1+|\beta_-|}^{1-|\beta_-|} d\beta_+ \ldots . \quad (32)$$

Now since $1 - |\beta_-| \ge 0$ and hence $-1 + |\beta_-| \le 0$ the integral over β_+ can be easily lifted with no influence on the integration domain in α_+, α_-. The problem of definition of the integration domain in α_+, α_- in principle is reduced to change of the variables in the integral

$$\int_{-a}^{a} d\alpha_1 \int_{-b}^{b} d\alpha_2 \delta(\alpha_1 + \alpha_2) \ldots, \quad (33)$$

where $a = 1 - |\beta_+ + \beta_-|$, $b = 1 - |\beta_+ - \beta_-|$. It is much simplified due to the fact that $\beta_+ = 0$ and thus $a = b \equiv 1 - |\beta_-|$. This gives

$$\int_{-a}^{a} d\alpha_1 \int_{-a}^{a} d\alpha_2 \delta(\alpha_1 + \alpha_2) \ldots = 2\int_{-a}^{a} d\alpha_- \int_{-a+|\alpha_-|}^{a-|\alpha_-|} d\alpha_+ \delta(\alpha_+) \ldots . \quad (34)$$

Now the integral over α_+ can be trivially performed with the help of δ function again producing no additional restrictions for the integration domain in α_- and β_-. The final result reads

$$H(x_1, x_2 = 2\xi - x_1, \xi) = \int_{-1}^{1} d\beta_- \int_{-1+|\beta_-|}^{1-|\beta_-|} d\alpha_- \delta(x_- - \beta_- - \alpha_-\xi) \times \underbrace{2F(\beta_-, -\beta_-, \alpha_-, -\alpha_-)}_{f(\beta_-, \alpha_-)}. \quad (35)$$

Certainly we just recovered the known Radyushkin's result for the double distribution representation of GPDs.

Let us just make a short summary of the crucial points.

(i) We started from the spectral representation for $H(x_1, x_2 = 2\xi - x_1, \xi)$ as the function of the skewness parameter ξ and of two longitudinal momentum fractions x_1, x_2 satisfying the condition $x_1 + x_2 = 2\xi$. The form of this spectral representation ensured the proper support properties in x_1, x_2 as well as the polynomiality property of the corresponding Mellin moments in x_1 and x_2. The spectral density was a certain quadruple rather than double distribution.

(ii) The constraint $x_1 + x_2 = 2\xi$ was taken into account by the introduction of two δ functions restricting the integration domain in the space of spectral variables.

(iii) The two superfluous integrations can be lifted with the help of two δ functions. This requires the special attention to the integration domain in the space of spectral parameters. This problem can be most easily solved by switching to the set of natural variables both in the space of spectral parameters and x_1, x_2 space.

(iv) In our toy exercise lifting the two integrations does not lead to any special restrictions on the remaining spectral parameters α_-, β_- and we just recover the usual Radyushkin's result for the double distribution representation of GPDs.

(v) We find the spectral representation (26) which is symmetric under the exchange of x_1 and x_2 suitable for the generalization to the multiparton case. The analysis of πN TDAs with the help of the approach discussed above is presented in the next subsection.

B. Spectral representation for πN TDAs

We are now about to apply the ideas described in the previous section to the case of πN TDAs. Let us consider πN TDA $H(x_1, x_2, x_3 = 2\xi - x_1 - x_2, \xi)$ as a function of light-cone momentum fractions x_1, x_2 and x_3 carried by three quarks. The three light-cone momentum fractions satisfy the condition $x_1 + x_2 + x_3 = 2\xi$. The support property in x_1, x_2, x_3 is given by

$$\begin{aligned} -1 + \xi \le x_1 \le 1 + \xi; \\ -1 + \xi \le x_2 \le 1 + \xi; \\ -1 + \xi \le x_3 \le 1 + \xi. \end{aligned} \tag{36}$$

In order to write down the spectral representation for $H(x_1, x_2, x_3 = 2\xi - x_1 - x_2, \xi)$ we introduce three sets of spectral parameters $\beta_{1,2,3}$, $\alpha_{1,2,3}$. The momentum fractions $x_{1,2,3}$ are supposed to have the following decomposition in terms of spectral parameters:

$$\begin{aligned} x_1 = \xi + \beta_1 + \alpha_1 \xi; \\ x_2 = \xi + \beta_2 + \alpha_2 \xi; \\ x_3 = \xi + \beta_3 + \alpha_3 \xi. \end{aligned} \tag{37}$$

In order to satisfy this constrain we require that

$$\beta_1 + \beta_2 + \beta_3 = 0; \qquad \alpha_1 + \alpha_2 + \alpha_3 = -1. \tag{38}$$

This allows to write down the following spectral representation for πN TDAs:

$$\begin{aligned} H(x_1, x_2, x_3 = 2\xi - x_1 - x_2, \xi) \\ = \left[\prod_{i=1}^{3} \int_{\Omega_i} d\beta_i d\alpha_i \right] \delta(x_1 - \xi - \beta_1 - \alpha_1 \xi) \\ \times \delta(x_2 - \xi - \beta_2 - \alpha_2 \xi) \delta(\beta_1 + \beta_2 + \beta_3) \\ \times \delta(\alpha_1 + \alpha_2 + \alpha_3 + 1) F(\beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2, \alpha_3). \end{aligned} \tag{39}$$

By Ω_i, $i = \{1, 2, 3\}$ we denote the usual domains in the parameter space:

$$\Omega_i = \{|\beta_i| \le 1; |\alpha_i| \le 1 - |\beta_i|\}; \tag{40}$$

and $F(\beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2, \alpha_3)$ is now a sextuple distribution. The spectral conditions (40) ensure the support properties (36). Obviously, the (n_1, n_2, n_3)-th Mellin moment in (x_1, x_2, x_3) of πN TDA is a polynomial of order $n_1 + n_2 + n_3$ of ξ:

$$\begin{aligned} \left[\prod_{i=1}^{3} \int_{-1+\xi}^{1+\xi} dx_i \right] \delta(x_1 + x_2 + x_3 - 2\xi) x_1^{n_1} x_2^{n_2} x_3^{n_3} \\ \times H(x_1, x_2, x_3 = 2\xi - x_1 - x_2, \xi) = P_{n_1+n_2+n_3}(\xi). \end{aligned} \tag{41}$$

In complete analogy with the previously considered case of usual GPDs in order to properly reduce the spectral representation in terms of sextuple distribution for πN TDA to that in terms of quadruple distribution we need to perform two integrations in

$$\left[\prod_{i=1}^{3} \int_{\Omega_i} d\beta_i d\alpha_i \right] \delta(\beta_1 + \beta_2 + \beta_3) \delta(\alpha_1 + \alpha_2 + \alpha_3 + 1) \ldots \tag{42}$$

employing δ functions and specify the integration limits in the remaining four integrals. This problem can be solved by introducing the appropriate natural variables.

Let us start with the integral

$$\int_{-1}^{1} d\beta_1 \int_{-1}^{1} d\beta_2 \int_{-1}^{1} d\beta_3 \delta(\beta_1 + \beta_2 + \beta_3). \tag{43}$$

In order to visualize the integration domain (43) it is natural to employ the barycentric coordinates. In these coordinates the domain selected by the conditions $|\beta_i| \le 1$ ($i \in \{1, 2, 3\}$) and $\beta_1 + \beta_2 + \beta_3 = 0$ is represented by a regular hexagon (confer Fig. 4). It is convenient to single out three domains inside this hexagon:

$$\begin{aligned} D_1 &: \{\beta_1 \ge 0, \beta_2 \le 0, \beta_3 \le 0\} \cup \{\beta_1 \le 0, \beta_2 \ge 0, \beta_3 \ge 0\}; \\ D_2 &: \{\beta_2 \ge 0, \beta_1 \le 0, \beta_3 \le 0\} \cup \{\beta_2 \le 0, \beta_1 \ge 0, \beta_3 \ge 0\}; \\ D_3 &: \{\beta_3 \ge 0, \beta_1 \le 0, \beta_2 \le 0\} \cup \{\beta_3 \le 0, \beta_1 \ge 0, \beta_2 \ge 0\}. \end{aligned} \tag{44}$$

Obviously,

$$\begin{aligned} \int_{-1}^{1} d\beta_1 \int_{-1}^{1} d\beta_2 \int_{-1}^{1} d\beta_3 \delta(\beta_1 + \beta_2 + \beta_3) \\ = \sum_{i=1}^{3} \int_{D_i} d\beta_1 d\beta_2 d\beta_3 \delta(\beta_1 + \beta_2 + \beta_3). \end{aligned} \tag{45}$$

Now in order to get rid of one of three integrations in (45) we should switch to the natural coordinates. There are three possible choices of the natural coordinates in (45). For the moment we are going to adopt the coordinates

$$\rho_3 = \frac{\beta_1 - \beta_2}{2}; \qquad \sigma_3 = \frac{\beta_3 - \beta_1 - \beta_2}{2}. \tag{46}$$

The constrained triple integral (43) can be then rewritten as

$$\int_{-1}^{1} d\sigma_3 \int_{-1+|\sigma_3|/2}^{1-|\sigma_3|/2} d\rho_3 \ldots. \tag{47}$$

In principle in a completely analogous way one may also employ the coordinates

$$\begin{aligned} \rho_1 &= \frac{\beta_2 - \beta_3}{2}; \qquad \sigma_1 = \frac{\beta_1 - \beta_2 - \beta_3}{2}; \\ \rho_2 &= \frac{\beta_3 - \beta_1}{2}; \qquad \sigma_2 = \frac{\beta_2 - \beta_3 - \beta_1}{2} \end{aligned} \tag{48}$$

yielding the result

$$\int_{-1}^{1} d\sigma_i \int_{-1+(|\sigma_i|/2)}^{1-(|\sigma_i|/2)} d\rho_i \ldots. \tag{49}$$

Now let us address the problem of computation of the constrained triple integral over α_i in (39):

$$\int_{-a}^{a} d\alpha_1 \int_{-b}^{b} d\alpha_2 \int_{-c}^{c} d\alpha_3 \delta(\alpha_1 + \alpha_2 + \alpha_3 + 1) \ldots, \tag{50}$$

where we introduced the notations

$$a \equiv 1 - |\beta_1|; \qquad b \equiv 1 - |\beta_2|; \qquad c \equiv 1 - |\beta_3|; \tag{51}$$

$(a \geq 0; a \leq 1,\ b \geq 0; b \leq 1,\ c \geq 0; c \leq 1)$.

Introducing the natural coordinates[3]

$$\omega_3 = \alpha_3; \qquad \nu_3 = \frac{\alpha_1 - \alpha_2}{2} \tag{52}$$

and employing the results of Appendix A we conclude that for $\beta_i \in D_1 \cup D_2 \cup D_3$ the constrained integral (50) can be rewritten as

$$\int_{-1+|\beta_3|}^{1-|\beta_1|-|\beta_2|} d\omega_3 \int_{-1+|\beta_1|+(1+\omega_3)/2}^{1-|\beta_2|-(1+\omega_3)/2} d\nu_3 \ldots. \tag{53}$$

Now let us put all together and write down the spectral representation for πN TDAs in terms of quadruple distributions. The important observation is that once we have chosen the variables σ_3, ρ_3 and ω_3, ν_3 to perform the constrained integration in β_1, β_2, β_3 and α_1, α_2, α_3 respectively the natural variables on which πN TDAs depends are

$$w_3 = \frac{x_3 - x_1 - x_2}{2}, \qquad v_3 = \frac{x_1 - x_2}{2}. \tag{54}$$

Expressing the β_i and α_i through σ_3, ρ_3, ω_3, ν_3 the two delta functions in the definition (39) can be traded for

$$\begin{aligned} &\delta(x_1 - \xi - \beta_1 - \alpha_1\xi)\delta(x_2 - \xi - \beta_2 - \alpha_2\xi)|_{x_1+x_2+x_3=2\xi} \\ &\quad = \delta(w_3 - \sigma_3 - \omega_3\xi)\delta(v_3 - \rho_3 - \nu_3\xi). \end{aligned} \tag{55}$$

Note that at the level of delta functions we achieved the "factorization" of w_3 and v_3 dependencies on the spectral parameters.

Thus in the natural spectral parameters (46) and (52) and quark-diquark coordinates (54) we recovered the form of the spectral representation of πN TDAs in terms of quadruple distributions:

$$\begin{aligned} H(w_3, v_3, \xi) &= \int_{-1}^{1} d\beta_1 d\beta_2 d\beta_3 \delta(\beta_1 + \beta_2 + \beta_3) \int_{-1+|\beta_1|}^{1-|\beta_1|} d\alpha_1 \int_{-1+|\beta_2|}^{1-|\beta_2|} d\alpha_2 \int_{-1+|\beta_3|}^{1-|\beta_3|} d\alpha_3 \delta(\alpha_1 + \alpha_2 + \alpha_3 + 1) \\ &\quad \times \delta(x_1 - \xi - \beta_1 - \alpha_1\xi)\delta(x_2 - \xi - \beta_2 - \alpha_2\xi) F(\beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2, \alpha_3) \\ &= \int_{-1}^{1} d\sigma_3 \int_{-1+(|\sigma_3|/2)}^{1-(|\sigma_3|/2)} d\rho_3 \int_{-1+|\sigma_3|}^{1-|\rho_3-(\sigma_3/2)|-|\rho_3+(\sigma_3/2)|} d\omega_3 \int_{-(1/2)+|\rho_3-(\sigma_3/2)|+(\omega_3/2)}^{(1/2)-|\rho_3+(\sigma_3/2)|-(\omega_3/2)} d\nu_3 \delta(w_3 - \sigma_3 - \omega_3\xi) \\ &\quad \times \delta(v_3 - \rho_3 - \nu_3\xi) F_3(\sigma_3, \rho_3, \omega_3, \nu_3), \end{aligned} \tag{56}$$

where

$$F_3(\sigma_3, \rho_3, \omega_3, \nu_3) \equiv F\left(\rho_3 - \frac{\sigma_3}{2}, -\rho_3 - \frac{\sigma_3}{2}, \sigma_3, \nu_3 - \frac{1+\omega_3}{2}, -\nu_3 - \frac{1+\omega_3}{2}, \omega_3\right). \tag{57}$$

Employing three possible sets of natural spectral parameters one can write down three equivalent spectral representations in terms of three sets of quark-diquark coordinates w_i, v_i with $i = 1, 2, 3$ defined in (11):

[3]There are two additional possible choices: $\omega_1 = \alpha_1$; $\nu_1 = \frac{\alpha_2 - \alpha_3}{2}$ and $\omega_2 = \alpha_2$; $\nu_2 = \frac{\alpha_3 - \alpha_1}{2}$.

$$H(w_i, v_i, \xi) = \int_{-1}^{1} d\sigma_i \int_{-1+(|\sigma_i|/2)}^{1-(|\sigma_i|/2)} d\rho_i \int_{-1+|\sigma_i|}^{1-|\rho_i-(\sigma_i/2)|-|\rho_i+(\sigma_i/2)|} d\omega_i \int_{-(1/2)+|\rho_i-(\sigma_i/2)|+(\omega_i/2)}^{(1/2)-|\rho_i+(\sigma_i/2)|-(\omega_i/2)} d\nu_i \delta(w_i - \sigma_i - \omega_i\xi) \times \delta(v_i - \rho_i - \nu_i\xi) F_i(\sigma_i, \rho_i, \omega_i, \nu_i), \quad (58)$$

where $F_3(\sigma_3, \rho_3, \omega_3, \nu_3)$ is defined in (57) and

$$F_1(\sigma_1, \rho_1, \omega_1, \nu_1) \equiv F\left(\sigma_1, \rho_1 - \frac{\sigma_1}{2}, -\rho_1 - \frac{\sigma_1}{2}, \omega_1, \nu_1 - \frac{1+\omega_1}{2}, -\nu_1 - \frac{1+\omega_1}{2}\right);$$
$$F_2(\sigma_2, \rho_2, \omega_2, \nu_2) \equiv F\left(-\rho_2 - \frac{\sigma_2}{2}, \sigma_2, \rho_2 - \frac{\sigma_2}{2}, -\nu_2 - \frac{1+\omega_2}{2}, \omega_2, \nu_2 - \frac{1+\omega_2}{2}\right). \quad (59)$$

The spectral representation (58) for πN TDA in terms of quadruple distribution is the main result of our paper. However this form of the result is still not very convenient for practical applications. In the next section we demonstrate that the spectral representation (58) satisfies the support properties of πN TDAs established in Sec. III. We also derive the explicit expressions for πN TDAs in the ERBL-like and DGLAP-like type I and II domains.

V. SUPPORT PROPERTIES OF πN TDAS AND THE SPECTRAL REPRESENTATION

In order to make our formulas more compact, in what follows we omit the indice i for the quark-diquark coordinates w_i and v_i and spectral parameters σ_i, ρ_i, ω_i, ν_i and the spectral densities F_i. Our subsequent analysis equally applies for all $i = 1, 2, 3$.

It is extremely instructive to check that each contribution into πN TDA in (58) satisfies the support properties which were established in Sec. III:

$$-1 \le w \le 1; \qquad -1 + |\xi - \xi'| \le v \le 1 - |\xi - \xi'| \quad (60)$$

with ξ' defined in (12). In particular this allows to check that $(N-n, n)$-th $(N \ge n \ge 0)$ Mellin moments of πN TDA in (w, v) indeed satisfy the polynomiality property:

$$\int_{-1}^{1} dw \int_{-1+|\xi-\xi'|}^{1-|\xi-\xi'|} dv w^{N-n} v^n H(w, v, \xi) = P_N(\xi), \quad (61)$$

where $P_N(\xi)$ is a polynomial of order N in ξ.

Let us first consider the case $\xi = 0$. Employing the first delta function we get $\sigma = w$ for $-1 \le w \le 1$ and 0 otherwise. This obviously ensures the first condition (60) for $\xi = 0$. Once the integral over σ is performed the dependence on v is introduced through

$$\int_{-1+(|w|/2)}^{1-(|w|/2)} d\rho \delta(v - \rho) \ldots . \quad (62)$$

The result of this integral is nonzero only for

$$-1 + \frac{|w|}{2} \le v \le 1 - \frac{|w|}{2}, \quad (63)$$

that is precisely the second condition (60) for $\xi = 0$.

Let us now show that the spectral representation (56) possesses the desired support properties for arbitrary value of $\xi \in (0; 1]$.[4]

First of all it is easy to see that the first one of the two conditions (60) is respected. Indeed the w dependence in (58) is introduced through the expression

$$\int_{-1}^{1} d\sigma \int_{-1+|\sigma|}^{1-|\rho-(\sigma/2)|-|\rho+(\sigma/2)|} d\omega \delta(w - \sigma - \omega\xi) \ldots . \quad (64)$$

From the inequalities (A8), (A11), and (A14) it follows that

$$-1 + |\sigma| \le 1 - \left|\rho - \frac{\sigma}{2}\right| - \left|\rho + \frac{\sigma}{2}\right| \le 1 - |\sigma|. \quad (65)$$

Thus in (64) we are integrating only over some part of the familiar "GPD square" $|\rho| \le 1 - |\sigma|$. This guarantees the vanishing of πN TDA for $|w| > 1$. One can in the usual way perform the integration over ω introducing the additional θ-function to take into the account the unusual upper limit in the integral over ω:

$$\theta\left(1 - \left|\rho - \frac{\sigma}{2}\right| - \left|\rho + \frac{\sigma}{2}\right| - \frac{w - \sigma}{\xi}\right) \equiv \theta(\cdots). \quad (66)$$

For $\xi > 0$ we get

[4] The final result for $\xi \in [-1; 0)$ is presented in Appendix B.

For $w \in (-\infty; -1)$: $H(w, v, \xi) = 0$; For $w \in [-1; -\xi]$:

$$H(w, v, \xi) = \frac{1}{\xi} \int_{((w-\xi)/(1+\xi))}^{((w+\xi)/(1-\xi))} d\sigma \int_{-1+(|\sigma|/2)}^{1-(|\sigma|/2)} d\rho \int_{-(1/2)+|\rho-(\sigma/2)|+((w-\sigma)/(2\xi))}^{(1/2)-|\rho+(\sigma/2)|-((w-\sigma)/(2\xi))} d\nu \delta(v - \rho - \nu\xi)\theta(\cdots)F\left(\sigma, \rho, \frac{w-\sigma}{\xi}, \nu\right);$$

For $w \in [-\xi; \xi]$:

$$H(w, v, \xi) = \frac{1}{\xi} \int_{((w-\xi)/(1+\xi))}^{((w+\xi)/(1+\xi))} d\sigma \int_{-1+(|\sigma|/2)}^{1-(|\sigma|/2)} d\rho \int_{-(1/2)+|\rho-(\sigma/2)|+((w-\sigma)/(2\xi))}^{(1/2)-|\rho+(\sigma/2)|-((w-\sigma)/(2\xi))} d\nu \delta(v - \rho - \nu\xi)\theta(\cdots)F\left(\sigma, \rho, \frac{w-\sigma}{\xi}, \nu\right);$$

For $w \in [\xi; 1]$:

$$H(w, v, \xi) = \frac{1}{\xi} \int_{((w-\xi)/(1-\xi))}^{((w+\xi)/(1+\xi))} d\sigma \int_{-1+(|\sigma|/2)}^{1-(|\sigma|/2)} d\rho \int_{-(1/2)+|\rho-(\sigma/2)|+((w-\sigma)/(2\xi))}^{(1/2)-|\rho+(\sigma/2)|-((w-\sigma)/(2\xi))} d\nu \delta(v - \rho - \nu\xi)\theta(\cdots)F\left(\sigma, \rho, \frac{w-\sigma}{\xi}, \nu\right);$$

For $w \in (1; \infty)$: $H(w, v, \xi) = 0.$ (67)

Now we are about to perform the integration over ν with the help of the last remaining δ function. The resulting domain of integration in σ and ρ is defined by the inequalities

$$-1 + \frac{|\sigma|}{2} \le \rho \le 1 - \frac{|\sigma|}{2}; \tag{68}$$

$$-\frac{1}{2} + \left|\rho - \frac{\sigma}{2}\right| + \frac{w-\sigma}{2\xi} \le \frac{v-\rho}{\xi} \le \frac{1}{2} - \left|\rho + \frac{\sigma}{2}\right| - \frac{w-\sigma}{2\xi}; \tag{69}$$

$$1 - \left|\rho - \frac{\sigma}{2}\right| - \left|\rho + \frac{\sigma}{2}\right| \ge \frac{w-\sigma}{\xi}; \tag{70}$$

as well as the integration limits in σ depending on the value of w (see (67)).

It can be shown that for $\xi \ge 0$ the two inequalities (69) are equivalent to

$$\begin{aligned} &\rho \le \frac{\sigma}{2} + \frac{v+\xi'}{1+\xi} \quad \text{for } v \ge -\xi'; \\ &\rho \le \frac{\sigma}{2} + \frac{v+\xi'}{1-\xi} \quad \text{for } v \le -\xi' \end{aligned} \tag{71}$$

together with

$$\begin{aligned} &\rho \ge -\frac{\sigma}{2} + \frac{v-\xi'}{1-\xi} \quad \text{for } v \ge \xi'; \\ &\rho \ge -\frac{\sigma}{2} + \frac{v-\xi'}{1+\xi} \quad \text{for } v \le \xi'. \end{aligned} \tag{72}$$

Analogously the inequality (70) for $\xi \ge 0$ is equivalent to

$$\begin{aligned} &\rho \le \frac{\sigma}{2\xi} + \frac{\xi'}{\xi} \quad \text{for } v \ge |\xi'|; \\ &\rho \ge -\frac{\sigma}{2\xi} - \frac{\xi'}{\xi} \quad \text{for } v \le -|\xi'|; \\ &\sigma \ge \frac{w-\xi}{1+\xi} \quad \text{for } \begin{cases} v \ge -\xi' \\ v \le \xi' \end{cases}; \\ &\sigma \ge \frac{w-\xi}{1-\xi} \quad \text{for } \begin{cases} v \le -\xi' \\ v \ge \xi' \end{cases}. \end{aligned} \tag{73}$$

The last step is to match the integration domain defined by the inequalities (68) and (71)–(73) with the explicit w-dependent limits of integration in σ (67). There are nine possibilities:

$$\{w \in [-1; -\xi], w \in [-\xi; -\xi], w \in [\xi; 1]\} \otimes \{v \in (-\infty; -|\xi'|], v \in [-|\xi'|; |\xi'|], v \in [|\xi'|; \infty)\}. \tag{74}$$

Let us consider in details the case

$$w \in [-1; -\xi]; \qquad v \in [\xi'; \infty). \tag{75}$$

The integration domain in (σ, ρ) plane is defined by the intersection of a domain specified by the inequalities (68) and (71)–(73):

$$\begin{aligned} &\rho \ge -\frac{\sigma}{2} + \frac{v-\xi'}{1-\xi}; \qquad \rho \le \frac{\sigma}{2} + \frac{v+\xi'}{1+\xi}; \\ &\rho \le \frac{\sigma}{2\xi} + \frac{\xi'}{\xi}; \qquad |\rho| \le 1 - \frac{|\sigma|}{2} \end{aligned} \tag{76}$$

with the strip

$$\frac{w-\xi}{1+\xi} \le \sigma \le \frac{w+\xi}{1-\xi}. \tag{77}$$

The domain defined by the inequalities (76) and (77) is presented on Fig. 6. By the thick solid lines we show the borders of the domain defined by the first two inequalities (76). The thin solid line is the border of the domain defined by the inequality $\rho \le 1 - \frac{\sigma}{2}$. The dashed line is the border

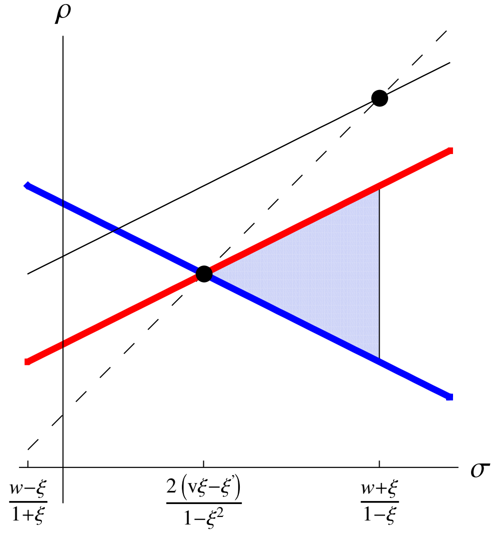

FIG. 6 (color online). The domain of integration in (σ, ρ) plane in Eq. (67) for $-1 \le w \le -\xi$ and $\xi' \le v \le 1 - \xi' + \xi$ defined by the inequalities (76) and (77). See explanations in the text.

of the domain defined by the inequality (73): $\rho \le \frac{\sigma}{2\xi} + \frac{\xi'}{\xi}$. The shaded area corresponds to the resulting domain of integration in (67) for $-1 \le w \le -\xi$ and $\xi' \le v \le 1 - \xi' + \xi$.

The abscissa of the apex of this triangular domain is

$$\sigma = \frac{2(v\xi - \xi')}{1 - \xi^2}. \tag{78}$$

One may check that for $v = \xi'$ the abscissa of the apex coincides with the left boundary of the strip (77):

$$\frac{2(v\xi - \xi')}{1 - \xi^2}\bigg|_{v=\xi'} = \frac{w - \xi}{1 + \xi}, \tag{79}$$

while for $v = 1 - \xi' + \xi$ it coincides with the right boundary of the strip (77):

$$\frac{2(v\xi - \xi')}{1 - \xi^2}\bigg|_{v=1-\xi'+\xi} = \frac{w + \xi}{1 - \xi}. \tag{80}$$

For $v > 1 - \xi' + \xi$ the apex of the triangular domain lies on the right of the strip (77) and hence has empty intersection with it. This makes the double integral (67) vanish for $v \ge 1 - \xi' + \xi$ and ensures the desired support property of $H(w, v, \xi)$.

The third inequality in (76) does not further restrict the domain since the apex of the triangular domain belongs to the line $\rho = \frac{\sigma}{2\xi} + \frac{\xi'}{\xi}$ and the triangular domain lies to the right of this line for $0 \le \xi \le 1$. The two inequalities (68) also do not impose additional restriction for the domain. Indeed one may check that the $\rho = \frac{\sigma}{2\xi} + \frac{\xi'}{\xi}$ intersects with $\rho = 1 - \frac{|\sigma|}{2}$ at $\sigma = \frac{w+\xi}{1-\xi}$.

The eight remaining cases (74) can be considered according to this pattern in a completely analogous way. One may check that the quadruple integral (58) for $H(w, v, \xi)$ for $\xi \ge 0$ reduces to the following expressions:

(i) For w and v outside the domain $-1 \le w \le 1$ and $-1 + |\xi - \xi'| \le v \le 1 - |\xi - \xi'|$ the integral vanishes.

(ii) For $w \in [-1; -\xi]$ and $\xi' \le v \le 1 - \xi' + \xi$ (DGLAP-like type I domain):

$$H(w, v, \xi) = \frac{1}{\xi^2} \int_{((2(v\xi-\xi'))/(1-\xi^2))}^{((w+\xi)/(1-\xi))} d\sigma \times \int_{-(\sigma/2)+((v-\xi')/(1-\xi))}^{(\sigma/2)+((v+\xi')/(1+\xi))} d\rho \times F\left(\sigma, \rho, \frac{w-\sigma}{\xi}, \frac{v-\rho}{\xi}\right). \tag{81}$$

(iii) For $w \in [-1; -\xi]$ and $-\xi' \le v \le \xi'$ (DGLAP-like type II domain):

$$H(w, v, \xi) = \frac{1}{\xi^2} \int_{((w-\xi)/(1+\xi))}^{((w+\xi)/(1-\xi))} d\sigma \times \int_{-(\sigma/2)+((v-\xi')/(1+\xi))}^{(\sigma/2)+((v+\xi')/(1+\xi))} d\rho \times F\left(\sigma, \rho, \frac{w-\sigma}{\xi}, \frac{v-\rho}{\xi}\right). \tag{82}$$

(iv) For $w \in [-1; -\xi]$ and $-1 + \xi' - \xi \le v \le -\xi'$ (DGLAP-like type I domain):

$$H(w, v, \xi) = \frac{1}{\xi^2} \int_{-((2(v\xi+\xi'))/(1-\xi^2))}^{((w+\xi)/(1-\xi))} d\sigma \times \int_{-(\sigma/2)+((v-\xi')/(1+\xi))}^{(\sigma/2)+((v+\xi')/(1-\xi))} d\rho \times F\left(\sigma, \rho, \frac{w-\sigma}{\xi}, \frac{v-\rho}{\xi}\right). \tag{83}$$

(v) For $w \in [-\xi; \xi]$ and $\xi' \le v \le 1 - \xi + \xi'$ (DGLAP-like type II domain):

$$H(w, v, \xi) = \frac{1}{\xi^2} \int_{((2(v\xi-\xi'))/(1-\xi^2))}^{((w+\xi)/(1+\xi))} d\sigma \times \int_{-(\sigma/2)+((v-\xi')/(1-\xi))}^{(\sigma/2)+((v+\xi')/(1+\xi))} d\rho \times F\left(\sigma, \rho, \frac{w-\sigma}{\xi}, \frac{v-\rho}{\xi}\right). \tag{84}$$

(vi) $w \in [-\xi; \xi]$ and $-\xi' \le v \le \xi'$ (ERBL-like domain):

$$H(w, v, \xi) = \frac{1}{\xi^2} \int_{((w-\xi)/(1+\xi))}^{((w+\xi)/(1+\xi))} d\sigma \times \int_{-(\sigma/2)+((v-\xi')/(1+\xi))}^{(\sigma/2)+((v+\xi')/(1+\xi))} d\rho \times F\left(\sigma, \rho, \frac{w-\sigma}{\xi}, \frac{v-\rho}{\xi}\right). \quad (85)$$

(vii) $w \in [-\xi; \xi]$ and $-1 + \xi - \xi' \leq v \leq -\xi'$ (DGLAP-like type II domain):

$$H(w, v, \xi) = \frac{1}{\xi^2} \int_{-((2(v\xi+\xi'))/(1-\xi^2))}^{((w+\xi)/(1+\xi))} d\sigma \times \int_{-(\sigma/2)+((v-\xi')/(1+\xi))}^{(\sigma/2)+((v+\xi')/(1-\xi))} d\rho \times F\left(\sigma, \rho, \frac{w-\sigma}{\xi}, \frac{v-\rho}{\xi}\right). \quad (86)$$

(viii) $w \in [\xi; 1]$ and $v \in [-\xi'; 1 - \xi + \xi']$: the result coincides with (84) as it certainly should be since this is the part of the same DGLAP type II domain. Note that this makes $H(w, v, \xi)$ a smooth function for $w = \xi$ as it should be since this line ($w_i = \xi \Leftrightarrow x_i = 2\xi$) does not correspond to any change of evolution properties of $H(w, v, \xi)$.

(ix) $w \in [\xi; 1]$ and $v \in [\xi'; -\xi']$ (DGLAP-like type I domain):

$$H(w, v, \xi) = \frac{1}{\xi^2} \int_{((w-\xi)/(1-\xi))}^{((w+\xi)/(1+\xi))} d\sigma \times \int_{-(\sigma/2)+((v-\xi')/(1-\xi))}^{(\sigma/2)+((v+\xi')/(1-\xi))} d\rho \times F\left(\sigma, \rho, \frac{w-\sigma}{\xi}, \frac{v-\rho}{\xi}\right). \quad (87)$$

(x) $w \in [\xi; 1]$ and $v \in [-1 + \xi - \xi'; \xi']$: the result again coincides with (86) since this is the part of the same DGLAP-like type II domain.

VI. RADYUSHKIN TYPE ANSATZ FOR πN TDAS

In this section we discuss what could be a possible approach for modelling of quadruple distributions $F(\sigma, \rho, \omega, \nu)$ occurring in the spectral representation (58).

Employing the analogy with the case of usual GPDs one may assume that the profile of $F(\sigma, \rho, \omega, \nu)$ in (σ, ρ) space is determined by the shape of the function $f(\sigma, \rho)$ to which πN TDA is reduced in the limit $\xi \to 0$. For the moment we put aside the complicated and interesting problem of the rigorous physical meaning of this limit. It will be discussed elsewhere. Thus, we suggest to employ the following factorized ansatz for quadruple distributions:

$$F(\sigma, \rho, \omega, \nu) = f(\sigma, \rho) h(\sigma, \rho, \omega, \nu), \quad (88)$$

where $h(\sigma, \rho, \omega, \nu)$ is a profile function normalized according to

$$\int_{-1+|\sigma|}^{1-|\rho-(\sigma/2)|-|\rho+(\sigma/2)|} d\omega \times \int_{-1+|\rho-(\sigma/2)|+((1+\omega)/(2))}^{1-|\rho+(\sigma/2)|-((1+\omega)/(2))} d\nu h(\sigma, \rho, \omega, \nu) = 1. \quad (89)$$

A possible model is to exploit further the analogy with the standard Radyushkin ansatz for the double distributions [24] and to assume that the (ω, ν) profile of $h(\sigma, \rho, \omega, \nu)$ is determined by the shape of the asymptotic form of the nucleon distribution amplitude:

$$\Phi^{\rm as}(y_1, y_2, y_3) = \frac{15}{4} y_1 y_2 y_3. \quad (90)$$

The DA (90) is defined for $y_{1,2,3} \in [0; 2]$ such that $y_1 + y_2 + y_3 = 2$.

In terms of quark-diquark variables $\tilde{\omega} = 1 - y_1 - y_2$ and $\tilde{\nu} = \frac{y_1 - y_2}{2}$ $\Phi^{\rm as}$ reads

$$\Phi^{\rm as}(\tilde{\omega}, \tilde{\nu}) = \frac{15}{4}(\tilde{\omega} + 1)\left(\tilde{\nu} + \frac{1-\tilde{\omega}}{2}\right)\left(-\tilde{\nu} + \frac{1-\tilde{\omega}}{2}\right). \quad (91)$$

Note that

$$\int_0^2 dy_1 dy_2 dy_3 \delta(2 - y_1 - y_2 - y_3) \Phi^{\rm as}(y_1, y_2, y_3) \equiv \int_{-1}^{1} d\tilde{\omega} \int_{-((1-\tilde{\omega})/(2))}^{((1-\tilde{\omega})/(2))} d\tilde{\nu} \Phi^{\rm as}(\tilde{\omega}, \tilde{\nu}) = 1. \quad (92)$$

$\Phi^{\rm as}(\tilde{\omega}, \tilde{\nu})$ is defined for

$$-1 \leq \tilde{\omega} \leq 1; \quad \text{and} \quad -\frac{1-\tilde{\omega}}{2} \leq \tilde{\nu} \leq \frac{1-\tilde{\omega}}{2}, \quad (93)$$

while $h(\sigma, \rho, \omega, \nu)$ is defined for

$$-1 + |\sigma| \leq \omega \leq 1 - \left|\rho - \frac{\sigma}{2}\right| - \left|\rho + \frac{\sigma}{2}\right|; \quad -\frac{1-\omega}{2} + \left|\rho - \frac{\sigma}{2}\right| \leq \nu \leq \frac{1-\omega}{2} - \left|\rho + \frac{\sigma}{2}\right|. \quad (94)$$

Thus it makes sense to employ the following substitution of the variables:

$$\tilde{\omega} = \frac{\omega + \frac{1}{2}(|\rho - \frac{\sigma}{2}| + |\rho + \frac{\sigma}{2}| - |\sigma|)}{1 - \frac{1}{2}(|\rho - \frac{\sigma}{2}| + |\rho + \frac{\sigma}{2}| + |\sigma|)}; \quad \tilde{\nu} = \frac{(1-\tilde{\omega})}{2} \frac{2\nu - |\rho - \frac{\sigma}{2}| + |\rho + \frac{\sigma}{2}|}{1 - \omega - |\rho - \frac{\sigma}{2}| - |\rho + \frac{\sigma}{2}|}. \quad (95)$$

This results in the following expression for the profile function $h(\sigma, \rho, \omega, \nu)$:

$$h(\sigma,\rho,\omega,\nu)=\frac{15}{16}\frac{(1+2\nu-\omega-2|\rho-\frac{\sigma}{2}|)(1-2\nu-\omega-2|\rho+\frac{\sigma}{2}|)(1-|\sigma|+\omega)}{(1-\frac{1}{2}(|\rho-\frac{\sigma}{2}|+|\rho+\frac{\sigma}{2}|+|\sigma|))^5}. \tag{96}$$

One may check that the profile function (96) satisfies the normalization condition (89). It is extremely interesting to note that in terms of the initial spectral parameters α_1, α_2, α_3 and β_1, β_2, β_3 satisfying $\alpha_1+\alpha_2+\alpha_3=-1$ and $\beta_1+\beta_2+\beta_3=0$ the profile function (96) can be rewritten in the very symmetric form:

$$h(\beta_1,\beta_2,\beta_3;\alpha_1,\alpha_2,\alpha_3)|_{\substack{\beta_1+\beta_2+\beta_3=0\\ \alpha_1+\alpha_2+\alpha_3=-1}} =\frac{15}{4}\frac{(1+\alpha_1-|\beta_1|)(1+\alpha_2-|\beta_2|)(1+\alpha_3-|\beta_3|)}{(1-\frac{1}{2}(|\beta_1|+|\beta_2|+|\beta_3|))^5}. \tag{97}$$

The inverse transformation (95) reads

$$\omega=\tilde{\omega}\Big(1-\frac{1}{2}\Big(\Big|\rho-\frac{\sigma}{2}\Big|+\Big|\rho+\frac{\sigma}{2}\Big|+|\sigma|\Big)\Big)-\frac{1}{2}\Big(\Big|\rho-\frac{\sigma}{2}\Big|+\Big|\rho+\frac{\sigma}{2}\Big|-|\sigma|\Big);$$
$$\nu=\tilde{\nu}\Big(1-\frac{1}{2}\Big(\Big|\rho-\frac{\sigma}{2}\Big|+\Big|\rho+\frac{\sigma}{2}\Big|+|\sigma|\Big)\Big)+\frac{1}{2}\Big(\Big|\rho-\frac{\sigma}{2}\Big|-\Big|\rho+\frac{\sigma}{2}\Big|\Big). \tag{98}$$

This allows to easily compute the integrals occurring in the calculation of $(N-n,n)$-th Mellin moments $(N\geq n\geq 0)$ in (w,v) of πN TDAs:

$$\int_{-1+|\sigma|}^{1-|\rho-(\sigma/2)|-|\rho+(\sigma/2)|}d\omega \times\int_{-1+|\rho-(\sigma/2)|+((1+\omega)/(2))}^{1-|\rho+(\sigma/2)|-((1+\omega)/(2))}d\nu\,\omega^{N-n}\nu^n h(\sigma,\rho,\omega,\nu). \tag{99}$$

In principle one may also think of a more intricate profile function. In fact any particular function $\Phi(\tilde{\omega},\tilde{\nu})$ normalized according to

$$\int_{-1}^{1}d\tilde{\omega}\int_{-((1-\tilde{\omega})/(2))}^{((1-\tilde{\omega})/(2))}d\tilde{\nu}\Phi(\tilde{\omega},\tilde{\nu})=1 \tag{100}$$

will define some profile function $h(\sigma,\rho,\omega,\nu)$ after the substitution (95).[5] E.g. taking $\Phi(\tilde{\omega},\tilde{\nu})\sim(\tilde{\omega}+1)^{b_1}\times(\tilde{\nu}+\frac{1-\tilde{\omega}}{2})^{b_2}(-\tilde{\nu}+\frac{1-\tilde{\omega}}{2})^{b_3}$ would lead to the natural generalization of the b parameter dependent Radyushkin's profile familiar for usual GPDs.

It is interesting also to consider the most simple possible profile with no distortion in (ω,ν) directions:

$$\Phi(\tilde{\omega},\tilde{\nu})=\delta(\tilde{\omega})\delta(\tilde{\nu}). \tag{101}$$

[5]However, one has to make certain assumptions on the end-point behavior of the function $f(\sigma,\rho)$ to which πN TDA is reduced in the limit $\xi\to 0$.

Contrary to the case of usual GPDs for which the counterpart of the profile (101) leads to ξ-independent ansatz the resulting TDA preserves the minimal necessary ξ dependence. Indeed $\tilde{\omega}=0$ and $\tilde{\nu}=0$ does not imply $\omega=0$ and $\nu=0$ and hence the ξ-dependence introduced through two δ-functions in (58) is preserved and generates the proper ξ-dependent domain of definition for the resulting πN TDA (20). Unfortunately, the model with the profile (101) turns out to be pathological since it leads to πN TDAs which are not continuous at the crossover lines $v=\pm\xi'$ and $w=-\xi$ separating ERBL-like and DGLAP-like type I, II domains. This makes impossible the calculation of the amplitude of the hard exclusive process in question given by convolution of πN TDA with the appropriate hard part (see [15]). Indeed the imaginary part of the corresponding amplitude is given by the values of πN TDA at the crossover lines $v=\pm\xi'$ and $w=-\xi$.

For the moment as a toy model we are going to employ the factorized ansatz (88) with the profile function (96). It is a good point now to discuss a possible model for the function $f(\sigma,\rho)$ that is the second ingredient of the factorized ansatz (88). In the limit $\xi\to 0$ πN TDA reduces to this function:

$$H(w,v,\xi=0)=f(w,v). \tag{102}$$

The requirements of convergence of integrals (81)–(87) for πN TDA impose some restriction on the behavior of the function $f(\sigma,\rho)$ on the border of its domain of definition. It turns out that $f(\sigma,\rho)$ should vanish at least as a certain power of the relevant variables at the borders of its domain of definition. Thus for the function $f(\sigma,\rho)$ we suggest the following simple form:

$$f(\sigma,\rho)=\theta(-1\leq\sigma\leq 1)\theta\Big(-1+\frac{|\sigma|}{2}\leq\rho\leq 1-\frac{|\sigma|}{2}\Big)\times\frac{40}{47}(1-\sigma^2)\Big((\rho-1)^2-\frac{\sigma^2}{4}\Big)\Big((\rho+1)^2-\frac{\sigma^2}{4}\Big). \tag{103}$$

In terms of the initial spectral parameters β_1, β_2, β_3 satisfying $\beta_1+\beta_2,+\beta_3=0$ (103) can be rewritten as

$$f(\beta_1,\beta_2,\beta_3)|_{\beta_1+\beta_2+\beta_3=0}=\frac{40}{47}\prod_{i=1}^{3}\theta(|\beta_i|\leq 1)(1-\beta_i^2). \tag{104}$$

The function $f(\sigma,\rho)$ vanishes on the border of its domain of definition and is normalized according to

$$\int_{-1}^{1}d\sigma\int_{-1+(|\sigma|/2)}^{1-(|\sigma|/2)}f(\sigma,\rho)=1. \tag{105}$$

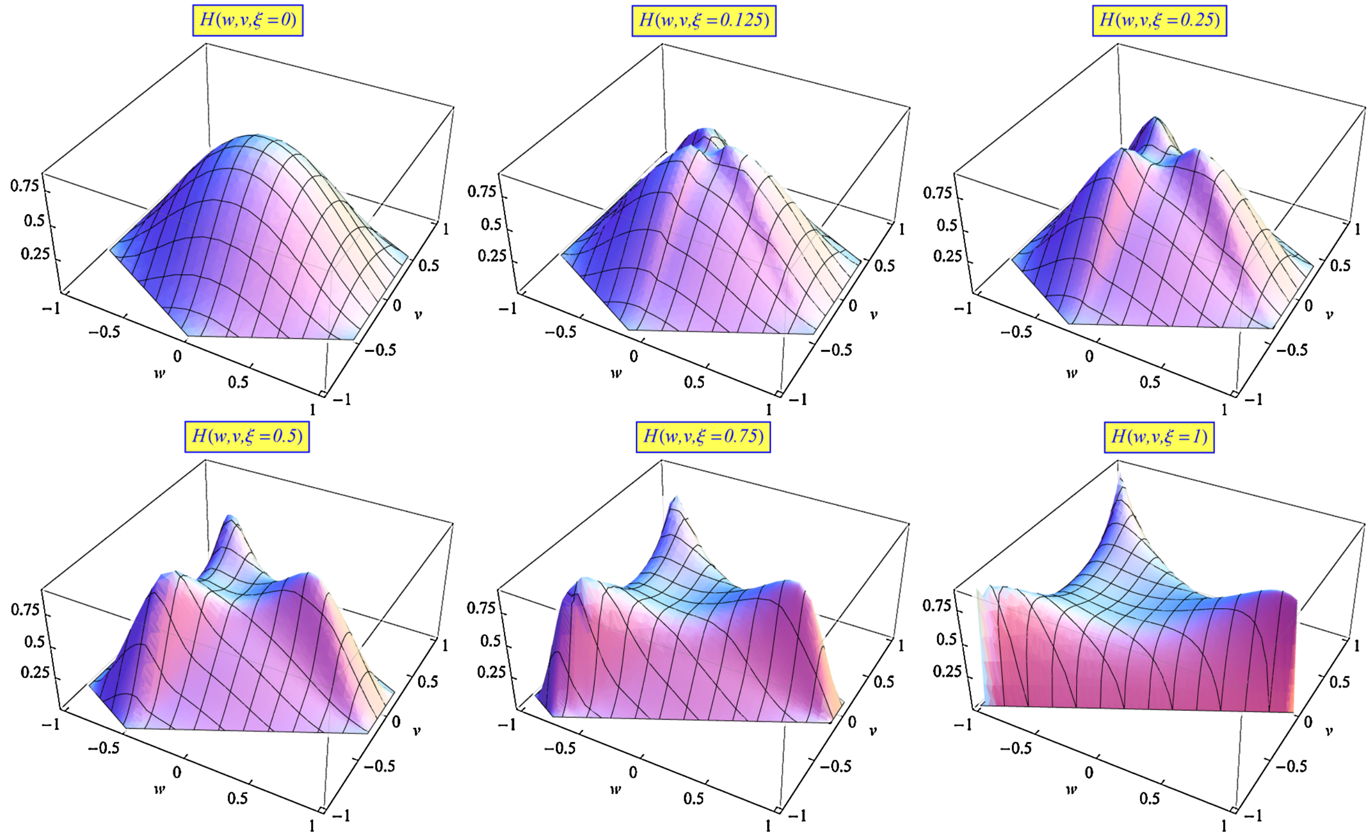

FIG. 7 (color online). The contribution into πN TDA $H(w_3, v_3, \xi) \equiv H(w, v, \xi)$ as a function of w and v for different values of ξ computed using the factorized ansatz (88) with the profile function (96) and $f(\sigma, \rho)$ given by (103).

Let us stress that we employ the normalization (105) only for our toy model. Advanced modelling of πN TDAs aiming the quantitative description of the physical observables would certainly require more complicated form of $f(\sigma, \rho)$.

The normalization for the nucleon to pion TDAs can be derived either from the soft pion limit or from the lattice calculations of several first Mellin moments of πN TDAs or from the comparison with the results of [18]. On the other hand it can be computed considering the light baryon exchange contributions into the Mellin moments of πN TDAs using the phenomenological values say of $g_{\pi NN}$ and $g_{\pi N\Delta}$ couplings. The normalization can also in principle be established directly form the experimental measurements of the cross-section once the scaling behavior would be found reasonable.

On Fig. 7 we show the results of the calculation of the contribution $H(w_3, v_3, \xi) \equiv H(w, v, \xi)$ as a function of w and v for different values of ξ computed with the help of the factorized ansatz (88) with the profile (96) and $f(\sigma, \rho)$ given by the toy model (103).

Note that for $\xi = 1$ the TDA $H(w, v, \xi)$ does not vanish at the corners of its domain of definition. This is potentially dangerous since this may lead to the break up of the factorization property of the hard exclusive process in question. Fortunately this problem is an artefact of our oversimplified toy model (104) for the forward limit of πN TDA. It was checked that taking $f(\sigma, \rho)$ that vanishes quadratically at the borders of the domain of definition

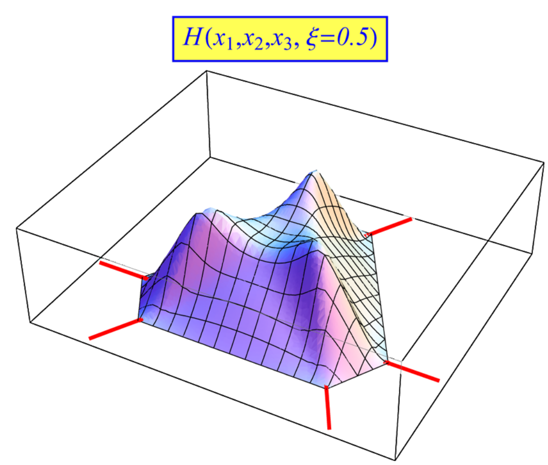

FIG. 8 (color online). πN TDA $H(x_1, x_2, x_3, \xi)$ as a function of x_1, x_2 and x_3 $(x_1 + x_2 + x_3 = 2\xi)$ for $\xi = 0.5$ in barycentric coordinates. By thick solid lines we show the continuation of the edges of the equilateral triangle that border the ERBL-like domain cf. Fig. 3.

$$f(\beta_1, \beta_2, \beta_3)|_{\beta_1+\beta_2+\beta_3=0} = \frac{4410}{3167}\prod_{i=1}^{3}\theta(|\beta_i| \le 1)(1-\beta_i^2)^2 \qquad (106)$$

leads to a vanishing πN TDA at the corners of its domain of definition for $\xi = 1$.

On Fig. 8 we show πN TDA $H(x_1, x_2, x_3, \xi)$ for $\xi = 0.5$ as a function of three dependent light-cone momentum fractions x_1, x_2 and x_3 ($x_1 + x_2 + x_3 = 2\xi$) in the barycentric coordinates. By thick solid lines we show the continuation of the edges of the equilateral triangle which form the ERBL-like domain cf. Fig. 3.

VII. CONCLUSIONS

The nonperturbative part of hard processes involving hadrons is encoded in various universal partonic distributions (parton distribution functions, fragmentation functions, distribution amplitudes and their generalizations). Waiting for a complete understanding of the dynamics of quark and gluon confinement in hadrons, one should model these distributions in agreement with general requirements of the underlying field theory such as Lorentz invariance and causality. Spectral representation of hadronic matrix elements offers an elegant way to address this program. The double distribution representation for GPDs became the basis for various successful phenomenological GPD models.

In this paper we introduced the notion of quadruple distributions and constructed the spectral representation for the transition distribution amplitudes involving three parton correlators which arise in the description of baryon to meson transitions. We also generalized Radyushkin's factorized ansatz for the case of quadruple distributions and provided an explicit expression for the corresponding profile function. Analogously to the case of GPDs the shape of the corresponding profile function is supposed to be fixed by the asymptotic form of the nucleon distribution amplitude. Our model also requires the knowledge of nucleon to meson TDAs in the forward limit as input quantities. Contrarily to the GPD case, the nucleon to meson TDAs suffer from the fact that there is no illuminating forward limit. This problem requires further investigation. For a moment, we suggest to employ a simple shape of nucleon to meson TDAs in the forward limit assuming that they are fixed by their behavior at the borders of their domain of definition. Our construction opens the way to quantitative modeling of baryon-meson and baryon-photon TDAs in their complete domain of definition.

Let us emphasize that for the moment we have not included any D-term like contributions to the spectral representation of the nucleon to meson TDAs in terms of quadruple distributions. Indeed the results of [36] and of Chapter 3.8 of [27] give us confidence that the eventual D-term like contributions to TDAs can be included by means of complementing the spectral density in (39) with additional terms proportional to powers of ξ. The subsequent analysis can be performed according to the same pattern.

Let us also point out that our method can be generalized for the case of 4-quark correlators important for the description of higher twist contributions.

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APPENDIX A: A USEFUL CONSTRAINED INTEGRAL

Let us consider the constrained triple integral

$$I(a,b,c) = \int_{-a}^{a} d\alpha_1 \int_{-b}^{b} d\alpha_2 \int_{-c}^{c} d\alpha_3 \delta(\alpha_1+\alpha_2+\alpha_3+1) \times f(\alpha_1,\alpha_2,\alpha_3), \qquad (A1)$$

where $a \ge 0; a \le 1,\ b \ge 0; b \le 1,\ c \ge 0; c \le 1$. We introduce the natural coordinates ω_3 and ν_3:

$$\alpha_1 = \nu_3 + \frac{-1-\omega_3}{2}; \quad \alpha_2 = -\nu_3 + \frac{-1-\omega_3}{2}; \quad \alpha_3 = \omega_3. \qquad (A2)$$

In the natural coordinates ω_3 and ν_3 the integration in (A1) is over the intersection of three stripes:

$$-c \le \omega_3 \le c; \qquad -a + \frac{1+\omega_3}{2} \le \nu_3 \le a + \frac{1+\omega_3}{2}; \quad -b - \frac{1+\omega_3}{2} \le \nu_3 \le b - \frac{1+\omega_3}{2}. \qquad (A3)$$

One may check that for $a \ge b$ the integral (A1) can be rewritten as

$$\begin{aligned} I(a,b,c) = {} & \int_{-1-a-b}^{-1-a+b} d\omega_3 \theta(\omega_3+c)\theta(c-\omega_3) \times \int_{-b-((1+\omega_3)/(2))}^{a+((1+\omega_3)/(2))} d\nu_3 f(\omega_3,\nu_3) \\ & + \int_{-1-a+b}^{-1+a-b} d\omega_3 \theta(\omega_3+c)\theta(c-\omega_3) \times \int_{-b-((1+\omega_3)/(2))}^{b-((1+\omega_3)/(2))} d\nu_3 f(\omega_3,\nu_3) \\ & + \int_{-1+a-b}^{-1+a+b} d\omega_3 \theta(\omega_3+c)\theta(c-\omega_3) \times \int_{-a+((1+\omega_3)/(2))}^{b-((1+\omega_3)/(2))} d\nu_3 f(\omega_3,\nu_3). \end{aligned} \qquad (A4)$$

Analogously for $b \geq a$ the integral (A1) can be rewritten as

$$\begin{aligned} I(a,b,c) = & \int_{-1-a-b}^{-1+a-b} d\omega_3 \theta(\omega_3 + c)\theta(c-\omega_3) \\ & \times \int_{-b-((1+\omega_3)/(2))}^{a+((1+\omega_3)/(2))} d\nu_3 f(\omega_3,\nu_3) \\ & + \int_{-1-b+a}^{-1+b-a} d\omega_3 \theta(\omega_3 + c)\theta(c-\omega_3) \\ & \times \int_{-a+((1+\omega_3)/(2))}^{a+((1+\omega_3)/(2))} d\nu_3 f(\omega_3,\nu_3) \\ & + \int_{-1+b-a}^{-1+a+b} d\omega_3 \theta(\omega_3 + c)\theta(c-\omega_3) \\ & \times \int_{-a+((1+\omega_3)/(2))}^{b-((1+\omega_3)/(2))} d\nu_3 f(\omega_3,\nu_3). \end{aligned} \quad (A5)$$

In order to be able to perform the integral (A1) we need to specify the intersection of three stripes (A3). The results (A4) and (A5) are obtained for arbitrary positive a, b and c. Let us now take into the account that

$$a = 1 - |\beta_1|; \quad b = 1 - |\beta_2|; \quad c = 1 - |\beta_3| \quad (A6)$$

with $|\beta_i| \leq 1$ and $\beta_1 + \beta_2 + \beta_3 = 0$.

(i) Let us first consider the case when β_is belong to the domain D_1 (44). In this domain we have $|\beta_1| = |\beta_2| + |\beta_3|$ and thus $a = b + c - 1$. So in the domain D_1 the following inequalities are respected:

$$0 \leq a \leq 1, \quad 0 \leq b \leq 1, \quad 0 \leq a - b + 1 \leq 1. \quad (A7)$$

One may check that these inequalities result in

$$c \geq -1 + a + b; \quad -c = -1 + b - a; \text{ and } b \geq a. \quad (A8)$$

Thus employing (A5) we get

$$I(a,b,c)|_{D_1} = \int_{-c}^{-1+a+b} d\omega_3 \times \int_{-a+((1+\omega_3)/(2))}^{b-((1+\omega_3)/(2))} d\nu_3 f(\omega_3,\nu_3). \quad (A9)$$

(ii) Analogously, in the domain D_2 we have $|\beta_2| = |\beta_1| + |\beta_3|$ and thus $b = a + c - 1$. The inequalities

$$0 \leq a \leq 1, \quad 0 \leq b \leq 1, \quad 0 \leq b - a + 1 \leq 1 \quad (A10)$$

result in

$$c \geq -1 + a + b; \quad -c = -1 + a - b; \text{ and } a \geq b. \quad (A11)$$

Employing (A4) we get

$$I(a,b,c)|_{D_2} = \int_{-c}^{-1+a+b} d\omega_3 \times \int_{-a+((1+\omega_3)/(2))}^{b-((1+\omega_3)/(2))} d\nu_3 f(\omega_3,\nu_3). \quad (A12)$$

(iii) Finally, let us consider the case when β_i belong to the domain D_3. In this domain we have $|\beta_3| = |\beta_1| + |\beta_2|$ and hence $c = a + b - 1$. Thus in the domain D_3 the following inequalities are respected:

$$0 \leq a \leq 1, \quad 0 \leq b \leq 1, \quad 0 \leq a + b - 1 \leq 1. \quad (A13)$$

One may check that in this domain

$$-c \geq -1 + b - a; \quad -c \geq -1 + a - b; \quad c = -1 + a + b. \quad (A14)$$

Thus independently of $a \geq b$ or $a \leq b$ the integral over the intersection of three stripes (A4) or (A5) is again reduced to

$$I(a,b,c)|_{D_3} = \int_{-c}^{-1+a+b} d\omega_3 \times \int_{-a+((1+\omega_3)/(2))}^{b-((1+\omega_3)/(2))} d\nu_3 f(\omega_3,\nu_3). \quad (A15)$$

APPENDIX B: CASE $\xi < 0$

For completeness in this Appendix we present the result for πN TDA $H(w, v, \xi)$ in the ERBL-like and DGLAP-like type I and II domains for the case $-1 \leq \xi < 0$ which is useful e.g. for $\bar{N}N \to \pi\gamma^*$ in the forward region [16].

(i) For w and v outside the domain $-1 \leq w \leq 1$ and $-1 + |\xi - \xi'| \leq v \leq 1 - |\xi - \xi'|$ the integral vanishes.

(ii) For $w \in [-1; \xi]$ and $\xi' \leq v \leq 1 - \xi' + \xi$ (DGLAP-like type II domain):

$$\begin{aligned} H(w,v,\xi) = & \frac{1}{\xi^2} \int_{((w+\xi)/(1-\xi))}^{((2(v\xi-\xi'))/(1-\xi^2))} d\sigma \\ & \times \int_{(\sigma/2)+((v+\xi')/(1+\xi))}^{-(\sigma/2)+((v-\xi')/(1-\xi))} d\rho \\ & \times F\left(\sigma, \rho, \frac{w-\sigma}{\xi}, \frac{v-\rho}{\xi}\right). \end{aligned} \quad (B1)$$

(iii) For $w \in [-1;\xi]$ and $-\xi' \le v \le \xi'$ (DGLAP-like type I domain):

$$H(w,v,\xi) = \frac{1}{\xi^2}\int_{((w+\xi)/(1-\xi))}^{((w-\xi)/(1+\xi))} d\sigma \times \int_{(\sigma/2)+((v+\xi')/(1+\xi))}^{-(\sigma/2)+((v-\xi')/(1+\xi))} d\rho \times F\left(\sigma,\rho,\frac{w-\sigma}{\xi},\frac{v-\rho}{\xi}\right). \quad \text{(B2)}$$

(iv) For $w \in [-1;\xi]$ and $-1+\xi'-\xi \le v \le -\xi'$ (DGLAP-like type II domain):

$$H(w,v,\xi) = \frac{1}{\xi^2}\int_{((w+\xi)/(1-\xi))}^{-((2(v\xi+\xi'))/(1-\xi^2))} d\sigma \times \int_{(\sigma/2)+((v+\xi')/(1-\xi))}^{-(\sigma/2)+((v-\xi')/(1+\xi))} d\rho \times F\left(\sigma,\rho,\frac{w-\sigma}{\xi},\frac{v-\rho}{\xi}\right). \quad \text{(B3)}$$

(v) For $w \in [\xi;-\xi]$ and $-\xi' \le v \le 1-\xi'+\xi$ (DGLAP-like type II domain) the result coincides with (B1).

(vi) $w \in [\xi;-\xi]$ and $\xi' \le v \le -\xi'$ (ERBL-like domain):

$$H(w,v,\xi) = \frac{1}{\xi^2}\int_{((w+\xi)/(1-\xi))}^{((w-\xi)/(1-\xi))} d\sigma \times \int_{(\sigma/2)+((v+\xi')/(1-\xi))}^{-(\sigma/2)+((v-\xi')/(1-\xi))} d\rho \times F\left(\sigma,\rho,\frac{w-\sigma}{\xi},\frac{v-\rho}{\xi}\right). \quad \text{(B4)}$$

(vii) $w \in [\xi;-\xi]$ and $-1+\xi'-\xi \le v \le \xi'$ (DGLAP-like type II domain): the result coincides with (B3).

(viii) $w \in [-\xi;1]$ and $v \in [-\xi';1-\xi+\xi']$ (DGLAP-like type I domain):

$$H(w,v,\xi) = \frac{1}{\xi^2}\int_{((w+\xi)/(1+\xi))}^{((2(v\xi-\xi'))/(1-\xi^2))} d\sigma \times \int_{(\sigma/2)+((v+\xi')/(1+\xi))}^{-(\sigma/2)+((v-\xi')/(1-\xi))} d\rho \times F\left(\sigma,\rho,\frac{w-\sigma}{\xi},\frac{v-\rho}{\xi}\right). \quad \text{(B5)}$$

(ix) $w \in [-\xi;1]$ and $v \in [\xi';-\xi']$ (DGLAP-like type II domain):

$$H(w,v,\xi) = \frac{1}{\xi^2}\int_{((w+\xi)/(1+\xi))}^{-((2(v\xi+\xi'))/(1-\xi^2))} d\sigma \times \int_{(\sigma/2)+((v+\xi')/(1-\xi))}^{-(\sigma/2)+((v-\xi')/(1-\xi))} d\rho \times F\left(\sigma,\rho,\frac{w-\sigma}{\xi},\frac{v-\rho}{\xi}\right). \quad \text{(B6)}$$

(x) $w \in [\xi;1]$ and $v \in [-1+\xi-\xi';\xi']$ (DGLAP-like type I domain):

$$H(w,v,\xi) = \frac{1}{\xi^2}\int_{((w+\xi)/(1+\xi))}^{((w-\xi)/(1-\xi))} d\sigma \times \int_{(\sigma/2)+((v+\xi')/(1-\xi))}^{-(\sigma/2)+((v-\xi')/(1+\xi))} d\rho \times F\left(\sigma,\rho,\frac{w-\sigma}{\xi},\frac{v-\rho}{\xi}\right). \quad \text{(B7)}$$

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